Portfolio Construction with Hierarchical Momentum

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KEY FINDINGS

- Integration of momentum and hierarchical clustering boosts cumulative and risk-adjusted returns and diversification.
- Clustering enhances portfolio stability, reduces drawdowns inherent in momentum portfolios, and encourages sparse allocations.
- Results are backed up by robust empirical validation through an extensive out-of-sample backtest and rigorous statistical tests, while accounting for transaction costs.

ABSTRACT

This article presents a portfolio construction approach that combines the hierarchical clustering of a large asset universe with the stock price momentum. On one hand, investing in high-momentum stocks enhances returns by capturing the momentum premium. On the other hand, hierarchical clustering of a high-dimensional asset universe ensures sparse diversification, stabilizes the portfolio across economic regimes, and mitigates the problem of increased drawdowns typically present in momentum portfolios. Moreover, the proposed portfolio construction approach avoids the covariance matrix inversion. An out-of-sample backtest on a non-survivorship-biased dataset of international stocks shows that, compared to the model-based and model-free benchmarks, hierarchical momentum portfolios achieve improved cumulative and risk-adjusted portfolio returns as well as decreased portfolio drawdowns net of transaction costs. The study further suggests that the unique characteristics of the hierarchical momentum portfolios arise because of both dimensionality reduction via clustering and momentum-based stock selection.

Efficient portfolio construction is one of the key problems in finance. The concept of a portfolio risk–return trade-off has been formalized by Markowitz (1952, 1959). This seminal approach is guaranteed to find the optimal mean–variance (MV) portfolio when the true asset means and covariances are known, which is practically impossible. Although tractable and theoretically appealing, the optimized portfolios are prone to unstable weights and concentrated allocations; see Frost and Savarino (1988), Best and Grauer (1991), and Chopra and Ziemba (1993). These issues get inflated in a high-dimensional setting where the precision of plug-in estimates deteriorates with the number of assets held in the portfolio; see Kan and Zhou (2007). Lopez de Prado (2016) advocates that the benefits of diversification can disappear because of estimation errors, which get amplified through the covariance matrix inversion typically present in the MV optimization setting. The amplification of errors worsens with the extent of ill-conditioning of the covariance matrix, which happens when the assets in the universe get more correlated—exactly when diversification is most desired (see Bailey and de Prado 2012).

MV portfolios constructed via the use of the sample mean and the sample covariance matrix generally perform poorly out of sample; see Michaud (1989). Consequently, various approaches aimed at overcoming the difficulty of portfolio selection with a large asset universe have been proposed in the existing literature. The most prevalent approaches are based on shrinkage estimation (Jorion 1986; Ledoit and Wolf 2003, 2004; De Nard 2022), imposing a factor structure for the covariance (Sharpe 1963; Chan, Karceski, and Lakonishok 1999), imposing constraints on the portfolio weights (Jagannathan and Ma 2003; Brodie et al. 2009; DeMiguel et al. 2009; Li 2015), resampling methods (Michaud 1989; Michaud and Michaud 2008), Bayesian approaches that incorporate a prior economic view (Black and Litterman 1990; Garlappi, Uppal, and Wang 2007; Lai, Xing, and Chen 2011), and modeling the asset universe using a network (Peralta and Zareei 2016; Li et al. 2018; Výrost, Lyócsa, and Baumöhl 2019).

Another strand of literature departing from the classical MV techniques investigates the presence of a hierarchical structure in financial markets. Mantegna (1999) showed that, by using only the information carried by the historical price series of stocks, it is possible to identify clusters that make economic sense, for example, stocks from the same industry. Stock clustering is performed via the construction of a hierarchical binary tree, also known as a dendrogram. Moreover, Brida and Risso (2010) show that clusters that make sense from an economic point of view can be recovered from the data of the German stock market. Tola et al. (2008) consider the problem of the statistical uncertainty of the correlation matrix and show that the use of clustering algorithms can improve the reliability of the portfolio in terms of the ratio between predicted and realized risk. Hierarchical trees, correlation-based trees, and networks obtained from a correlation matrix are also studied in Tumminello, Lillo, and Mantegna (2010).

Lopez de Prado (2016) investigated how information arising from networks can be used in portfolio selection and introduced an approach called hierarchical risk parity. Utilizing the dendrogram structure, quasi-diagonalization and recursive bisection are employed to compute the portfolio weights, without the necessity of covariance matrix inversion. Molyboga (2020) tests this approach by using more realistic assumptions for institutional investors and shows that the resulting portfolio achieves improved Sharpe ratios and reduced drawdowns. A generalization of Lopez de Prado (2016) is presented by Raffinot (2018), who proposed to perform the allocation of capital top down in the constructed tree by splitting the wealth into equal parts at each branching of the tree. Moreover, Puerto, Rodríguez-Madrena, and Scozzari (2020) propose a framework that combines clustering and portfolio optimization in a single step based on solving a mixed-integer linear programming problem.

Momentum strategies are investment approaches that attempt to exploit the persistence of asset performance, commonly known as momentum, to generate excess returns. The underlying premise of these strategies is that assets that have performed well in the recent past are likely to continue performing well, whereas assets that have performed poorly are likely to continue performing poorly. There are two primary types of momentum strategies: cross-sectional momentum and time-series momentum (TM). Cross-sectional momentum (Jegadeesh and Titman 1993; Kwon and Satchell 2018) is based on the persistence of performance ranking, where past winners tend to become future winners. On the other hand, TM (Moskowitz, Ooi, and Pedersen 2012) is based on the persistence of the performance sign, where past positive performers tend to continue performing positively in the future.

Vast empirical evidence shows that, in short to medium time horizons, the best-performing stocks tend to continue to perform well and vice versa (Chan, Jegadeesh, and Lakonishok 1996; Rouwenhorst 1998; Jegadeesh and Titman 2001, 2011). Furthermore, Asness et al. (2014) provide a review of common myths casting doubt on the validity of momentum-based strategies and provide evidence refuting each of them. Zhu and Yung (2016) investigate the coexistence and interaction of momentum and short-term reversal strategies. Another innovative approach to momentum investing is the use of deep neural networks introduced by Lim, Zohren, and Roberts (2019), which avoids explicit definitions of the trend estimator and the position sizing rule. Practical implementations of outperforming momentum-based strategies include the work of Polbennikov, Desclée, and Dubois (2020), who present realistic and implementable strategies that use quantitative signals related to momentum and value. Recent research also explores the optimal combination of time-series and cross-sectional momentum; see Schmid and Wirth (2021). Lastly, Pedersen, Babu, and Levine (2021) investigate issues with standard MV optimization and propose an enhancement involving shrinkage, which they apply to TM and industry momentum strategies and document strong outperformance compared to various benchmarks.

An approach that combines clustering with momentum is presented in Lu et al. (2018), where the authors employ clustering methods to construct central, peripheral, and dispersed portfolios. They further suggest a strategy that switches between these portfolios based on momentum and market trend prediction and manages to outperform the Markowitz portfolio on the universe of Chinese stocks. A related approach based on high-frequency data is also presented in León et al. (2017) and Wang et al. (2022). Although momentum is traditionally associated with individual stocks, some studies have also explored its connection with standard equity factors. For instance, Gupta and Kelly (2019) show that exploiting factor momentum by buying recent top-performing factors and selling poor-performing factors leads to significant excess performance compared to traditional stock momentum. Similar results have been reported by Babu et al. (2020) and Arnott et al. (2021). Momentum strategies, however, also carry higher risks, especially in volatile markets or during sudden market downturns, as highlighted by Daniel and Moskowitz (2016). Therefore, it is crucial to combine momentum strategies with effective risk management techniques to ensure robust and stable performance over time. Our approach is based on the idea of combining momentum with hierarchical clustering (HC) to ensure sparse diversification and limit risk while still capturing the momentum premium.

This article presents a portfolio construction approach that combines stock price momentum with HC of a high-dimensional asset universe. By selecting high-momentum stocks, investors aim to capture the momentum premium and boost returns. However, to limit risk and ensure sparse diversification, we employ HC of the asset universe to identify a sparse subset of assets. Specifically, we first obtain the hierarchical structure of the market and then select the stock with the highest momentum score from each cluster. The wealth is then equally split between the chosen assets, bypassing the covariance matrix inversion. We focus on constructing long-only portfolios because shorting is not always practically implementable, particularly for smaller investors. All in all, we propose a practical and replicable approach that can be used by a broad range of investors that offers a novel way to balance a momentum-based strategy with an effective risk management technique based on HC, leading to potentially more robust and stable investment performance over time.

We conduct an extensive out-of-sample backtest to empirically validate the performance of our proposed hierarchical momentum (HM) investment strategy. Our results demonstrate that the HM portfolio achieves improved cumulative and risk-adjusted returns net of transaction costs compared to both model-based and model-free benchmarks. Although momentum portfolios are often associated with increased drawdowns, we show that the HM strategy manages to capture the momentum premium without increasing risk. In fact, our approach exhibits the lowest volatility and portfolio drawdowns because of the dimensionality reduction and diversification obtained through HC. Moreover, comparing the HM portfolio to a naive momentum portfolio and a clustering portfolio based on Raffinot (2018) indicates that the unique characteristics of this portfolio arise because of both clustering and momentum-based stock selection. Furthermore, compared to the existing literature, we impose stricter constraints on the tested portfolios (e.g., realistic rebalancing frequency, a sensible number of assets in the portfolios, and transaction costs) and perform an out-of-sample backtest on a non-survivorship-biased dataset of international stocks.

METHODOLOGY

The main objective of this section is to introduce a novel portfolio construction algorithm applicable in a high-dimensional setting without the need for asset covariance matrix inversion. The algorithm consumes a set of asset returns as an input, performs dimensionality reduction via the construction of a dendrogram, and outputs a sparse momentum-based portfolio.

The methodology builds upon Mantegna (1999), Lopez de Prado (2016), and Raffinot (2018), who established that there exists a persistent hierarchical structure in the market that can be uncovered by clustering. In contrast to these papers, however, we use the hierarchical structure of the market to derive a portfolio that takes advantage of the momentum premium while alleviating the typical momentum drawback of excessive risk.

The HM portfolio construction approach can be summarized in two main steps:

1. HC: Apply a chosen distance function to a high-dimensional dataset of asset returns and compute a market distance matrix. Using the computed distance matrix, derive a binary tree (i.e., dendrogram) describing the hierarchical structure of the market.
2. HM portfolio: Construct the portfolio weights by combining the market's hierarchical structure with the assets' momentum scores, leading to a sparse and diversified momentum-based portfolio.

Hierarchical Clustering

First, one needs to define a distance matrix between financial assets based on the information contained in their historical return time series. In accordance with the existing literature, we employ a distance function derived from the Pearson correlation coefficient between asset returns. In particular, our correlation estimation approach utilizes (a rolling window of) five years of weekly asset returns expressed in a corresponding local currency.

