Technische Universität Berlin

Combinatorial Properties of Rectangulations

Masterarbeit

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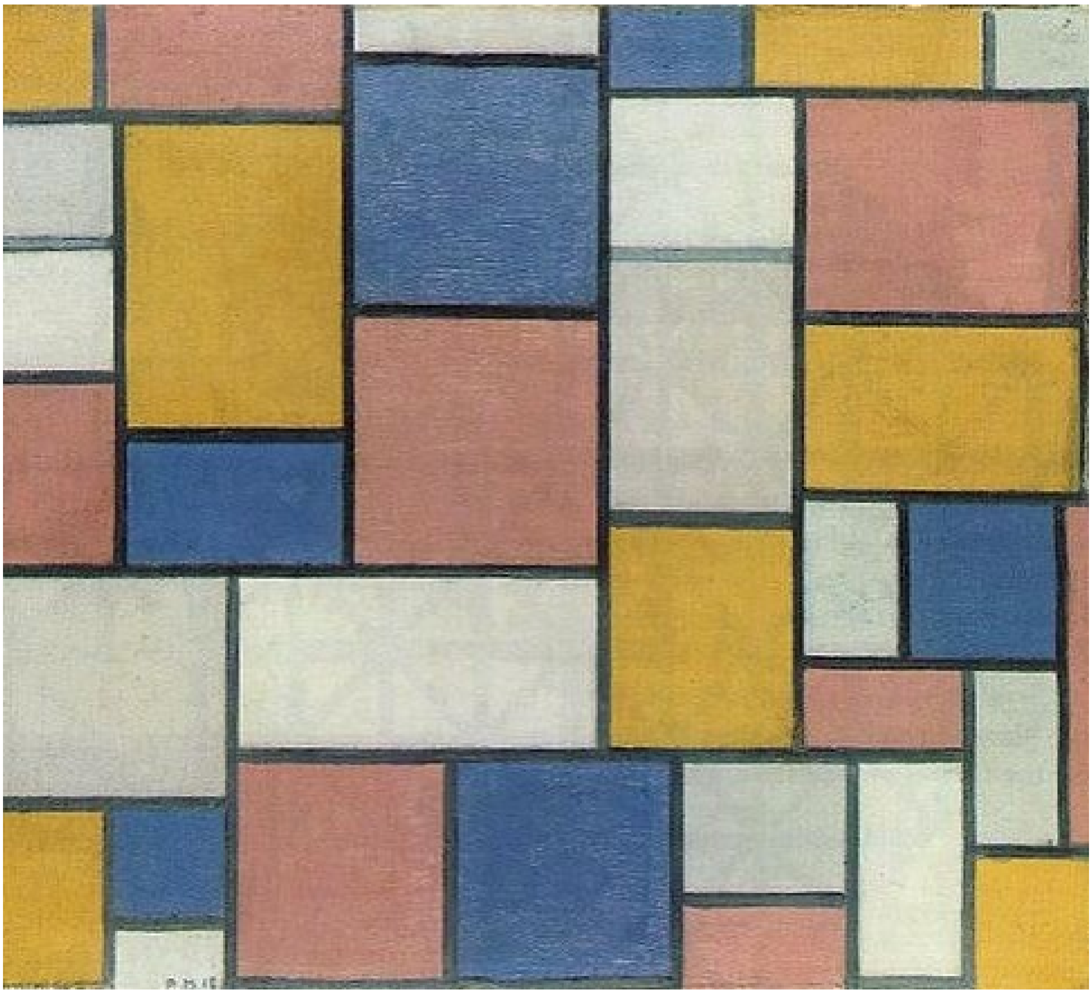

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Cover image: Piet Mondrian, *Composition with Color Planes and Gray Lines*, 1918 [1]

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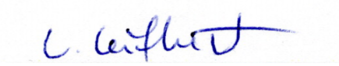
Erklärung der Selbstständigkeit

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den 30. August 2021



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Lilli Leifheit

Zusammenfassung in deutscher Sprache

Gegenstand dieser Arbeit sind Zerlegungen von Rechtecken in kleinere, im Inneren disjunkte Rechtecke, von denen nicht vier einen gemeinsamen Punkt enthalten. Solche Zerlegungen stehen in Bijektion zu diversen kombinatorischen Objekten. Die aus [1],[7],[10] bekannten Bijektionen zu Baxter-Permutationen, Zwillingspaaren von binären Bäumen und Tripeln von sich nicht schneidenden Gitterpfaden werden vorgestellt und bewiesen. Des Weiteren werden drei spezielle Klassen von Rechteckzerlegungen untersucht, die sich durch das Vermeiden gewisser Muster in ihren Segmenten auszeichnen. Es wird gezeigt, dass die Bijektion zu Baxter-Permutationen auf diese Klassen eingeschränkt werden kann, und Bijektionen zu gewissen Klassen von Muster-vermeidenden Permutationen erzeugt. Ein neues Ergebnis ist die Bijektion zwischen Rechteckzerlegungen, die „einseitig“ sind, und Permutationen, die die Muster 2-41-3, 3-14-2, 2-14-3, 3-41-2 vermeiden. Im letzten Teil der Arbeit werden Zählresultate für diese Klassen von Rechteckzerlegungen präsentiert: Wie viele kombinatorisch nicht äquivalente Zerlegungen in n Rechtecke mit den beschriebenen Eigenschaften gibt es jeweils?

Contents

Introduction and Motivation

Rectangulations are partitions of rectangles into axis-aligned, interior disjoint rectangles, no four of which intersect. They have applications in integrated circuit design, i.e., the arrangement of several components on a microchip, but also in geography: they can be used to create abstract maps, so called cartograms, where countries are represented by rectangles and only adjacency of countries is preserved. The area of the rectangle is often chosen to represent certain properties such as population size.
We will later see a sufficient condition for a rectangulation to allow arbitrary area-assignments to its rectangles [8].
Rectangulations are of combinatorial interest; they are known to be in bijection with a variety of combinatorial objects such as Baxter permutations [1], [2], twin pairs of binary trees [7] or triples of non-intersecting paths [10]. Many more bijections may be found in [10].
We will introduce and prove several of the afore mentioned bijections for rectangulations in the second chapter. Further, we will study three special classes of rectangulations that are characterized by certain pattern avoidance properties of their segments. As we will see, they are in bijection with certain pattern avoiding classes of permutations. The bijection between one-sided rectangulations and (2-41-3, 3-14-2, 2-14-3, 3-41-2) - avoiding permutations is possibly a new result. An overview of enumeration results obtained from these bijections is given in the fourth chapter.

Literature

We will use definitions, concepts and proofs from various sources and combine them with own proofs. Moreover we show that some concepts by different authors are equivalent. The main sources of this thesis are "A bijection between permutations and floorplans" by Ackerman, Barequet and Pinter, where the presented bijection between rectangulations and Baxter permutations can be found [1]; "Orders induced by segments in floorplan partitions and (2-14-3,3-41-2)-avoiding permutations" by Asinowski et al. - here, a new equivalence of rectangulations is introduced and several classes of pattern avoiding permutations are studied [2]; and "Bijections for Baxter families and related objects" by Felsner et al. [10]. An earlier proof of the bijection between rectangulations and Baxter

permutations using twin binary trees stems from "Stack words, standard tableaux and baxter permutations" by Dulucq and Guibert [7]. Further we use the work of Eppstein et al. [8] and the work of Wimer et al. [17] on area-universal rectangulations.
More research has been done in this field, there are results on flip graphs for rectangulations by Cardinal et al. who consider diagonal rectangulations [5] and a generalization of rectangulations to higher dimensions by Asinowski et al. [3]. There is a recent work on algorithmic generation of different classes of rectangulations by Merino and Mütze: "Combinatorial generation via permutation languages III, rectangulations" [13].

1 Basics

In this chapter, we introduce some of the main definitions and concepts developed in the literature on rectangulations.

1.1 Definitions

Definition 1.1 (rectangulation [5]) A *rectangulation* is a partition of a rectangle into axis-aligned subrectangles such that no four rectangles meet in one point.

One can interpret a rectangulation as a graph where the vertices are the corners of the rectangles, and the edges are the line segments between vertices. The inclusion-maximal line segments are often referred to as *segments* or *walls*. The condition that no four rectangles meet in one point is equivalent to the condition that segments do not cross. Segments only ever meet in T-joints. We say a segment *touches* another segment if its endpoint meets the interior of the other segment. A *subrectangulation* is a rectangular subgraph of a rectangulation. Rectangles are minimal subrectangulations. Rectangulations are also known as floorplans or rectangular layouts.
When indicating a particular rectangulation, we usually refer to its entire equivalence class under the equivalence relation introduced subsequently:

Definition 1.2 (neighbourhood relations [2]) A rectangle A in a rectangulation R is said to be a *left neighbour* of a rectangle B in R if there is a vertical segment S in R such that the right boundary of A and the left boundary of B are contained in S.
A is said to be an *above neighbour* of B if there is a horizontal segment S in R such that the lower boundary of A and the upper boundary of B are contained in S. Right and below neighbours are defined similarly.

Figure 1.1 depicts the two neighbourhood relations. It should be stressed that neighbouring rectangles need not intersect.

a) A left neighbour of B b) A above neighbour of B

Figure 1.1: Illustration of neighbourhood relations

1.2 Equivalences

In this thesis two rectangulations are considered equivalent if there is a labelling of their rectangles that maintains neighbourhood relations.

Definition 1.3 (equivalence [2]) Let R_1, R_2 be two rectangulations. R_1 and R_2 are *equivalent* ($R_1 \sim R_2$), if there is a bijection $f\colon R_1 \to R_2$ such that for all A, B rectangles of R_1,

$$A \text{ is a left neighbour of } B \Leftrightarrow f(A) \text{ is a left neighbour of } f(B)$$
$$\text{and } A \text{ is an above neighbour of } B \Leftrightarrow f(A) \text{ is an above neighbour of } f(B)$$

The *equivalence class* of R is $[R]$. Define $\mathcal{R}_n$ as the set containing all such equivalence classes $\mathcal{R}_n := \{[R] \mid R \text{ rectangulation with n rectangles }\}$

The brackets $[R]$ are usually omitted in the following.

Figure 1.2: Two equivalent rectangulations

The rectangulations depicted in Figure 1.2 are considered equivalent. This equivalence was introduced by Asinowski et al. in [2] and is referred to as *R-equivalence* and its equivalence classes as *mosaic floorplans*.
A coarser notion of equivalence was studied by Asinowski et al. who named it *S-equivalence*. It is defined via neighbourhood relations of segments instead of rectangles.

A stricter notion of equivalence (sometimes called *strong equivalence*) is obtained by the additional requirement that adjacency of rectangles be preserved. In our notion (called *weak equivalence* if we need to differentiate),  and  are equivalent, in the strong equivalence they are not.
An overview and classification of several other definitions of equivalence relations on rectangulations in the literature can be found in the Appendix.
If not stated otherwise, the equivalence we will be referring to is the equivalence defined in Definition 1.3. It has the advantage that each equivalence class contains a diagonal rectangulation.

Definition 1.4 (diagonal rectangulation [5]) A rectangulation is called *diagonal* if all its rectangles intersect the top-left-to-bottom-right diagonal.

Proposition 1.5 ([10], [5]) *For each rectangulation R there is a diagonal rectangulation R_d with $R_d \sim R$.*

We will only sketch the proof here; later, this result will be obtained as a corollary from Theorem 2.13.

Proof. In a diagonal rectangulation, since every rectangle intersects the top-left-to-bottom-right diagonal, all top right corners of rectangles are above the diagonal, and all bottom left corners are below it. There are four types of intersections of segments (vertices): $\vdash$, $\dashv$, $\perp$, $\top$. The top right corner of a rectangle can be either of type $\dashv$ or $\top$, the bottom left corner can be either of type $\perp$ or $\vdash$. Consequently, in a diagonal rectangulation, all $\dashv$-vertices and all $\top$-vertices are above the diagonal, and $\vdash$-vertices and all $\perp$-vertices are below the diagonal. In particular, "along every vertical segment, all $\dashv$-vertices are above all $\vdash$-vertices, and along every horizontal segment, all $\perp$-vertices are to the left of all $\top$-vertices" [13, p.6].
Conversely, if in R along every vertical segment, all $\dashv$-vertices are above all $\vdash$-vertices, and along every horizontal segment, all $\perp$-vertices are to the left of all $\top$-vertices, then we can find an equivalent diagonal rectangulation R_d by moving the segments without changing the order of the vertices along a segment nor the adjacency of the rectangles in R.
If there is an occurrence of $\substack{\vdash\\ \dashv}$ or $\top\!\perp$, i.e., there is a segment along which a $\vdash$-vertex is above a $\dashv$-vertex, or a $\top$-vertex is to the left of a $\perp$-vertex, it needs to be resolved by means of *wall slides* in order to obtain a diagonal rectangulation. A wall slide is a flip in a rectangulation that converts $\substack{\vdash\\ \dashv}$ into $\substack{\dashv\\ \vdash}$ and $\top\!\perp$ into $\perp\!\top$ by "exchanging the order of two vertices along a segment" [5, p.7] respectively.

Figure 1.3: Illustration of the two kinds of wall slides

The following construction assures that a wall slide is a local change: For any occurrence of $\dashv\!\vdash$, draw a horizontal line that extends from the left boundary of the rectangulation to the vertical segment and lies shortly below the left horizontal segment and but above any lower horizontal segments. It will thus be intersected only by vertical segment. Similarly, draw a second horizontal line that extends from the right boundary to the vertical segment and lies shortly above the right horizontal segment and below any higher horizontal segments. Then insert two blocks of the same height (say twice the distance between the two horizontal segments) into the rectangulation, replacing the two lines. Any vertical segment intersecting one of the lines is prolonged such that it crosses the corresponding block. For an illustration of this construction see Figure 1.4. We obtain an equivalent rectangulation where the order of the two horizontal segments along the vertical segment is inverted. Proceed similarly for an occurrence of $\top\!\!\perp$. R_d can be obtained from R in finitely many steps by resolving all occurrences of $\dashv\!\vdash$ and $\top\!\!\perp$.

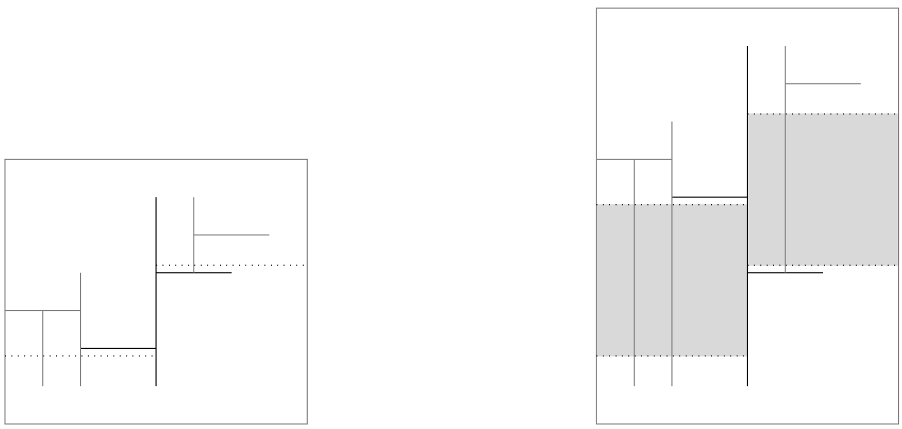


Figure 1.4: More detailed construction of a wall slide

□

Figure 1.2 shows two rectangulations that are weakly equivalent but not strongly equivalent. They can be transformed into one another with one wall slide. The second one is diagonal.

