

# Maintaining GAC via UP on CNF encodings from BDDs and sDNNFs

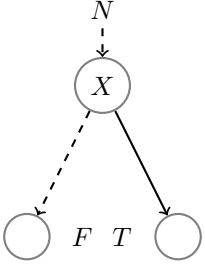
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## 1 Definitions

**Definition 1** We are working with the following structures:



- Syntax:

- BDD node  $N$  is tuple:  $\langle N, X, T, F \rangle$
- Set of nodes  $\mathcal{N}$
- Set of variables  $\mathcal{V}$  with total order  $\prec$  and successor operations ( $s(X) = X+1$ )

- Rules:

**SINK** Let  $M = \max(\mathcal{V}) + 1$ , there are two distinct unique sinks  $0, 1 \in \mathcal{N}$  with:

$$\begin{aligned} \langle 0, M, 1, 0 \rangle \\ \langle 1, M, 1, 0 \rangle \end{aligned}$$

**EXIST**  $\forall N \in \mathcal{N}$

$$\exists T, F \in \mathcal{N} \exists X \in \mathcal{V}. \langle N, X, T, F \rangle$$

**UNIQUE**  $\forall T, F \in \mathcal{N} \forall X \in \mathcal{V} \exists N_1, N_2 \in \mathcal{N}$

$$\langle N_1, X, T, F \rangle \wedge \langle N_2, X, T, F \rangle \text{ then } N_1 = N_2$$

**REDUNDANT**  $\forall H \in \mathcal{N} \forall X \in \mathcal{V}$

$$\neg \exists N \in \mathcal{N} \text{ s.t. } \langle N, X, H, H \rangle$$

**ORDER**  $\forall N, T, F \in \mathcal{N} \forall X, Y_1, Y_2 \in \mathcal{V}$

$$\begin{aligned} (\langle N, X, T, F \rangle \wedge \langle T, Y_T, *, * \rangle \wedge \langle F, Y_F, *, * \rangle) \\ \rightarrow (X \prec Y_T \wedge X \prec Y_F) \end{aligned}$$

- Semantics:

**FORMULA**

$$N \rightarrow ((X \rightarrow T) \wedge (\neg X \rightarrow F)) \quad \square$$

**Definition 2** The tuple  $(\mathcal{V}, \mathcal{N}, R)$  is

**BDD** if it obeys SINK and EXIST

**ROBDD** if it is a BDD and obeys UNIQUE, REDUNDANT and ORDER  $\square$

**Definition 3**  $[N_1, \dots, N_n]$  is a path iff  $\forall N_i$  it holds  $N_i \neq 0$  and  $((\neg X \wedge \langle N_i, X, *, N_{i+1} \rangle) \vee (X \wedge \langle N_i, X, N_{i+1}, * \rangle))$   $\square$

NOTE IN THIS DEFINITION X IS FREE, if X has domain 0,1 we could use both descendants

**Definition 4**

- $N$  is reachable  $P(N)$ , iff there exists a path  $[R, \dots, N]$
- $N$  is consistent  $C(N)$ , iff there exists a path  $[N, \dots, 1]$   $\square$

**Definition 5** BDD encoding is GAC wrt scope  $(X_1, X_2, \dots, X_n)$  and their current domains IFF  $\exists N_1, N_2, \dots, N_l \in \mathcal{N}$  such that  $\forall X_1, X_2, \dots, X_n$  it holds  $\bigwedge_{i=1}^l (C(N_i) \wedge P(N_i))$ .  $\square$

## 2 BDD GAC Encoding

**Definition 6** Let  $(\mathcal{V}, \mathcal{N})$  be a ROBDD (possible shared). The set  $\mathcal{N}$  contains the additional variables to describe a path  $N_1, N_2, \dots, N_n$  through the BDD. To construct a path we have several options to prepare it for the SAT solver, with different strength:

**ROOT** Root is reachable, the 1 sink is true and the 0 sink is false:

$$R \wedge 1 \wedge \neg 0$$

**BDD1**  $N$  is reachable and  $X = 1$  ( $X = 0$ ) then  $T$  ( $F$ ) is reachable

$$\begin{aligned}(N \wedge X) &\rightarrow T \\ (N \wedge \neg X) &\rightarrow F\end{aligned}$$

**BDD2** 1)  $N$  is reachable then one of the descendants is reachable. 2) If both descendants are not reachable, then  $N$  is not reachable

$$\begin{aligned}N &\rightarrow (F \vee T) \\ (\neg F \wedge \neg T) &\rightarrow \neg N\end{aligned}$$