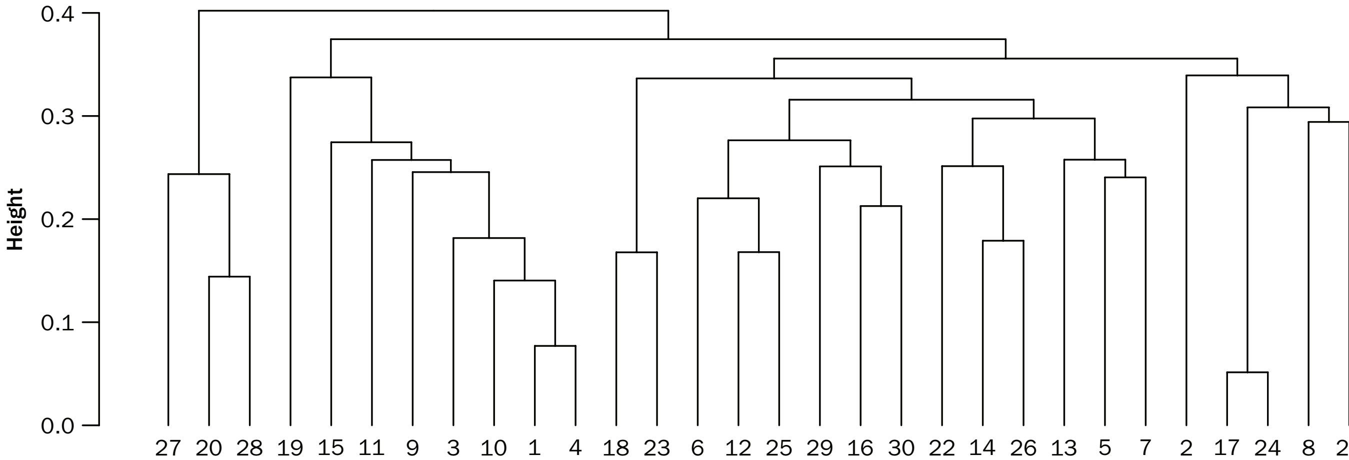
Consider a correlation ρ_{ij} between assets A_i and A_j. Because correlation by itself is not a distance, we follow the literature and define $d(A_i, A_j)$, the correlation distance between assets A_i and A_j, by

$$d(A_i, A_j) := \sqrt{\frac{1}{2}(1-\rho_{ij})}$$

For a universe of N assets, an $N \times N$ distance matrix $\mathbf{D}$ is constructed from the elements $d(A_i, A_j)$. Note that the correlation distance is small (large) among assets with a large positive (negative) correlation. This is a desired property because assets with a large positive correlation can be treated as substitutes and should consequently be clustered together.

EXHIBIT 1

An Example of a Dendrogram



The next step is to use the constructed distance matrix and determine a binary tree describing the hierarchical structure of the market. We employ the (agglomerative) HC, a clustering method that recursively combines the elements of a set bottom up; see Rokach and Maimon (2005). The result of this clustering approach is a dendrogram; see Exhibit 1.

The first step of the algorithm identifies the pair of assets A_i and A_j with the shortest distance and clusters them together. In the next step, the two rows (columns) of the distance matrix corresponding to these clustered assets are replaced by a single row (column) exhibiting the distances between all the remaining assets and the newly formed cluster. This distance is computed using the so-called linkage criterion. Following Raffinot (2018), we employ the average linkage because of its simplicity and robustness. Thereby, the distance between the newly formed cluster $\{A_i, A_j\}$ and any of the other assets A_k is computed as the average of the distances between the asset A_k and the two assets A_i and A_j:

$$d(\{A_i, A_j\}, A_k) = \frac{d(A_i, A_k) + d(A_j, A_k)}{2} \tag{1}$$

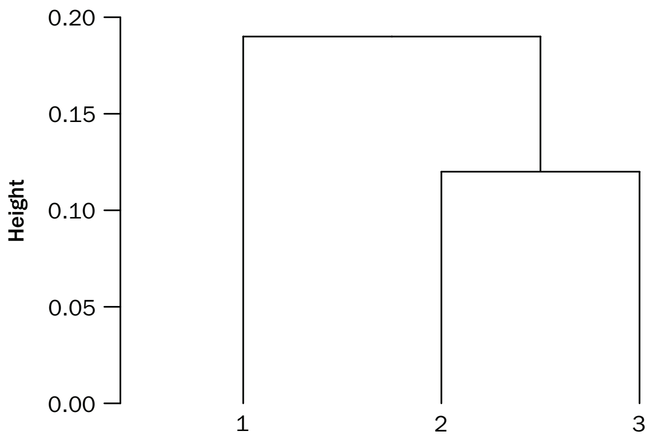
The formula extends analogously to the general case of a distance between two clusters with an arbitrary number of assets. There, the distance is equal to the average of the pairwise distances between the individual assets from the two separate clusters. The clustering procedure is iterated until the last two clusters are combined, registering the steps in a so-called linkage matrix, such as in Nielsen (2016).

The following example demonstrates the HC algorithm with average linkage on a set of assets $\{A_1, A_2, A_3\}$ with the distance matrix $\mathbf{D}_1$ given by

$$\boldsymbol{D}_1 = \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{pmatrix} 0 & 0.15 & 0.23 \\ 0.15 & 0 & 0.12 \\ 0.23 & 0.12 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array}$$

EXHIBIT 2

Dendrogram Corresponding to the Hierarchical Structure from the Example



Because assets A_2 and A_3 exhibit the shortest relative distance of 0.12, they get clustered together. The adjusted distance matrix $\mathbf{D}_2$, computed according to Equation 1, becomes

$$\mathbf{D}_2 = \begin{array}{cc} \mathbf{1} & \{\mathbf{2,3}\} \\ \left(\begin{array}{cc} 0 & 0.19 \\ 0.19 & 0 \end{array}\right) & \begin{array}{c} \mathbf{1} \\ \{\mathbf{2,3}\} \end{array} \end{array}$$

In the last step, asset A_1 and the cluster $\{A_2, A_3\}$ are merged.

The clustering steps are recorded in a linkage matrix. This matrix has two columns and, for a set of N assets, it has $N - 1$ rows. Row l contains the indexes c_1, c_2 of the two elements that were clustered in step l of the algorithm. If the entry $c_i < 0$, it refers to the asset $-c_i$. If the entry $c_i > 0$, it refers to the cluster that was formed in step c_i. The linkage matrix $\mathbf{L}$ corresponding to the earlier example is given by

$$\boldsymbol{L} = \begin{pmatrix} -2 & -3 \\ -1 & 1 \end{pmatrix}$$

and the resulting dendrogram is shown in Exhibit 2.

HM Portfolio

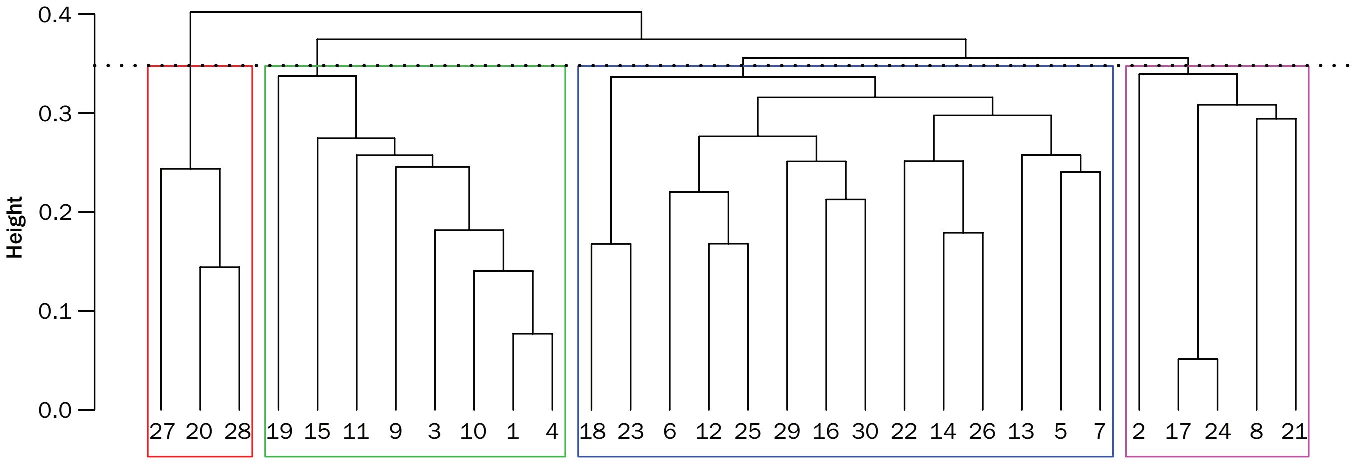
The final step of our approach takes into account the dendrogram describing the hierarchical structure of the market and determines the portfolio weights. In the construction of portfolio weights, we follow the two main objectives: (1) utilizing the momentum premium to boost returns while integrating both cross-sectional and TM and (2) taking into account the hierarchical structure of the market in order to ensure sparse diversification and limit risk—addressing the typical problem of increased drawdowns of momentum portfolios.

To determine the portfolio weights, we first employ the constructed dendrogram to derive a partitional clustering of the market with, for example, $n \in \mathbb{N}$ clusters. To do so, we slice the dendrogram horizontally at a height $h \in \mathbb{R}_+$, splitting the tree into n subtrees representing the n desired clusters. Note that, for every n, such that $1 \leq n \leq N$, there exists a height h that enables partitional clustering with exactly n clusters. Exhibit 3 provides an example of partitional clustering on a given dendrogram for $n = 4$. The corresponding h is depicted with a dotted line.

Next, we employ the n-cluster partition of the market with clusters C_i for $i = 1, \ldots, n$. We define the momentum score M_k of an asset A_k as its cumulative total return (i.e., including dividends) over the last year. Then, within each cluster, we select the asset with the highest momentum score M_i^*—notice the connection to the cross-sectional momentum. Specifically, we identify these assets as $M_i^* = max_{k \in C_i} M_k$ for $i = 1, \ldots, n$. Sparsity (i.e., dimensionality reduction) is facilitated through the choice of n. Portfolio weights are then constructed such that the wealth is split equally between the chosen assets, while assets with a negative momentum score, $M_i^* < 0$, are given a zero weight—notice the connection to the TM. Meaning, if $m \in \mathbb{N}$ out of n initially chosen assets have a negative momentum score M_i^*, the wealth is split equally between

EXHIBIT 3

Partitional Clustering for $n = 4$



the remaining $n - m$ assets. Consequently, the proposed approach yields a long-only asset allocation strategy and, for a sufficiently large n, prevents the issue of portfolio concentration. We focus on constructing long-only portfolios because shorting is not always realistically implementable, particularly for smaller investors, and because the goal of the article is to provide a practical and replicable approach that can be used by a broad range of investors.