1.3 Orders on rectangulations

In this section, we fix a rectangulation $R \in \mathcal{R}_n$. Our goal is to construct two linear orders on the rectangles in R [2]. Let the relation A *is to the left of* B (A $\longleftarrow$ B) be the transitive and reflexive closure of A *is a left neighbour of* B. This is a partial order; we cannot possibly obtain circles like A *is to the left of* B *and* B *is to the left of* A.
Similarly, define the relations

$$\begin{aligned} A \longrightarrow B \ &: A \textit{ is to the right of } B \\ A \uparrow B \ &: A \textit{ is above } B \\ A \downarrow B \ &: A \textit{ is below } B \end{aligned}$$

as the transitive and reflexive closure of the corresponding neighbourhood relations.
Any two distinct rectangles in a rectangulation are in exactly one relation (left / right/ above/ below). Therefore left-or-below ($\swarrow$) and left-or-above ($\nwarrow$) are linear orders:

$$A \swarrow B \ :\Leftrightarrow \ A = B, \text{ or } A \longleftarrow B, \text{ or } A \downarrow B$$

$$A \nwarrow B \ :\Leftrightarrow \ A = B, \text{ or } A \longleftarrow B, \text{ or } A \uparrow B$$

Remark 1.6.

1. Since equivalence maintains neighbourhood relations, the order of the rectangles in equivalent rectangulations is identical.

2. For a diagonal rectangulation, the order $\nwarrow$ is identical to the order in which the rectangles intersect the the top-left-to-bottom-right diagonal.

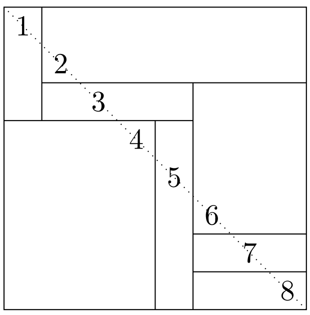


Figure 1.5: A diagonal rectangulation

Proposition 1.7 ([1, Observation 3.5]) *If two blocks are direct successors in one of the orders, then they share a common segment, i.e. they are neighbours.*

Proof. Let A, B be direct successors in the $\swarrow$-order, $A \swarrow B$. Suppose they are not neighbours: Either they are not comparable (this is not possible in a linear order) or they are in a relation - without loss of generality say A is to the left of B. Thus, there is a chain of left neighbours $A_1, A_2, ..., A_m$ relating A and B :

$$A \longleftarrow A_1 \longleftarrow A_2 \longleftarrow ... \longleftarrow A_m \longleftarrow B$$

But then $A_1, A_2, ..., A_m$ must be listed between A and B by the $\swarrow$-order, they are not successors. □

The converse of Remark 1.7 is of course not true; a rectangle may have more than four neighbours, but there can only be two direct neighbours (successor and predecessor) in each of the orders. Therefore neighbouring rectangles are not necessarily mapped to neighbouring letters in the corresponding permutation.

Remark 1.8 ([1, Observation 3.4]). If A is smaller than B in $\swarrow$ and in $\nwarrow$, then A is to the left of B. If A is smaller than B in $\swarrow$ and greater than B in $\nwarrow$, then A is below B.

Example 1.9. In Figure 1.6 we have

$$A \swarrow B \swarrow C \swarrow D \swarrow E \swarrow F \quad \text{and}$$
$$B \nwarrow F \nwarrow C \nwarrow A \nwarrow E \nwarrow D.$$

In particular,

$$C \swarrow E \quad \text{and} \quad C \nwarrow E \quad \text{thus} \quad C \longleftarrow E$$
$$A \swarrow C \quad \text{and} \quad C \nwarrow A \quad \text{thus} \quad A \downarrow C$$

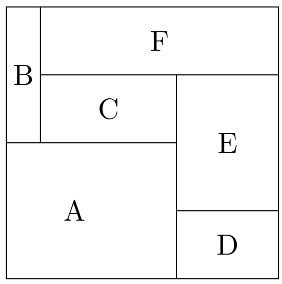


Figure 1.6

Ackerman et al. describe these orders as *block deletion orders* [1]:
Delete the rectangle in the top left hand corner and extend the adjacent rectangles to fill the void (either to the left or to the top, depending on the type). Repeat until only one rectangle remains. Record the order in which the rectangles are deleted. Incidentally this order coincides with the $\nwarrow$-order.
The $\swarrow$-order can alternatively be characterized as block deletion order from the bottom left hand corner.

Example 1.10. Figure 1.7 depicts a block deletion executed in the rectangulation from the previous example.

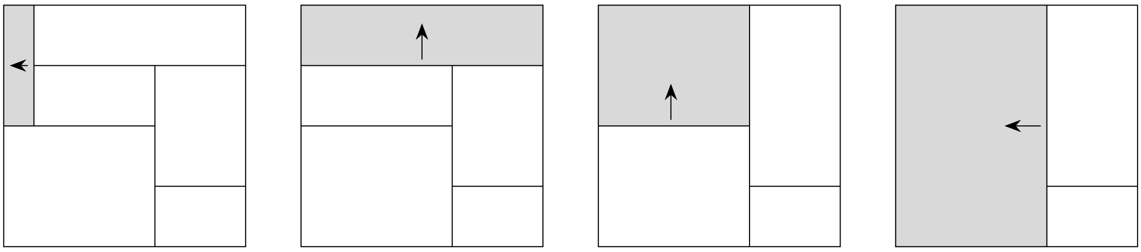


Figure 1.7: Block deletion from the top left hand corner

2 Bijections

In this chapter, some combinatorial objects are introduced and proven to be in bijection with rectangulations. The first bijection concerns a certain class of pattern avoiding permutations.

2.1 Baxter permutations

Definition 2.1 (pattern avoiding permutation [2]) A permutation $\pi = (a_1\, a_2\, \ldots\, a_n) \in \mathcal{S}_n$ *contains* a pattern $\tau = (b_1\ b_2\ \ldots\ b_k) \in \mathcal{S}_k$ if there are $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(a_{i_1}\ a_{i_2}\ \ldots\ a_{i_k})$ is order isomorphic to τ, i.e. $a_{i_l} < a_{i_m}$ if and only if $b_l < b_m$. Additionally, we use the dash notation to indicate whether the letters of the pattern are required to be adjacent: π contains a pattern $\tau = b_1 \ldots b_j - b_{j+1} \ldots b_k \in \mathcal{S}_k$ if it contains $\tau = (b_1\ b_2\ \ldots\ b_k)$ and whenever there is no dash between two letters b_j, b_{j+1} of τ then the corresponding letters $a_{i_j}, a_{i_{j+1}}$ are adjacent in π. A permutation π *avoids* a pattern τ if it does not contain τ.

Example 2.2.

1. A permutation π avoids the pattern 54321 if there are no four consecutive descents in π.

2. $\pi = (7\,6\,3\,1\,5\,2\,8\,4)$ contains the pattern 3-14-2 realized by the subsequence (7 2 8 4) and (3 1 5 2), and the pattern 2-41-3 realized by the subsequence (3 5 2 4).

Definition 2.3 (Baxter permutation [10]) A *Baxter permutation* is a permutation $\pi = (a_1\ a_2\ \ldots\ a_n) \in \mathcal{S}_n$ for which there are no $i, j, k \in [n],\ i < j,\ j+1 < k$, such that

$$a_j < a_k < a_i < a_{j+1} \quad \text{or} \quad a_{j+1} < a_i < a_k < a_j$$

The set of all Baxter permutations on $[n]$ is denoted by $\mathcal{B}_n$.

In the dash notation, Baxter permutations avoid 2-41-3 and 3-14-2. The number of Baxter permutations on $[n]$ is the n-th Baxter number $B(n)$. For example $B(4) = 22$, since for $n = 4$, the only non-Baxter permutations are (3 1 4 2) and (2 4 1 3). Chung et al. [6]

have shown that

$$B(n) = \sum_{r=0}^{n-1} \frac{\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}}{\binom{n+1}{1}\binom{n+1}{2}}.$$

At the end of this chapter we will see an alternative way of counting these numbers.

Definition 2.4 ($\Phi\colon \mathcal{R}_n \to \mathcal{S}_n$ [2]) Let $R_i,\ i = 1, \ldots, n$ be the rectangles of $R \in \mathcal{R}_n$ with $R_i \nwarrow R_j \Leftrightarrow i \le j$, then

$$R \mapsto \pi = (\pi_1\ \pi_2\ \ldots\ \pi_n) \text{ with } \pi_i \le \pi_j \Leftrightarrow R_i \swarrow R_j.$$

The mapping Φ can be interpreted as follows: For a rectangulation R, label the rectangles of R according to the order $\swarrow$, and then read the labels according to order $\nwarrow$.

Remark 2.5. Φ is well defined: The order of the rectangles is the same for each rectangulation of an equivalence class $R \in \mathcal{R}_n$ (cf. Remark 1).

Example 2.6. Applying Φ to the following rectangulation yields the permutation (1 6 3 2 5 4).

labelled according to $\swarrow$ labelled according to $\nwarrow$

Figure 2.1: Illustration of the mapping Φ

A mapping similar to Φ can be found in [2], [1], [5]. Our definition of Φ was adopted from Asinowski et al. [2], whereas the mappings described in [5] and [1] can be interpreted as labelling the rectangles in the order $\nwarrow$ first. Their bijections result in the inverse permutation - this is only plausible, since by exchanging the orders, they exchange the role of i and π_i. As a consequence, rectangulations that are reflected with respect to a horizontal axis yield inverse permutations, since this reflection too exchanges the two orders. Self-inverse Baxter permutations are obtained from horizontally symmetric rectangulations. Note that the inverse of a Baxter permutation is also Baxter since the forbidden patterns 2-41-3 and 3-14-2 are inverse to each other.

Example 2.7. Horizontal reflection results in the inverse permutation: The reflection of the previous example yields the permutation $\pi^{-1} = (1\ 4\ 3\ 6\ 5\ 2)$.

labelled according to $\nwarrow$ labelled according to $\swarrow$

Figure 2.2: Reflection of Figure 2.1

Proposition 2.8 *Under the mapping Φ from Definition 2.4, Remark 1.8 translates to: Ascents in permutations are the image of vertical segments, descents are the image of horizontal segments.*

Proof. An ascent in a permutation $\Phi(R) = \pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in \mathcal{S}_n$ consists of two consecutive letters π_i, π_{i+1} with $\pi_i < \pi_{i+1}$. Hence in the pre-image R of π under ϕ the corresponding rectangles labelled π_i, π_{i+1} satisfy $\pi_i \swarrow \pi_{i+1}$ (since $\pi_i \leq \pi_{i+1}$) and $\pi_i \nwarrow \pi_{i+1}$ (since $i \leq i+1$). Therefore $\pi_i \longleftarrow \pi_{i+1}$ (cf. Remark 1.8). Since they are successors in the $\nwarrow$-order (the indices are i and $i+1$), Remark 1.7 affirms that they share a common segment, and since π_i is to the left of π_{i+1}, the common segment is vertical.
Similarly, a descent consists of two letters π_i, π_{i+1} with $\pi_i > \pi_{i+1}$. Thus, $\pi_{i+1} \swarrow \pi_i$ and $\pi_i \nwarrow \pi_{i+1}$ and therefore $\pi_i \downarrow \pi_{i+1}$ and with Remark 1.7 they share a common horizontal segment. ☐

Lemma 2.9 ([1, Lemma 3.6]) *The permutation obtained by applying Φ to a rectangulation $R \in \mathcal{R}_n$ is a Baxter permutation on $[n]$.*

Proof. Let $R \in \mathcal{R}_n$ and $\pi = \Phi(R)$ the permutation obtained by applying Φ.
Suppose $\pi = (\pi_1\ \ldots\ \pi_n)$ is not Baxter. Then there are $i < j < k$ s.t. either $\pi_{j+1} < \pi_i < \pi_k < \pi_j$ or $\pi_j < \pi_k < \pi_i < \pi_{j+1}$. W.l.o.g. we find ourselves in the first situation. Choose π_i, π_k s.t. $\pi_k = \pi_i + 1$. Why do we always find such π_i, π_k? In π all values that are greater than π_{j+1} and smaller than π_j are either to the left of π_j or to the right of π_{j+1} and since there are π_i and π_k, both sides are non-empty. Thus, there need to be two consecutive values that lie on different sides of π_{j+1} and π_j. This implies (by Remark 2.8):

1. The rectangle labelled π_i is above π_{j+1} and left of π_j.
2. The rectangle labelled π_k is below π_j and right of π_{j+1}.
3. The rectangle labelled π_i is left of π_k and some segment s_1 meets both of them.
4. The rectangle labelled π_j is above π_{j+1} and some segment s_2 meets both of them.

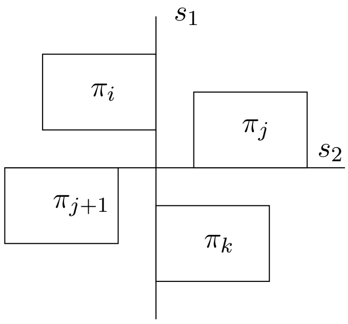


Figure 2.3

As we can see in Figure 2.3, s_1 and s_2 need to intersect. This yields a contradiction. The second case is treated similarly. □

We have shown that Φ maps rectangulations to Baxter permutations. The next step is to show that it is injective:

Lemma 2.10 ([1, Lemma 3.7]) Φ *is injective.*

Proof. [1] We proceed by induction on the number of rectangles n.
For $n = 1$ there is only one rectangulation and only one permutation, so Φ is clearly injective. Assume that Φ is injective on $\mathcal{R}_n$ for a fixed n. Let $R_1, R_2 \in \mathcal{R}_{n+1}$ with $\Phi(R_1) = \Phi(R_2) =: \pi$. Due to the definition of the $\swarrow$-order, the top right hand rectangle is labeled $n+1$ in both R_1 and R_2. Recall the block deletion from Example 1.10 and delete the rectangle $n+1$ in R_1 and R_2 with block deletion from the top right hand corner. Denote the resulting rectangulations by R_1^{-n+1}, R_2^{-n+1}. The order relations among the remaining rectangles in R_1^{-n+1} and R_2^{-n+1} are not altered by this operation: All rectangles are still adjacent to the same segments they were before, except the ones that were touching the segment that was deleted with $n+1$ – these are now adjacent to the boundary instead, and the only neighbour they have lost is $n+1$. Therefore it holds $\Phi(R_1^{-n+1}) = \Phi(R_2^{-n+1})$ and this is the permutation obtained be deleting $n+1$ in π. Now the induction hypothesis assures that $R_1^{-n+1} = R_2^{-n+1}$. Examine the way $n+1$ is situated in R_1 and R_2. The bottom left hand corner of the rectangle $n+1$ can be either $\vdash$ or $\perp$, but it must be identical in R_1 and R_2: else, choose a rectangle a to the left of $n+1$ and a rectangle b below $n+1$ in both rectangulations. In a corner of type $\vdash$ a would be to the left of b while for type $\perp$ a would be above b, which would result in different permutations. Also in both R_1 and R_2 the same sets of rectangles need to be to the left of $n+1$ and below $n+1$, or else $n+1$ would not be in the same position in $\Phi(R_1)$ and $\Phi(R_2)$. Consequently, R_1 and R_2 are identical locally around $n+1$ and they are identical when $n+1$ is deleted, so they are the same.

□

There exist several mappings from Baxter permutations to rectangulations. Here we present an algorithm by Ackerman et al. [1].

Definition 2.11 ($\Psi : \mathcal{B}_n \to \mathcal{R}_n$ [1]) The mapping is given by the following algorithm.

Let $\pi = (\pi_1 \ \dots \ \pi_n)$ be a Baxter permutation.

