A SAT Encoding for the AtMostSeqCard Constraint

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1 Introduction

Give description of the car sequencing problem and the straight forward encoding in IP/CNF.

The naive CNF and IP encoding of the car sequencing benchmark is far from optimal. In this paper we will show gradually how to come up with a better encoding.

2 Motivation

We are seeking an encoding that enforces GAC on the recently proposed AtMostSeqCard constraint ([Siala et al., 2012]). This constraint is not as expressive as the Sequence constraint but is more suited for some benchmark problems and has a linear filtering algorithm. Here we will show that there is a compact CNF encoding that shows good results in the benchmark set of the CSPLIB. Furthermore we will try to improve the bounds on the set of hard instance.

3 Encoding of one AtMostSeqCard

We will first show how to encode a cardinality constraint with a counter encoding (ref , [Eén and Sörensson, 2006] ?) and then integrate the AtMostSeq by reusing the auxiliary variables of the counter encoding.

Over this whole section we will work with the following notation. Given a set of consecutive positions $P = \{1 \dots n\}$ and a property that holds at a position $i \in P$ iff the boolean variable x_i is true.

3.1 Encoding of Counters

We want to encode the following cardinality constraint

$$\sum_{i \in \{1 \dots n\}} x_i = d$$

where d is a fixed value. We call the encoding for such a cardinality constraint a counter encoding because we count exactly d occurrences over positions $\{1 \dots n\}$. The idea is to encode cumulative sums (as in [Brand et al., 2007] and an order encoding ([Tamura et al., 2009]) of the auxiliary variables.

We introduce the following variables (representing cumulative sums):

- $y_{i,j}$ is true iff in the positions $1 \dots i$ the property holds at least j times.

The following formula clarifies the relationship between x and y.

$$y_{i,j} \iff (j \leq \sum_{l=0}^{i} x_l)$$

Fig. 1. The variables $y_{i,j}$ with an upper bound d of two over a sequence of 10. By preprocessing the variables corresponding to the cells containing $U(L)$ and above(below) are set to false (true). The question mark identifies unassigned variables of the counter encoding

There are two types of binary clauses that relate the variables y among each other and two types of ternary clauses that coordinate y with the variables x.

$$\bigwedge_{i \in \{0 \dots n-1\}} \bigwedge_{j \in \{0..d+1\}} \neg y_{i,j} \vee y_{i+1,j} \tag{1}$$

$$\bigwedge_{i \in \{1..n\}} \bigwedge_{j \in \{1..d+1\}} \neg y_{i,j} \vee y_{i-1,j-1} \tag{2}$$

These clauses restrict the structure of the auxiliary variables to consist of a counter. Now we need to relate these variables to x. First we restrict the counter not to increase if x_i is false.

$$\bigwedge_{i \in \{1 \dots n\}} \bigwedge_{j \in \{0..d+1\}} x_i \vee \neg y_{i,j} \vee y_{i-1,j} \tag{3}$$

Second we define clauses that push the counter up if x_i is true.

$$\bigwedge_{i \in \{0 \dots n-1\}} \bigwedge_{j \in \{0..d\}} \neg x_{i+1} \vee \neg y_{i,j} \vee y_{i+1,j+1} \tag{4}$$

Now we finally have to "initialize" the counter.

$$y_{n,d} \wedge \left(\bigwedge_{i \in \{0 \dots n\}} y_{i,0} \right) \wedge \neg y_{0,1} \wedge \left(\bigwedge_{i \in \{0 \dots n\}} \neg y_{i,d+1} \right) \tag{5}$$

It should be mentioned that with (6) we instruct the counter to measure exactly d occurences. It is easy to change the intializer to at most d or at least d occurences.

Fig. 2. Taking the previous example and let x_i be true for position 2 and 7. Then the resulting assignment to the counter variable is given in this table.

