Notes on PSD and timing delays in PTAs

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1 From Power Spectral Density to Timing Delays

In PTA analyses, stochastic processes such as a gravitational-wave background, red noise, or chromatic noise are modelled in terms of a *power spectral density* (PSD), $S(f)$, defined in units of $\mathrm{s}^2\,\mathrm{Hz}^{-1}$. The PSD describes how variance in timing residuals is distributed as a function of Fourier frequency.

1.1 Power Spectral Density

For a stochastic process in timing residuals, the PSD $S(f)$ is defined such that the total variance in the residuals is obtained by integrating over frequency,

$$\sigma^2 = \int_0^\infty S(f)\,df. \tag{1}$$

The PSD itself does not represent a timing delay; rather, it gives the *variance per unit frequency*.

For a gravitational-wave background, the timing-residual PSD is related to the characteristic strain spectrum $h_c(f)$ via

$$S_{\mathrm{GW}}(f) = \frac{h_c(f)^2}{12\pi^2 f^3}, \tag{2}$$

where

$$h_c(f) = A\left(\frac{f}{f_{\mathrm{ref}}}\right)^{\alpha}, \qquad \alpha = \frac{3-\gamma}{2}. \tag{3}$$

Here A is the dimensionless strain amplitude at the reference frequency f_{ref}, and γ is the power spectral index.

1.2 Finite Datasets and Discrete Frequencies

PTA datasets have a finite timespan T. As a result, the Fourier frequencies are discrete,

$$f_k = \frac{k}{T}, \qquad k = 1, 2, \ldots \tag{4}$$

with a frequency resolution

$$\Delta f = \frac{1}{T}. \tag{5}$$

The contribution to the variance from a single Fourier frequency bin is approximately

$$\mathrm{Var}(f_k) \simeq S(f_k)\,\Delta f. \tag{6}$$

1.3 RMS Timing Delay per Fourier Mode

Rather than plotting the PSD directly, it is often more intuitive to consider the root-mean-square (RMS) timing amplitude associated with each Fourier mode. This is defined as

$$\delta t(f_k) \equiv \sqrt{S(f_k)\,\Delta f}. \tag{7}$$

This quantity has units of seconds and represents the characteristic timing delay contributed by a single Fourier frequency bin.

For the gravitational-wave background, inserting the PSD gives

$$\delta t_{\rm GW}(f) = \sqrt{\frac{A^2}{12\pi^2} f_{\rm ref}^{\gamma-3} f^{-\gamma} \frac{1}{T}}. \tag{8}$$

This expression is the quantity commonly plotted in PTA analyses (e.g., Figure 1 of NANOGrav GWB searches), where violin plots or credible bands show the posterior distribution of $\delta t(f_k)$ at each Fourier frequency.

1.4 Interpretation

The RMS delay $\delta t(f_k)$ should be interpreted as the timing amplitude associated with an individual Fourier mode, not as the total RMS of the timing residuals. The total residual variance is obtained by summing contributions from all modes,

$$\sigma_{\rm tot}^2 = \sum_k S(f_k)\,\Delta f. \tag{9}$$

Large values of $\delta t(f_k)$ at low frequencies therefore do not imply equally large total residuals, but instead reflect the steep red spectra typical of PTA noise processes and gravitational-wave backgrounds.