Note that one could also search for the optimal value of the hyperparameter n. In the appendix, however, we show that the proposed approach is robust to the choice of n and such an optimization could only provide a minor improvement while increasing the probability of overfitting. Furthermore, various weighting schemes, such as momentum or inverse-variance weighting, could also be used to construct the portfolio after the stocks with the highest momentum scores from each cluster have been determined. However, the momentum weighting could lead to excessive volatility and portfolio concentration, whereas with the inverse-variance weighting, stocks with the highest momentum scores could receive the lowest portfolio weight, which could hamper the portfolio performance. Moreover, the alternative weighting schemes could substantially increase portfolio turnover. Therefore, we decided to employ simple equal weighting in our analysis.

EMPIRICAL ANALYSIS

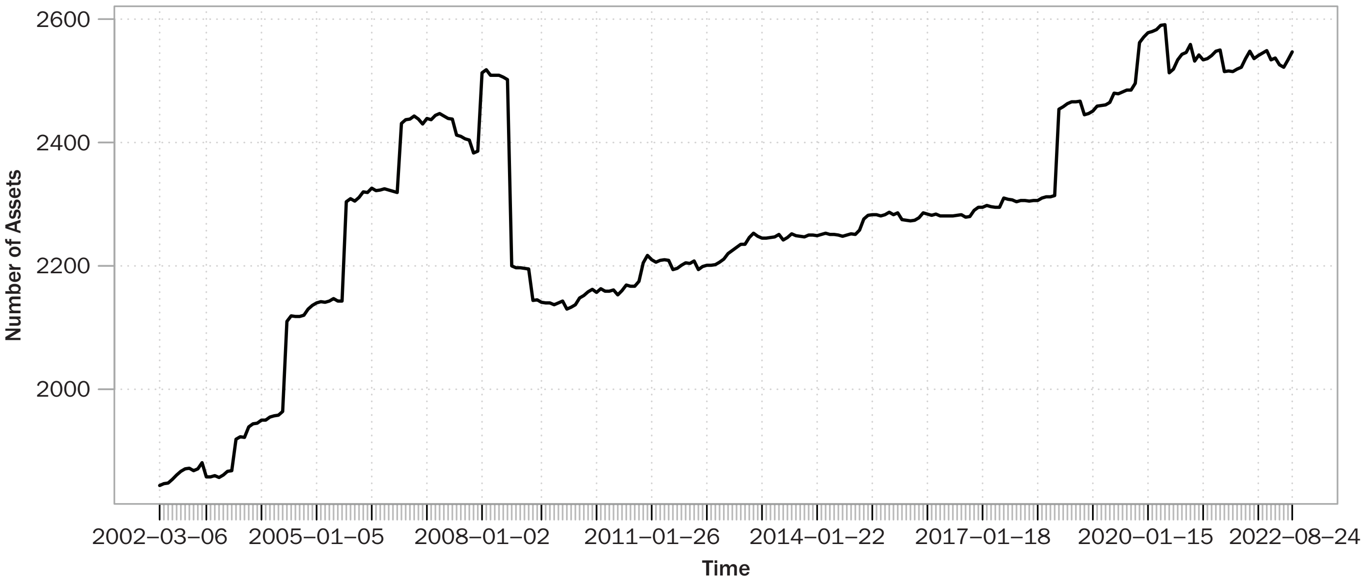
Data

The empirical analysis utilizes a high-dimensional dataset of stocks included in the MSCI All Country World Index (ACWI) for the period from June 1997 to August 2022. These stocks represent an extensive asset universe consisting of large- and mid-cap stocks across 23 developed and 24 emerging markets. The data are sourced from Bloomberg and MSCI.

We change the size of the asset universe over time by taking into consideration when different assets enter and exit the universe. Thereby, we only allow investment into assets that are, at a given time, members of the MSCI ACWI index. Hence, no forward-looking information (e.g., a stock entering the index or an index member exiting the index in the future) is exploited in the analysis, which is relevant for the replicability of our method in real-life investing. Furthermore, we only allow investment

EXHIBIT 4

Number of Eligible Assets in the Employed Dataset



into assets for which (1) at least five years of historical returns are available and (2) trading has not been halted for more than two weeks in the last five years. In this way, we construct a high-dimensional non-survivorship-biased dataset of eligible assets. As shown by Alves and Filipe (2021), the momentum effect in their dataset disappears when correcting for survivorship bias. To avoid such forward-looking bias that drives momentum-based strategies, it is key to work with the assembled non-survivorship-biased dataset.

Exhibit 4 displays the number of eligible assets from the MSCI ACWI index over the time of the backtest. The distributions of yearly logarithmic returns of the eligible assets, showing the dynamics of the market returns across different economic regimes, are depicted in Exhibit A1 in the appendix.

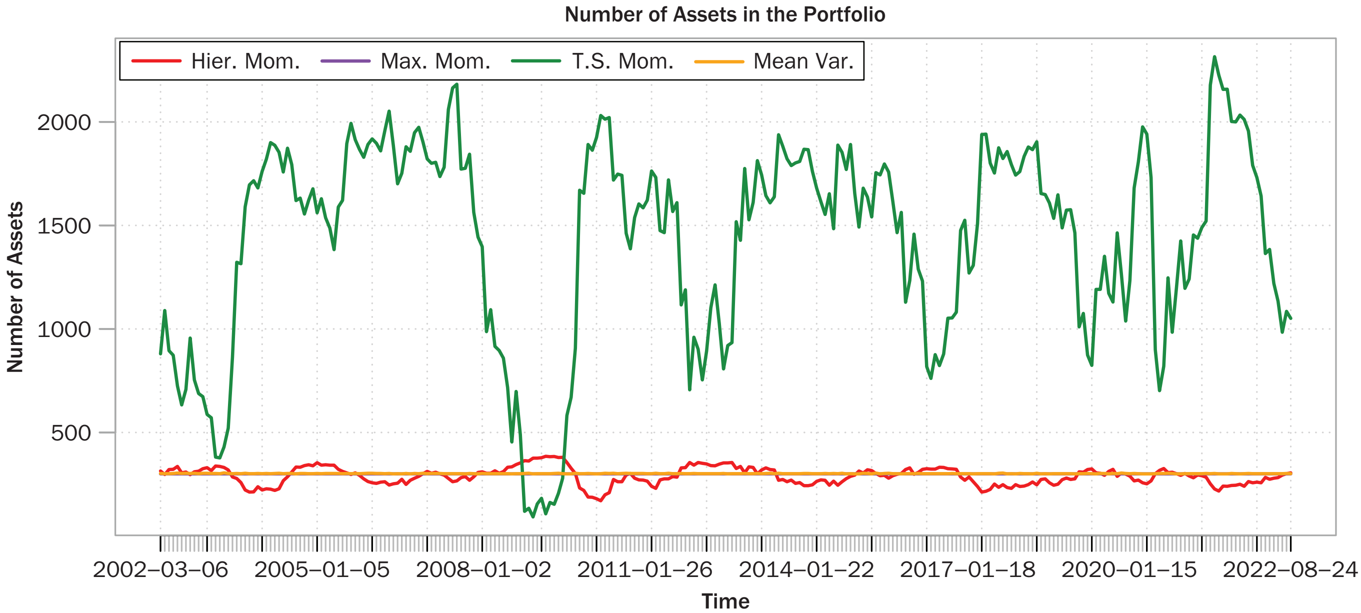
Out-of-Sample Backtest

Next, we consider an out-of-sample backtest of the proposed HM portfolio construction strategy and compare its performance to various benchmark strategies. We study five model-based portfolio construction strategies:

1. HM: The baseline model presented in the previous section
2. Maximum momentum (MM): A model that does not utilize the hierarchical structure of the asset universe and simply invests in the equally weighted portfolio of top *n* assets based on the momentum score
3. TM: A model that does not utilize the hierarchical structure of the asset universe and invests in the equally weighted portfolio of all assets with a positive momentum score
4. Hierarchical Raffinot (HR): A model that utilizes the hierarchical structure of the asset universe to find a diversified weighting by distributing wealth equally to each cluster hierarchy. Referring to Exhibit 3, this corresponds to $\frac{1}{2}$ of wealth in the red cluster, $\frac{1}{4}$ of wealth to the green cluster, and $\frac{1}{8}$ of wealth

EXHIBIT 5

Number of Selected Assets for the Investigated Model-Based Portfolio Allocation Strategies



in the blue and purple clusters. The same wealth splitting is then further employed within each cluster; see Raffinot (2018) for more details.

5. MV: Long-only Markowitz portfolio with risk-aversion parameter $\lambda = 3$ and maximum portfolio weight constraint equal to $1/n$ to avoid portfolio overconcentration and impose a comparable number of assets to the HM and MM models.

Moreover, we compare the model-based strategies to two model-free strategies, namely the equally weighted portfolio and capitalization-weighted portfolio of all eligible assets.

Although the model-free strategies, as well as the HR strategy, assign weights to the entire eligible asset universe consisting of $N \in \mathbb{N}$ assets, as depicted in Exhibit 4, the number of stocks in the HM strategy is controlled at the step of slicing the tree by specifying the chosen number of clusters.

Exhibit 5 shows the final number of selected assets over time for the HM, MM, TM, and MV strategies. We chose $n = 300$ for the MM and MV strategies. The corresponding number of chosen assets by construction stays constant over time for the MM strategy and nearly constant for the MV strategy, where this number is controlled via the maximum portfolio weight constraint equal to $1/n$. The number of selected assets varies over time for the HM and TM strategies because assets with a negative momentum score are given a zero weight. Although we do not restrict the number of assets in the TM strategy, we choose $n = 450$ clusters for the HM strategy. That yields approximately 300 assets in the HM portfolio over time and ensures the comparability of the model-based strategies; see Exhibit 5. In the appendix, we show that the empirical results are robust to the alternative choices of n. Moreover, notice that the number of assets in the HM portfolio increased during the Global Financial Crisis of 2008–2009. Because correlations generally increase in a crisis period, the increased diversification is a favorable feature of the HM strategy.

