SAT Encodings for the Car Sequencing Problem

Valentin Mayer-Eichberger and Toby Walsh

NICTA and University of New South Wales
Locked Bag 6016, Sydney NSW 1466, Australia
{valentin.mayer-eichberger,toby.walsh}@nicta.com.au

Abstract. Car sequencing is a well known NP-complete problem. This paper introduces encodings of this problem into CNF based on sequential counters. We demonstrate that SAT solvers are powerful in this domain and report new lower bounds for the benchmark set in the CSPlib.

1 Introduction

Car sequencing is the first benchmark in the constraint programming library (prob001 in CSPLib [5]). The associated decision problem (is there a production sequence for cars on the assembly line satisfying the sliding capacity constraints?) has been shown to be NP-complete. To date, however, it has not received much attention from the SAT community. This is disappointing as we show here that SAT is a very effective technology to solve such problems. We introduce several CNF encodings for this problem that demonstrate the strength of SAT solvers. Furthermore, we identify new lower bounds by a preprocessing technique and by SAT solving.

The paper is organised as follows. First we formally introduce the car sequencing problem. Then in Section 2 we define the CNF encodings and discuss their properties. A direct method to compute lower bounds is explained in Section 3. In the experimental evaluation in Section 4 we compare the CNF encodings and show results on the lower bounds.

1.1 Car Sequencing

Car sequencing deals with the problem of scheduling cars along an assembly line with capacity constraints for different stations (e.g. radio, sun roof, air-conditioning, etc). Cars are partitioned into classes according to their requirements. Let C and O be disjoint sets of classes and options. To each class $k \in C$ there is given a demand of cars d_k to be scheduled. Each option $l \in O$ is limited to u_l occurrences on each subsequence of length q_l (denoted as a capacity constraint u_l/q_l), i.e. no more than u_l cars with option l can be sequenced among q_l consecutive cars. To each class $k \in C$ we denote the set O_k of options it requires and for convenience to each option $l \in O$ we denote C_l the set of classes that require this options. A solution is a valid sequence of all cars.

Let n be the length of the total sequence. The following pseudo Boolean model is a basis for our translation to CNF. We use 0/1 variables x_i^m to denote that an object(class or option) m is at position i. For the sequence it must hold:

– Demand constraints: $\forall k \in C$

$$\sum_{1}^{n} x_i^k = d_k$$

– Capacity constraints: $\forall l \in O$ with ratio u_l/q_l

$$\bigwedge_{i=0}^{n-q_l} (\sum_{j=1}^{q_l o} x_{i+j}^l \leq u_l)$$

And in all positions $i \in \{1 \dots n\}$ of the sequence it must hold:

– Link between classes and options: for each $k \in C$ and

$$\forall l \in O_k : \quad x_i^k - x_i^l \leq 0$$

$$\forall l \notin O_k :: \quad x_i^k + x_i^l \leq 1$$

– Exactly one car:

$$\sum_{k \in C} x_i^k = 1$$

Example 1. Given classes $C = \{1, 2, 3\}$ and options $O = \{a, b\}$. The demands (number of cars) for the classes are $3, 2, 2$ and capacity constraints on options are $1/2$ and $1/5$, respectively. Class 1 has no restrictions, class 2 requires option a and class 3 needs options $\{a, b\}$. The only legal sequence for this problem is $[3, 1, 2, 1, 2, 1, 3]$, since class 2 and 3 cannot be sequenced after another and class 3 need to be at least 5 positions apart.

Car sequencing in the CSPlib contains a selection of benchmark problems of this form ranging from 100 to 400 cars. Over the years different approaches have been used to solve these instances, among them constraint programming, local search and integer programming [9][6][7][3][10].

Car sequencing has also been treated as an optimisation problem, although the literature does not agree on a common optimisation function and several versions have been proposed. There are several variations for the optimisation function minimising the number of violated capacity constraints. However, in this paper we use the definition of [8]. Fortuitously for us, this transforms easily to a decision problem and SAT solving can be directly applied. An unsatisfiable car sequencing problem can be made solvable by adding empty slots (also called dummy cars) to the sequence. The task is then to minimise the number of dummy cars needed for a valid sequence. A lower bound of lb to an instance is shown by proving unsatisfiability with $lb - 1$ dummy cars.