1. Draw a rectangle and label it π_1.
2. Construct an $n \times n$ grid within the rectangle.
3. **for** $i = 2$ to n **do**
4. **if** $\pi_i < \pi_{i-1}$ **then**
5. Slice the bottom-right rectangle by a horizontal segment at the i-th level
6. Name the new rectangle below the new segment π_i
7. **while** $\pi_i < \pi_k$, π_k being the rectangle to the left of π_i, **do**
8. Extend π_i leftwards at the expense of π_k
9. **else**
10. Slice the bottom-right rectangle by a vertical segment at the i-th level
11. Name the new rectangle to the right of the new segment π_i
12. **while** $\pi_i > \pi_k$, π_k being the rectangle above π_i, **do**
13. Extend π_i upwards at the expense of π_k

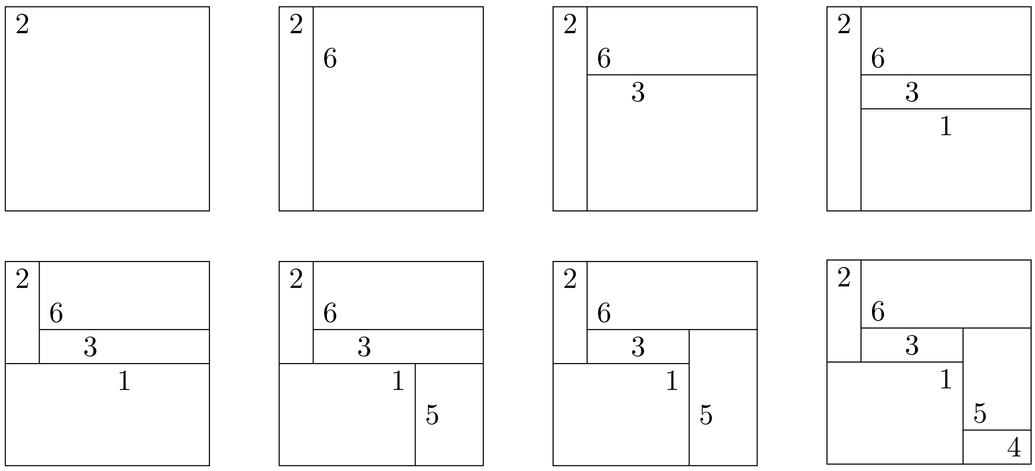


Figure 2.4: Applying the algorithm to $\pi = (2\ 6\ 3\ 1\ 5\ 4)$

Remark 2.12. The algorithm produces indeed a rectangulation with n blocks.

Theorem 2.13 ([1, Theorem 1]) *Φ and Ψ are inverse and a bijection between $\mathcal{B}_n$ and $\mathcal{R}_n$.*

Proof. [1] Let π be a Baxter permutation and $R = \Psi(\pi)$. Let $\pi' = \Phi(R)$. It is enough to show $\pi = \pi'$.
We remark that the order in which the rectangles are inserted by Ψ is the $\nwarrow$-order: each rectangle is inserted to the right or below the previous ones. Therefore the rectangles of R – if we pass them in the $\nwarrow$-order - are labelled $\pi_1, \ldots, \pi_n$.
It remains to show that the labels of the rectangles of R in the $\swarrow$-order are $1, 2, \ldots, n$.
We proceed by induction: Clearly, the block labelled 1 is smaller than all other blocks in the $\swarrow$-order: no other rectangle could have been inserted below it in the course of Ψ, and no rectangle can be left of it.
Suppose all blocks labelled $1, 2, \ldots, k$, for a fixed k, are visited by $\swarrow$ in the order corresponding to their labels. We show that the next rectangle in $\swarrow$ is the one labelled $k+1$. There are two cases: $k+1$ precedes k in π or k precedes $k+1$ in π. Assume the first case holds. Then

$$\pi = (\ldots \ k+1 \ \ A \ \ B \ \ k \ \ldots)$$

where A is a sequence of integers greater than $k+1$ and B is a sequence of integers smaller than k – both may be empty – because of π being a Baxter permutation. According to the algorithm, $k+1$ is inserted first, then to its right all elements of A, followed by all elements of B under a horizontal segment, and then k, which is inserted to the right of the last element in B.

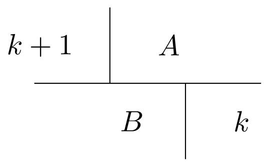


Figure 2.5

Let a be the rightmost element in A, b the leftmost element in B if the respective sequences are non-empty. As to the surroundings of $k+1, A, B, k$ there are several possibilities: the top right hand corner of k could be either as shown in Figure 2.6(a) or as shown in Figure 2.6(b).
In the first case, the rectangle labelled c must have been inserted after k, so c must be after k in π, and it must be greater than k, for k and c are separated by a vertical segment, and smaller than a, or if A is empty, smaller than $k+1$, since it has not been extended upwards. The latter is obviously impossible for an integer. If A is not empty, and B is not empty, then $k+1, a, b, c$ form an occurrence of the forbidden pattern 2-41-3 in π. If

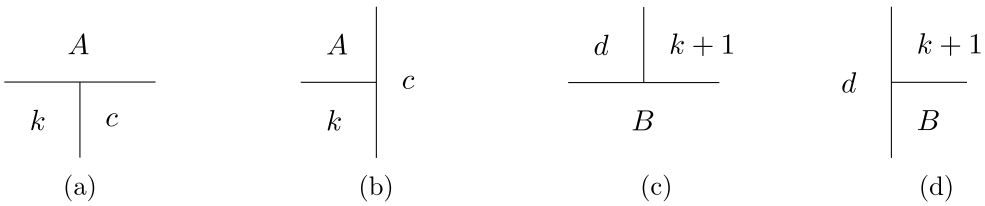


Figure 2.6

B is empty, $k+1, a, k, c$ form an occurrence of the forbidden pattern 2-41-3.
We conclude that the case of Figure 2.6(b) must hold.
Similarly, the bottom-left-hand corner of $k+1$ could be either Figure 2.6(c) or Figure 2.6(d): In case 2.6(c), the rectangle labelled d was inserted before $k+1$, so it must be located before $k+1$ in π. Moreover, d it must be smaller than $k+1$, for the segment is vertical, and greater than b, or if B is empty, greater than k, since it has not been extended upwards. The latter is again impossible for an integer. If B is not empty, and A is not empty, then d, a, b, k form an occurrence of the forbidden pattern 2-41-3 in π. If A is empty, $d, k+1, b, k$ form an occurrence of the forbidden pattern 2-41-3.
We conclude that case 2.6(d) must hold. Thus, $k+1, A, B, k$ are enclosed by two parallel vertical segments:

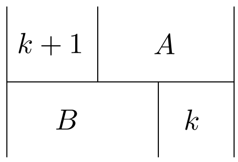


Figure 2.7

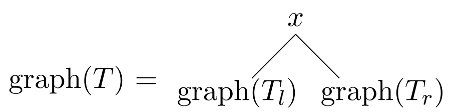
Therefore, we can now ascertain that $k+1$ follows k in the $\swarrow$-order, completing the induction.
The case k precedes $k+1$ is analogous and therefore omitted. $\square$

Remark 2.14. This proof shows that each equivalence class of rectangulations contains one rectangulation – namely the one constructed by the algorithm Ψ – which is diagonal: Each rectangle intersects the $\nwarrow$- diagonal.

2.2 Twin pairs of binary trees

There is another way to describe the mapping from Baxter permutations to rectangulations, and it uses twin pairs of binary trees.

Definition 2.15 (Binary tree) The empty set and a single node are *binary trees* as well as any ordered triple (T_l, x, T_r) where T_l, T_r are binary trees, and x is a single node. T_l is called the *left subtree* or *left child* of x, T_r is called the *right subtree* or *right child* of x and x is called *root*. The *graph of a binary tree* $T = (T_l, x, T_r)$ is defined recursively as well:



The graph of a binary tree is a circle-free connected graph where every node has degree at most 3, the root has degree at most 2. Nodes with degree 1 are called *leaves*, the others *inner nodes*. In the following we will not always distinguish between a binary tree and its graph.

Definition 2.16 (Canopy [5]) A *canopy* of a binary tree T with $n+2$ leaves is a 0-1-word obtained from its leaf pattern by labelling each leaf from left to right, except the first and the last one:

$$\operatorname{cnp}(T)_i = \begin{cases} 1, & \text{if the } (i+1)\text{st leave of } T \text{ is a left leaf} \\ 0, & \text{if the } (i+1)\text{st leave of } T \text{ is a right leaf} \end{cases} \quad , \ i = 1, ..., n$$

This is the same concept as the *reduced fingerprint* in [10].

Definition 2.17 (Twin pair of binary trees [7]) A pair of binary trees (T, S) with the same number of leaves $n+2$ is said to be *twin* if their canopies are the reverse of each other:

$$T \text{ and } S \text{ are twin } \Leftrightarrow \operatorname{cnp}(T)_i = \operatorname{cnp}(S)_{n+1-i} \quad , \ i = 1, \ldots, n$$

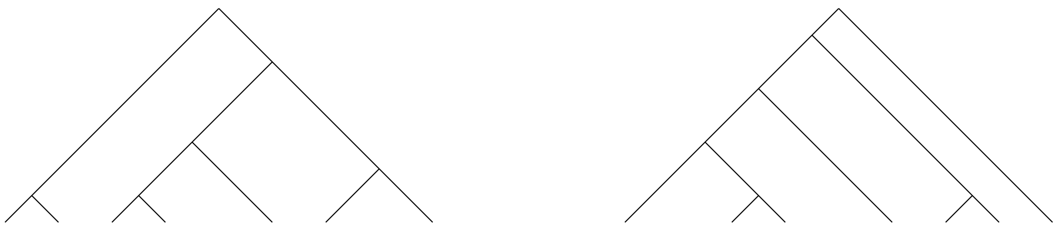


Figure 2.8: a binary tree with canopy 01001

Figure 2.9: a binary tree with canopy 10010

Figure 2.10: A twin pair of binary trees

There is a very straight forward bijection between rectangulations and twin pairs of binary trees. We have seen that each equivalence class of rectangulations contains a diagonal representative (cf. Proposition 1.5). We obtain a pair of binary trees from a rectangulation by cutting along the main diagonal. Vertices in the rectangulation are mapped to inner nodes, points of intersection with the diagonal to leaves. Since every rectangle intersects the diagonal, there are no cycles in the graphs on either side of the diagonal. Since segments only meet in T-joints, every node has degree at most 3. Left leaves in the upper tree touch left leaves in the lower tree, but the lower tree is upside down. Therefore the canopies are complementary.
The inverse mapping is given by glueing twin binary trees at their leaves such that the lower tree forms the bottom left hand corner and the upper tree forms the top right hand corner. As the canopies match, this will yield a rectangulation. Glueing left leaves to left leaves and right leaves to right leaves results in straight lines crossing the diagonal.

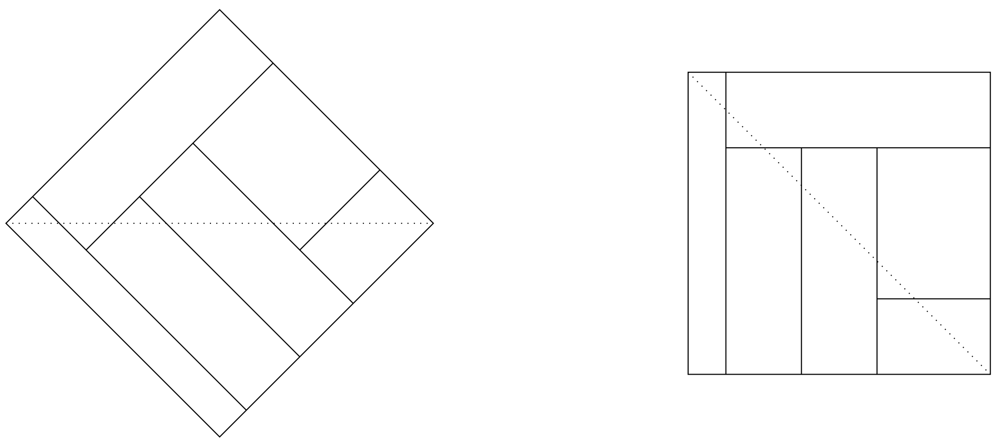


Figure 2.11: Glueing a twin pair of binary trees

If we attempt to glue a pair of trees with non-matching canopies, the result is not a rectangulation.

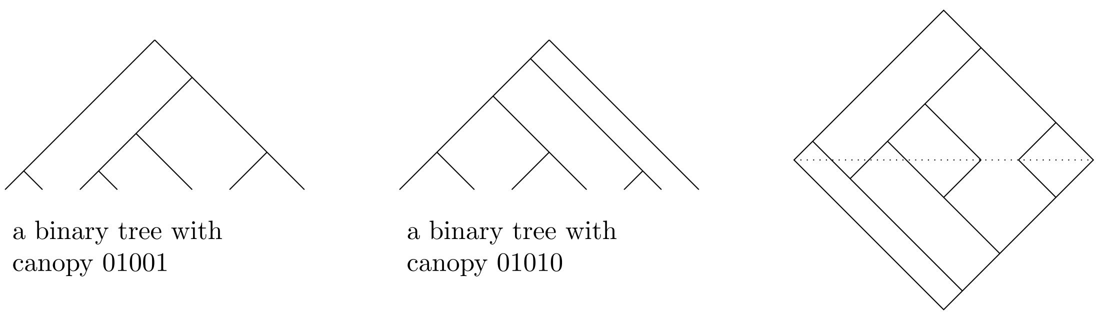


Figure 2.12: Glueing a non-twin pair of binary trees

We can endow a permutation with a twin pair of binary trees:

Definition 2.18 (maxtree, mintree [10]) Let $\pi \in \mathcal{S}_n$. The *maxtree* $\mathrm{Max}(\pi)$ is recursively defined:

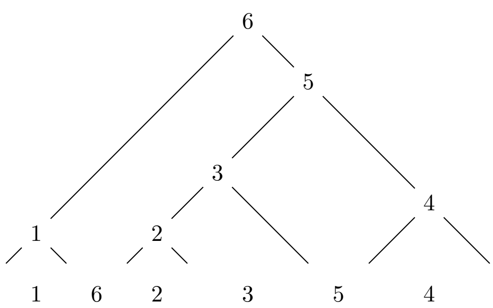
$$\mathrm{Max}(\pi) = \begin{cases} \text{unlabeled one-node tree} & \text{, for } \pi = \emptyset \\ \begin{array}{ccc} & x & \\ / & & \backslash \\ \mathrm{Max}(\pi_l) & & \mathrm{Max}(\pi_r) \end{array} & \text{, for } \pi = (\pi_l\ x\ \pi_r) \end{cases}$$

where x is the maximal element of π and π_l and π_r are the subsequences of π to the left and to the right of x. Similarly,

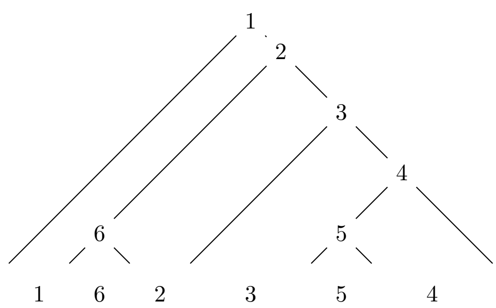
$$\mathrm{Min}(\pi) = \begin{cases} \text{unlabeled one-node tree} & \text{, for } \pi = \emptyset \\ \begin{array}{ccc} & x & \\ / & & \backslash \\ \mathrm{Min}(\pi_l) & & \mathrm{Min}(\pi_r) \end{array} & \text{, for } \pi = (\pi_l\ x\ \pi_r) \end{cases}$$

where x is the minimal element of π and π_l and π_r are the subsequences of π to the left and to the right of x.

Example 2.19. Let $\pi = (1\ 6\ 2\ 3\ 5\ 4)$, then $\mathrm{Max}(\pi)$ and $\mathrm{Min}(\pi)$ are:



$\mathrm{Max}(\pi)$



$\mathrm{Min}(\pi)$

Remark 2.20. ([10])

1. The max- and min-tree are full binary trees: every inner node has two children.
2. We can identify the gap between two adjacent leaves with the inner node above it – the least common ancestor of the two leaves.
3. Labelling the gaps between leaves with the label of the corresponding inner nodes gives rise to a permutation $(\pi_1\ \dots\ \pi_n)$.
4. The ith leaf v_i in $\mathrm{Max}(\pi)$ corresponds to the pair (π_{i-1}, π_i) and v_i is a left leaf if and only if (π_{i-1}, π_i) is a descent.