**BDD3**

$$\begin{aligned}(N \wedge T) &\rightarrow X \\ (N \wedge F) &\rightarrow \neg X\end{aligned}$$

**PARENT** Let  $P_1, P_2 \dots P_k$  be all parents of node  $N$ . If non of parents is reachable then  $N$  not reachable. Let  $P_1, P_2 \dots P_k$  with  $(\langle P_i, *, N, * \rangle \vee \langle P_i, *, *, N \rangle)$  then

$$\bigwedge_{i=1}^k \neg P_i \rightarrow \neg N$$

**TOPDOWN** Let  $N_1, N_2 \dots N_k$  with  $\langle N_i, X, T_i, F_i \rangle$  and let  $Q_1, Q_2 \dots Q_l$  s.t.  $\exists Y, Z \in \mathcal{V}$  with  $Y \prec X \prec Z$  s.t.  $\langle Q_i, Z, *, * \rangle$  and  $\exists P \in \mathcal{N}$  s.t.  $\langle P, Y, Q_i, * \rangle$  or  $\langle P, Y, *, Q_i \rangle$  then

$$\bigwedge_{i=1}^l \neg Q_i \rightarrow \left( \left( \bigwedge_{i=1}^k \neg F_i \rightarrow X \right) \wedge \left( \bigwedge_{i=1}^k \neg T_i \rightarrow \neg X \right) \right)$$

**MAX** Max one is reachable in each level: Let  $N_1, N_2 \dots N_k$  with  $\langle N_i, X, *, * \rangle$

$$\bigwedge_{i \prec j} \neg N_i \vee \neg N_j$$

### 3 LOCAL KNOWLEDGE rules

I guess we can change this section to work only with sDNF, then ROBDD are a special case of it. so the rules have to be rewritten (possible simpler). (rule for local knowledge, rule for OR node, rule for AND node, rule for leaf)

This is a more general approach (could work for sDNF as well). We model intersection and union with additional local knowledge variables. Node variables mean again *being on the final path*.

**Definition 7** We define the occurring literals of a sDNF subtree with root  $N$  as follows

$$lit(N) = \begin{cases} \{L\} & \text{if } N \text{ is leaf and } L \text{ is its literal} \\ \bigcup_i lit(F_i) & F_1 \dots F_n \text{ children of } N \end{cases} \quad \square$$

Now for each node  $N$  we introduce variables  $N_Y$  for each  $Y \in lit(N)$ . They correspond to:  $N_Y$  means that literal  $Y$  is known in node  $N$ .

**GLOBAL** Local knowledge at the root is global knowledge.

$$\forall Y \in lit(R) \quad R_Y \rightarrow Y$$

**INTERSECTION** Let  $N$  be an OR node. Implications are propagated up, if they are known in all descendants  $F_1 \dots F_n$ .  $\forall Y \in lit(N)$

$$\left( \bigwedge_{i=1}^n F_{i,Y} \right) \rightarrow N_Y$$

**UNION** Let  $N$  be an AND node. Implications are propagated up, if they are known in one of the descendants  $F_1 \dots F_n$ .  $\forall Y \in lit(N)$

$$\left( \bigvee_{i=1}^n F_{i,Y} \right) \rightarrow N_Y$$

**QUESTION FOR JEAN: WHICH SDDNDDDFS BLABLABLA excludes this ?**

**Example 1** does it work for:

$$(X \vee Y) \wedge (\neg X \vee Y) \quad \square$$

## 4 SPECIAL RULES

This section contains rules for BDD encodings representing special classes of boolean functions:

**PBC WE NEED EXTRA SYMBOLS AND BOTTOM UP RULES FOR CONSISTENCY** The BDD represents a PBC. Then let  $N_1, N_2 \dots N_k$  with  $\langle N_i, X, *, * \rangle$  and  $C(N_i)$  implies  $C(N_{i+1})$  (follows from being PBCs, and level property corresponding logical function), then

$$\bigwedge_{i=1}^{n-1} C(N_i) \rightarrow C(N_{i+1})$$

## 5 UPBDD Encoding Rules

This section contains rules for the UPBDD theories. No change in strength of UP but faster and smaller encoding.

## 6 Conclusion

We showed that the two standard SAT encodings of PBC do not maintain GAC and proposed an encoding that does.