SAT Encodings for the Car Sequencing Problem

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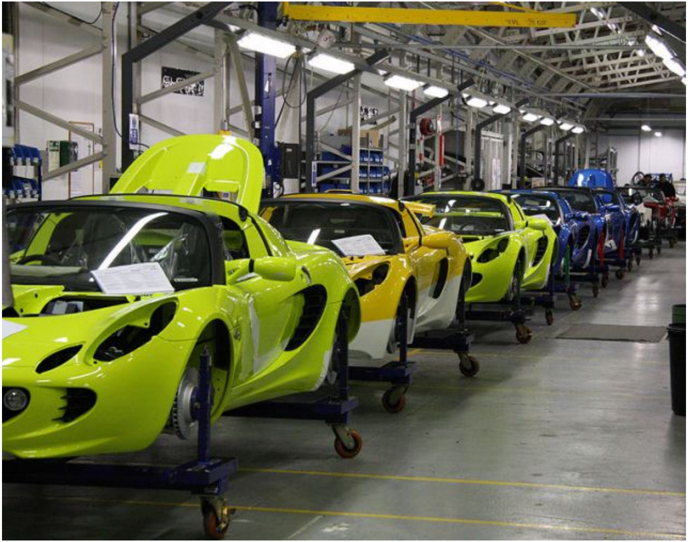
Car Sequencing



Picture from Wikipedia

Car Sequencing from the CSPLib Benchmark

Assemble a production line of cars such that capacity constraints on the workstations are not exceeded.
Notation:

- Set of Classes C
- Demand d_i for class $i \in C$
- Set of Options O
- Capacity constraint with ratio u_l/q_l for option $l \in O$
- Set $O_i \subseteq O$ that is required by class $i \in C$

Car Sequencing: Example

- $C = \{1, 2, 3\}$ with demand $3, 2, 2$
- $O = \{a, b\}$ with capacity constraints $1/2$ and $1/5$
- Class 1 no restriction
- Class 2 requires option a
- Class 3 requires option a and b

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	3	1	2	1	2	1	3
a	1	-	1	-	1	-	1
b	1	-	-	-	-	-	1

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	3	1	2	1	2	1	3
a	1	-	1	-	1	-	1
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Car Sequencing is NP-Complete

PB Model

- Boolean variable c_i^k: car $k \in C$ is at position i
- Boolean variable o_i^l: option $l \in O$ is at position i
- Demand constraints: $\forall k \in C$

$$\sum_{i=1}^{n} c_i^k = d_k$$

- Capacity constraints: $\forall l \in O$ with ratio u_l/q_l

$$\bigwedge_{i=0}^{n-q_l} (\sum_{j=1}^{q_l} o_{i+j}^l \leq u_l)$$

PB Model

And in all positions $i \in \{1 \dots n\}$ of the sequence it must hold:

- Link between classes and options: for each $k \in C$ and

$$\forall l \in O_k : \; c_i^k - o_i^l \leq 0$$
$$\forall l \in O \setminus O_k : \; c_i^k + o_i^l \leq 1$$

- Exactly one car:

$$\sum_{k \in C} c_i^k = 1$$

Modelling in CNF

- The PB model is the mostly used model in CP,IP and local search!
- This model with standard translation to CNF (minisat+,clasp ...) has bad performance
- Choose the right cardinality translation
- More redundant constraints:

$$d_l = \sum_{i=1}^{n} o_i^l = \sum_{k \in C_l} d_k$$

- Global constraint: Cardinality + Sequence

Sequential Counter: Variables

- Translation of Boolean Cardinality:

$$\sum_{i \in \{1 \dots n\}} x_i = d$$

- x_i is true iff the object is at position i
- $s_{i,j}$ is true iff in the positions $0, 1 \dots i$ the object exists at least j times

Sequential Counter

$\forall i \in \{1 \ldots n\} \; \forall j \in \{0 \ldots d+1\}$:

$$\neg s_{i-1,j} \vee s_{i,j} \tag{1}$$

$$x_i \vee \neg s_{i,j} \vee s_{i-1,j} \tag{2}$$

$\forall i \in \{1 \ldots n\} \forall j \in \{1 \ldots d+1\}$:

$$\neg s_{i,j} \vee s_{i-1,j-1} \tag{3}$$

$$\neg x_i \vee \neg s_{i-1,j-1} \vee s_{i,j} \tag{4}$$

$$s_{0,0} \wedge \neg s_{0,1} \wedge s_{n,d} \wedge \neg s_{n,d+1} \tag{5}$$

Sequential Counter: Example

$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10
3			U	U	U	U	U	U	U	U	U
2		U	?	?	?	?	?	?	?	?	L
1	U	?	?	?	?	?	?	?	?	L	
0	L	L	L	L	L	L	L	L	L		