2 Modelling the Car Sequencing Problem

In this section we introduce different ways to model the car sequencing problem in CNF. The basic building blocks are cardinality constraints of the form $\sum_{i\in\{1\dots n\}} x_i = d$ and $\sum_{i\in\{1\dots n\}} x_i \leq d$.

First we will show how to translate cardinality constraints by a variant of the sequential counter encoding proposed by [11]. This can be used to model the demand and the capacity constraints. Then we show how to enforce a global view by integrating capacity constraint into the sequential counter.

2.1 Sequential Counter Encoding

We will show how to encode a cardinality constraint of the form $\sum_{i\in\{1\dots n\}} x_i = d$ where x_i are Boolean variables and d is a fixed demand. The key idea is to introduce auxiliary variables to represent cumulative sums.

In this section we use two types of variables, for each position i

- x_i is true iff an object (class or option) is at position i
- $s_{i,j}$ is true iff in the positions $0, 1 \dots i$ the object exists at least j times (for technical reasons $0 \leq j \leq d+1$).

The following set of clauses (1) to (5) define the sequential counter encoding (SC):

$\forall i \in \{1 \dots n\}\ \forall j \in \{0 \dots d+1\}$:

$$\neg s_{i-1,j} \vee s_{i,j} \tag{1}$$

$$x_i \vee \neg s_{i,j} \vee s_{i-1,j} \tag{2}$$

$\forall i \in \{1 \dots n\} \forall j \in \{1 \dots d+1\}$:

$$\neg s_{i,j} \vee s_{i-1,j-1} \tag{3}$$

$$\neg x_i \vee \neg s_{i-1,j-1} \vee s_{i,j} \tag{4}$$

$$s_{0,0} \wedge \neg s_{0,1} \wedge s_{n,d} \wedge \neg s_{n,d+1} \tag{5}$$

The variables $s_{i,j}$ represent the bounds for cumulative sums for the sequence $x_1 \dots x_i$. The encoding is best explained by visualising $s_{i,j}$ as a two dimensional grid with positions (horisontal) and cumulative sums (vertical). The binary clauses (1) and (3) ensures that the counter (i.e. the variables representing the cumulative sums) is monotonically increasing. Clauses (2) and (4) control the interaction with the variables x_i. If x_i is true, then the counter has to increase at position i whereas if x_i is false an increase is prevented at position i. The conjunction (5) sets the initial values for the counter to start counting at 0 and ensures that the total sum at position n is equal to d.

Example 2. The auxiliary variables $s_{i,j}$ with $d = 2$ over a sequence of 10 variables. The cells with $U(L)$ and above(below) are set to false (true) by preprocessing. The question mark identifies yet unassigned variables. Left: before variable assignments, right: after variable assignment x_2 and x_7 to true.

3			U	U	U	U	U	U	U	U	U
2		U	?	?	?	?	?	?	?	?	L
1	U	?	?	?	?	?	?	?	?	L	
0	L	L	L	L	L	L	L	L	L		
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10

3			U	U	U	U	U	U	U	U	U
2		U	0	0	0	0	0	1	1	1	L
1	U	0	1	1	1	1	1	1	1	L	
0	L	L	L	L	L	L	L	L	L		
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10
x_i	0	0	1	0	0	0	0	1	0	0	0

The encoding in [11] follows a similar idea and focuses on $\leq$ expressions. SC can be adapted to represent such constraints by removing $s_{n,d}$ from the conjunction (5). Then the counter accepts all assignments to $x_1 \dots x_n$ with up to d variables set to true. Comparing the resulting clauses there are certain differences between the encoding proposed here and the one in [11]. We post twice as many clauses and ensure uniqueness by the auxiliary variables, i.e. we ensure the same model count.