5. In $\mathrm{Min}(\pi)$ the ith leaf w_i is a left leaf if and only if (π_{i-1}, π_i) is an ascent.

Proposition 2.21 ([10, Proposition 6.2]) *For $\pi \in \mathcal{S}_n$, $(\mathrm{Max}(\pi),\ \mathrm{Min}(\rho(\pi)))$ is a twin pair of binary trees with $n+1$ leaves. Here $\rho(\pi)$ denotes the reverse sequence of π.*

Proof. Reversing the permutation turns descents (π_{i-1}, π_i) into ascents $(\rho(\pi)_{n+1-i}, \rho(\pi)_{n+2-i})$ and thus with Remark 2.20 the leaf patterns of π and $\rho(\pi)$ are complementary. $\square$

Of course, the mapping from permutations to twin binary trees from Proposition 2.21 can be extended to a mapping from permutations to rectangulations by omitting the labels of the min- and maxtree and applying the glueing described earlier. Denote the resulting rectangulation by $R(\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi)))$.

Example 2.22. Let $\pi = (1\ 6\ 2\ 3\ 5\ 4)$, then the corresponding rectangulations is:

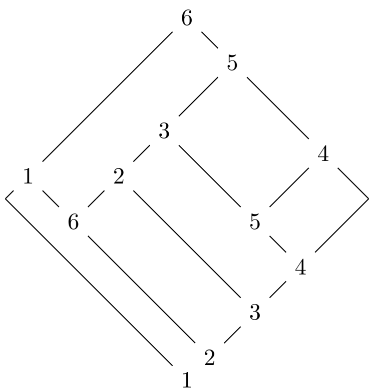


Figure 2.13: $\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi))$

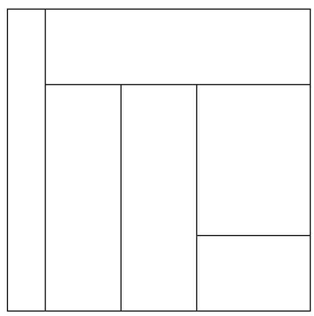
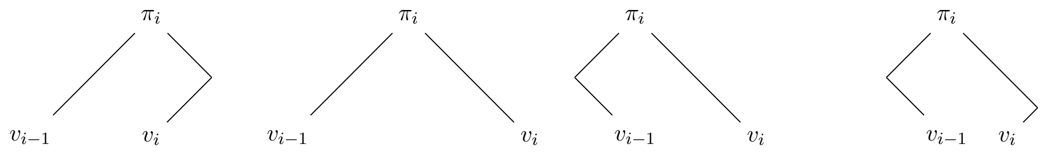


Figure 2.14: $R(\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi)))$

Remark 2.23. When glueing $\mathrm{Max}(\pi)$ and $\mathrm{Min}(\rho(\pi))$, the labelling of the nodes in both trees extends naturally to a labelling of the rectangles. The top right hand corner of any rectangle inherits from $\mathrm{Max}(\pi)$ the same label as the bottom left hand corner does from $\mathrm{Min}(\rho(\pi))$.

We have already labelled the gaps between two leaves with the label of the least common ancestor of the leaves (cf. Remark 2.20). The gaps are labelled $(\pi_1\ \dots\ \pi_n)$ from left to right in $\mathrm{Max}(\pi)$ and $(\pi_n\ \dots\ \pi_1)$ in $\mathrm{Min}(\rho(\pi))$ so that they match when the trees are glued together. The node in $\mathrm{Max}(\pi)$ from which the label is inherited constitutes the top right hand corner of the emerging rectangle. Similarly in $\mathrm{Min}(\rho(\pi))$ the node providing the label constitutes the bottom left hand corner of the emerging rectangle.
Depending on whether the leaves are left or right leaves, the space enclosed between them and their least common ancestor will have one of the following four shapes:



a) left left b) left right c) right right d) right left

Figure 2.15: Different shapes enclosed between leaves

Note that a shape a) in $\mathrm{Max}(\pi)$ will be glued to a shape a) in $\mathrm{Min}(\rho(\pi))$, likewise for c), whereas shape b) in $\mathrm{Max}(\pi)$ will be glued to d) in $\mathrm{Min}(\rho(\pi))$ and vice versa.

Theorem 2.24 *The mapping $\Theta : \mathcal{S}_n \to \mathcal{R}_n$, $\pi \mapsto R(\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi)))$ is injective when restricted to the Baxter permutations B_n and inverse to Φ from Definition 2.4.*

The proof of this theorem is very parallel to the one of Theorem 2.13.

Proof. Let $\pi \in B_n$, and $R = \Theta(\pi) = R(\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi)))$, and $\overline{\pi} = \Phi(R)$. We want to show that $\overline{\pi} = \pi$. From Remark 2.23 it is known that in $\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi))$ the rectangles are labelled $(\pi_1, \dots, \pi_n)$ in the order in which they intersect the main diagonal, that is in the $\nwarrow$-order. Hence it suffices to show that the labelling of the rectangles inherited from π (and "forgotten" by Θ) is the same as the labelling constructed by Φ by listing the rectangles according to the $\swarrow$-order, and then it follows that its image under Φ is equal to π.
Therefore, we will show that the labels of the rectangles in $\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi))$ read in the $\swarrow$-order give the sequence $1, \dots, n$.
Our aim is to show that for all $k \in [n]$, $k+1$ follows k immediately in the $\swarrow$-order. There are several case distinctions in this proof. Firstly, we examine the case '$k+1$ precedes k in π', i.e.,

$$\pi = (\dots\ k+1\ \ A\ \ B\ \ k\ \dots)$$

where A is a possibly empty sequence of integers greater than $k+1$ and B is a possibly empty sequence of integers smaller than k – this follows from π being a Baxter permutation.
Suppose there is a rectangle c, $c \neq k, k+1$, with $k \swarrow c \swarrow k+1$. There are two possible scenarios :
<u>Case I:</u> c is to the right of k and below $k+1$.

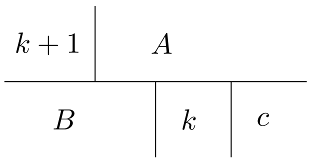


Figure 2.16: case I in R

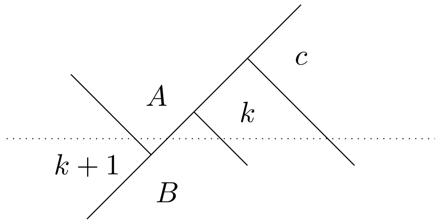


Figure 2.17: case I in $\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi))$

Since in $\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi))$, the rectangles intersect the diagonal in the order $(\pi_1, \ldots, \pi_n)$, it follows from Figure 2.17 that c is to the right of k in π; $\pi = (\ldots\ k+1\ A\ B\ k\ c\ \ldots)$.

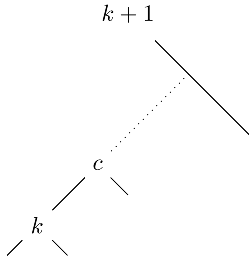


Figure 2.18: A empty

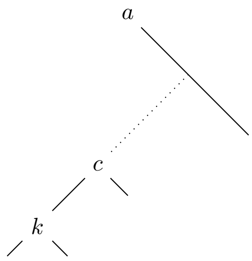


Figure 2.19: A non-empty

Suppose A is empty. Then the constellation in $\mathrm{Max}(\pi)$ needs to be as shown in Figure 2.18 It follows that $k < c < k+1$ because k is a child of c while c is a child of $k+1$ in $\mathrm{Max}(\pi)$. This is not possible since c is an integer.

Suppose A is not empty. Let a be the rightmost element of A. We obtain from Figure 2.19 that $k < c < a$.

In π, a is adjacent to either the leftmost element b in B, or to k if B is empty. Thus, $(k+1\ a\ b\ c)$ or $(k+1\ a\ k\ c)$ form an occurrence of the pattern 2-41-3. This is impossible since π is a Baxter permutation.

The second case is somewhat similar:

<u>Case II:</u> c is to the left of $k+1$ and above k.

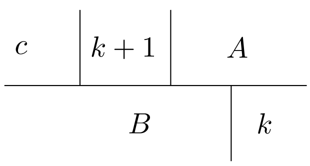


Figure 2.20: case II in R

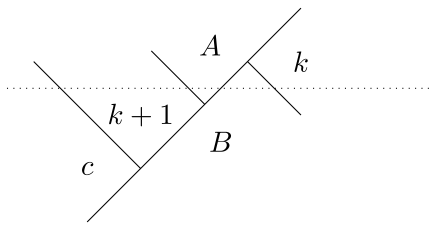


Figure 2.21: case II in $\mathrm{Max}(\pi) \cup \mathrm{Min}(\rho(\pi))$

We deduce from Figure 2.21 that c must be located before k+1 in π, i.e., $\pi = (\ldots\ c\,k+1\ A\ B\ k\ \ldots)$. B must be non-empty, because otherwise $k < c < k+1$: As we can see in Figure 2.22,

$k+1$ would be a child of c in $\mathrm{Min}(\pi)$ and c would be a child of k if B were empty.

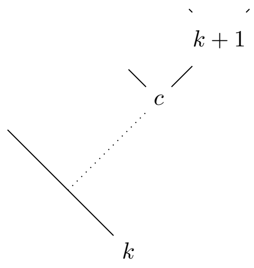


Figure 2.22: B empty in $\mathrm{Min}(\rho(\pi))$

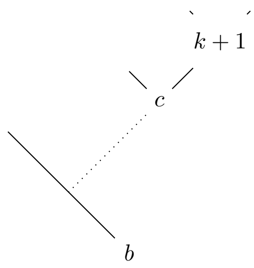


Figure 2.23: B non-empty in $\mathrm{Min}(\rho(\pi))$

If B is non-empty, it follows from Figure 2.23 that $c > b$, b being the leftmost element in B.
We remark that b and a, if it exists, or b and $k+1$, if A is empty, are adjacent in π, and thus $(c\ a\ b\ k)$ or respectively $(c\ k+1\ b\ k)$ form an occurrence of the pattern 2-41-3. This is impossible since π is a Baxter permutation. We conclude that there can be no rectangle c with $k \swarrow c \swarrow k+1$.
In the case 'k precedes $k+1$ in π' the configuration as depicted in Figure 2.24 emerges with an occurrence of the pattern 3-14-2. The argumentation is similar and thus omitted.

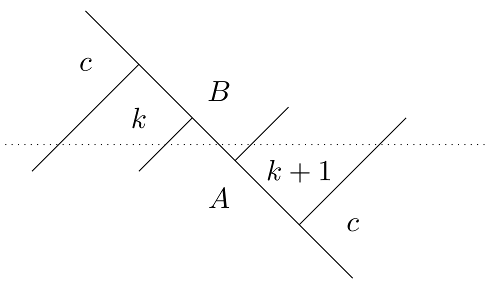


Figure 2.24: case k precedes $k+1$

□

Remark 2.25.

1. It follows that Θ from Theorem 2.2 and Ψ from Theorem 2.13 are the same.

2. This also provides a bijection between Baxter permutations and pairs of twin binary trees. Such a bijection was first described by Dulucq et al. [7].

2.3 Triples of non-intersecting paths

In this section we present a bijection between twin pairs of binary trees and triples of non-intersecting lattice paths adopted from [10]. A similar bijection was first introduced by Dulucq and Guibert [7].

In a first step we associate a binary tree T with two 0-1-strings: The first one is the canopy $\mathrm{cnp}(T)$, containing information about the leaves (cf. Definition 2.16). The second one encodes information about the inner nodes of T. In [10] it is called the reduced bodyprint $\hat{\beta}(T)$:

Definition 2.26 (in-order, bodyprint [10]) The *in-order* is an ordering of the nodes of a binary tree, defined recursively as in-order(T) = in-order(T_l), r, in-order(T_r) where r is the root of T and T_l and T_r are its left and right subtrees respectively. Let T be a binary tree and let $x_1, \ldots, x_n$ be the inner nodes of T in in-order. Then the *bodyprint* $\beta(T)$ is determined by

$$\beta(T)_i = \begin{cases} 1 & x_i \text{ is a right child or the root} \\ 0 & x_i \text{ is a left child} \end{cases} \qquad i = 1, \ldots, n$$

We can omit the last entry since it is always 1, and obtain the *reduced bodyprint* $\hat{\beta}(T)$.

For a binary tree T with $n+1$ leaves both the canopy and the reduced bodyprint have length $n-1$.

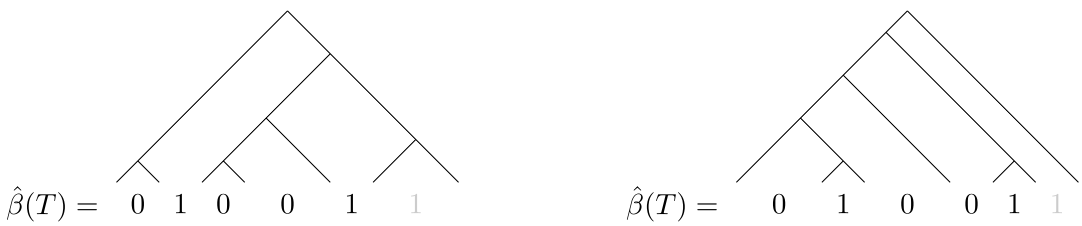


Figure 2.25: Two binary trees with reduced bodyprint 01001

Lemma 2.27 ([10, Lemma 4.2]) *Let T be a binary tree with k left leaves and $l = n-k+1$ right leaves. Then*

$$\sum_{i=1}^{n-1} \mathrm{cnp}(T)_i = \sum_{i=1}^{n-1} \hat{\beta}(T)_i = k-1$$

and

$$\sum_{i=1}^{j} \mathrm{cnp}(T)_i \geq \sum_{i=1}^{j} \hat{\beta}(T)_i \quad \text{for all } j = 1, \ldots, n-1$$

Proof. There is a pairing between inner nodes that are left children, and the right leaves of T without the rightmost right leaf, given by the maximal line segments of slope $\searrow$. The rightmost right leaf would be paired with the root which is not a left child. Right leaves correspond to 0-entries in $\mathrm{cnp}(T)$, and inner nodes that are left children correspond to 0-entries in $\hat{\beta}(T)$. The number of 1-entries in both strings is $(n-1)-(n-k)=k-1$: in $\mathrm{cnp}(T)$ the rightmost right leaf is ignored. All other $n-k$ right leaves are paired with inner nodes that are left children and contribute a 0-entry in $\hat{\beta}(T)$. This proves the first identity.
For the second one, let $v_0, ..., v_n$ be the leaves from left to right, $x_1, ..., x_n$ the inner nodes in in-order. Then v_i determines $\mathrm{cnp}(T)_i$ and x_i determines $\hat{\beta}(T)_i$. By identifying the inner nodes x_i with the gap below them, the in-order of the inner nodes corresponds to the left-to-right order of the gaps. For a pair $(v_i,\, x_j)$ in the pairing, v_i is the rightmost leaf below x_j. The gap below x_j ends at the leftmost leaf v_j of the right subtree of x_j, therefore $j \leq i$. This provides a pairing of the left inner nodes and right leaves that has the property that the index of the corresponding 0-entry in $\hat{\beta}(T)$ is always at most the index of the corresponding 0-entry in $\mathrm{cnp}(T)$. This yields the second identity. $\square$

Definition 2.28 ([10]) Let $\left\langle {n \atop k} \right\rangle$ be the set with cardinality $\binom{n}{k}$ containing all 0-1-strings of length n with k 1-entries. For $a, b \in \left\langle {n \atop k} \right\rangle$ we define a *dominates* b as:

$$a \geq b \quad \Leftrightarrow \quad \sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i \quad \text{for all } j = 1, ..., n$$

Remark 2.29 ([10])*.* For a binary tree T with $l+1$ left and $k+1 = n-l+1$ right leaves, the canopy and the bodyprint satisfy $\mathit{cnp}(T), \hat{\beta}(T) \in \left\langle {n \atop k} \right\rangle$ and $\mathit{cnp}(T) \geq \hat{\beta}(T)$.