3.2 Extending to AtMostSeqCard

Given a sequence of boolean variables among exactly d have to be true and each window of size q cannot contain more than u true variables. This is the AtMostSeqCard constraint:

$$\text{AtMostSeqCard}(u, q, d, [x_1, \ldots, x_n]) \iff (\sum_{i=1}^{n} x_i = d) \wedge \bigwedge_{i=1}^{n-q} (\sum_{l=1}^{q} x_{i+l} \leq u)$$

To archive GAC on this constraint we need to take the counter encoding of the previous section and add the following binary clauses:

$$\bigwedge_{\substack{i \in \{1 \ldots n\} \\ i-q \geq 0}} \bigwedge_{\substack{j \in \{1 \ldots d+1\} \\ j-u \geq 0}} \neg y_{i,j} \vee y_{i-q,j-u} \tag{6}$$

This seems suprising!

Theorem 1. *The clauses of counter encoding (1) to (5) with the clauses of (6) are logically equivalent to the AtMostSeqCard.*

Proof. The proof uses ideas from the decompositions of the Sequence Constraint and the encoding by cumulative sums, see [Brand et al., 2007]

Size of Encoding: Let $s = n \cdot d -$ upper and lower triangle, then we generate s auxiliary variables and $3 \cdot s$ binary clauses and $2 \cdot s$ tenery clauses. The term $n \cdot d$ dominates. The precise number of variables can be computed by the hight d and the slope u/q and a little bit of algebra, which leads to $d \cdot (u - q + \frac{d \cdot q}{u})$. So the number of variables s should be

$$s = d \cdot n - d \cdot (u - q + \frac{d \cdot q}{u})$$

For example the AtMostSeqCard($u = 4, q = 8, d = 12, n = 22$) would have $12 \cdot 22 - 12 \cdot (4 - 8 + \frac{12 \cdot 8}{4}) = 24$ variables.

Conjecture 1. The clauses (1) to (6) enforce GAC on the AtMostSeqCard on every partial assignment with the failed literal test.

The power of the binary clauses are best shown in the example [Siala et al., 2012].

4 Encoding of Carsequencing

We need to relate the cars and options as in the problem specification. This can be done in two ways. First the straight forward way.

Fig. 3. Here we analyse the initial state of variables for a constraint with $u = 4, q = 8, d = 12, n = 22$. In this example we see that already x_7, x_8, x_{15} and x_{16} need to be set to false. Notice that for this particular case we only need 24 boolean variables to create the GAC encoding.

Fig. 4. State of the auxiliary variables for $u = 4, q = 8, d = 12, n = 22$ and choices x_1 and x_{13} to true and x_{12}, x_{14} and x_{21} to false. Notice the amount of propagation due to the clauses of the AtMostSeqCard constraint, also notice that variable x_1 was a redundant choice. Normal font= choice, bold = propagated, ()= earlier propagated, green arrow positve implication, red arrow negative implication by (6).

4.1 Relating Cars and Option directly

Let C be the index set identifying a class and O be the index set for options. The instance for a car sequencing problem is given by a mapping $m : C \to 2^O$, relating to each class a set of options. For each class we have a cardinality constraint and for each option a AtMostSeq constraint. From this we can construct for each option and for each class a AtMostSeqCard constraint (since all cars have to be assigned).

Each such AtMostSeqCard constraint is encoded into SAT. In addition we need to relate classes and options on each position. This is done by the following clauses. Let $m' : O \to 2^C$ be the mapping relating to each option the corresponding classes.

$$\bigwedge_{i \in \{1 \ldots n\}} \bigwedge_{\substack{c \in C \\ o \in m(c)}} \neg x_{i,c} \vee x_{i,o} \tag{7}$$

and the reverse

$$\bigwedge_{i \in \{1 \ldots n\}} \bigwedge_{o \in O} \left(\neg x_{i,o} \vee \bigvee_{c \in m'(o)} x_{i,c} \right) \tag{8}$$

Notice in an ASP encoding this would be modelled by one rule and the completion semantics covers for the reverse case.