2.2 The Capacity Constraint

We now show how to translate the interleaving capacity constraints to CNF. Each subsequence of length q can have at most u true assignments. Thus, the capacity constraints are a sequence of cardinality expressions.

$$\bigwedge_{i=0}^{n-q} (\sum_{l=1}^{q} x_{i+l} \leq u)$$

We will translate this expression to CNF in two ways. The straight forward way is to encode a sequential counter for each subsequence separately. This will introduce independent auxiliary variables for each subsequence. The second way is more elaborate and is explained in this section.

The idea is to encode a more global view into the demand constraint by integrating the capacity of each subsequence into the counter. We intend to encode the global view on the following expression:

$$(\sum_{i=1}^{n} x_i = d) \wedge \bigwedge_{i=0}^{n-q} (\sum_{l=1}^{q} x_{i+l} \leq u)$$

Interestingly, we can reuse the auxiliary variables of the SC encoding and impose the following set of clauses:

$\forall i \in \{q \dots n\}$, $\forall j \in \{u \dots d+1\}$:

$$\neg s_{i,j} \vee s_{i-q,j-u} \tag{6}$$

The clauses restrict the internal counting not to exceed the capacities constraints. The following example will demonstrate the way these clauses work.

Example 3. See tables in Figure 1 for a visualisation of the variables. We construct the grid for a capacity constraint 4/8 and a demand of $d = 12$ on a sequence of 22 variables. UP will force x_7, x_8, x_{15} and x_{16} to be false prior to search. The second table examines the variables after decisions x_1 and x_{13} to true and x_{12}, x_{14} and x_{21} to false. Notice the amount of propagation due to the clauses (6), the interesting two cases are shown by an arrow. (Configuration of this example taken from [10]).

2.3 Link Cars and Options

For a complete CNF translation of car sequencing we need to link classes and options. This is done by the following clauses.

$\forall i \in \{1 \dots n\}$:

$$\bigwedge_{\substack{k \in C \\ l \in O_k}} \neg x_i^k \vee x_i^l \tag{7}$$

$$\bigwedge_{\substack{k \in C \\ l \notin O_k}} \neg x_i^k \vee \neg x_i^l \tag{8}$$

$$\bigwedge_{l \in O} \left(\neg x_i^l \vee \bigvee_{k \in C_l} x_i^k \right) \tag{9}$$

Clause (7) and (8) follow directly from the pseudo Boolean model, whereas we add (9) to ensure more propagation when e.g. branched negatively on a set of variables of classes such that a support for an option is lost and its variable should be false.

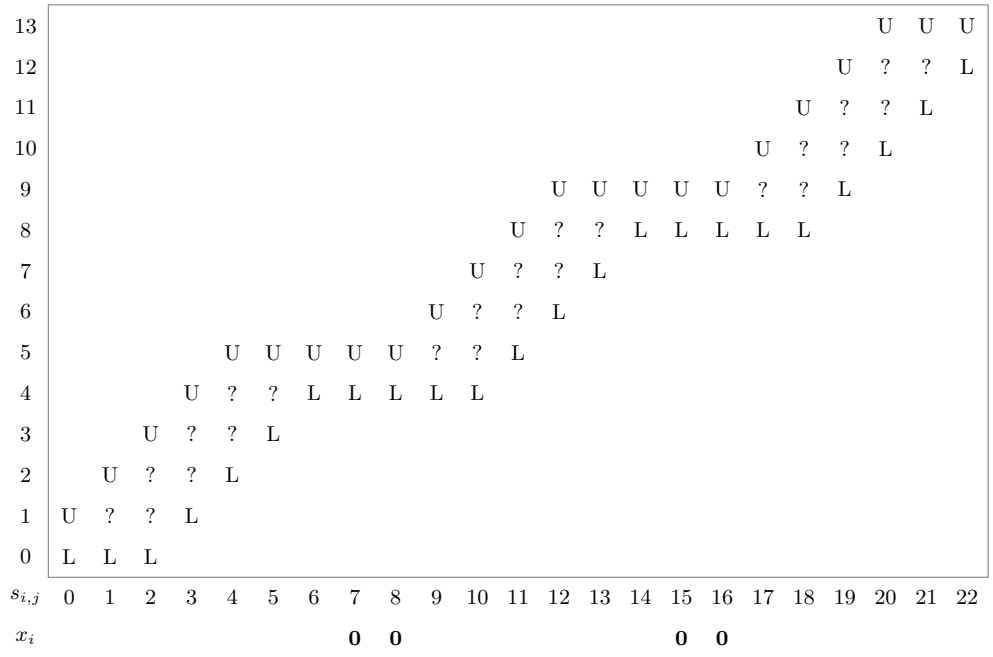
Furthermore, for each position an additional sequential counter is used to restrict the number of cars to exactly one.

