UNIVERSITY OF COPENHAGEN
DEPARTMENT OF MATHEMATICAL SCIENCES

MSc Project Outside Course Scope

Trace inequalities and Stahl's theorem

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Title and subtitle:	[Trace inequalities and Stahl's theorem]
Description:	The recently proven BMV conjecture (now known as Stahl's theorem) represents a class of Hermitian matrix Trace formulae in terms of the Laplace transform of a positive measure, and yields a number of interesting trace inequalities. In this project we detail a proof of the theorem and discuss corollaries using Bernstein's theorem for monotone functions.
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1 Bernstein's theorem on monotone functions

Definition 1. A function $f \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^\infty((0,\infty))$ is said to be completely monotone if $(-1)^n f^{(n)}(t) \geq 0$ for all $n \geq 0$ and $t > 0$.

Theorem 1.1. *f is completely monotone if and only if it is the Laplace transform of a finite Borel measure μ on $[0,\infty)$, i.e. if and only if*

$$\forall x \in [0,\infty), \quad f(x) = \int_{[0,\infty)} e^{-xt} d\mu(t).$$

Proof. We proceed with the forward implication. Let $t > 0$ be arbitrary. Since $(-1)^n g^{(n)}$ is non-negative and non-increasing, averaging and iterating yields:

$$\begin{aligned}
(-1)^n g^{(n)}(t) &\leq \frac{2}{t}\int_{t/2}^{t} (-1)^n g^{(n)}(t)\, dt \\
&= \frac{2}{t}|g^{(n-1)}(t) - g^{(n-1)}(t/2)| \\
&\leq \frac{2}{t}(|g^{(n-1)}(t)| + |g^{(n-1)}(t)|) \\
&\leq \frac{1}{t^n}\sum_{k=0}^{n} C_k g(t/2^k) \leq \frac{1}{t^n}\sum_{k=0}^{n} C_k g(0),
\end{aligned}$$

where $C_k \in \mathbb{R}$ for all k. In particular this means that $g^{(n)}(t) = o(1/t^n)$ for $t \to \infty$. Now, since $g(+\infty) := \lim_{t\to\infty} g(t)$ exists, iteratively integrating by parts (the boundary terms vanish due to the previous calculation) yields:

$$\begin{aligned}
g(x) - g(+\infty) &= -\int_x^\infty g'(t)\, dt \\
&= -(t-x)g'(t)\,|_x^\infty + \int_x^\infty (t-x)g''(t)\, dt \\
&= \frac{(-1)^{n+1}}{n!}\int_x^\infty (t-x)^n g^{(n-1)}(t)\, dt \\
&= \frac{(-1)^{n+1}}{(n-1)!}\int_{x/n}^\infty (1-\frac{x}{tn})^n (nt)^n g^{(n+1)}(nt)\, dt \\
&= \int_0^\infty \varphi_n(xt)\, d\sigma_n(t),
\end{aligned}$$

where $\varphi_n(x) = (1 - x/n)^n \mathbb{1}_{[0,n]}(x)$ and

$$\sigma_n(t) = \frac{1}{(n-1)!}\int_{1/t}^\infty (ns)^n |g^{(n+1)}(ns)|\, ds.$$

Now, $\sigma_n(t)$ is increasing for all $n \in \mathbb{N}$, and integration by parts again yields:

$$\lim_{t\to\infty} \sigma_n(t) = \frac{1}{(n-1)!}\int_0^\infty (ns)^n |g^{(n+1)}(ns)|\, ds = g(0) - g(+\infty) < \infty,$$

which means that σ_n has an uniform upper bound, and hence admits a convergence subsequence $\sigma_{n_k} \to \sigma$ to some non-negative function. Monotone convergence then yields:

$$g(x) - g(+\infty) = \lim_{k\to\infty} \int_0^\infty \varphi n_k(xt)\, d\sigma_{n_k}(t) = \int_0^\infty d\sigma(t),$$

and rearranging with $\mu := \sigma + g(+\infty)\delta_0$ completes the proof (the converse follows directly from Leibniz integral rule). $\square$

2 Stahl's theorem

Theorem 2.1. *Suppose that $A, B \in \mathbb{R}^{n\times n}$ are Hermitian matrices, with B positive definite. Then the function $f(t) = tr(e^{A-Bt})$ is the Laplace transform of a finite Borel measure on $[0,\infty)$, i.e.*

$$\forall t \in [0,\infty), \quad f(t) = \int_{[0,\infty)} e^{-xs} d\mu(s).$$

We may freely assume $B = \mathrm{diag}(b_n, \dots, b_1)$ is diagonal with decreasing eigenvalues $b_n > \dots > b_1 > 0$. Indeed, B is unitarily diagonalizable by some matrix T_0, and we easily see that e^{A-Bt} has the same eigenvalues as $T_0 * e^{A-Bt} T_0$. Furthermore, since $A - Bt$ is also Hermitian, it is unitarily diagonalizable by some matrix T_1, i.e. $T_1 * (A - Bt)T_1 * = \mathrm{diag}(\lambda_1(t), \dots, \lambda_n(t))$, with eigenvalues determined by the multi-variable equation $\det(\lambda I - A + tB) = 0$. Taking t out of the determinant and substituting $x = 1/t$, $y = \lambda/t$ we obtain an equation of the form