4.2 Alternative: no $x_{i,k}$ variables!

Here we will show that there is an encoding of the car sequencing problem that does not use at all the variables $x_{i,k}$. The encoding builds entirely on the auxiliary variables $y_{i,j,k}$ and their relationship. This is rather surprising. Here the idea:

For each position in the sequence, the sum of true y for all cars that have option o need to be the same size as the number of true y for o. As a formula, for all positions i and options $o \in O$ it has to hold:

$$\sum_j y_{i,j,o} = \sum_{c \in m'(o)} \sum_j y_{i,j,c}$$

Encoding this restriction should be enough to fully grasp the car sequencing problem. The construction can be done by a sorting network (e.g. [Batcher, 1968]) and the size should be $n \cdot log^2(d)$ in the hight d of the counter. For pure cardinality constraints this can be improved in a CNF encoding as in [Asín et al., 2011], but here we are faced with an equvialence which needs a full sorter. Maybe the recursive structure of the cummulative sum can be exploited in this construction, since in position $i-1$ we have already almost have sorted the sequence with up to one difference.

In this encoding we even have a stronger notion of propagation, because also on all partial assignment on $y_{i,j,k}$ we archive GAC (this is not the case in the direct connection between cars and options).

5 Evaluation

Best of results that can be robustly (standard heuristics) archived by current sat solvers. I compared newest version of minisat, lingeling, cryptominisat, glucose and clasp and they all consistently find solutions within 1h runtime.

Table 1.

	set1	**set2**	**set3**	**set4**
sat	70	4	0	7
unsat	0	0	4	13
unknown	0	0	1	10

This is by far better than most papers evaluating the car sequencing problem on some specialized algorithm (e.g. branch and bound) or special constraint (CP) or optimization (IP).

For the set 4 a more detailed view is interesting as the benchmark targets the optimization version of the car sequencing problem.

We can solve 7 satisfiable instances and prove 13/23 instances to be unsatisfiable.

6 Extensions

– Optimizations: there are two definitions of the cost function for the car sequencing problem. First is to allow arbitrary cars without any options and minimize the number of cars with options. And second is to minimize the number of windows that exceed the capacity constraint on their options. It would be interesting to compare both definition and to evaluate against

Table 2. Solutions to the benchmark proposed in [Gravel et al., 2005] with minimum violations found on the target function (violated capacity of options per window) by a local search method and compared to solutions on the decision version SAT encoding with lingeling (LING).

name	**min**	**LING**	**sec**
200-01	0	SAT	189.9
300-01	0	SAT	315.7
400-01	1	?	-
200-02	2	?	-
300-02	12	?	-
400-02	16	?	-
200-03	4	UNSAT	70.2
300-03	13	UNSAT	873.0
400-03	9	UNSAT	88.1
200-04	7	UNSAT	19.7
300-04	7	UNSAT	33.6
400-04	19	UNSAT	83.2
200-05	6	UNSAT	543.7
300-05	29	UNSAT	9.7
400-05	0	SAT	2146.1
200-06	6	?	-
300-06	2	?	-
400-06	0	SAT	605.4
200-07	0	SAT	30.2
300-07	0	SAT	122.6
400-07	4	?	-
200-08	8	?	-
300-08	8	UNSAT	65.9
400-08	4	?	-
200-09	10	UNSAT	350.4
300-09	7	?	-
400-09	5	UNSAT	220.9
200-10	19	UNSAT	9.9
300-10	21	UNSAT	18.3
400-10	0	SAT	468.1

published results in the literature. There are still gaps between known upper and lower bounds.
- There is a natural extension of the AtMostSeqCard constraint that to a cyclic version and in the same and natural way we can extend the encoding given above. It would be interesting to find good benchmarks.
- The Sequence constraint consists of a sequence of among constraints and we should compare this encoding to the known CNF encodings and filtering algorithms in the literature.

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