EXHIBIT 6

Out-of-Sample Portfolio Performance

	Model-Based Strategies					Model-Free Strategies	
	HM	MM	TM	HR	MV	Equally Weighted	Market Cap
Cum. Ret.	**659.74%**	585.45%	408.32%	253.79%	596.34%	449.19%	274.52%
Mean Ret.	**10.38%**	10.32%	8.53%	7.85%	10.32%	9.15%	7.50%
Volatility	**12.87%**	15.86%	13.26%	19.46%	15.35%	15.00%	16.02%
Sharpe Rat.	**0.81**	0.65	0.64	0.40	0.67	0.61	0.47
Sortino Rat.	**1.06**	0.86	0.86	0.57	0.89	0.82	0.63
Max. Draw.	**–50.75%**	–59.94%	–57.54%	–61.17%	–58.98%	–59.53%	–58.38%
Turnover	635.25%	643.15%	393.32%	1137.73%	641.67%	87.23%	–

NOTES: This exhibit compares the performance of different model-based and model-free portfolio construction strategies based on an out-of-sample backtest. The exhibit presents results for key performance measures, namely, realized cumulative portfolio return, annualized portfolio return mean, volatility, Sharpe and Sortino ratios, maximum drawdown, and turnover. The results are presented with a correction for transaction costs of 20 basis points.

Throughout the momentum-based strategies, we compute the momentum scores as the assets' cumulative total returns in their local currencies over the last year. For the MV strategy, we estimate the employed mean vectors using one year of weekly logarithmic total asset returns. Given the additivity of logarithmic returns, the estimation in the MV strategy is consistent with the momentum-based strategies. On the other hand, the hierarchical (or covariance for MV) structure utilizes five years of weekly logarithmic asset returns. The portfolio performance is then expressed in US dollars, while no currency hedging is performed. However, note that our approach also allows for simple inclusion of a currency overlay strategy managing the currency risk of the internationally diversified portfolio, as presented in Ulrych and Vasiljević (2020) and Polak and Ulrych (2021). The portfolios are rebalanced on a monthly frequency and transaction costs of 20 basis points are assumed. All realized performance metrics that follow are expressed net of transaction costs.

Exhibit 6 presents the out-of-sample realized portfolio performance metrics (expressed in annualized terms) for the analyzed model-based portfolio construction strategies in comparison to the model-free benchmarks. It is striking to observe that the proposed HM strategy outperforms all the other strategies in terms of pure returns (i.e., cumulative and mean return), risk-adjusted returns (i.e., Sharpe and Sortino ratios), and risk (i.e., volatility and maximum drawdown). Moreover, the HM strategy produces the lowest turnover among the four model-based strategies. By design, the model-free strategies display substantially lower turnover compared to the model-based strategies. Note that the market capitalization-weighted portfolio is by construction a zero-turnover strategy unless the constituents of the universe change, which has minimal effect. An interested reader can observe the portfolio performance metrics before transaction costs and the relative weights of investment in various sectors (as computed by MM and HM strategies) in the appendix.

Overall, as depicted in Exhibit 6, the model-based strategies tend to outperform the model-free benchmarks, with the exception of the HR strategy. This shows that employing only the hierarchical structure of the market to construct portfolios is not practical nor well-performing—the portfolio is likely to become excessively concentrated and always invests in the whole available asset universe. On the other hand, the MM (as well as TM) strategy produces a reasonably well-performing portfolio, with the well-known problem of possible large drawdowns. Our empirical analysis shows that this problem is alleviated when momentum is combined with HC—the realized maximum drawdown is decreased by 9.19%, which is equivalent to a relative

EXHIBIT 7

Sharpe Ratio Difference Test: P-Values

	HM	MM	TM	HR	MV	Equally Weighted
MM	**6.48%**					
TM	**1.07%**	88.98%				
HR	**12.01%**	36.96%	38.17%			
MV	**9.29%**	15.75%	68.15%	32.38%		
Equally Weighted	**3.84%**	73.03%	66.56%	47.27%	61.37%	
Market Cap	**0.31%**	17.23%	6.12%	90.27%	13.34%	4.73%

NOTES: This exhibit displays the results of the robust hypothesis test for the difference in Sharpe ratios between two investment strategies proposed by Ledoit and Wolf (2008). The null hypothesis assumes that the Sharpe ratios of the two strategies are equal. A P-value of *x*% indicates that the null hypothesis can be rejected at the *x*% significance level.

reduction of 15.33%. Not only the maximum drawdown is reduced, but also the whole distribution of drawdowns is improved; see Exhibit A3 in the appendix. Hence, the outperformance of the HM strategy is driven by considering HC and momentum-based stock selection jointly.

In order to statistically compare the risk-adjusted performance of the investment strategies from Exhibit 6, we perform a test for the difference in Sharpe ratios for each pair of the considered strategies. To this end, we use the robust performance hypothesis testing approach proposed by Ledoit and Wolf (2008). This test assumes that, under the null hypothesis, the Sharpe ratios of the two strategies are equal.

As shown in Exhibit 7, many pairs of strategies do not exhibit statistically significant differences in their Sharpe ratios. This is expected, as the Sharpe ratio estimates are known to be highly variable. We find that the HM strategy, however, has statistically significantly different Sharpe ratios than all other strategies at the 10% level of significance, except for the HR strategy against which it attains a P-value of 12%. Furthermore, the HM strategy consistently exhibits the lowest P-values compared to all other strategies. Exhibit 7 demonstrates that the proposed HM strategy effectively improves the risk-adjusted portfolio performance as measured by the Sharpe ratio. For interested readers, we provide the results from a robust test for differences in variances in the appendix.

Next, we perform time-series regressions of the excess returns generated by the analyzed investment strategies against the explanatory variables from the five-factor model of Fama and French (2015) with the additional momentum variable. This approach allows us to assess how much of the strategies' performance can be explained by these well-studied factors. The factor data, available at a daily frequency, are freely available online. The regression includes six factors, as described in the following:

- Market (MKT): The excess return of the market portfolio over the risk-free rate
- Size (SMB): The difference in returns between small and large companies
- Value (HML): The difference in returns between high book-to-market (i.e., value) and low book-to-market (i.e., growth) stocks
- Profitability (RMW): The difference in returns between diversified portfolios of stocks with robust and weak profitability
- Investment (CMA): The difference in returns between diversified portfolios of stocks of low- and high-investment firms, also known as conservative and aggressive
- Momentum (MOM): The difference in returns between stocks with high and low prior returns

EXHIBIT 8
Time-Series Regressions

Variable	HM	MM	TM	HR	MV	Equally Weighted
Alpha (in %)	1.87*	0.07	–0.46	0.88	0.18	1.13
	(1.69)	(0.05)	(–0.51)	(0.227)	(0.125)	(1.26)
MKT	0.86***	1.02***	0.91***	0.61***	0.99***	0.98***
	(144.95)	(129.74)	(187.29)	(29.35)	(129.61)	(204.46)
SMB	0.57***	0.70***	0.51***	0.54***	0.68***	0.59***
	(45.86)	(42.54)	(50.53)	(12.39)	(42.24)	(58.96)
HML	0.13***	0.23***	0.21***	0.07	0.21***	0.27***
	(8.80)	(12.40)	(17.96)	(1.39)	(11.69)	(23.51)
RMW	0.22***	0.08***	0.19***	0.34***	0.13***	0.18***
	(12.29)	(3.41)	(13.19)	(5.39)	(5.69)	(12.48)
CMA	–0.16***	–0.29***	–0.09***	0.06	–0.28***	–0.21***
	(–7.76)	(–10.76)	(–5.23)	(0.80)	(–10.51)	(–12.60)
MOM	0.21***	0.48***	0.23***	0.08***	0.46***	0.06***
	(28.54)	(48.76)	(38.40)	(2.92)	(48.18)	(–9.40)
Adjusted R-squared	0.84	0.82	0.90	0.16	0.82	0.93

NOTES: This exhibit presents the results of time-series regressions of the daily excess returns of various investment strategies on daily excess returns of a set of explanatory factor variables from Fama and French (2015). The sample period spans from June 1997 to August 2022. Alpha represents the intercept of the daily excess return regression and is expressed in annualized percentage points. We report the coefficient estimates, along with their associated *t*-statistics. Coefficients that reject the null hypothesis of zero at the 10%, 5%, and 1% significance level are marked with one, two, or three asterisks, respectively. The values in the brackets are the *t*-statistics, which are mentioned in the caption.

The results of the time-series regressions of the excess returns produced by the studied investment strategies on the aforementioned factor returns are reported in Exhibit 8. Alpha denotes the regression intercept and is reported in annualized form. The corresponding *t*-statistics are presented below the coefficient estimates. We use the capitalization-weighted market returns (described earlier) as the MKT factor, given that our asset universe is larger compared to Fama and French (2015).

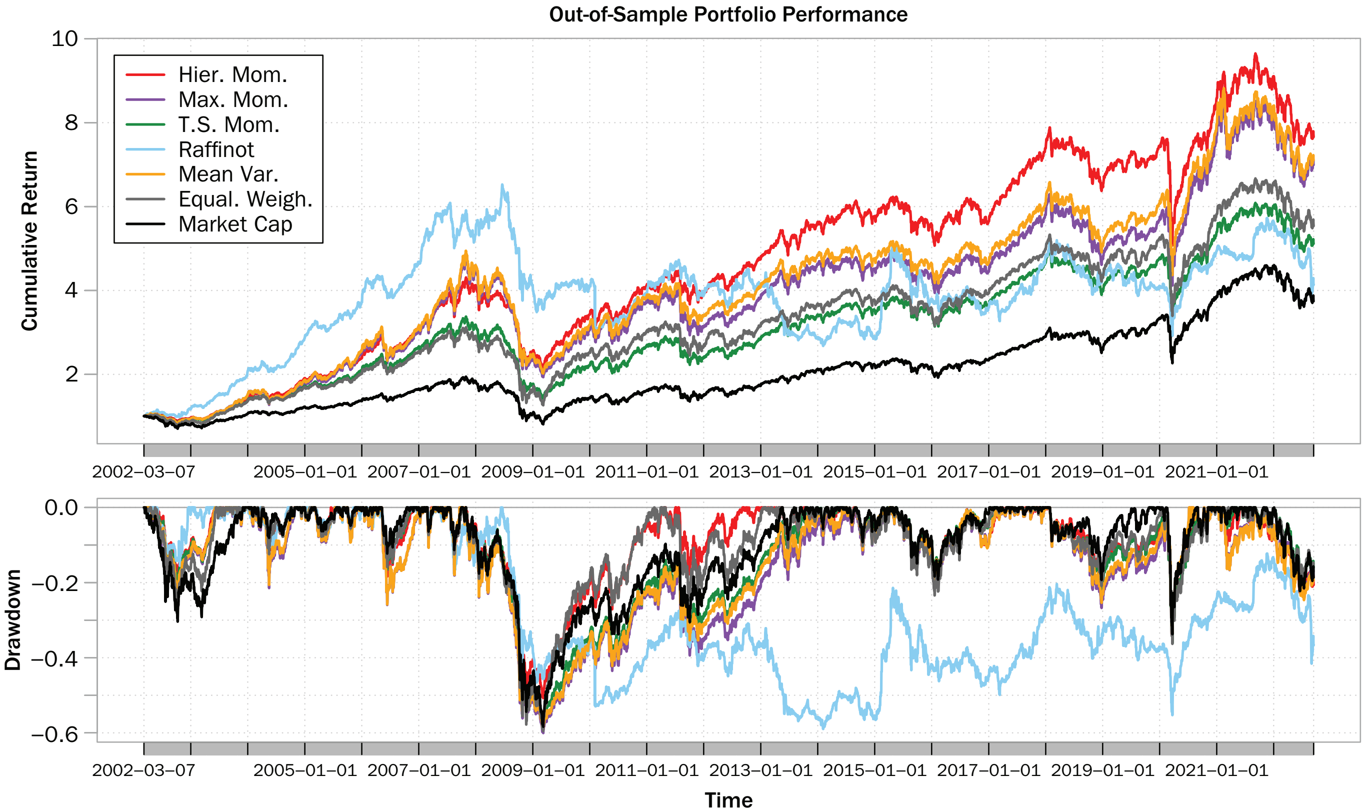
The main objective of this analysis is to understand which factors contribute to the performance of each investment strategy. The momentum-based strategies are expected to have positive loadings on the momentum factor, which is indeed the case, and these loadings are highly statistically significant. Given that the mean estimator of the MV strategy reflects momentum (and the maximum portfolio weight constraint is equal to $1/n$), the same holds for the MV strategy.