Theorem 2.30 ([10, Theorem 4.4]) *There is a bijection between binary trees T with $l+1$ left and $k+1 = n-l+1$ right leaves and pairs of strings $(a, b) \in \left\langle {n \atop k} \right\rangle \times \left\langle {n \atop k} \right\rangle$ with $a \geq b$ given by $a = \mathrm{cnp}(T)$ and $b = \hat{\beta}(T)$.*

Proof. We have seen that for a binary tree the canopy and the bodyprint have the required properties (cf. Remark 2.29). We will show by induction on n that for each $(a, b) \in \left\langle {n \atop k} \right\rangle \times \left\langle {n \atop k} \right\rangle$ with $a \geq b$ exists a unique T with $a = \mathrm{cnp}(T)$ and $b = \hat{\beta}(T)$. As a base case observe that for $n = 1$ there are only two possibilites: Either $a = b = 0$ which uniquely determines the tree or $a = b = 1$ with the corresponding tree .
For an arbitrary but fixed $n \in \mathbb{N}$ suppose that there is a unique T with $a = \mathrm{cnp}(T)$ and $b = \hat{\beta}(T)$ for each $(a, b) \in \left\langle {n-1 \atop k} \right\rangle \times \left\langle {n-1 \atop k} \right\rangle$ with $a \geq b$.
Now let $(a, b) \in \left\langle {n \atop k} \right\rangle \times \left\langle {n \atop k} \right\rangle$ with $a \geq b$.
If $a = 0^{n-k}1^k$ then $b = a$ and they uniquely determine the tree in Figure 2.26.

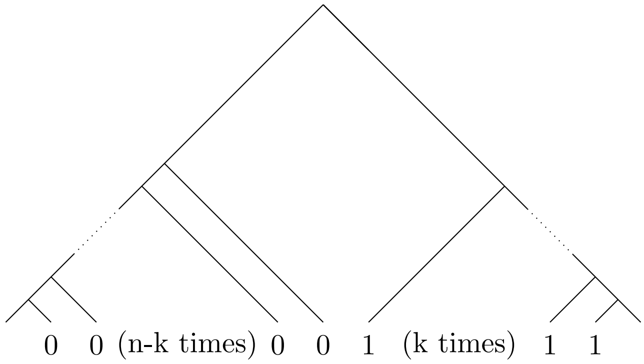


Figure 2.26: tree corresponding to $a = b = 0^{n-k}1^k$

Else, there is an $i \in [n]$ such that $a_{i-1}a_i = 10$. Write $a = a'\ a_{i-1}\ a_i\ a''$ and $b = b'\ b_i\ b''$ and define

$$a^* = \begin{cases} a'\ 1\ a'' & \text{if } b_i = 0 \\ a'\ 0\ a'' & \text{if } b_i = 1 \end{cases} \qquad \text{and} \qquad b^* = b'\ b''$$

This amounts to removing either a 0-entry or a 1-entry from both strings. It holds $a^* \geq b^*$ and a and b are of length $n-1$. The induction hypothesis provides a unique tree T^* with $a^* = \operatorname{cnp}(T^*)$ and $b^* = \hat{\beta}(T^*)$. By converting the ith leaf of T^* into an inner node with two children one constructs a tree T with $\operatorname{cnp}(T) = a$ and $\hat{\beta}(T) = b$. T is the unique tree with this property: In any tree with this property the leaves v_{i-1}, v_i are a left leaf followed by a right leaf, hence they are children of the inner node x_i and removing them yields the tree T^*.

□

We will next consider a twin pair of binary trees (T, S). As before, both T and S can be mapped bijectively to a pair of 0-1-strings $a_T \geq b_T$ and $a_S \geq b_S$ where $a_T = cnp(T)$, $b_T = \hat{\beta}(T)$, $a_S = \operatorname{cnp}(S)$, $b_S = \hat{\beta}(S)$. Since T and S are twin, one has $a_T = \rho(a_S)$. It holds $\rho(b_S) \geq \rho(a_S)$ and hence

$$\rho(b_S) \geq \rho(a_S) = a_T \geq b_T$$

Since a_T and b_T uniquely determine T and a_S and b_S uniquely determine S, it is clear that the triple $\rho(b_S),\ \ \rho(a_S) = a_T,\ b_T$ suffices to uniquely define (T, S). This yields the following theorem.

Theorem 2.31 ([10]) *There is a bijection between twin pairs of binary trees (T, S) with $l+1$ left and $k+1 = n-l+1$ right leaves and triples of 0-1-strings $(a, b, c) \in \left\langle {n \atop k} \right\rangle^3$ with $a \geq b \geq c$ given by $a = \rho(\hat{\beta}(S)),\ b = \operatorname{cnp}(T)$ and $c = \hat{\beta}(T)$.*

Further, there is a natural bijection between 0-1-strings and north-east lattice paths:

Definition 2.32 (lattice path) The *integer lattice* $\mathcal{L} = (V, E)$ consists of the vertices $V = \mathbb{Z}^2$ and edges $E = \left\{ \left(\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right], \left[\begin{smallmatrix} i+1 \\ j \end{smallmatrix}\right] \right), \left(\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right], \left[\begin{smallmatrix} i \\ j+1 \end{smallmatrix}\right] \right) \mid \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] \in \mathbb{Z}^2 \right\}$. All vertical and horizontal edges of length 1. A *north-east lattice path* is a path on the integer lattice using only north- and eastward steps obtained by addition of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ respectively.

The bijection is given by identifying 0-1-strings with north-east lattice paths where 1 corresponds to an eastward step and 0 corresponds to a northward step.
If string a dominates string b, the corresponding path α is above path β at all times, in particular they do not intersect if their starting points are chosen appropriately. This leads to the following reformulation of Theorem 2.31.

Theorem 2.33 ([10, Theorem 5.5]) *There is a bijection between twin pairs of binary trees with $k+1$ left leaves and $l+1$ right leaves and triples of non-intersecting north-east lattice paths (P_1, P_2, P_3), where P_1 is from $(0,2)$ to $(k, l+2)$, P_2 is from $(1,1)$ to $(k+1, l+1)$ and P_3 is from $(2,0)$ to $(k+2, l)$.*

Example 2.34. Figure 2.27 illustrates the bijection.

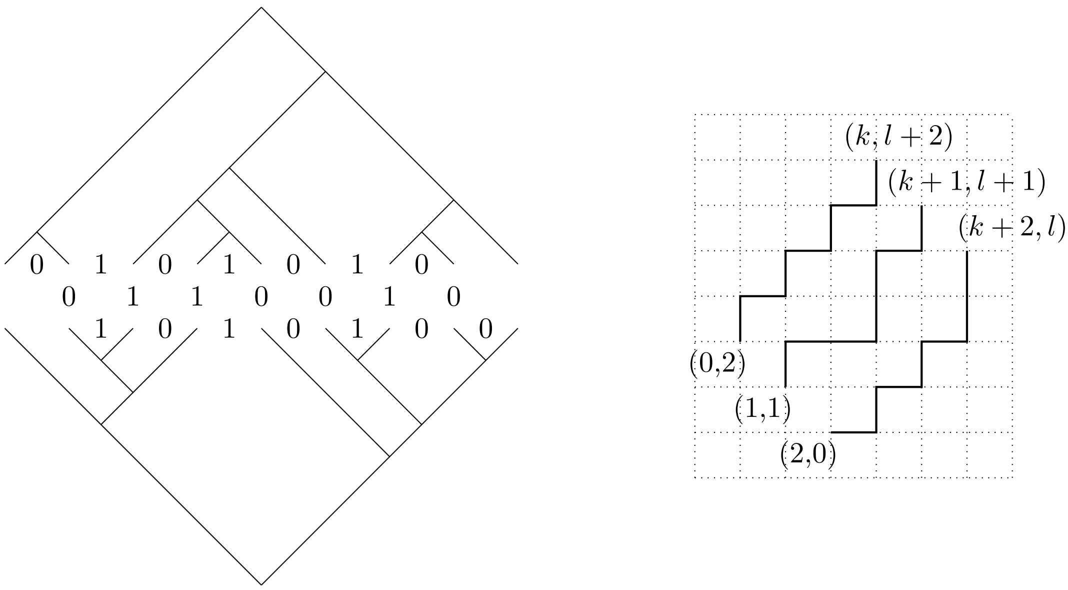


Figure 2.27

Having established a bijection to triples of non-intersecting paths we can apply the following special case of the Lemma of Lindström-Gessel-Viennot and count them.

Theorem 2.35 (Lindström-Gessel-Viennot [12]) *Let G be a directed acyclic graph and let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two n-tuples of vertices such that every n-tuple* *of non-intersecting paths* P_1, ..., P_n *from* A *to* B *takes* a_i *to* b_i *for each* i. *Let* $w(a,b)$ *count the number of paths from* a *to* b *and*

$$M = \begin{pmatrix} w(a_1,b_1) & w(a_1,b_2) & \cdots & w(a_1,b_n) \\ w(a_2,b_1) & w(a_2,b_2) & \cdots & w(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ w(a_n,b_1) & w(a_n,b_2) & \cdots & w(a_n,b_n) \end{pmatrix}$$

Then $\det(M)$ *is the number of non-intersecting* n*-tuples of paths from* A *to* B.

For the integer lattice $\mathcal{L}$, the number of north-east-paths between two points $a = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $b = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \in V(\mathcal{L})$ is $w(a,b) = \binom{(b_1-a_1)+(b_2-a_2)}{b_1-a_1}$. Each path from a to b needs to be of length $(b_1 - a_1) + (b_2 - a_2)$ and we can choose where to place the northward steps. Hence, the next theorem follows directly from Lindström-Gessel-Viennot.

Theorem 2.36 ([10, Theorem 5.6]) *The number of twin pairs of binary trees with* $k+1$ *left leaves and* $l+1$ *right leaves is*

$$\det(M_{k,l}) = \det \begin{pmatrix} \binom{k+l}{k} & \binom{k+l}{k-1} & \binom{k+l}{k-2} \\ \binom{k+l}{k+1} & \binom{k+l}{k} & \binom{k+l}{k-1} \\ \binom{k+l}{k+2} & \binom{k+l}{k+1} & \binom{k+l}{k} \end{pmatrix} = 2\frac{(k+l)!(k+l+1)!(k+l+2)!}{k!(k+1)!(k+2)!l!(l+1)!(l+2)!} = \Theta_{k,l}$$

Thanks to the bijection of between twin pairs of binary trees and rectangulations, $\Theta_{k,l}$ is also the number of rectangulations with k horizontal and l vertical segments.
Consequently, the number of twin binary trees with $n+2$ leaves and the number of rectangulations with $n+1$ rectangles is $B(n+1) = \sum_{k=0}^{n} \Theta_{k,n-k}$.

3 Special classes of rectangulations

This chapter examines special classes of rectangulations. They are characterized by certain constellations such as  or  which they avoid. An extensive analysis of pattern avoiding rectangulations can be found in [13].

Definition 3.1 (pattern avoiding rectangulation) A rectangulation $R \in \mathcal{R}_n$ *avoids* a pattern $P \in \mathcal{R}_m$ if no subset of the segments of R can be mapped bijectively to the segments of P such that the pairwise contact relations of the segments are preserved: If a segment s_1 touches another segment s_2 in R from below, above, left or right, then the image of s_1 does the same to the image of s_2 in P.

3.1 Guillotine Rectangulations

Definition 3.2 (guillotine rectangulation [2]) A *guillotine rectangulation* R either consists of just one rectangle, or there is a segment ("principal cut") that splits R into two subrectangulations that are also guillotine.

Guillotine rectangulations are also known as slicing floorplans or sliceable floorplans since they are obtained by "recursively splitting a rectangle in two by a vertical or horizontal cut" [5, p.2].
The following proposition is well known [1].

Proposition 3.3 *Guillotine floorplans are precisely those rectangulations who avoid the* windmill patterns  *and* .

Proof. In a guillotine rectangulation, any segment s is a principal cut of some subrectangulation A, splitting it into two guillotine subrectangulations B and C. The segments that touch s are contained in the subrectangulations B and C. The segments that s touches are part of the boundary of A (see Figure 3.1).
If a rectangulation R contains the patterns  or , it is not guillotine. If one of the segments (s_1) forming the windmill was a principal cut, say of the subrectangulation A,

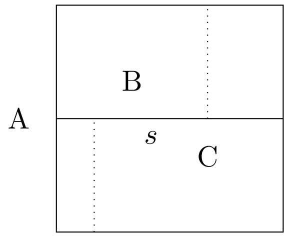


Figure 3.1

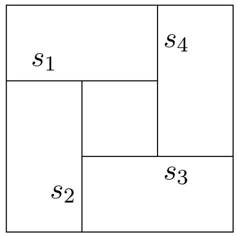
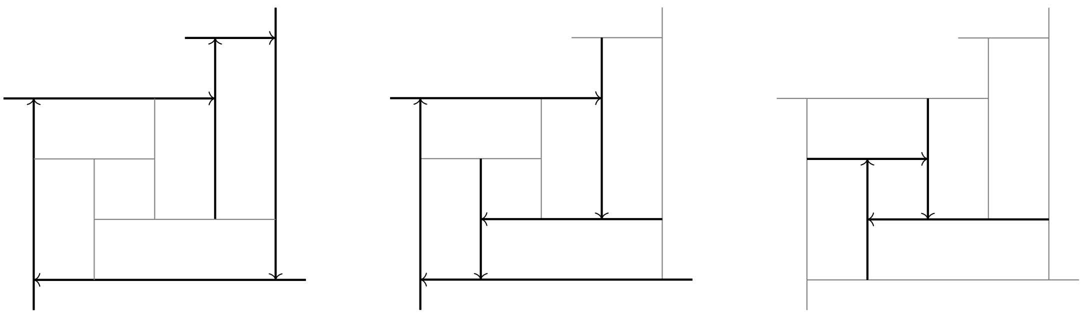


Figure 3.2

then the segment touching it (s_2) would be part of a smaller subrectangulation B contained in A, the segment s_3 touching s_2 would be part of a smaller subrectangulation C contained in B, and s_4 touching s_3 would be part of a smaller subrectangulation D contained in C. However, s_1 touches s_4, thus A must be contained in D which yields a contradiction.
It remains to show that any rectangulation that is not guillotine contains one of the windmill patterns. Let R be not guillotine. We may assume that R does not contain a principal cut, since otherwise we can reduce the argumentation to the two subrectangulations created by the principal cut. At least one of them needs to be non-guillotine, and if it contains a windmill then so does R.
Principal cuts are the only segments that do not touch any other inner segment because they extend from one boundary to the other. Hence, in a rectangulation without principal cuts, every inner segment touches another inner segment. Starting with any segment and following the chain of contacts we will find a circle eventually, since there are finitely many segments, and the chain cannot have an endpoint. Let c be a circle in R which is minimal with regard to the area it encloses. If c consists of four segments, then we have found a windmill: A circle of four segments touching each other is a windmill.



(a) c with concave vertex (b) smaller circle inscribed in c (c) 4-circle inscribed in c

Figure 3.3

Suppose c consists of more than four segments. Since in a rectangulation there are only right angles, four of which already sum up to 360 degrees, a circle consisting of more than four segments needs to have a concave vertex. In a concave vertex there is one end of a segment that is directed towards the center of the circle. Any path formed by following a chain of contacts starting with this segment needs to either end in a circle or meet the boundary of the circle c at some point, thus creating a shortcut and contradicting the minimality of c. We conclude that any rectangulation containing a circle needs to contain a 4-circle. $\square$

Example 3.4. Figure 3.4 shows a non-guillotine rectangulation: there is one principal cut S, but the sub-rectangulation to the left of S is not guillotine as none of the segments extends from one boundary to the other. It contains the pattern .
The rectangulation in figure 3.5 is guillotine.