$$0 = \det(yI + B - xA) = \prod_{i=1}^n (y + b_j - xa_{j,j}) + O(x^2),$$

which implies that we may consider $\lambda(t)$ as a multi-valued function with n holomorphic branches; its natural domain is a Riemann surface S with n sheets laid over the Riemann sphere. Each branch λ_j satisfies $\lambda_j(t) = -b_j t + a_{j,j} + o(1/t)$ for $t \to \infty$. By definition of trace we have that:

$$f(t) = \sum_{j=1}^n e^{\lambda_j(t)}.$$

We need a technical result on these functions:

Lemma 2.2. *Let $\lambda_1, \dots, \lambda_n$ be the n holomorphic branches defined above, and let $g \in \mathcal{H}(\mathbb{C})$. Then*

$$h(z) := \sum_{j=1}^{n} g(\lambda_j)$$

is an entire function.

Proof. Indeed, the function holomorphic in an open set containing $\mathbb{R}$, and since it uniquely extends everywhere on $\mathbb{C}$, it follows from the monodromy principle that f is entire. $\square$

We now define an explicit formula for the positive measure in Stahl's theorem:

$$d\mu(s) := \left(\sum_{j=1}^{n} e^{a_{j,j}} \delta_{b_j}(s) + w(s) \right) ds,$$

where ds is the Lebesgue measure and

$$w(s) := -\sum_{j:b_j<s} \mathrm{res}_\infty e^{\lambda_j(\zeta)+s\zeta} = \frac{1}{2\pi i} \sum_{j:b_j<s} \int_C e^{\lambda_j(\zeta)+s\zeta} d\zeta,$$

for some sufficiently large circle C centered at the origin.
Proof of equality: First observe that Lemma 2.2 implies that $w(s) = 0$ for $s > b_n$ (and trivially also for $s < b_n$). We directly compute the Laplace transform of w:

$$\int_0^\infty e^{-st} w(s)\, ds = \sum_{k=1}^{n-1} \int_{b_k}^{b_{k+1}} e^{-st} w(s)\, ds =: \sum_{k=1}^{n-1} I_k(t).$$

Let $t > 0$ be fixed. Since λ_j is holomorphic in some open set containing $\mathbb{R}$ we may deform, without changing the value of $I_k(t)$, the circle C in such a way so that it does not contain the positive ray. Changing the order of summation and integration yields:

$$\begin{aligned}
\sum_{k=1}^{n-1} I_k(t) &= \sum_{k=1}^{n-1} \int_{b_k}^{b_{k+1}} \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)+s(\zeta-t)}\, d\zeta\, ds \\
&= \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)} \int_{b_j}^{b_n} e^{s(\zeta-t)}\, ds\, d\zeta \\
&= \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)} \big(e^{b_n(\zeta-t)} - e^{b_j(\zeta-t)}\big) \frac{d\zeta}{\zeta - t}.
\end{aligned}$$

Now, Lemma 2.2 again implies that:

$$\sum_{j=1}^{n-1} \int_{C'} e^{\lambda_j(\zeta)} e^{b_n(\zeta - t)} \frac{d\zeta}{\zeta - t} = 0.$$

Writing $\lambda_j(zeta) = -b_j\zeta + a_{j,j} + r_j(\zeta)$ where $r_j(\zeta) \to 0$ for $\zeta \to \infty$, applying the Cauchy integral formula on the unbounded domain defined by C' then yields for all $j = 1, \ldots, n-1$:

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{C'} e^{\lambda_j(\zeta)+b_j(\zeta-t)}\frac{d\zeta}{\zeta-t} &= -\frac{e^{a_{j,j}-b_jt}}{2\pi i}\int_{C'} e^{r_j(zeta)}\frac{d\zeta}{\zeta-t} \\
&= e^{a_{j,j}-b_jt}(e^{r_j(t)} - \lim_{\zeta\to\infty} e^{r_j(\zeta)}) \\
&= e^{\lambda_j(t)} - e^{-b_jta_{j,j}}.
\end{aligned}$$