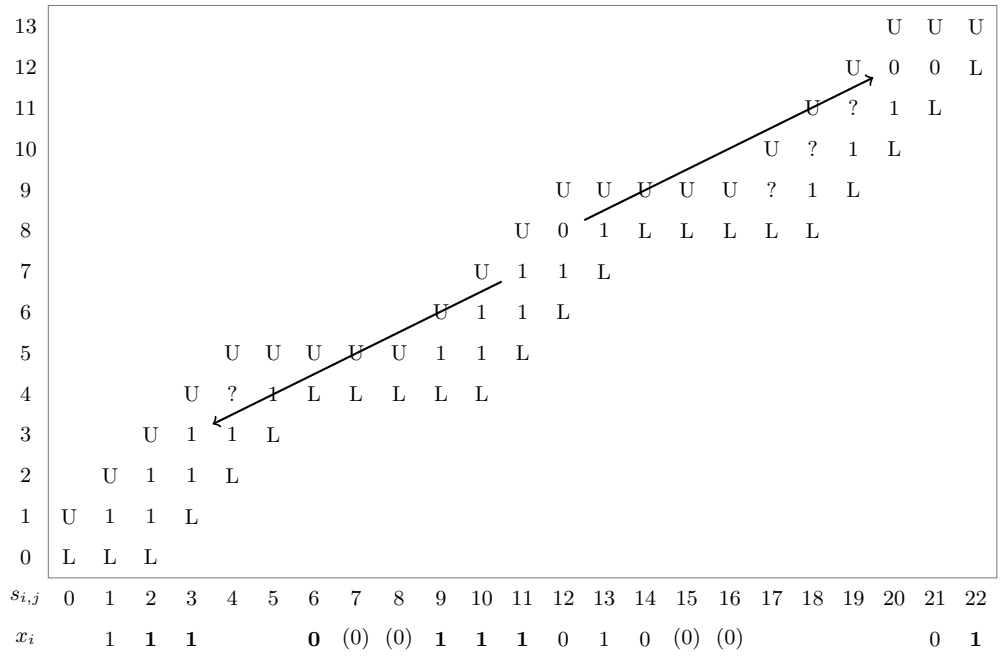
2.4 The Complete Model

For each class we are given a cardinality constraint by the demands and for each option a capacity rule. From this we can identify for each option $l \in O$ its implicit demand by summing up the demand of all classes that require this options.

$$d_l = \sum_{k \in C_l} d_k$$

Fig. 1. The grid of variables of Example 3. Notation for bottom row: normal font = decision variable, bold = propagated, in brackets = propagated previously.





Moreover, we can determine for each class the strictest capacity constraint from all its options. Thus, classes and options are translated to CNF in a uniform way, by a demand constraint and by interleaving capacity constraints. The link between classes and options for all models is encoded by clauses (7),(8) and (9). For our experimental section we define three models E1, E2 and E3 that vary in the way the demand and capacity constraints are translated.

- E1 translates demand and capacity constraints separately by the clauses (1) to (5).
- E2 translates these constraints by the integrated way using clauses (1) to (6).
- E3 combines both E1 and E2.