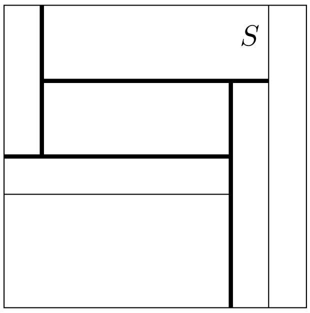


Figure 3.4: not a guillotine rectangulation

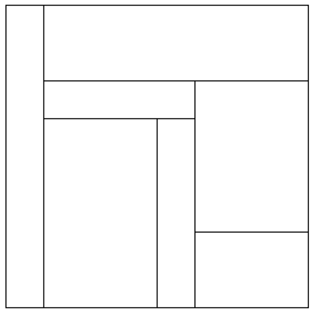


Figure 3.5: a guillotine rectangulation

As we will see, guillotine rectangulations are in bijection with a subclass of Baxter permutations.

Definition 3.5 (separable permutation [1]) Let $\sigma = (\sigma_1\ \sigma_2\ \ldots\ \sigma_n)$ and $\tau = (\tau_1\ \tau_2\ \ldots\ \tau_m)$ be two permutations. $\pi = (\pi_1\ \pi_2\ \ldots\ \pi_{n+m})$ is said to be the result of *concatenating*

$$\tau \textit{ above } \sigma, \text{ if } \pi_i = \sigma_i \text{ for } 1 \le i \le n \text{ and } \pi_{n+i} = n + \tau_i \text{ for } 1 \le i \le m.$$
$$\tau \textit{ below } \sigma, \text{ if } \pi_i = m + \sigma_i \text{ for } 1 \le i \le n \text{ and } \pi_{n+i} = \tau_i \text{ for } 1 \le i \le m.$$

A permutation π is *separable* if either

1. $\pi = (1)$ or

2. There are two separable permutations σ and τ such that π is the concatenation of τ above or below σ.

Separable permutations are perhaps best characterized by their *graphs*, defined for a permutation π as the point set $\{(i, \pi_i) \mid i \in [n]\}$ [2].
Either a separable permutation has size 1, or its graph can be split into two non-empty blocks B_1, B_2 which are themselves the graphs of two separable permutations, and all points of B_1 are either above and to the left of all points of B_2 or below and to the left.

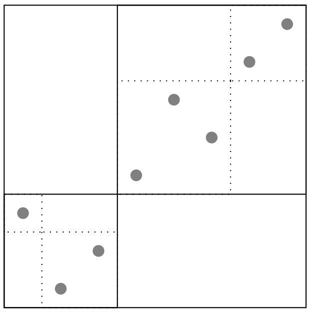


Figure 3.6: a separable permutation with an ascending structure

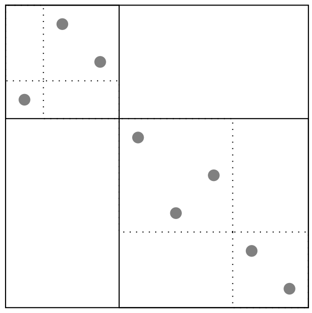


Figure 3.7: a separable permutation with a descending structure

Separable permutations are known to be those who avoid the pattern 2-4-1-3 and 3-1-4-2 [4]. In particular they form a subclass of Baxter permutations – recall that Baxter permutations avoid the pattern 2-41-3 and 3-14-2, which is a less constrictive condition.

Theorem 3.6 ([1, Theorem 1]) *There is a bijection between guillotine rectangulations and separable permutations obtained by restricting Φ from Theorem 2.4 to guillotine rectangulations.*

Proof. Let $R \in \mathcal{R}_n$ be a guillotine rectangulation. We use induction on the number of rectangles n to show that $\pi = \Phi(R)$ is separable. The claim is trivially true for $n = 1$ and $\pi = (1)$. Assume it holds for all rectangulations with fewer than n rectangles. For $n > 1$, let s be the segment that splits R into two guillotine rectangulations R_a, R_b. W.l.o.g. assume s is horizontal and R_a above R_b. Then all m rectangles of R_a are smaller than all $n - m$ rectangles of R_b in the $\nwarrow$-order and greater in the $\swarrow$-order. Therefore π can be written as

$$\pi = ((n - m + \alpha_1) \ldots (n - m + \alpha_m)\ \beta_1 \ldots \beta_{n-m})$$

and π is the concatenation of $\pi_b = (\beta_1 \ \ldots \ \beta_{n-m}) = \Phi(R_b)$ below $\pi_a = (\alpha_1 \ \ldots \ \alpha_m) = \Phi(R_a)$. By induction π_a and π_b are separable, therefore π is separable. Similarly, if s is vertical, then all m rectangles of R_a are smaller than all $n - m$ rectangles of R_b in both orders and π is the concatenation of π_b above π_a.

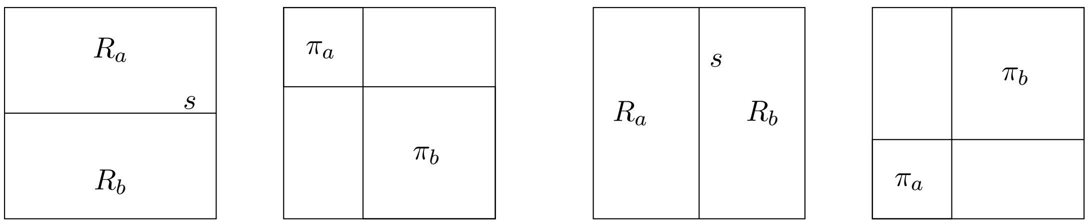


Figure 3.8: R_b below R_a corresponds to π_b concatenated below π_a, R_b to the right of R_a corresponds to π_b concatenated above π_a

Let $\pi \in \mathcal{S}_n$ be separable and we aim to show that $R = \Phi^{-1}(\pi)$ is guillotine. If $n > 1$, then π is the concatenation of two separable permutations π_a and π_b.
If π_b is concatenated above π_a, then all rectangles belonging to π_a are smaller in both orders than all rectangles belonging to π_b, that is all rectangles belonging to π_a are to the left of all rectangles belonging to π_b - they are separated by a vertical segment.
If π_b is concatenated above π_a, the separating segment is horizontal.

$\square$

3.2 One-sided Rectangulations

One-sided rectangulations were introduced by Eppstein, Mumford, Speckmann, and Verbeek in their work about *area-universal layouts* [8] . They are called one-sided, because for each maximal segment the segments touching it lie on at most one side.

Definition 3.7 (one-sided rectangulation) A rectangulation is *one-sided* if it avoids the patterns and .

Example 3.8. The first rectangulation in Figure 3.9 is one-sided, the second one not since it contains the pattern . Both are not guillotine.

(a) a one-sided rectangulation (b) not a one-sided rectangulation

Figure 3.9: Illustration of one-sidedness

Definition 3.9 (area-universal [8]) A rectangulation R is *area-universal*, if for each assignment of areas to its rectangles $a\colon R \to \mathbb{R}_{>0}$, there exists a rectangulation R' that is equivalent to R whose rectangles have the specified areas. We say R' *realizes* the area assignment a.

Clearly the notion of equivalence is crucial to this definition. With the equivalence we have used so far - which we will call weak equivalence in this section - any rectangulation is (almost) area-universal (cf. Proposition 3.10).
Wimer et al. use a notion of equivalence that is defined via a pair of planar graphs whose vertices represent the vertical and horizontal segments of the rectangulations [17].

See the appendix for a proof that this equivalence corresponds to our weak equivalence. The following result is also proven in [9].

Proposition 3.10 ([17, Theorem 3]) *For any rectangulation R and assignment of areas $a\colon R \to \mathbb{R}_{>0}$, there always exists a unique (up to a uniform stretch transformation), possibly degenerate, rectangulation R' that is weakly equivalent to R and realizes a.*

Definition 3.11 (equivalence for degenerate rectangulations) A rectangulation is *degenerate* if four rectangles meet in one point. A degenerate rectangulation R_D is *equivalent* to a non-degenerate rectangulation R if there is a sequence of non-degenerate rectangulations R_n that are equivalent to R and converge towards R_D.

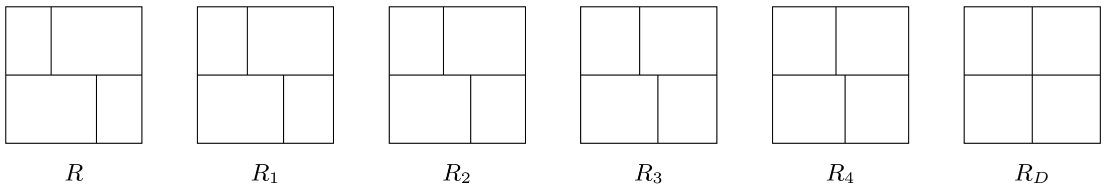


Figure 3.10: A sequence of rectangulations converging towards R_D

Example 3.12. For the rectangulation R in Figure 3.11 and the area assignment

$$w\colon R \to \mathbb{R}_{>0}\ ,\ w(A) = 1,\ w(B) = 2,\ w(C) = 2,\ w(D) = 1$$

there exists a weakly equivalent rectangulation R' that realizes w, however there is no strongly equivalent one. For the area assignment

$$w'\colon R \to \mathbb{R}_{>0}\ ,\ w'(A) = 1.5,\ w'(B) = 1.5,\ w'(C) = 1.5,\ w'(D) = 1.5$$

there exists only a degenerate rectangulation R'' that realizes w'.

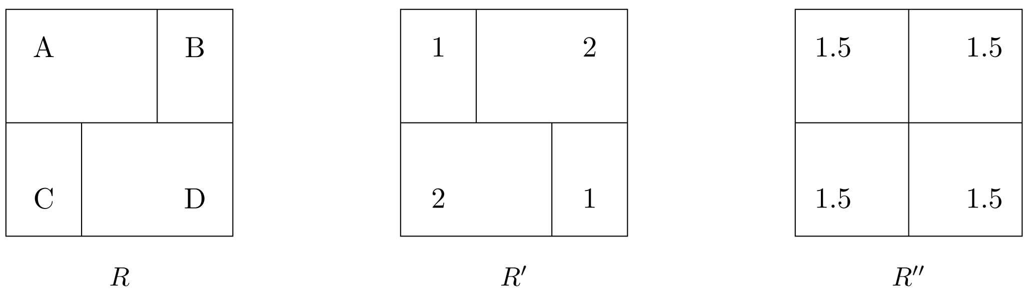


Figure 3.11: Some rectangulations can not realize all area assignments

Lemma 3.13 *A degenerate rectangulation can not be equivalent to a one-sided rectangulation with the same number of rectangles.*

Proof. An intersection of two segments in a degenerate rectangulation is the limit of

1. three segments forming one of the patterns , , , . These do not occur in one-sided rectangulations by definition.
2. four segments forming one of the patterns , , or where the central rectangle is contracted. This results in a rectangulation with fewer rectangles.

□

Lemma 3.14 ([8, Theorem 2]) *Every rectangulation that is weakly equivalent to R is strongly equivalent to R if and only if R is one-sided.*

Proof. Let R be a one-sided rectangulation and R' weakly equivalent to R. Then R' is one-sided too. For every pair of rectangles A, B in R that are adjacent, their images A', B' in R' are adjacent as well, because in a one-sided rectangulation adjacent rectangles lie on opposite sides of a maximal segment which constitutes the side of at least one of them. There can be no wall slides. Consequently R' is strongly equivalent to R.
If R is not one-sided, there is a maximal segment in R which is touched on both sides by adjoining segments. We can construct a rectangulation R' that is weakly equivalent to R but not strongly equivalent to R by performing a wall slide along this segment. □

One-sided rectangulations can be characterized by the property that they are area-universal under the strong equivalence. This is a result by Eppstein et al. however, their definition of equivalence is slightly flawed as is discussed in the appendix.

Theorem 3.15 ([8, Theorem 2]) *The following properties of a rectangulation R are equivalent:*

1. *R is area-universal with respect to strong equivalence.*
2. *R is one-sided.*

Proof. [8] Let R be a one-sided rectangulation. By Proposition 3.10 for any given area assignment w there exists a rectangulation R' that realizes w and is weakly equivalent to R. By Lemma 3.13, R' is not degenerate and by Lemma 3.14, it is strongly equivalent to R too. Thus, R is area-universal with respect to strong equivalence.
Suppose R is not one-sided. By Lemma 3.14 there is a rectangulation R' that is weakly equivalent to R but not strongly equivalent to R. Let w be the area assignment given by the areas of R'. By Proposition 3.10 R' is the only rectangulation that is weakly equivalent to R and realizes w up to affine transformation. There can be no rectangulation that is strongly equivalent to R and realizes w; thus, R is not area-universal with respect to strong equivalence. □

Lemma 3.14 states that the equivalence classes of one-sided rectangulations are the same for both notions of equivalence. Hence, the following result is of interest independently of the equivalence one uses. To our knowledge, it is a new result.

Theorem 3.16 *One-sided rectangulations are in bijection with Baxter permutations avoiding the patterns* 2-14-3 *and* 3-41-2*, by a restriction of the bijection* Φ *described in Theorem 2.4.*

Proof. Recall two remarks from the previous chapters:

1.7 If two rectangles are direct successors in one of the orders $\nwarrow$ or $\swarrow$ then they share a segment.

2.8 Ascents in $\phi(R)$ correspond to vertical segments in R, descents in $\phi(R)$ correspond to horizontal segments in R.

Let $R \in \mathcal{R}_n$. Suppose $\pi = \Phi(R)$ is a Baxter permutation containing the pattern 2-14-3 realized by $\pi_i, \pi_j, \pi_{j+1}, \pi_k,\ i < j < k$, with $\pi_j < \pi_i < \pi_k < \pi_{j+1}$. Then we can deduce the following about R:

1. π_j is a left neighbour of π_{j+1} , i.e., they share a common vertical segment.
2. π_i is above π_j and π_{j+1} is above π_k .
3. π_i and π_j are to the left of π_{j+1} and π_k

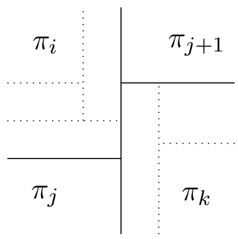


Figure 3.12

Thus, in R there is a vertical segment between π_j and π_{j+1} to both sides of which there are abutting horizontal segments, making R not one-sided. The case π contains 3-41-2 is similar and yields that R contains the pattern .
On the other hand, suppose a rectangulation R is not one-sided, i.e., contains w.l.o.g. .
Let the rectangles adjacent to this pattern be labeled $\pi_i, \pi_j, \pi_k, \pi_l$ with $i < j < k < l$ and $\pi_j < \pi_i < \pi_l < \pi_k$. Thus, we already know that the corresponding permutation $\pi = \phi(R)$ contains the pattern 2-1-4-3, but it remains to show that it does even contain 2-14-3. In other words we need to show $k = j+1$. Suppose there is π_x with $j < x < k$. The rectangles surrounding the pattern can be divided into four areas according to their order relations:

A $= \{\pi_x \mid x < i\}$

B $= \{\pi_x \mid j < x < k \text{ and } \pi_x > \pi_k\}$

C $= \{\pi_x \mid j < x < k \text{ and } \pi_x < \pi_j\}$

D $= \{\pi_x \mid x > l\}$

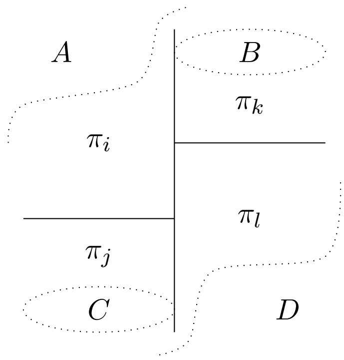


Figure 3.13

We observe that π_x must be in either B or C.
If $\pi_x \in B$, then $\pi_i, \pi_j, \pi_x, \pi_l$ form again the pattern 2-1-4-3. If $x = j + 1$ we are done, else suppose there is $\pi_{\overline{x}}$ with $j < \overline{x} < x$.
If $\pi_x \in C$, then $\pi_i, \pi_x, \pi_k, \pi_l$ form again the pattern 2-1-4-3. If $k = x + 1$ we are done, else suppose there is $\pi_{\overline{x}}$ with $x < \overline{x} < k$.
As both j and k are finite integers bounded by n, there are only finitely many numbers x with $j < x < k$. Therefore, by repeating the process for a finite number of times, we will obtain $\pi_i, \pi_{\overline{j}}, \pi_{\overline{k}}, \pi_l$ with $\overline{k} = \overline{j} + 1$. This is an ocurrence of the pattern 2-14-3.
The case R contains is similar and thus omitted; here we obtain the pattern 3-41-2. $\square$

3.3 Rectangulations that are guillotine and one-sided

Similar to one-sided rectangulations, rectangulations that are guillotine and one-sided have an interesting universality property: Nathenson and Tóth have shown that under the strong equivalence, a rectangulation is aspect-ratio-universal if and only if it is one-sided and guillotine [14]. The aspect ratio denotes the ratio of height to width of a rectangle.

