SAT Encodings for the Car Sequencing Problem

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08/07/2013 at Pragmatics of SAT

Car Sequencing

Source Wikipedia

- Cars require different options (air-conditioning, sun-roof, etc.)
- Is there a production sequence for cars on the assembly line satisfying the sliding capacity constraints?
- CSPLib Benchmark Nr. 1
- CP, IP, local search

Car Sequencing: Example

- $C = \{1, 2, 3\}$ with demand $d_1 = 3, d_2 = 2, d_3 = 2$
- $O = \{a, b\}$ with capacity constraints $1/2$ and $1/5$
- Class 1 no restriction
- Class 2 requires option a
- Class 3 requires option a and b

Sequence of cars	3	1	2	1	2	1	3
Option a	1	-	1	-	1	-	1
Option b	1	-	-	-	-	-	1

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Car Sequencing is NP-Complete

PB Model

- Boolean variable c_i^k: car $k \in C$ is at position i
- Boolean variable o_i^l: option $l \in O$ is at position i
- Demand constraints: $\forall k \in C$

$$\sum_{i=1}^{n} c_i^k = d_k$$

- Capacity constraints: $\forall l \in O$ with ratio u_l/q_l

$$\bigwedge_{i=0}^{n-q_l} \left(\sum_{j=1}^{q_l} o_{i+j}^l \leq u_l\right)$$

PB Model

And in all positions $i \in \{1 \ldots n\}$ of the sequence it must hold:

- Link between classes and options: for each $k \in C$ and

$$\forall l \in O_k : \; c_i^k - o_i^l \leq 0$$
$$\forall l \in O \setminus O_k : \; c_i^k + o_i^l \leq 1$$

- Exactly one car:

$$\sum_{k \in C} c_i^k = 1$$

Modelling in CNF

- The PB model is the mostly used model in CP,IP and local search!
- This model with standard translation to CNF (minisat+,clasp ...) has bad performance
- Choose the right cardinality translation
- Uniform treatment of classes and options:

$$\sum_{i=1}^{n} o_i^l = d_l = \sum_{k \in C_l} d_k$$

- Global constraint: Cardinality + Sequence

Sequential Counter: Variables

- Translation of Boolean Cardinality:

$$\sum_{i \in \{1 \dots n\}} x_i = d$$

- x_i is true iff the object is at position i
- $s_{i,j}$ is true iff in the positions $0, 1 \dots i$ the object exists at least j times

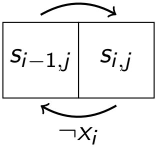
Sequential Counter: Example

3			U	U	U	U	U	U	U	U	U
2		U	?	?	?	?	?	?	?	?	L
1	U	?	?	?	?	?	?	?	?	L	
0	L	L	L	L	L	L	L	L	L		
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10

Setting x_2 and x_7 to 1:

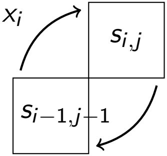
3			U	U	U	U	U	U	U	U	U
2		U	0	0	0	0	0	1	1	1	L
1	U	0	1	1	1	1	1	1	1	L	
0	L	L	L	L	L	L	L	L	L		
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10
x_i	0	0	1	0	0	0	0	1	0	0	0

Sequential Counter



$$\neg s_{i-1,j} \vee s_{i,j}$$

$$x_i \vee \neg s_{i,j} \vee s_{i-1,j}$$



$$\neg x_i \vee \neg s_{i-1,j-1} \vee s_{i,j}$$

$$\neg s_{i,j} \vee s_{i-1,j-1}$$

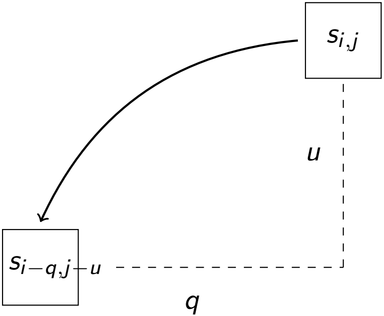
- This idea can translate all cardinality constraints

Demand Constraint + Capacity Constraint

$$(\sum_{i=1}^{n} x_i = d) \wedge \bigwedge_{i=0}^{n-q} (\sum_{l=1}^{q} x_{i+l} \leq u)$$

Demand Constraint + Capacity Constraint

$$(\sum_{i=1}^{n} x_i = d) \wedge \bigwedge_{i=0}^{n-q} (\sum_{l=1}^{q} x_{i+l} \leq u)$$



$$\neg s_{i,j} \vee s_{i-q,j-u}$$

Capacity Constraints: Example

Capacity constraint $4/8$, demand $d = 12$ on a sequence of 22 variables:

13																					U	U	U
12																				U	?	?	L
11																			U	?	?	L	
10																		U	?	?	L		
9													U	U	U	U	U	?	?	L			
8												U	?	?	L	L	L	L	L				
7											U	?	?	L									
6										U	?	?	L										
5					U	U	U	U	U	?	?	L											
4				U	?	?	L	L	L	L	L												
3			U	?	?	L																	
2		U	?	?	L																		
1	U	?	?	L																			
0	L	L	L																				
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
x_i								**0**	**0**							**0**	**0**						

Capacity Constraints: Example

Partial Assignment: x_1 and x_{13} to true and x_{12}, x_{14} and x_{21} to false.