3 Lower Bounds by Preprocessing

The idea to this method goes back to the proof in [4] to show a lower bound of 2 for the instance 19/97. Here we will restate this technique to apply the method on all problems from the benchmark in [7].

We start with instance 300-04 as an example. The demands and options are given in Table 2.

Table 1. Overview of options and demands for instance 300-04

There are two classes, 21 and 23, that require options 0, 1, 2 and 4 and sum of demands is 9. First observation is that all other classes share at least one option with these two classes. Secondly cars of class 21 and 23 have to be put at least 5 apart, so they cannot share a neighbour. Furthermore, they cannot be neighbour to any of the classes that have a $1/q$ restriction. This leaves us with the classes that only share the option 1 and for each car at most one adjacent car can have restriction 2/3. Since the first and the last car in the sequence can have any neighbour with that restriction, the number of cars that share no option is at least $9-2=7$. Since there are no such cars, the lower bound for this problem is 7.

A similar argument can be made for classes 21, 22, 23 that share options 0, 1 and 2. Here the collective demand is 20 and the supply of cars that have neither of these options is $20-13=7$. This gives a lower bound of 5, which is weaker than the first case.

We unify the two cases into a method that can then be used to compute lower bounds. Given a set of options $B \subseteq O$ that contain the following capacity constraint: at least one of the form $1/q$ where $q \geq 3$ and at most one with $2/r$ where $r \geq 3$ and arbitrary many $1/s$ for $s \geq 2$. If the total demand for this set

is d_B, then there have to be at least $d_B - 2$ cars that do not require any of these options in B. The reason for this is as in the example above. Cars that have all options in B are at least 3 positions apart and thus cannot share an adjacent car. Cars that share at least one option with B can at most occupy one side since the weakest restriction is $2/r$ and consequently for a valid sequence cars that contain no option in B are needed. Edge cases (start and end of the sequence) are removed and thus we need $d_B - 2$ cars with no options in B. A lower bound is then the difference of demand and availability of such cars.

4 Evaluation

First we will compare the three different SAT encodings of Section 2 on the CSPlib instances. Then we show our results for lower bounds. Our focus is on the 9 traditional instances plus the 30 hard instances proposed by [7]. We leave out the set of 70 easy instances, as all of them can be solved in under 2 second and the difference among the models is minimal.

We have written a command line tool that generates CNF in DIMACS format from a problem description as provided by the CSPlib. It can on request translate the specification by different sets of clauses and compute the lower bounds from the preprocessing as defined in Section 3. It is freely available at `github.com/vale1410/car-sequencing`.

For our experiments we choose the well-known SAT solver Minisat [1] of version 2.2.0, that represents a canonical implementation of state-of-the-art CDCL solvers. All experiments are done on Linux 64bit, Intel Xeon CPUs at 2.27GHs.

We show in Table 2 the results for the selected benchmark on models E1,E2 and E3 with a timeout of 1800sec. Most of the instances can be solved in under 60 seconds and there are only 11 instances in the benchmark that cannot be solved at all. Furthermore, 13 of the harder problems can be shown unsatisfiable. The difference between the models is smaller than we expected. There is a tendency of model E3 to solve SAT instances faster than the other two models, whereas model E1 is stronger in finding UNSAT proofs.

Table 3 shows the best known lower bounds (LB) and upper bounds (UB) found by the preprocessing and SAT solving. The column LB(pre) contains the lower bounds determined by the preprocessing in Section 3. The next 4 columns show the lower bounds and upper bounds and the time to compute the last instance. For the bounds the number of additional empty slots ranges from 0 to the best known upper bound in literature. Each run was limited to 1800 seconds and we picked the best results from among model E1 to E3. In the last two columns we show the bounds that have been published previously to the best of our knowledge.

Note that upper and lower bounds in the literature are subjected to different definitions of the optimisation goal and cannot directly be compared. Our result show that in some cases the different definitions lead to the same upper and lower bound, in others we find that they are incomparable. See 400-03 for conflicting lower and upper between the definitions, this was also reported in [3]. On the

Table 2. SAT solving on the three different models.