Definition 3.17 (aspect-ratio-universal [14]) A rectangulation R is *aspect-ratio-universal*, if for each assignment of aspect ratios to its rectangles, there exists a rectangulation equivalent to R whose rectangles have the specified aspect ratios.

Example 3.18. The aspect ratio 1 describes a square. The windmill rectangulations can not be drawn with squares only: For R from Figure 3.14 there is no equivalent rectangulation that realizes the aspect-ratio assignment $a(A) = a(B) = a(C) = a(D) = a(E) = 1$. If A is square, then its side length plus the side length of E determines the side length of B which in turn determines C and so on.

Figure 3.14: Some rectangulations can not realize all aspect ratio assignments

Theorem 3.19 ([14]) *A rectangulation is aspect-ratio-universal if and only if it is one-sided and guillotine.*

Again, there is a subclass of Baxter permutations that is in bijection with guillotine and one-sided rectangulations:

Definition 3.20 (separable-by-point permutation [2]) A permutation π of $[n]$ is *separable-by-point* if it is empty, or its graph can be split into three blocks B_1, B_2, B_3 such that

1. B_2 consists of one point,
2. the sub-permutations corresponding to B_1, B_3 are themselves separable-by-point (thus may be empty),
3. Either all points of B_1 are left and below B_2 and all points of B_3 are to the right and above B_2 (ascending structure) or all points of B_1 are left and above B_2 and all points of B_3 are to the right and below B_2 (descending structure).

Equivalently, π is *separable-by-point* if it is empty, or there are two separable permutations σ and τ such that π is the concatenation of τ above (1) above σ or τ below (1) below σ.

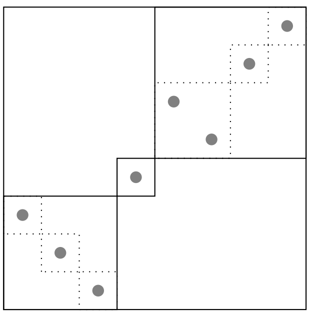


Figure 3.15: a separable-by-point permutation with an ascending structure

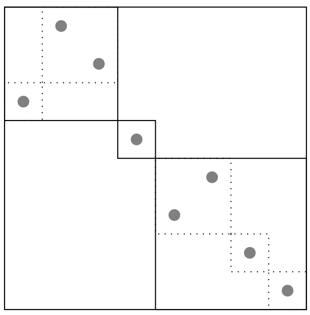


Figure 3.16: a separable-by-point permutation with a descending structure

Remark 3.21.

1. Separable-by-point permutations are separable.

2. Asinowski et al. have shown that separable-by-point permutations are exactly those permutations that avoid the patterns 2-4-1-3, 3-1-4-2, 2-14-3, 3-41-2 [2].

Theorem 3.22 *There is a bijection between rectangulations that are guillotine and one-sided and separable-by-point permutations. It is obtained by restricting* Φ *from Theorem 2.4 to guillotine and one-sided rectangulations.*

Theorem 3.22 can be obtained as a corollary from Theorem 3.16 and Theorem 3.6. Alternatively, there is a direct proof which we will sketch in the following.

Proof. A rectangulation R that is guillotine and consists of more than one rectangle needs to have a principal cut, i.e., a segment s that extends from one boundary to the other. If R is additionally one-sided, any segments touching s need to be on one side of s. Let the resulting subrectangulation on this side of s be denoted by A. On the other side of s there is either just one rectangle N or a parallel segment s' separates N from a subrectangulation B. Any subrectangulations of R – in particular A and B – are themselves guillotine and one-sided since these properties are hereditary. Thus, guillotine and one-sided rectangulations are always of the form depicted in figure 3.17 and the corresponding permutation is separable-by-point. The point corresponds to the rectangle N.

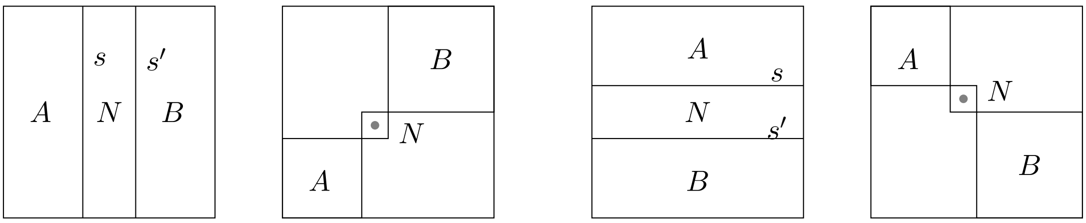


Figure 3.17: If s is vertical, π has an ascending structure, if s is horizontal, π has a descending structure.

□

4 Enumeration

This chapter will provide enumeration results for the distinct classes of rectangulations we have encountered so far. To this end we will exploit the bijections we have introduced in the previous chapters.

4.1 Baxter numbers count general rectangulations

As we have seen in the second chapter (in two ways), rectangulations can be counted by the Baxter numbers

$$B(n) = \sum_{r=0}^{n-1} \frac{\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}}{\binom{n+1}{1}\binom{n+1}{2}}$$

as proven by Chung et al. in [6] and by Felsner et al. in [10]. The first few numbers in this sequence are 1, 2, 6, 22, 92, 422, 2074, 10754, ... and the sequence can be found in the Online Encyclopedia of Integers Sequences OEIS [15] under A001181. The growth rate of the Baxter numbers is 8^n [2].

4.2 Schröder numbers count guillotine rectangulations

Recall that guillotine rectangulations correspond to separable permutations. West [16] has shown that the number of separable permutations on $[n]$ is the $(n-1)$st Schröder number S_{n-1}. The generating function $G(x)$ of (S_n) is

$$G(x) = \sum_{n=0}^{\infty} S_n x^n = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}$$

The first few numbers in this sequence are 1, 2, 6, 22, 90, 394, 1806, 8558, ... and the sequence can be found in the Online Encyclopedia of Integers Sequences OEIS [15] under A006318. The growth rate of the Schröder numbers is $\approx 5.8284^n$ [2].
The nth Schröder number also describes the number of dissections of an $n+3$-gon with a designated vertex that is not part of any cut; the number of lattice paths from $(0,0)$ to (n,n) who use only the steps $(1,0)$, $(0,1)$ and $(1,1)$ and do not rise above the diagonal; the number of ways of inserting parentheses into $n+1$ terms; and more.

4.3 Counting one-sided and guillotine rectangulations

As we have seen in the previous chapter, one-sided and guillotine rectangulations are in bijection with separable-by-point permutations. Asinowski et al. have found a generating function for separable-by-point permutations [2, Proposition 6.5], namely

$$H(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{1}{x} G(x(1-x)) = \frac{1 - x + x^2 - \sqrt{1 - 6x + 7x^2 - 2x^3 + x^4}}{2x}$$

The first few numbers in this sequence are 1, 2, 6, 20, 70, 254, 948, 3618,14058,55432 ... and the sequence can be found in the Online Encyclopedia of Integers Sequences OEIS [15] under A078482.
Furthermore, they have analysed the asymptotic behaviour of h_n which is

$$h_n \sim \left(\frac{2}{1 - \sqrt{8\sqrt{2} - 11}} \right)^n n^{-3/2}$$

This yields a growth rate of $\approx 4.5465^n$ [2].

Recursion formula

Guillotine rectangulations can be represented by an ordered tree, where the leaves correspond to empty rectangles, and inner nodes to rectangles that are divided by prinicipal cuts into the rectangles that are represented by their children. One-sidedness in rectangulations is represented by the condition that between subtrees with more than one leaf at the same level there needs to be at least one leaf.

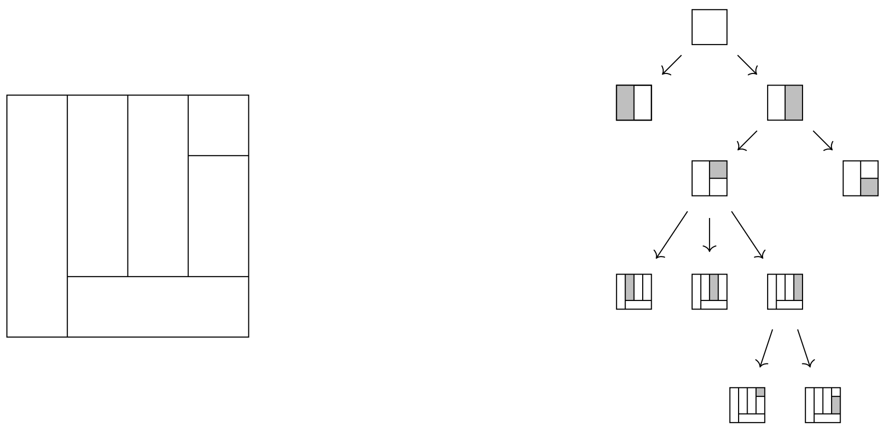


Figure 4.1: Illustration of the ordered tree associated with a guillotine and one-sided rectangulation

Let T_n be the number of such trees with $n \geq 2$ leaves. Since we need to distinguish between rectangulations with a horizontal and a vertical principal cut, it is $h_n = 2T_n, n \geq 2$.

Proposition 4.1 *T_n satisfies the following recursion formula:*

$$T_n = 3\ T_{n-1} + T_{n-2} + \sum_{i=2}^{n-3} 2\ T_i\ T_{n-i-1}.$$

Proof. $T_1 = T_2 = 1$. Assume the root has ≥ 2 children. The leftmost subtree is either

1. a leaf: The remaining subtrees have $n-1$ leaves, either in a single tree or in several trees (an ordered forest). The number of forests with $n-1$ leaves is the same as the number of trees with $n-1$ leaves, T_{n-1}. A forest can be turned into a tree with the same number of leaves by adding a root.

   This case contributes $2\ T_{n-1}$.

2. a tree with $i \geq 2$ leaves: Then the next subtree is a leaf. The remaining leaves are either in a single subtree with $n-i-1$ leaves or a forest with $n-i-1$ leaves. Either way there are T_{n-i-1} possibilities.

   This case contributes $2\ T_i\ T_{n-i-1}$ for all $i \in [2, n-3]$.

3. a tree with $i = n-2$ leaves: It is followed by two single leaves.

   This case contributes T_{n-2}.

4. a tree with $i = n-1$ leaves: It is followed by a single leaf.

   This case contributes T_{n-1}.

□

4.4 Counting one-sided rectangulations

We can obtain the first few values of the sequence $\mathcal{O}_n$ counting one-sided rectangulations by exploiting the bijection of Theorem 3.16 and implementing an algorithm that counts permutations avoiding 2-41-3, 3-14-2, 2-14-3, 3-41-2, called one-sided permutations in the following. The pseudocode for the algorithm is given below, and the python code can be found in the Appendix.

Why is it sufficient to generate candidates for one-sided permutations of length n out of one-sided permutations of length $n-1$? Suppose we have a $p \in \mathcal{S}_{n-1}$ that contains either one of these patterns: 2-41-3, 3-14-2, 2-14-3, 3-41-2. Then inserting n in any place does not change the relation of the objects that constitute this pattern, except if we insert n between 4 and 1, then the condition that they be direct successors is violated. However in this case n takes the role of 4 and the pattern is preserved. Therefore no permutation generated from a permutation that is not one-sided by inserting n can be one-sided.

1. Let $p_{1,1} = (1)$, $P_1 = \{p_{1,1}\}$
2. **for** n = 2, ..., 20:
3. $\quad O_n := 0$
4. $\quad$ **for** $p \in P_{n-1}$
5. $\qquad$ **for** $i = 1, ..., n$ **do**:
6. $\qquad p_{n,i} :=$ insert n in p in position i
7. $\qquad$ **if** $p_{n,i}$ does not contain patterns 2-41-3, 3-14-2, 2-14-3, 3-41-2 **do**:
8. $\qquad\quad$ increase O_n by 1
9. $\qquad\quad$ add $p_{n,i}$ to P_n
10. **return** $O_2, \ldots, O_{20}$

Remark 4.2. The algorithm yielded the sequence 1, 2, 6, 20, 72, 274, 1088, 4470, 18884, 81652, 360054, 1614618, 7346688, 33856008, ...

The same sequence was found by Merino and Mütze in [13] by a generating algorithm for pattern avoiding rectangulations.

4.5 Results

The next table provides an overview of the different classes of rectangulations we have considered and their combinatorial properties.

rectangulation	permutation	forbidden	patterns	sequence	OEIS
all	Baxter	2-41-3		1, 2, 6, 22, 92,	A001181
		3-14-2		422, 2074, 10754, ...	
guillotine	separable	2-4-1-3		1, 2, 6, 22, 90,	A006318
		3-1-4-2		394, 1806, 8558, ...	
one-sided		2-41-3		1, 2, 6, 20, 72,	
		3-14-2		274, 1088, 4470, ...	
		2-14-3			
		3-41-2			
guillotine	separable-	2-4-1-3		1, 2, 6, 20, 70,	A078482
and one-sided	by-point	3-1-4-2		254, 948, 3618, ...	
		2-14-3			
		3-41-2			

Appendix A

Equivalences

This section reviews and compares several equivalence notions used in [1], [2], [5] , [8], [9] and [17] with the purpose to show that some of them coincide and they form essentially two sets of equivalence classes.

R-equivalence [2] Two rectangulations are equivalent if there is a bijection of their rectangles that preserves neighbourhood relations of the rectangles (cf. Definition 1.3).

strong R-equivalence Two rectangulations are considered strongly equivalent if there is a bijection of their rectangles that preserves neighbourhood relations and adjacency of the rectangles.

seg-room-equivalence [1] Given a rectangulation R, a segment s *supports* a rectangle A, if in R s contains one of the edges of A. s and A hold a *top-, left, right-, or bottom-seg-room relation* if s support A from the respective direction. Two rectanglations are equivalent if there is a labeling of their rectangles such that they hold the same seg-room relations.

In [8], the authors study two notions of equivalence:

dual graph [8] Let R be a rectangulation. Let G be the dual graph of R; G has a vertex for every rectangle of R and an edge between two vertices if the corresponding rectangles are adjacent.

extended dual graph [8] Extend R with four rectangles at the top, bottom, left and right side of R such that the corners of the extended rectangulation belong to the top and bottom rectangle. Let $E(G)$ be the dual graph of the extended rectangulation.

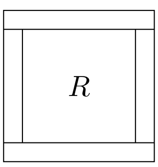


Figure A.1: How to extend a rectangulation

regular edge labeling [8] A regular edge labeling is a orientation and colouring of the edges of $E(G)$ where an edge is coloured blue and directed from left to right, if the corresponding rectangles share a vertical segment, and it is coloured red and directed from bottom to top, if the corresponding rectangles share a horizontal segment.

Figure A.2: An extended rectangulation with a regular edge labeling

strong equivalence [8] Two rectangulations are equivalent if they induce the same regular edge labeling on the same extended dual graph.

order-equivalence [8] For a rectangulation R define a partial order on the vertical maximal segments in which $s_1 \leq s_2$ if there exists an x-monotone curve that has its left endpoint in s_1, its right endpoint in s_2, and does not cross any horizontal maximal segment. Define in a symmetric way a partial order on the horizontal maximal segments. R and R' are order-equivalent, if their rectangles and maximal segments correspond one-to-one in a way that preserves these partial orders.