If a strategy's excess returns are fully explained by the given factors, then the strategy has not generated any unique alpha or risk-adjusted returns. If the strategy's excess returns are not fully explained by the factors, however, then the strategy has generated alpha, and one can conclude that the strategy has outperformed the benchmark. Exhibit 8 shows that, except for the proposed HM strategy, none of the studied investment strategies attains a statistically significant positive alpha. The annualized alpha of 1.87% for the HM strategy is statistically significant at the 10% level.

Caution should be exercised when interpreting these results because of the unique characteristics of the HM strategy. The strategy's long-only nature, in contrast to the long–short factors employed in the time-series regressions, imposes constraints that may limit its alpha potential. However, the HM strategy gains an

EXHIBIT 9

Cumulative Portfolio Returns and Realized Drawdowns



NOTES: The upper part of this exhibit portrays the cumulative portfolio returns for the analyzed portfolio optimization strategies. The lower part depicts the realized drawdowns of the analyzed portfolio construction strategies. All reported values are adjusted for transaction costs.

advantage from utilizing global stocks, which expands the asset universe and presents greater opportunities for capturing momentum effects. This advantage enhances the alpha potential of the HM strategy. To account for the differences in our dataset and the asset universe from Fama and French (2015), we repeat the time-series regressions in the appendix, employing factor portfolio return data from Jensen, Kelly, and Pedersen (2023) encompassing 93 countries. Notably, the HM strategy again demonstrates a statistically significant alpha, further underscoring its potential to deliver superior risk-adjusted returns.

Last, the upper part of Exhibit 9 depicts the cumulative portfolio returns of the investigated model-based and model-free portfolio allocation strategies. Observe the outperformance of the model-based strategies in comparison to the model-free benchmarks, again except for the concentrated and hence risky HR strategy. Moreover, the HM strategy tends to outperform the rest of the strategies, especially after the Global Financial Crisis. This outperformance is driven by both dimensionality reduction and diversification via HC (i.e., stabilizing the portfolio across different economic regimes) and momentum-based stock selection (i.e., capturing the momentum premium).

It is worth noting that the out-of-sample outperformance of the HM strategy in terms of cumulative returns does not come at the expense of increased risk. The lower part of Exhibit 9 shows the realized portfolio drawdowns. Although the MM and HR strategies exhibit substantial drawdowns, the HM strategy mitigates this problem and demonstrates on average noticeably lower drawdowns. This shows that HC enables risk mitigation by constructing a sparse and diversified portfolio.

CONCLUSION

Sparse diversification is essential for constructing portfolios that outperform the market. The global asset universe presents a high-dimensional setting for which the standard portfolio optimization methods based on asset return covariance matrix estimation and inversion underperform out of sample. Dimensionality reduction methods provide a low-dimensional representation of the high-dimensional setting while retaining meaningful information from the original data.

This article presents a portfolio construction framework in which dimensionality reduction and sparse diversification are performed by exploiting a hierarchical structure of the global equity market. The hierarchical structure of the market is recovered via HC, a clustering method that recursively combines financial assets with the shortest distance. This distance is defined through a metric based on correlation. After the hierarchical structure of the market has been obtained, momentum information is incorporated into the portfolio construction step by selecting the stock with the highest momentum score from each cluster. By doing so, we aim to seize the momentum premium and boost portfolio returns. The resulting portfolio weights are determined by splitting the wealth equally between the chosen assets. Therefore, the presented method provides a practically implementable and computationally feasible alternative to the standard approaches that suffer in the high-dimensional setting.

In the out-of-sample backtest, we compare the proposed HM strategy to various model-based and model-free benchmarks. We demonstrate that the presented HM portfolio outperforms all benchmarks in terms of improved cumulative and risk-adjusted portfolio performance. In particular, the proposed HM investment strategy significantly improves the Sharpe ratio as measured by the robust performance hypothesis testing approach from Ledoit and Wolf (2008). This shows that exploiting both clustering and momentum contributes to the outperformance of the HM strategy. This outperformance is hence driven by capturing growth potential via investing in high-momentum stocks while diversifying and stabilizing the portfolio across economic regimes via HC. Moreover, the out-of-sample outperformance of the HM strategy does not come at the expense of increased risk, measured by realized volatility and portfolio drawdowns. The diversification driven by HC mitigates the risk of increased drawdowns otherwise present in momentum-based portfolio strategies.

The empirical results show that a distance measure based on correlation carries valuable information about the hierarchical structure of a high-dimensional universe of non-survivorship-biased assets and that this information can be uncovered using HC. Moreover, the hierarchical structure is persistent enough to allow for the construction of sparse and diversified portfolios. Enhancing such an approach with an asset selection criterion based on momentum information allows for the construction of a significantly outperforming portfolio obtained without the high-dimensional covariance matrix inversion and the corresponding error amplification.

APPENDIX

ADDITIONAL EXHIBITS

EXHIBIT A1

Distribution of Yearly Logarithmic Returns of the Eligible Assets over Time

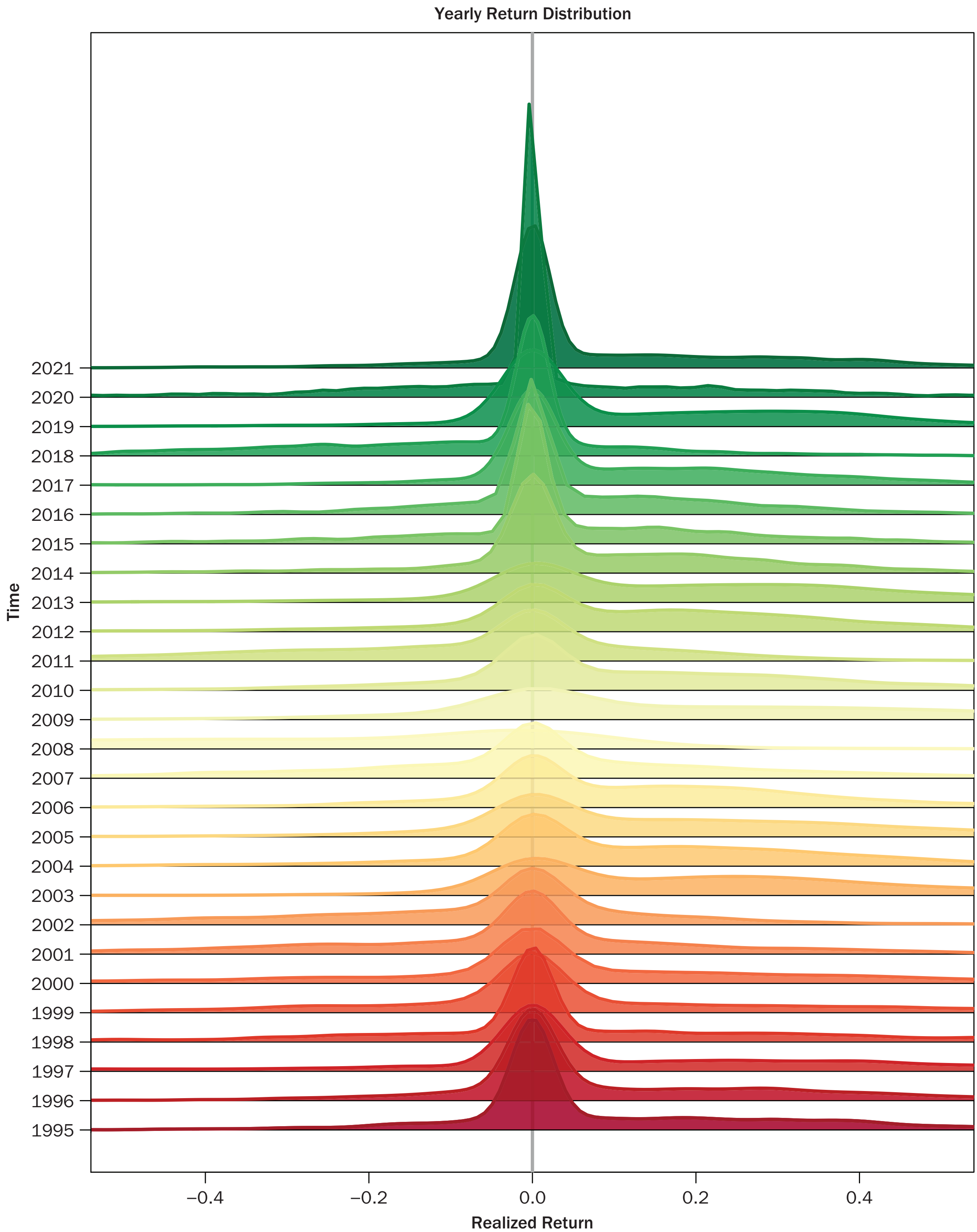
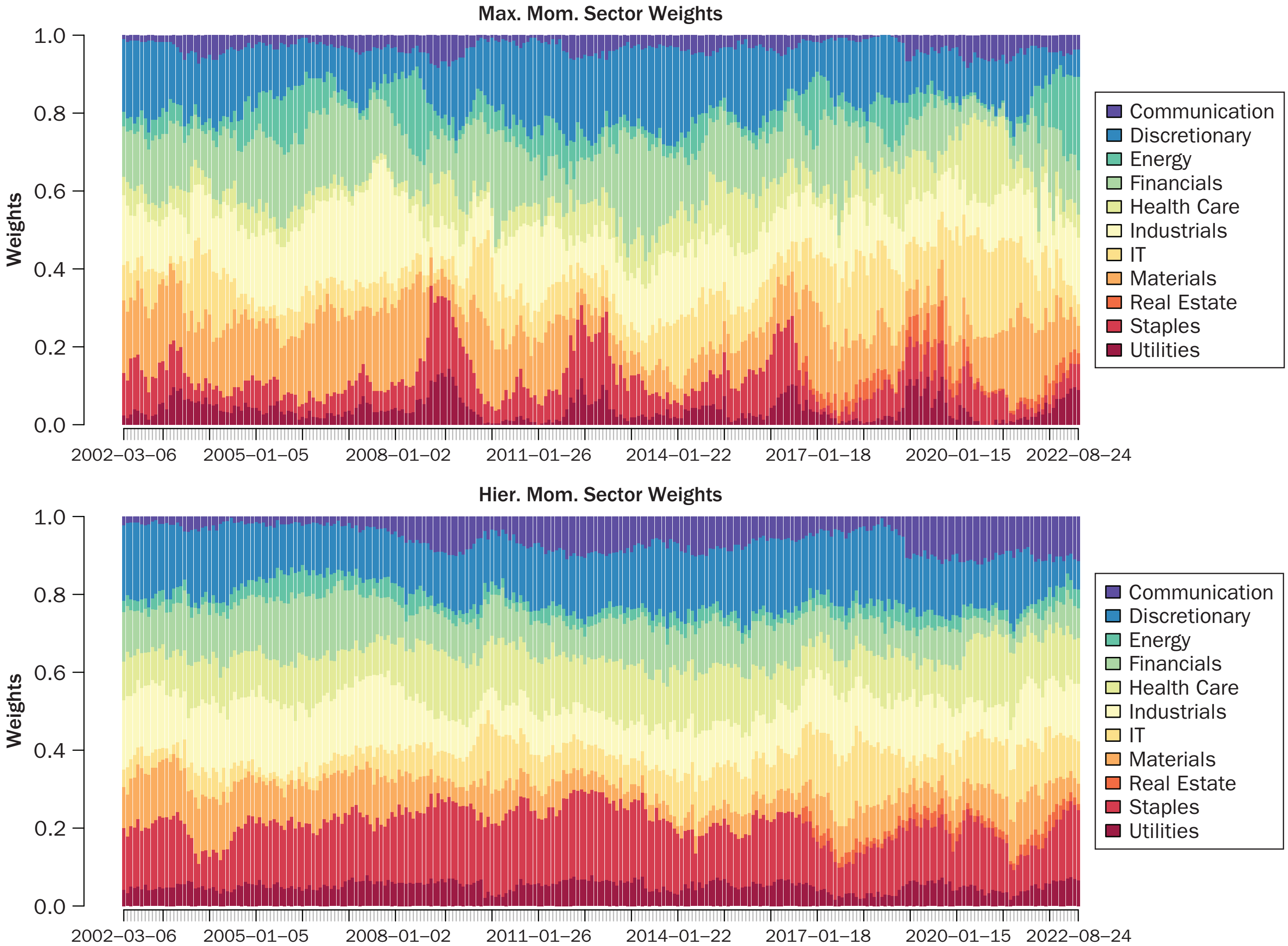