13																				U	U	U
12																			U	0	0	L
11																		U	?	1	L	
10																	U	?	1	L		
9													U	U	U	U	U	?	1	L			
8												U	0	1	L	L	L	L	L				
7											U	1	1	L									
6										U	1	1	L										
5					U	U	U	U	U	1	1	L											
4				U	?	1	L	L	L	L	L												
3			U	1	1	L																	
2		U	1	1	L																		
1	U	1	1	L																			
0	L	L	L																				
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
x_i		1	**1**	**1**			**0**	(0)	(0)	**1**	**1**	**1**	0	1	0	(0)	(0)					0	**1**

Discussion: Related Work

- Sinz: Sequential Counter CNF [6]
- Een and Soerensson: Translation through BDDs to CNF [3]
- Bacchus: Decomposition through DFAs to CNF [1]
- Brand et al: Decomposition to cumulative sums for CP [2]
- Siala et al: Linear time propagator for CP [5]

A Trick for Lower Bounds ([4])

class	0	1	2	3	4	5	6	7	8	9	10	11
demand	9	4	22	2	1	62	31	4	24	4	3	36
0: 1/2	-	-	-	-	-	-	-	-	-	-	-	x
1: 2/3	-	-	-	-	-	x	x	x	x	x	x	-
2: 1/3	-	-	x	x	x	-	-	-	x	x	x	-
3: 2/5	-	x	-	-	x	-	x	x	-	-	x	-
4: 1/5	x	x	-	x	x	-	-	x	-	x	x	-

class	12	13	14	15	16	17	18	19	20	**21**	22	**23**
demand	3	25	3	8	5	2	6	21	5	**7**	11	**2**
0: 1/2	x	x	x	x	x	x	x	x	x	**x**	x	**x**
1: 2/3	-	-	-	-	-	-	x	x	x	**x**	x	**x**
2: 1/3	-	-	-	x	x	x	-	-	-	**x**	x	**x**
3: 2/5	-	x	x	-	-	x	-	x	x	-	x	**x**
4: 1/5	x	-	x	-	x	x	x	-	x	**x**	-	**x**

- Class 21 and 23 have option 0,1,2,4 with a total demand of 9
- All other classes share at least one option with 21 and 23
- Potential neighbours of 21 and 23?

Results on CSPLib

	E1	E2	E3	ASP	PB
#solved UNSAT	**17**	15	**17**	10	8
#fastest UNSAT	**5**	4	4	0	4
#solved SAT	**11**	**11**	**11**	7	2
#fastest SAT	0	4	**7**	0	0

- More propagation important for SAT instances, not so much for UNSAT
- Combination of SAT and the Trick shows many lower bounds (additional empty cars)

Conclusions and Future Work

- SAT can be very competitive on CP benchmarks
- SAT is very strong on proving lower bounds
- Global Constraints motivate for encodings
- Choosing the right encoding of cardinality constraints is crucial

Future work:

- Comparison to CP and IP
- Theoretical analysis of the decompositions and usage in other domains
- Exponential encoding in the number of options?

End

Thank you very much

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Backupslides

Sequential Counter: Comparison to Sinzs AtMost

$\forall i \in \{1 \ldots n\}\ \forall j \in \{0 \ldots d+1\}$:

$$\neg s_{i-1,j} \lor s_{i,j} \tag{1}$$

$$x_i \lor \neg s_{i,j} \lor s_{i-1,j} \tag{2}$$

$\forall i \in \{1 \ldots n\} \forall j \in \{1 \ldots d+1\}$:

$$\neg s_{i,j} \lor s_{i-1,j-1} \tag{3}$$

$$\neg x_i \lor \neg s_{i-1,j-1} \lor s_{i,j} \tag{4}$$

$$s_{0,0} \land \neg s_{0,1} \land \neg s_{n,d+1} \tag{5}$$

SAT instances

Inst(SAT)	E1	E2	E3	ASP	PB
4/72	0.14	0.17	**0.05**	6.78	-
16/81	0.38	**0.08**	0.14	13.25	-
41/66	0.06	**0.04**	0.04	2.55	123.41
26/82	0.95	**0.16**	0.21	80.06	654.80
200-01	30.43	47.70	**15.35**	1141.87	-
200-07	6.32	2.39	**1.46**	1478.01	-
300-01	143.76	10.62	**5.98**	810.23	-
300-07	59.86	28.39	**8.24**	-	-
400-05	623.30*	768.97	846.41	-	-
400-06	445.36	24.79	**16.29**	-	-
400-10	884.91	18.99	**13.50**	-	-