Setting x_2 and x_7 to 1:

$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10
3			U	U	U	U	U	U	U	U	U
2		U	0	0	0	0	0	1	1	1	L
1	U	0	1	1	1	1	1	1	1	L	
0	L	L	L	L	L	L	L	L	L		

Sequential Counter: Related Work

- Carsten Sinz Sequential Counter [4]
- Fahim Bacchus translation of AMONG by the Regular constraint [1]
- Translation through BDDs [2]

Capacity Constraints: More Global

$$(\sum_{i=1}^{n} x_i = d) \wedge \bigwedge_{i=0}^{n-q} (\sum_{l=1}^{q} x_{i+l} \leq u)$$

$\forall i \in \{q \ldots n\}, \forall j \in \{u \ldots d+1\}$:

$$\neg s_{i,j} \vee s_{i-q,j-u} \tag{6}$$

Capacity Constraints: Example

Capacity constraint $4/8$, demand $d = 12$ on a sequence of 22 variables:

13																					U	U	U
12																				U	?	?	L
11																			U	?	?	L	
10																		U	?	?	L		
9													U	U	U	U	U	?	?	L			
8												U	?	?	L	L	L	L	L				
7											U	?	?	L									
6										U	?	?	L										
5					U	U	U	U	U	?	?	L											
4				U	?	?	L	L	L	L	L												
3			U	?	?	L																	
2		U	?	?	L																		
1	U	?	?	L																			
0	L	L	L																				
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
x_i								**0**	**0**							**0**	**0**						

Capacity Constraints: Example

Partial Assignment: x_1 and x_{13} to true and x_{12}, x_{14} and x_{21} to false.

13																					U	U	U
12																				U	0	0	L
11																			U	?	1	L	
10																		U	?	1	L		
9													U	U	U	U	U	?	1	L			
8												U	0	1	L	L	L	L	L				
7											U	1	1	L									
6										U	1	1	L										
5					U	U	U	U	U	1	1	L											
4				U	?	1	L	L	L	L	L												
3			U	1	1	L																	
2		U	1	1	L																		
1	U	1	1	L																			
0	L	L	L																				
$s_{i,j}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
x_i		1	**1**	**1**			**0**	(0)	(0)	**1**	**1**	**1**	0	1	0	(0)	(0)					0	**1**

A Trick for Lower Bounds ([3])

Table: Overview of options and demands for instance 300-04

class	0	1	2	3	4	5	6	7	8	9	10	11
demand	9	4	22	2	1	62	31	4	24	4	3	36
0: 1/2	-	-	-	-	-	-	-	-	-	-	-	x
1: 2/3	-	-	-	-	-	x	x	x	x	x	x	-
2: 1/3	-	-	x	x	x	-	-	-	x	x	x	-
3: 2/5	-	x	-	-	x	-	x	x	-	-	x	-
4: 1/5	x	x	-	x	x	-	-	x	-	x	x	-

class	12	13	14	15	16	17	18	19	20	**21**	22	**23**
demand	3	25	3	8	5	2	6	21	5	**7**	11	**2**
0: 1/2	x	x	x	x	x	x	x	x	x	**x**	x	**x**
1: 2/3	-	-	-	-	-	-	x	x	x	**x**	x	**x**
2: 1/3	-	-	-	x	x	x	-	-	-	**x**	x	**x**
3: 2/5	-	x	x	-	-	x	-	x	x	-	x	**x**
4: 1/5	x	-	x	-	x	x	x	-	x	**x**	-	**x**

Results: Solved Instances

	E1	E2	E3	ASP	PB
#solved UNSAT	**17**	15	**17**	10	8
#fastest UNSAT	**5**	4	4	0	4
#solved SAT	**11**	**11**	**11**	7	2
#fastest SAT	0	4	**7**	0	0

Conclusion and Future Work

- SAT can be very competitive on CP benchmarks
- SAT is very strong on proving lower bounds
- Global Constraints motivate for encodings
- Choosing the right encoding of cardinality constraints is crucial

Future work:

- Exponential encoding in the number of options?
- Theoretical analysis of the decompositions and usage in other domains