EXHIBIT A2

Sector Allocation of the Analyzed Investment Strategies



To provide a graphical description of the dataset, Exhibit A1 depicts the distributions of yearly logarithmic returns from 1995 to 2021. In order to focus on the core part of the distribution, we cut the extreme tails of the distributions. This exhibit allows us to grasp the dynamics of the market returns across the different economic regimes experienced over time. Notice how diverse those distributions are. Some years (e.g., 1999 and 2011) show reasonably symmetric distributions. More often, those distributions are skewed and have either a heavy left tail (e.g., 2002, 2011, 2018) or a heavy right tail (e.g., 2003, 2013, 2019). The most extreme distributions are observed in 2008 and 2020.

Exhibit A2 portrays the portfolio weights invested in various sectors as computed by MM and HM strategies over the time of the backtest. Notice the increased stability of the HM strategy driven by the portfolio construction based on HC.

In addition to the lower part of Exhibit 9 illustrating the realized portfolio drawdowns, Exhibit A3 depicts the portfolio drawdown boxplot. Observe especially the tails of the drawdowns for different portfolio construction strategies. In particular, the HM investment strategy tends to reduce more risk in comparison to the other analyzed strategies. This risk reduction is driven by sparse diversification designed via HC. This exhibit shows that the improved cumulative and risk-adjusted portfolio performance of the HM strategy does not come at the expense of increased risk.

EXHIBIT A3

Boxplot of Portfolio Drawdowns Given the Analyzed Asset Allocation Strategies

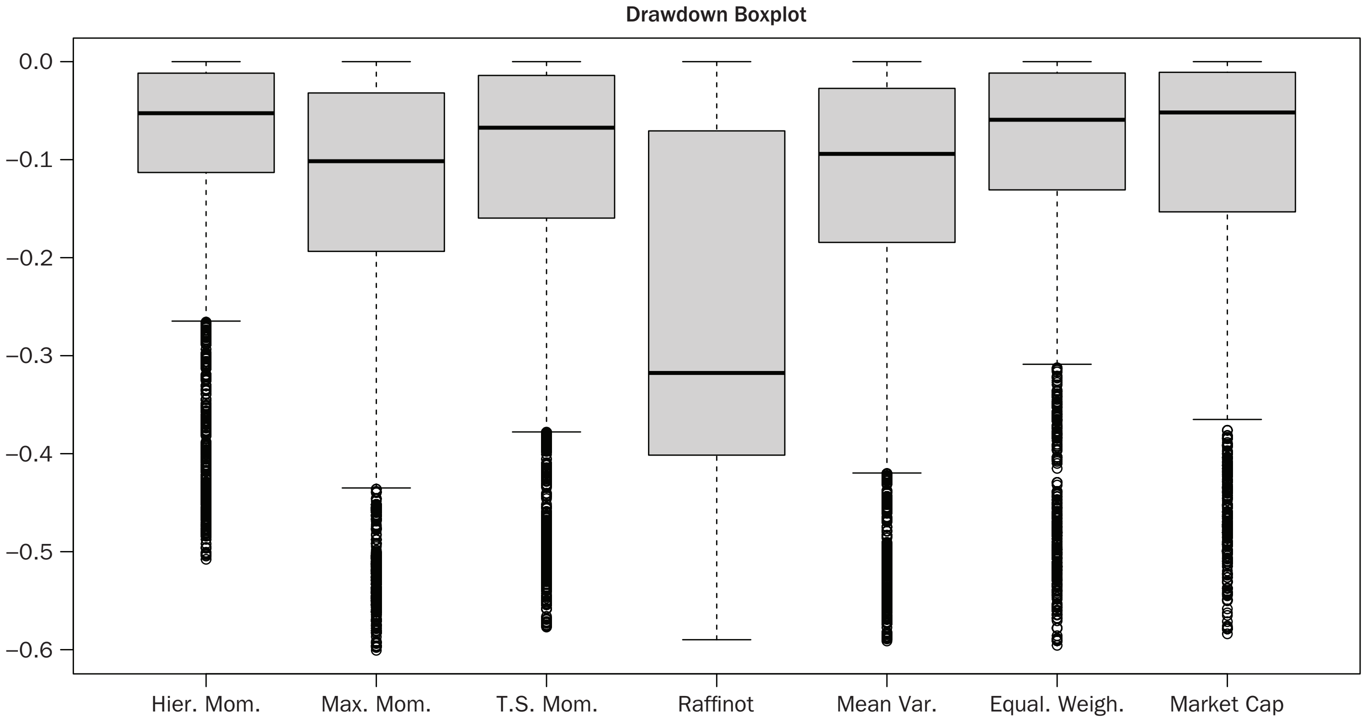


EXHIBIT A4

Out-of-Sample Portfolio Performance

	Model-Based Strategies					Model-Free Strategies	
	HM	MM	TM	HR	MV	Equally Weighted	Market Cap
Cum. Ret.	**910.05%**	814.39%	506.73%	488.48%	828.35%	472.05%	274.52%
Mean Ret.	**11.72%**	11.68%	9.37%	10.24%	11.67%	9.34%	7.50%
Volatility	**12.87%**	15.86%	13.25%	19.45%	15.35%	15.00%	16.02%
Sharpe Rat.	**0.91**	0.74	0.71	0.53	0.76	0.62	0.47
Sortino Rat.	**1.20**	0.97	0.94	0.74	1.00	0.84	0.63
Max. Draw.	**–49.75%**	–59.19%	–56.53%	–56.07%	–58.21%	–59.37%	–58.38%
Turnover	635.25%	643.15%	393.32%	1137.73%	641.67%	87.23%	–

NOTES: This exhibit compares the performance of different model-based and model-free portfolio construction strategies based on an out-of-sample backtest. The exhibit presents results for key performance measures, namely, realized cumulative portfolio return, annualized portfolio return mean, volatility, Sharpe and Sortino ratios, maximum drawdown, and turnover. In comparison with Exhibit 6, the results here are presented without adjustment for transaction costs.

Exhibit A4 presents the out-of-sample realized portfolio performance metrics before transaction costs for the analyzed model-based portfolio construction strategies in comparison to the model-free benchmarks. It is equivalent to Exhibit 6 presented in the main part of the article, which shows the performance results after a correction for transaction costs of 20 basis points.

In Exhibit A5, we employ the robust performance hypothesis testing approach proposed by Ledoit and Wolf (2011) to statistically compare the variances of each pair of investment strategies from Exhibit 6. Similar to the Sharpe ratio test from Exhibit 7, the null hypothesis assumes that the variances of the two strategies are equal.

EXHIBIT A5

Variance Difference Test: P-Values

	HM	MM	TM	HR	MV	Equally Weighted
MM	**0.00%**					
TM	**0.01%**	0.00%				
HR	**1.99%**	24.27%	3.19%			
MV	**0.00%**	0.00%	0.00%	17.60%		
Equally Weighted	**0.00%**	0.38%	0.00%	14.17%	22.60%	
Market Cap	**0.00%**	65.56%	0.00%	30.57%	3.80%	0.00%

NOTES: This exhibit displays the results of the robust hypothesis test for the difference in variances between two investment strategies proposed by Ledoit and Wolf (2011). The null hypothesis assumes that the variances of the two strategies are equal. A P-value of *x*% indicates that the null hypothesis can be rejected at the *x*% significance level.