Instance	Result	E1	E2	E3
4/72	SAT	0.2	0.2	0.1
6/76	UNSAT	6.5	8.2	18.3
10/93	UNSAT	8.8	17.3	17.5
16/81	SAT	0.3	0.1	0.1
19/71	-	-	-	-
21/90	UNSAT	157.9	93.8	288.3
36/92	UNSAT	23.5	54.2	57.9
41/66	SAT	0.0	0.0	0.1
26/82	SAT	0.6	0.1	0.1
200-01	SAT	76.5	68.1	7.5
200-02	-	-	-	-
200-03	UNSAT	26.9	20.5	35.1
200-04	UNSAT	116.2	372.2	15.5
200-05	UNSAT	515.5	-	1381.4
200-06	-	-	-	-
200-07	SAT	8.1	2.2	0.8
200-08	-	-	-	-
200-09	UNSAT	192.9	-	-
200-10	UNSAT	2.7	5.7	5.1

Instance	Result	E1	E2	E3
300-01	SAT	180.9	8.0	38.1
300-02	-	-	-	-
300-03	UNSAT	613.0	-	1276.7
300-04	UNSAT	4.9	44.4	3.7
300-05	UNSAT	0.6	3.3	0.8
300-06	-	-	-	-
300-07	SAT	173.6	15.8	8.6
300-08	UNSAT	34.7	864.1	62.9
300-09	-	-	-	-
300-10	UNSAT	1.2	17.0	1.7
400-01	-	-	-	-
400-02	-	-	-	-
400-03	UNSAT	52.9	66.2	49.8
400-04	UNSAT	16.0	301.7	11.1
400-05	SAT	318.3	1041.0	900.7
400-06	SAT	462.6	23.8	2.8
400-07	-	-	-	-
400-08	-	-	-	-
400-09	UNSAT	93.8	700.1	32.0
400-10	SAT	674.2	3.1	25.4

other hand instance 300-5 indicates that our version of the optimisation function allows smaller upper bounds due to a large gap between the two results. For most instances there is still a large gap between lower and upper bound, and room for improvement.

The method for computing lower bound from Section 3 is powerful and reports in many of the instances higher lower bounds than the SAT approach. It also confirms some of the upper bounds to be the exact solutions.

5 Conclusion and Future Work

We have introduced CNF encodings for the car sequencing problem based on sequential counters and demonstrated that SAT solvers perform well on the instances of the CSPlib. For one version of the optimisation problem we have shown lower and upper bounds and provide the SAT community with a set of hard benchmarks. Our approach is still work in progress. In the following we identify our next steps and future work.

The success of our method is based on the way we encode sequential counters. To better identify the advantages of this type of translation we need to compare it to other CNF encodings for cardinality constraints, as parallel counters and sorting networks [11][2]. We will also analyse the properties of the encodings

Table 3. Lower and upper bounds found by preprocessing (pre), by the SAT solving and the best known.

	LB (pre)	LB (SAT)	sec	UB (SAT)	sec	LB* (known)	UB* (known)
4/72		0	0.07	0	0.07	0	0
6/76		6	209.77	6	0.10	1	6
10/93		1	18.93	3	0.53	1	3
16/81		0	-	0	0.07	0	0
19/71	2	-	-	2	1.50	2	2
21/90	2	1	93.80	2	0.11	1	2
36/92		1	38.55	1	0.07	1	2
41/66		0	-	0	0.06	0	0
26/82		0	-	0	0.14	0	0
200-01		0	-	0	7.46		0
200-02	2	-	-	2	0.86		2
200-03		3	1323.05	3	110.33		3
200-04	7	1	17.59	7	1.04		7
200-05		1	639.42	3	39.63		6
200-06	6	-	-	6	0.69		6
200-07		0	-	0	0.69		0
200-08	8	-	-	8	20.17		8
200-09	10	1	189.14	10	2.32		10
200-10	17	16	213.88	17	3.51		19
300-01		0	-	0	24.83		0
300-02		-	-	6	39.89		12
300-03	13	2	872.06	13	77.76		13
300-04	7	6	795.68	7	12.20		7
300-05	2	12	1145.39	16	1247.82		27
300-06	2	-	-	2	1559.76		2
300-07		0	-	0	6.33		0
300-08	8	1	102.26	8	1.01		8
300-09	7	-	-	7	141.56		7
300-10	3	9	863.15	13	115.67		21
400-01		-	-	-	-		1
400-02	15	-	-	15	112.36		15
400-03		10	1531.53				9
400-04	19	5	25.85	19	222.68		19
400-05		0	-	0	302.19		0
400-06		0	-	0	2.76		0
400-07		-	-	-	-		4
400-08	4	-	-	-	-		4
400-09		4	1253.59	5	53.49		5
400-10		0	-	0	27.05		0