In [9] we reencounter the concept of regular edge labeling, although here it is called transversal structure. Again, two concepts of equivalence are introduced.

strong equivalence [9] Two floorplans are strongly equivalent if they induce the same transversal structure.

segment contact graph [9] The segment contact graph of a rectangulation R is the bipartite planar graph whose vertices are the maximal segments of R and edges correspond to contacts between segments.

separating decomposition [9] The separating decomposition is an orientation and colouring of the edges of the segment contact graph, where vertices are coloured white if they correspond to vertical segments and black if they correspond to horizontal segments. Edges are oriented from the segment that touches to the segment that is touched. Edges are coloured red if they correspond to the right contact of a horizontal segment or the bottom contact of a vertical segment, blue otherwise.

weak equivalence [9] Two floorplans are weakly equivalent if they induce the same separating decomposition.

Figure A.3: A rectangulation and its separating decomposition

In [17] two plane graphs are introduced.

***H*-graph and *V*-graph** [17] The H-graph of a rectangulation R is a directed acyclic plane graph where the vertices represent the maximal horizontal segments. For each rectangle in R there is an edge in the H-graph that is directed from the segment that contains the bottom of the rectangle towards the segment that contains the upper side of the rectangle. Analogously, the V-graph of a R is the directed acyclic plane graph where the vertices represent the maximal vertical segments. For each rectangle in R there is an edge in the V-graph that is directed from the segment that contains the left side of the rectangle towards the segment that contains the right side of the rectangle. The H-graph and V-graph have a unique source and sink which correspond to the left and right boundary segment (bottom and top, respectively). They are dual to each other with the exception that the vertices corresponding to the boundary segments both lie on the outer face of the other graph.

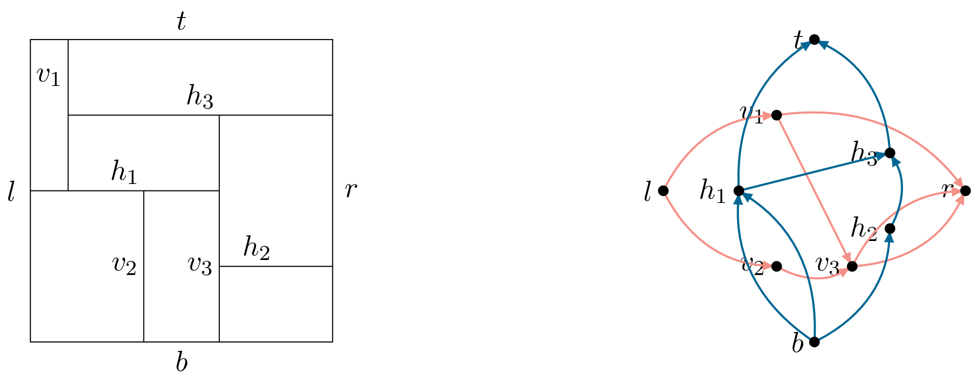


Figure A.4: A rectangulation and its H-graph and V-graph

***H*-*V*-equivalence** Two rectangulations are considered equivalent if their H-graphs and V-graphs are isomorphic as embedded graphs.

We attempt to reduce confusion with the following proposition.

Proposition A.1 *The same equivalence classes are engendered by*

1. *seg-room-equivalence [1]*
2. *R-equivalence [2]*
3. *weak equivalence via separating decompositions [9]*
4. *H-V-equivalence [17]*

A second set of equivalence classes is induced by

a. *strong R-equivalence*

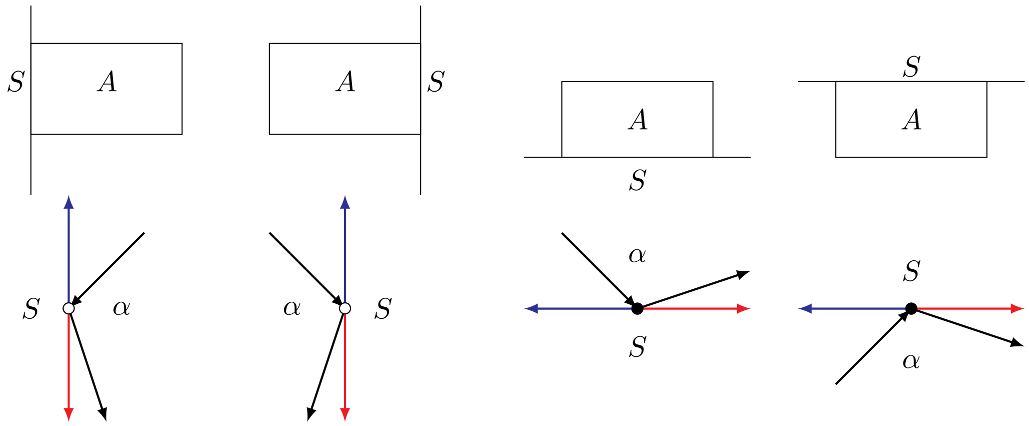
b. *strong equivalence via transversal structure or regular edge labeling [9] , [8]*

The second type of equivalence implies the first one.

Proof.
$\underline{1 \Rightarrow 2}$
The neighbourhood relation A *is a left neighbour of* B can be expressed as the seg-room relations S *supports* A *from the right* and S *supports* B *from the left* for some segment S. Similarly, A is an above neighbour of B, if there is a segment that supports A from below and B from above. Since the seg-room-equivalence preserves seg-room-relations, it preserves neighbourhood relations as well.
$\underline{2 \Rightarrow 1}$
Let R and R' be two rectangulations that are R-equivalent. The R-equivalence induces a one-to-one-correspondence of the rectangles of R and R' that preserves neighbourhood relations. Based on this, we construct a one-to-one correspondence of the segments of R and R' that preserves seg-room-relations: Let S be a vertical segment in R. Define a class A of all rectangles that hold a right-seg-room relation with S, and the class B of all rectangles that hold a left-seg-room relation with S. Per construction, all rectangles in A are left neighbours of all rectangles of B. Let A' and B' be the classes in R' containing the images of the rectangles in A and B respectively. All rectangles in A' are left neighbours of all rectangles in B', hence there must be a segment S' in R' that supports A' from the right and B' from the left. Define S' as mate of S. This yields a one-to-one correspondence of the segments and rectangles in R and R' that preserves seg-room-relations.

$3 \Rightarrow 1$

The faces of the segment contact graph of a rectangulation R are of degree 4 and in bijection with the rectangles of R [9]. Let A be a rectangle in R and α the corresponding quadrilateral face in the separating decomposition $G_{seg}(R)$. A vertical segment S supporting A in R from the left corresponds to a white vertex S incident to the quadrilateral face α in $G_{seg}(R)$, with the property that α lies between the blue outgoing edge and the red outgoing edge of S in clockwise order. Similarly for S supporting A from the right / above / below:



(a) S supports A from the left

(b) S supports A from the right

(c) S supports A from below

(d) S supports A from above

Figure A.5

Thus, seg-room relations can be deduced from the separating decomposition.

$1 \Rightarrow 3$

For the converse note that: the four types of segment contacts $\vdash$, $\dashv$, $\perp$, $\top$ that determine the separating decomposition can be expressed by seg-room relations as "S_1 supports B and C from above, and segment S_2 supports B from the right and C from the left" etc.

$1 \Rightarrow 4$

In the V-graph of a rectangulation R there is an edge from a vertex corresponding to a segment S_1 to a vertex corresponding to a segment S_2 if in R there is a rectangle A such that the left side of A is contained in S_1 and the right side of A is contained in S_2. This is a seg-room-relation and thus preserved under seg-room-equivalence. A similar statement holds for the H-graph.

$4 \Rightarrow 1$

Draw the V-graph above the H-graph such that

1. each vertex of the V-graph lies on one face of the H-graph, except for the two vertices corresponding to the boundary segments which lie on the outer face

2. each edge of the V-graph intersect exactly one edge of the H-graph
3. the analogue holds for the vertices and edges of the H-graph
4. the source of the V-graph is the leftmost node and its sink is the rightmost node
5. the source of the H-graph is the lowest node and its sink is the highest node

This determines a unique way of drawing them. An intersection of an edge (v_1, v_2) of the V-graph with an edge (h_1, h_2) of the H-graph represents the rectangle enclosed by the segments v_1, v_2, h_1, h_2 (see Figure A.6). Thus, if two rectangulations have the same V-and H-graph, then they induce the same seg-room-relations.

Figure A.6

<u>$a \Leftrightarrow b$</u>
The regular edge labeling is a labeling of the edges of the extended dual graph. In the dual graph, two vertices share an edge if the corresponding rectangles are adjacent. Therefore this equivalence preserves adjacency. We can define a relation on the edges of the regular edge labeling: $(u, v) \sim (x, y)$ if the colour of the edges is the same and $u = x$ or $v = y$. The classes generated by the transitive closure of this relation correspond to the segments of the rectangulation [9].

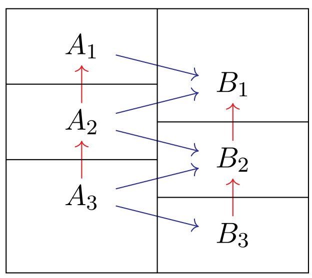


Figure A.7

Let S be a vertical segment in a rectangulation R and E the class of (blue) edges that corresponds to S in the regular edge labeling of R. Then the class of vertices that constitute the starting points of edges in E corresponds to all rectangles $A_1, \ldots, A_l$ whose

right side is contained in S. Similarly, the class of vertices that constitute the end points of edges in E corresponds to all rectangles $B_1, \dots, B_k$ whose left side is contained in S. Therefore A_i is a left neighbour of B_j, for all $i = 1, \dots, l, j = 1, \dots, k$, see Figure A.7.
We obtain that A is a left neighbour of B in R if and only if in the regular edge labeling there is a blue class of edges whose starting points include A and whose end points contain B; and A is an above neighbour of B in R if and only if in the regular edge labeling there is a red class of edges going from B to A.
$\underline{a \Rightarrow 2}$
This is clear by definition. □

The problem with order-equivalence

Eppstein et al. define order-equivalence as *"For a rectangulation R define a partial order on the vertical maximal segments in which $s_1 \leq s_2$ if there exists an x-monotone curve that has its left endpoint in s_1, its right endpoint in s_2, and does not cross any horizontal maximal segment. Define in a symmetric way a partial order on the horizontal maximal segments. This partial order can be defined by a directed acyclic multigraph that has a vertex per maximal segment and an edge from the segment on the left boundary of each rectangle to the segment on the right boundary of the rectangle; this graph is an st-planar graph, a planar directed acyclic graph in which the unique source and the unique sink are both on the outer face. The dual of this st-planar graph defines in a symmetric way a partial order on the horizontal segments. We say that L and L' are order-equivalent if their rectangles and maximal segments correspond one-to-one in a way that preserves these partial orders."* [8, p.541]
If we take this definition literally, the two rectangulations R_1, R_2 in Figure A.8 are order-equivalent. Denote the *st*-planar multigraph whose vertices correspond to the horizontal maximal segments by H-graph and analogously the graph derived from the vertical maximal segments by V-graph.
The H-graphs of R_1 and R_2 are embedded differently, but they define the same partial order on the horizontal segments of R_1 and R_2. The V-graphs are not isomorphic as multigraphs, but they induce the same partial order on the vertical segments of R_1 and R_2. One can show that seg-room-equivalence implies order-equivalence, but the converse is not true, since the above rectangulations R_1, R_2 are not seg-room-equivalent.
However, we suspect that this coarser notion of equivalence was not intended by the authors. Indeed, it would provoke a contradiction to several of their statements such as *"For any layout L and weight function w at most one order-equivalent layout L' (up to affine transformation) realizes w as a rectangular cartogram."* [8, Lemma 3]. Observe that the area of corresponding rectangles in R_1 and R_2 is the same. Also *"L is one-*

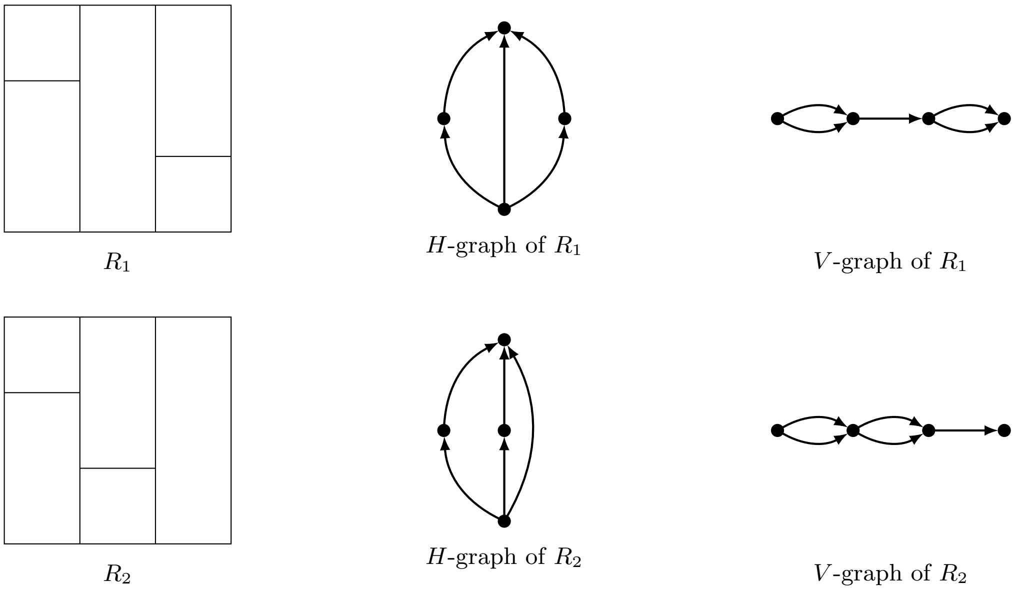


Figure A.8

sided if and only if every layout that is order-equivalent to L is [strongly] equivalent to L" [8, Theorem 1] is proven wrong by R_1 which is one-sided, and order-equivalent but not strongly equivalent to R_2 .

Appendix B

Counting one-sided rectangulations, python code

```
def contains2143(n,i,p):                # p permutation, i index of n in p
    if i == 0:
        return 0
    min= p[i-1]
    left = n
    for x in range(min+1,n):
        if p.index(x) < i-1:
            left = x
            break
    if left== n:
        return 0
    right = n
    for x in range(left+1,n):
        if p.index(x) > i:
            right = x
            break
    if right == n:
        return 0
    return 1


def contains2413(n,i,p):                # p permutation, i index of n in p
    if len(p) == i+1:
        return 0
    min = p[i+1]
    left = n
```

```
    for x in range(min+1,n):
        if p.index(x) < i:
            left = x
            break
    if left == n:
        return 0
    right = n
    for x in range(left+1,n):
        if p.index(x) > i+1:
            right = x
            break
    if right == n:
        return 0
    return 1

def contains3142(n,i,p):                # p permutation, i index of n in p
    if i == 0:
        return 0
    min = p[i-1]
    right = n
    for x in range(min+1,n):
        if p.index(x) > i:
            right = x
            break
    if right == n:
        return 0
    left = n
    for x in range(right+1,n):
        if p.index(x) < i-1:
            left = x
            break
    if left == n:
        return 0
    return 1

def contains3412(n,i,p):                # p permutation, i index of n in p
    if len(p) == i+1:
        return 0
```

```
    min = p[i+1]
    right = n
    for x in range(min+1,n):
        if p.index(x) > i+1:
            right = x
            break
    if right == n:
        return 0
    left = n
    for x in range(right+1,n):
        if p.index(x) < i:
            left = x
            break
    if left == n:
        return 0
    return 1

def is_onesided(n, P):
    count = 0
    P_n = []
    for p in P:
        for i in range(n):
            s = list(p)
            s.insert(i,n)
            if (contains2143(n,i,s)==0) & (contains3142(n,i,s)==0)
            & (contains2413(n,i,s) == 0) & (contains3412(n,i,s) ==0):
                count += 1
                P_n.append(s)
    print('count=', count)
    return P_n

def main():
    p=[1]
    P=[p]
    for n in range(2,20):
        P = is_onesided(n,P)
if __name__== "__main__":
  main()
```

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