UNSAT instances

Inst(UNSAT)	E1	E2	E3	ASP	PBO
6/76	72.55	117.28	**57.55**	929.87	289.68
10/93	11.40	**6.48**	11.08	331.31	-
21/90	119.83	**74.01**	95.18	-	-
36/92	**16.97**	18.67	41.34	277.63	-
200-03	137.21	**24.02**	30.81	-	-
200-04	69.64	475.76	16.83	-	**1.84**
200-05	**254.03**	1337.39*	1172.38*	-	-
200-09	358.81	-	504.26	-	**3.77**
200-10	**2.10**	3.36	2.53	3.91	2.32
300-03	**99.03**	-	214.41	949.55	-
300-04	3.17	30.57	**2.03**	46.60	4.40
300-08	18.52	799.16	50.98	123.22	**13.54**
300-05	**0.37**	2.73	0.62	-	-
300-10	1.08	25.15	**0.96**	1282.09	904.31
400-03	37.05	**30.31**	31.47	-	-
400-04	13.03	185.32	**6.33**	130.33	14.47
400-09	25.75	470.60	32.90	557.60	**5.19**

lower bounds

	LB (pre)	LB (SAT)	sec	UB (SAT)	sec	LB* (known)	UB* (known)
4/72		0	0.07	0	0.07	0	0
6/76		6	209.77	6	0.10	1	6
10/93		1	18.93	3	0.53	1	3
16/81		0	-	0	0.07	0	0
19/71	2	-	-	2	1.50	2	2
21/90	2	1	93.80	2	0.11	1	2
36/92		1	38.55	1	0.07	1	2
41/66		0	-	0	0.06	0	0
26/82		0	-	0	0.14	0	0
200-01		0	-	0	7.46		0
200-02	2	-	-	2	0.86		2
200-03		3	1323.05	3	110.33		3
200-04	7	1	17.59	7	1.04		7
200-05		1	639.42	3	39.63		6
200-06	6	-	-	6	0.69		6
200-07		0	-	0	0.69		0
200-08	8	-	-	8	20.17		8
200-09	10	1	189.14	10	2.32		10
200-10	17	16	213.88	17	3.51		19
300-01		0	-	0	24.83		0
300-02		-	-	6	39.89		12
300-03	13	2	872.06	13	77.76		13
300-04	7	6	795.68	7	12.20		7
300-05	2	12	1145.39	16	1247.82		27
300-06	2	-	-	2	1559.76		2
300-07		0	-	0	6.33		0
300-08	8	1	102.26	8	1.01		8
300-09	7	-	-	7	141.56		7
300-10	3	9	863.15	13	115.67		21
400-01		-	-	-	-		1
400-02	15	-	-	15	112.36		15
400-03		10	1531.53				9
400-04	19	5	25.85	19	222.68		19
400-05		0	-	0	302.19		0

Size

Length	Variables					Clauses				
	E1	E2	E3	ASP	PB	E1	E2	E3	ASP	PB
100	27	21	27	90	14	91	83	99	243	54
200	73	60	73	335	33	256	259	291	946	127
300	136	118	136	747	48	495	524	573	2195	197
400	223	199	223	1308	63	827	907	972	3879	271

Link between Cars and Options

$\forall i \in \{1 \ldots n\}$:

$$\bigwedge_{\substack{k \in C \\ l \in O_k}} \neg c_i^k \vee o_i^l \tag{6}$$

$$\bigwedge_{\substack{k \in C \\ l \notin O_k}} \neg c_i^k \vee \neg o_i^l \tag{7}$$

$$\bigwedge_{l \in O} \left(\neg o_i^l \vee \bigvee_{k \in C_l} c_i^k \right) \tag{8}$$

Example for non GAC of E2

Example

Let $n = 5$, $d = 2$ with a capacity constraint of $1/2$, and let x_3 be true, then unit propagation does not force x_2 nor x_4 to false. Setting them to true will lead to a conflict through clauses (12) and (**??**) on positions 2, 3 and 4.

3				U	U	U
2		U	U	.	.	L
1	U	.	.	L	L	
0	L	L	L			
$s_{i,j}$	0	1	2	3	4	5
x_i		.	.	1	.	.

Sequential Counter

$\forall i \in \{1 \ldots n\} \ \forall j \in \{0 \ldots d+1\}$:

$$\neg s_{i-1,j} \lor s_{i,j} \tag{9}$$

$$x_i \lor \neg s_{i,j} \lor s_{i-1,j} \tag{10}$$

$\forall i \in \{1 \ldots n\} \forall j \in \{1 \ldots d+1\}$:

$$\neg s_{i,j} \lor s_{i-1,j-1} \tag{11}$$

$$\neg x_i \lor \neg s_{i-1,j-1} \lor s_{i,j} \tag{12}$$

$$s_{0,0} \land \neg s_{0,1} \land s_{n,d} \land \neg s_{n,d+1} \tag{13}$$