EXHIBIT A6

Jensen–Kelly–Pedersen Regression

Variable	HM	MM	TM	HR	MV	Equally Weighted
Alpha (in %)	4.14***	2.94*	1.81	2.44	3.00*	3.70***
	(3.20)	(1.83)	(1.64)	(0.62)	(1.91)	(3.36)
Market	0.69***	0.70***	0.76***	0.53***	0.70***	0.73***
	(73.52)	(59.68)	(94.26)	(18.43)	(60.97)	(90.37)
Accruals	–0.25***	–0.16**	–0.10*	–0.12	–0.13*	–0.22***
	(–4.25)	(–2.19)	(–1.92)	(–0.67)	(–1.78)	(–4.43)
Debt Issuance	–0.61***	–0.54***	–0.35***	–1.11***	–0.60***	–0.48***
	(–5.75)	(–4.10)	(–3.85)	(–3.39)	(–4.64)	(–5.29)
Investment	0.08	0.15*	0.14**	0.07	0.10	0.03
	(1.24)	(1.87)	(2.52)	(0.36)	(1.26)	(0.48)
Low Leverage	–0.85***	–1.19***	–1.04***	–0.06	–1.12***	–1.02***
	(–11.65)	(–13.15)	(–16.78)	(–0.29)	(–12.64)	(–16.36)
Low Risk	–0.39***	–0.90***	–0.40***	0.06	–0.79***	–0.64***
	(–14.30)	(–26.57)	(–17.24)	(0.75)	(–24.03)	(–27.76)
Momentum	0.27***	0.69***	0.29***	0.11*	0.66***	–0.01
	(12.87)	(26.34)	(15.94)	(1.77)	(25.87)	(–0.38)
Profit Growth	0.27***	0.52***	0.14***	0.44**	0.47***	0.08
	(4.76)	(7.35)	(2.79)	(2.53)	(6.82)	(1.61)
Profitability	–0.18***	0.23***	0.17***	0.07	0.19**	–0.08
	(–2.61)	(2.68)	(2.98)	(0.35)	(2.34)	(–1.36)
Quality	0.38***	0.14*	0.18***	–0.17	0.14*	0.35***
	(5.79)	(1.70)	(3.34)	(–0.87)	(1.84)	(6.31)
Seasonality	0.39***	0.23**	0.21***	0.94***	0.28***	0.17**
	(4.63)	(2.14)	(2.95)	(3.61)	(2.69)	(2.32)
Short-Term Reversal	–0.06	–0.13***	0.04	0.08	–0.14***	0.06**
	(–1.53)	(–2.95)	(1.34)	(0.75)	(–3.09)	(2.07)
Size	0.26***	0.41***	0.42***	0.39***	0.39***	0.26***
	(7.06)	(9.19)	(13.78)	(3.46)	(8.86)	(8.29)
Value	–0.41***	–0.74***	–0.66***	–0.04	–0.69***	–0.26***
	(–6.82)	(–9.87)	(–12.81)	(–0.21)	(–9.37)	(–4.96)
Adjusted R-squared	0.79	0.79	0.86	0.15	0.78	0.89

NOTES: This exhibit presents the results of time-series regressions of the daily excess returns of various investment strategies on daily excess returns of a set of explanatory factor variables from Jensen, Kelly, and Pedersen (2023). The sample period spans from June 1997 to August 2022. Alpha represents the intercept of the daily excess return regression and is expressed in annualized percentage points. We report the coefficient estimates, along with their associated *t*-statistics. Coefficients that reject the null hypothesis of zero at the 10%, 5%, and 1% significance level are marked with one, two, or three asterisks, respectively. The values in the brackets are the t-statistics, which are mentioned in the caption.

EXHIBIT A7

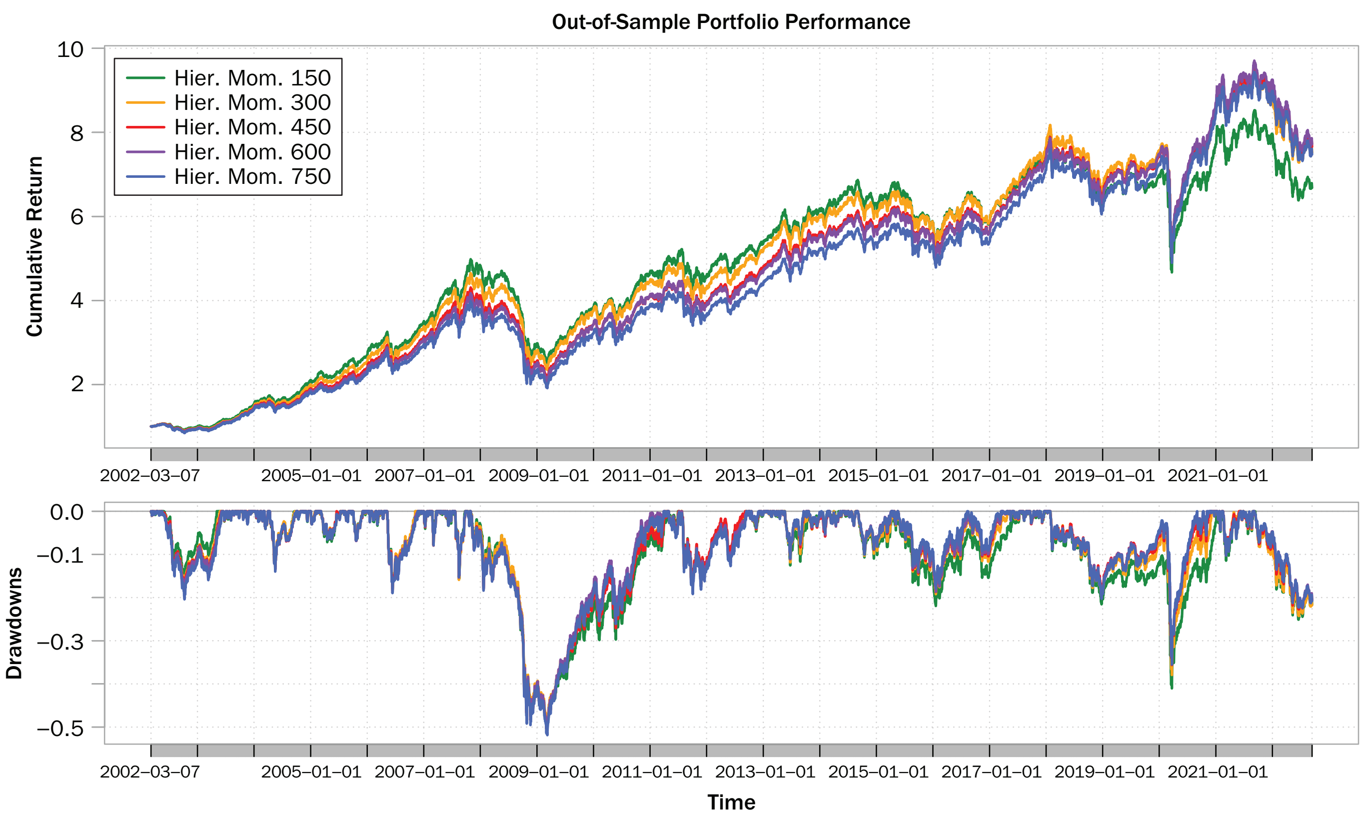
Out-of-Sample Portfolio Performance

	HM n = 150	HM n = 300	HM n = 450	HM n = 600	HM n = 750
Cum. Return	562.41%	644.07%	659.74%	**664.42%**	641.53%
Mean Return	9.73%	10.27%	10.38%	**10.43%**	10.31%
Volatility	12.86%	**12.84%**	12.87%	13.05%	13.23%
Sharpe Ratio	0.76	0.80	**0.81**	0.80	0.78
Sortino Ratio	1.00	1.05	**1.06**	1.05	1.03
Maximum Drawdown	–49.58%	**–49.40%**	–50.75%	–50.64%	–51.87%
Turnover	851.19%	705.71%	635.25%	587.54%	**552.49%**

NOTES: Observe the realized cumulative portfolio return, annualized portfolio return mean, volatility, Sharpe and Sortino ratios, maximum drawdown, and turnover of the HM portfolio optimization strategy given a different number of chosen clusters. The out-of-sample backtest results presented here are corrected for transaction costs.

EXHIBIT A8

Cumulative Portfolio Returns and Realized Drawdowns for the HM Portfolio Allocation Strategy

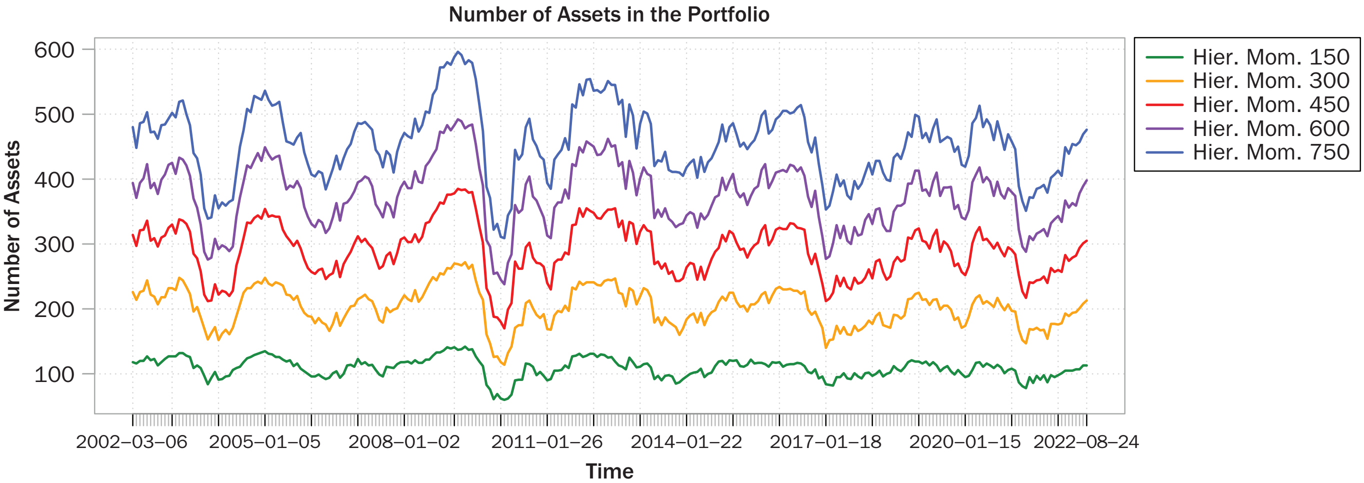


NOTES: The upper part of this exhibit portrays the cumulative portfolio returns for the HM portfolio allocation strategy with different numbers of chosen clusters. The lower part depicts the realized drawdowns of the analyzed portfolio construction strategies. All reported values are adjusted for transaction costs.

Compared to the Sharpe ratio test, the difference in variances between two investment strategies can be more easily distinguished statistically because the variance is a less noisy measure. As shown in Exhibit A5, many pairs of strategies exhibit statistically significant differences in their variances. Notably, the HM strategy has statistically

EXHIBIT A9

Number of Selected Assets for the HM Portfolio Allocation Strategy with Different Numbers of Chosen Clusters



significantly different variances with extremely low P-values close to 0%, except for the HR strategy against which it still has a low P-value of 2%. The exhibit demonstrates that the proposed HR strategy effectively reduces the risk as measured by the variance.