End

Thank you very much

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SAT instances

Inst(SAT)	E1	E2	E3	ASP	PB
4/72	0.14	0.17	**0.05**	6.78	-
16/81	0.38	**0.08**	0.14	13.25	-
41/66	0.06	**0.04**	0.04	2.55	123.41
26/82	0.95	**0.16**	0.21	80.06	654.80
200-01	30.43	47.70	**15.35**	1141.87	-
200-07	6.32	2.39	**1.46**	1478.01	-
300-01	143.76	10.62	**5.98**	810.23	-
300-07	59.86	28.39	**8.24**	-	-
400-05	623.30*	768.97	846.41	-	-
400-06	445.36	24.79	**16.29**	-	-
400-10	884.91	18.99	**13.50**	-	-

UNSAT instances

Inst(UNSAT)	E1	E2	E3	ASP	PBO
6/76	72.55	117.28	**57.55**	929.87	289.68
10/93	11.40	**6.48**	11.08	331.31	-
21/90	119.83	**74.01**	95.18	-	-
36/92	**16.97**	18.67	41.34	277.63	-
200-03	137.21	**24.02**	30.81	-	-
200-04	69.64	475.76	16.83	-	**1.84**
200-05	**254.03**	1337.39*	1172.38*	-	-
200-09	358.81	-	504.26	-	**3.77**
200-10	**2.10**	3.36	2.53	3.91	2.32
300-03	**99.03**	-	214.41	949.55	-
300-04	3.17	30.57	**2.03**	46.60	4.40
300-08	18.52	799.16	50.98	123.22	**13.54**
300-05	**0.37**	2.73	0.62	-	-
300-10	1.08	25.15	**0.96**	1282.09	904.31
400-03	37.05	**30.31**	31.47	-	-
400-04	13.03	185.32	**6.33**	130.33	14.47
400-09	25.75	470.60	32.90	557.60	**5.19**

lower bounds

	LB (pre)	LB (SAT)	sec	UB (SAT)	sec	LB* (known)	UB* (known)
4/72		0	0.07	0	0.07	0	0
6/76		6	209.77	6	0.10	1	6
10/93		1	18.93	3	0.53	1	3
16/81		0	-	0	0.07	0	0
19/71	2	-	-	2	1.50	2	2
21/90	2	1	93.80	2	0.11	1	2
36/92		1	38.55	1	0.07	1	2
41/66		0	-	0	0.06	0	0
26/82		0	-	0	0.14	0	0
200-01		0	-	0	7.46		0
200-02	2	-	-	2	0.86		2
200-03		3	1323.05	3	110.33		3
200-04	7	1	17.59	7	1.04		7
200-05		1	639.42	3	39.63		6
200-06	6	-	-	6	0.69		6
200-07		0	-	0	0.69		0
200-08	8	-	-	8	20.17		8
200-09	10	1	189.14	10	2.32		10
200-10	17	16	213.88	17	3.51		19
300-01		0	-	0	24.83		0
300-02		-	-	6	39.89		12
300-03	13	2	872.06	13	77.76		13
300-04	7	6	795.68	7	12.20		7
300-05	2	12	1145.39	16	1247.82		27
300-06	2	-	-	2	1559.76		2
300-07		0	-	0	6.33		0
300-08	8	1	102.26	8	1.01		8
300-09	7	-	-	7	141.56		7
300-10	3	9	863.15	13	115.67		21
400-01		-	-	-	-		1
400-02	15	-	-	15	112.36		15
400-03		10	1531.53				9
400-04	19	5	25.85	19	222.68		19
400-05		0	-	0	302.19		0

Size

Length	Variables					Clauses				
	E1	E2	E3	ASP	PB	E1	E2	E3	ASP	PB
100	27	21	27	90	14	91	83	99	243	54
200	73	60	73	335	33	256	259	291	946	127
300	136	118	136	747	48	495	524	573	2195	197
400	223	199	223	1308	63	827	907	972	3879	271

Link between Cars and Options

$\forall i \in \{1 \ldots n\}$:

$$\bigwedge_{\substack{k \in C \\ l \in O_k}} \neg c_i^k \vee o_i^l \tag{7}$$

$$\bigwedge_{\substack{k \in C \\ l \notin O_k}} \neg c_i^k \vee \neg o_i^l \tag{8}$$

$$\bigwedge_{l \in O} \left(\neg o_i^l \vee \bigvee_{k \in C_l} c_i^k \right) \tag{9}$$

Example for non GAC of E2

Example

Let $n = 5$, $d = 2$ with a capacity constraint of $1/2$, and let x_3 be true, then unit propagation does not force x_2 nor x_4 to false. Setting them to true will lead to a conflict through clauses (4) and (6) on positions 2, 3 and 4.

3				U	U	U
2		U	U	.	.	L
1	U	.	.	L	L	
0	L	L	L			
$s_{i,j}$	0	1	2	3	4	5
x_i		.	.	1	.	.