Summation over j, and comparing to the measure μ defined above completes the proof of equality.
Proof of positivity: Recall first that we only need to show $w(s) \geq 0$, and that this is trivial for $s \notin [b_1, b_n]$. So let s be fixed with $b_k < s < b_{k+1}$. Recall the natural domain of the full derivative function λ is a Riemann surface S with n sheets over the Riemann sphere $\overline{\mathbb{C}}$. Let $\pi : S \to \overline{\mathbb{C}}$ be the projection map. The asymptotic expressions above imply that there exists an $R > 0$ such that the functions $\lambda_j(\zeta) + s\zeta = (s - b_j)\zeta + \ldots$ are holomorphic for $|\zeta| > R$ and with positive derivatives for $j \leq k$ and negative derivatives for $j > k$. In particular, $g(p) := \lambda(p) + s\pi(p)$ for $p \in S$ is a meromorphic 1-form. Increasing R as necessary, we may also assume that $\text{sign}(Im(\lambda_j(\zeta) + s\zeta)) = \text{sign}(Im\zeta)$ for $|\zeta| > R/2$ and $j \leq k$ and likewise $\text{sign}(Im(\lambda_j(\zeta) + s\zeta)) = -\text{sign}(Im\zeta)$ for $j > k$. We choose the circle $C = \{\zeta \in \mathbb{C} : |\zeta| = R\}$ to define the function w. Consider now the open sets:

$$\begin{aligned}
D^+ &= \{p \in S : |\pi(p)| < R,\, Im(\pi(p)) > 0,\, Im(\lambda(p) + s\pi(p)) > 0\}, \\
D^- &= \{p \in S : |\pi(p)| < R,\, Im(\pi(p)) < 0,\, Im(\lambda(p) + s\pi(p)) < 0\},
\end{aligned}$$

and the set $D = \text{Int}(\overline{D^-} \cup \overline{D^+})$. The pre-image $\pi^{-1}(C)$ consists of n disjoint circles $C_j \subseteq S$, which we label in accordance with the branch functions λ_j. It follows from our choice of $R > 0$ that $C_j \in \partial D$ for $j \leq k$ and $C_j \cap \overline{D} = \emptyset$ for $j > k$. Let $D_1 \subseteq D$ be a (connected) component of D. It is a Riemann surface of finite type, and hence its boundary can be parameterized by a family smooth curves $(\gamma_n)_{n=1}^{\infty}$. We need a technical result on these boundary curves:

Lemma 2.3. *No curve $\gamma \in (\gamma_n)_{n=1}^{\infty}$ can project into the open upper or lower half-plane.*

Proof. Suppose without loss of generality that *gamma* projects into the upper half-plane. The Cauchy-Riemann equations directly imply that $\text{Re}g$ has constant tangential derivative along γ, which contradicts the fact that $\text{Re}g$ is single-value on the closed curve γ. □

Since λ is holomorphic on D_1, the maximum principle ensures that there is a circle C_j with $C_j \subseteq D_1$. In order to prove positivity we assert that:

$$w(s) = \sum_{\substack{D_1 \subseteq D \\ D \text{ connected}}} \sum_{j : C_j \subseteq \partial D_1} \int_{C_j} e^{\lambda(p) + s\pi(p)} \, d\pi(p),$$

so it suffices to prove that each integral of this form is positive. The part of ∂D_1 that projects into C is exactly the circle C_j. The remainder is by definition projected into $\{\zeta : |\zeta| < R\}$, and since g is holomorphic in D_1 we have:

$$\int_{\partial D_1} e^{g(p)} d\pi = 0,$$

and it therefore suffices to show negativity of the part of ∂D_1 which projects into $\{\zeta : |\zeta| < R\}$. To show this, let γ be an arbitrary boundary curve which projects into $\{\zeta : |\zeta| < R\}$. By Lemma 2.3, γ intersects the real line and is therefore split into two pieces γ^+ and γ^- which project into the upper and lower half-plane, respectively. Furthermore, since γ is mapped to itself by the involution induced by complex conjugation, we also have that the pieces γ^+ and γ^- are symmetric. Writing out $e^g = e^{\mathrm{Re} g}(\cos \mathrm{Im} g + i \sin \mathrm{Im} g)$ and since $\mathrm{Im} g = 0$ and $\mathrm{Re} g$ is increasing by construction, we conclude that $t \mapsto e^{g(\gamma(t))}$ is real and increasing. Combining all this, and using integration by parts, we conclude that:

$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma e^{g(p)} \, d\pi(p) &= \frac{1}{2\pi i} \left(\int_{\gamma^+} + \int_{\gamma^-} \right) (d\xi(t) + i d\eta(t)) \\
&= \frac{1}{\pi} \int_{\gamma^+} e^{g(\gamma(t))} \, d\eta(t) \\
&= -\frac{1}{\pi} \int_{\gamma^+} \eta(t) \, d(e^{g(\gamma(t))}) < 0.
\end{aligned}$$

Summation over all curves γ which project into $\{\zeta : |\zeta| < R\}$ then yields the desired negativity, completing the proof.

The Bernstein theorem then implies the positivity $\mathrm{Tr}(e^{A-Bt}) \geq 0$, and similar inequalities for the (higher) derivatives. In particular, the trace derivative formula implies that if $AB = BA$ then we have $\mathrm{Tr}(Be^{A-Bt}) \geq 0$. It is worth noting that inequalities of this type has applications in eigenvalue problems in mathematical physics, although this is beyond the scope of this report.