on a more theoretical level and evaluate how the auxiliary variables are used in branching and conflict clauses.

Our analysis lack comparison to other closely related paradigms as constraint programming, pseudo Boolean solvers and answer set programming. We plan to conduct experiments with these approaches in a controlled environment to identify better strength and weakness of either approach.

The set of car sequencing benchmarks in the CSPlib contain the same type of capacity constraints for all options and we plan to collect instances from other sources and to generate new instances of greater variety. The benchmark referenced in [12] contains industrial instances with rather complex definition for the optimisation function and they report poor results for constraint programming approaches. A common definition of the optimisation problem is important for the research community such that lower and upper bounds can be compared properly.

We believe there is a generalisation to the method in Section 3 to arbitrary sets of options. Our results show evidence that such a structure can be beneficial in determining lower bounds. The perspective of analysing subsets of options and their corresponding demands might also lead to interesting redundant constraints that can heavily improve solving times. The generation of such constraints would be based on an exponential method in the number of options, but typically this numbers is very small compared to the length of the sequence.

References

1. Niklas Eén and Niklas Sörensson. An Extensible SAT-solver. In *SAT*, pages 502–518, 2003.
2. Niklas Eén and Niklas Sörensson. Translating Pseudo-Boolean Constraints into SAT. *JSAT*, 2(1-4):1–26, 2006.
3. Bertrand Estellon, Frédéric Gardi, and Karim Nouioua. Large neighborhood improvements for solving car sequencing problems. *RAIRO - Operations Research*, 40(4):355–379, 2006.
4. Ian P. Gent. Two Results on Car-sequencing Problems. In *Report APES-02-1998*, 1998.
5. Ian P. Gent and Toby Walsh. $\mathrm{CSP}_{\mathrm{LIB}}$: A Benchmark Library for Constraints. In *CP*, pages 480–481, 1999.
6. Jens Gottlieb, Markus Puchta, and Christine Solnon. A Study of Greedy, Local Search, and Ant Colony Optimization Approaches for Car Sequencing Problems. In *EvoWorkshops*, pages 246–257, 2003.
7. M. Gravel, C. Gagné, and W. L. Price. Review and Comparison of Three Methods for the Solution of the Car Sequencing Problem. *The Journal of the Operational Research Society*, 56(11):1287–1295, 2005.
8. Laurent Perron and Paul Shaw. Combining Forces to Solve the Car Sequencing Problem. In *CPAIOR*, pages 225–239, 2004.
9. Jean-Charles Régin and Jean-Francois Puget. A Filtering Algorithm for Global Sequencing Constraints. In *CP*, pages 32–46, 1997.
10. Mohamed Siala, Emmanuel Hebrard, and Marie-José Huguet. An Optimal Arc Consistency Algorithm for a Chain of Atmost Constraints with Cardinality. In *CP*, pages 55–69, 2012.

11. Carsten Sinz. Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. In *CP*, pages 827–831, 2005.
12. Christine Solnon, Van-Dat Cung, Alain Nguyen, and Christian Artigues. The car sequencing problem: Overview of state-of-the-art methods and industrial case-study of the ROADEF '2005 challenge problem. *European Journal of Operational Research*, 191(3), 2008.