Lastly, to account for the differences in our dataset and the asset universe from Fama and French (2015), we repeat the time-series regressions from Exhibit 8 with factor portfolio return data from Jensen, Kelly, and Pedersen (2023), which covers 93 countries and is freely available online. Specifically, we utilize the world region, which combines the developed, emerging, and frontier market regions. The weighting scheme employed is the capped value weighted approach, as recommended by Jensen, Kelly, and Pedersen (2023). We incorporate all available themes (i.e., factors), as specified in Exhibit A6. The construction details of these factors can be found in Jensen, Kelly, and Pedersen (2023).

Exhibit A6 presents the results from the daily time-series regressions using the aforementioned factors. Similar to the previous analysis, we observe positive loadings on the momentum factor from the momentum-based strategies, including the MV strategy. Notably, the annualized alpha of the HM strategy increased to 4.14%, and it is now statistically significant even at the 1% level. Although the asset universe from Jensen, Kelly, and Pedersen (2023) is not a perfect match for our sample, it is closer compared to the one from Fama and French (2015). We again used the capitalization-weighted market returns as the market factor to partially address any issues arising from the differences in the asset universes. Based on our findings, we have strong statistical evidence to support the superiority of the proposed HM strategy over the other investment strategies studied, even after transaction costs.

SENSITIVITY ANALYSIS

Here, we analyze how the number of chosen clusters in the HM portfolio construction strategy affects the out-of-sample portfolio performance. Exhibit A7 presents the out-of-sample realized portfolio performance metrics (expressed in annualized terms) for the HM portfolio construction strategy given the number of chosen clusters n. We present the results for $n = 150$, 300, 450, 600, and 750. Essentially all portfolio performance metrics (e.g., especially volatility, Sharpe and Sortino ratio, and maximum drawdown) remain stable across the various choices of n. Only the cumulative and mean returns tend to increase slightly with the increasing number of clusters, except for $n = 750$. On the other hand, it is intuitively clear that the turnover decreases with an increasing number of clusters because the smaller part of the portfolio has to be rebalanced when one is invested in a larger number of assets (i.e., a trade-off between portfolio sparsity and turnover). All in all, Exhibit A7 shows that the number of clusters does not affect the portfolio outperformance in comparison to Exhibit 6.

Similar reasoning can be observed also in Exhibit A8, which depicts the cumulative portfolio returns and drawdowns of the HM portfolio construction strategy analyzed with different amounts of chosen clusters. In comparison to Exhibit 9, notice that all the strategies perform similarly and the performance of the HM strategy is in general not driven by the chosen number of clusters. Finally, we present the number of chosen assets in the HM-based portfolio for different choices of n in Exhibit A9.

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REFERENCES

Alves, C. F., and E. M. Filipe. 2021. “Survivorship Bias on the Momentum Effect: Evidence from the Portuguese Market.” *BOHR International Journal of Finance and Market Research* 1 (1): 9–16.

Arnott, R. D., M. Clements, V. Kalesnik, and J. T. Linnainmaa. 2021. “Factor Momentum.” *SSRN* 3116974.

Asness, C., A. Frazzini, R. Israel, and T. Moskowitz. 2014. “Fact, Fiction, and Momentum Investing.” *The Journal of Portfolio Management* 40 (5): 75–92.

Babu, A., A. Levine, Y. H. Ooi, L. H. Pedersen, and E. Stamelos. 2020. “Trends Everywhere.” *Journal of Investment Management* 18 (1): 52–68.

Bailey, D. H., and M. L. de Prado. 2012. “The Sharpe Ratio Efficient Frontier.” *Journal of Risk* 15 (2): 13.

Best, M. J., and R. R. Grauer. 1991. “On the Sensitivity of Mean–Variance-Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results.” *The Review of Financial Studies* 4 (2): 315–342.

Black, F., and R. Litterman. 1990. “Asset Allocation: Combining Investor Views with Market Equilibrium.” *Goldman Sachs Fixed Income Research* 115 (1): 7–18.

Brida, J. G., and W. A. Risso. 2010. “Hierarchical Structure of the German Stock Market.” *Expert Systems with Applications* 37 (5): 3846–3852.

Brodie, J., I. Daubechies, C. De Mol, D. Giannone, and I. Loris. 2009. “Sparse and Stable Markowitz Portfolios.” *Proceedings of the National Academy of Sciences* 106 (30): 12267–12272.

Chan, L. K., N. Jegadeesh, and J. Lakonishok. 1996. “Momentum Strategies.” *The Journal of Finance* 51 (5): 1681–1713.

Chan, L. K., J. Karceski, and J. Lakonishok. 1999. “On Portfolio Optimization: Forecasting Covariances and Choosing the Risk Model.” *The Review of Financial Studies* 12 (5): 937–974.

Chopra, V. K., and W. T. Ziemba. 1993. “The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice.” *The Journal of Portfolio Management* 19 (2): 6–11.

Daniel, K., and T. J. Moskowitz. 2016. “Momentum Crashes.” *Journal of Financial Economics* 122 (2): 221–247.

DeMiguel, V., L. Garlappi, F. J. Nogales, and R. Uppal. 2009. “A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms.” *Management Science* 55 (5): 798–812.

De Nard, G. 2022. “Oops! I Shrunk the Sample Covariance Matrix Again: Blockbuster Meets Shrinkage.” *Journal of Financial Econometrics* 20 (4): 569–611.

Fama, E. F., and K. R. French. 2015. “A Five-Factor Asset Pricing Model.” *Journal of Financial Economics* 116 (1): 1–22.

Frost, P. A., and J. E. Savarino. 1988. “For Better Performance: Constrain Portfolio Weights.” *The Journal of Portfolio Management* 15 (1): 29–34.

Garlappi, L., R. Uppal, and T. Wang. 2007. “Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach.” *The Review of Financial Studies* 20 (1): 41–81.

Gupta, T., and B. Kelly. 2019. “Factor Momentum Everywhere.” *The Journal of Portfolio Management* 45 (3): 13–36.

Jagannathan, R., and T. Ma. 2003. “Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps.” *The Journal of Finance* 58 (4): 1651–1683.

Jegadeesh, N., and S. Titman. 1993. “Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency.” *The Journal of Finance* 48 (1): 65–91.

——. 2001. “Profitability of Momentum Strategies: An Evaluation of Alternative Explanations.” *The Journal of Finance* 56 (2): 699–720.

——. 2011. “Momentum.” *Annual Review of Financial Economics* 3 (1): 493–509.

Jensen, T. I., B. T. Kelly, and L. H. Pedersen. 2023. “Is There a Replication Crisis in Finance?” *The Journal of Finance* 78: 2465–2518.

Jorion, P. 1986. “Bayes–Stein Estimation for Portfolio Analysis.” *Journal of Financial and Quantitative Analysis* 21 (3): 279–292.

Kan, R., and G. Zhou. 2007. “Optimal Portfolio Choice with Parameter Uncertainty.” *Journal of Financial and Quantitative Analysis* 42 (3): 621–656.

Kwon, O. K., and S. Satchell. 2018. “The Distribution of Cross Sectional Momentum Returns.” *Journal of Economic Dynamics and Control* 94: 225–241.

Lai, T. L., H. Xing, and Z. Chen. 2011. “Mean–Variance Portfolio Optimization When Means and Covariances Are Unknown.” *The Annals of Applied Statistics* 5 (2A): 798–823.

Ledoit, O., and M. Wolf. 2003. “Improved Estimation of the Covariance Matrix of Stock Returns with an Application to Portfolio Selection.” *Journal of Empirical Finance* 10 (5): 603–621.

——. 2004. “Honey, I Shrunk the Sample Covariance Matrix.” *The Journal of Portfolio Management* 30 (4): 110–119.

——. 2008. “Robust Performance Hypothesis Testing with the Sharpe Ratio.” *Journal of Empirical Finance* 15: 850–859.

——. 2011. “Robust Performances Hypothesis Testing with the Variance.” *Wilmott* 55: 86–89.

León, D., A. Aragón, J. Sandoval, G. Hernández, A. Arévalo, and J. Niño. 2017. “Clustering Algorithms for Risk-Adjusted Portfolio Construction.” *Procedia Computer Science* 108: 1334–1343.

Li, J. 2015. “Sparse and Stable Portfolio Selection with Parameter Uncertainty.” *Journal of Business & Economic Statistics* 33 (3): 381–392.

Li, Y., X. Jiang, Y. Tian, S.-P. Li, and B. Zheng. 2018. "Portfolio Optimization Based on Network Topology." *Physica A: Statistical Mechanics and Its Applications* 515: 671–681.

Lim, B., S. Zohren, and S. Roberts. 2019. "Enhancing Time-Series Momentum Strategies Using Deep Neural Networks." *The Journal of Financial Data Science* 1 (4): 19–38.

Lopez de Prado, M. 2016. "Building Diversified Portfolios That Outperform Out of Sample." *The Journal of Portfolio Management* 42 (4): 59–69.

Lu, Y. N., S. P. Li, L. X. Zhong, X. F. Jiang, and F. Ren. 2018. "A Clustering-Based Portfolio Strategy Incorporating Momentum Effect and Market Trend Prediction." *Chaos, Solitons & Fractals* 117: 1–15.

Mantegna, R. 1999. "Hierarchical Structure in Financial Markets." *The European Physical Journal B* 11 (1): 193–197.

Markowitz, H. 1952. "Portfolio Selection." *The Journal of Finance* 7 (1): 77–91.

——. 1959. *Portfolio Selection*. New Haven: Yale University Press.

Michaud, R. O. 1989. "The Markowitz Optimization Enigma: Is 'Optimized' Optimal?" *Financial Analysts Journal* 45 (1): 31–42.

Michaud, R. O., and R. O. Michaud. 2008. *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation*. New York: Oxford University Press.

Molyboga, M. 2020. "A Modified Hierarchical Risk Parity Framework for Portfolio Management." *The Journal of Financial Data Science* 2 (3): 128–139.

Moskowitz, T. J., Y. H. Ooi, and L. H. Pedersen. 2012. "Time Series Momentum." *Journal of Financial Economics* 104 (2): 228–250.

Nielsen, F. 2016. "Hierarchical Clustering." In *Introduction to HPC with MPI for Data Science* 195–211. Cham: Springer.

Pedersen, L. H., A. Babu, and A. Levine. 2021. "Enhanced Portfolio Optimization." *Financial Analysts Journal* 77 (2): 124–151.

Peralta, G., and A. Zareei. 2016. "A Network Approach to Portfolio Selection." *Journal of Empirical Finance* 38: 157–180.

Polak, P., and U. Ulrych. 2021. "Dynamic Currency Hedging with Non-Gaussianity and Ambiguity." Research Paper No. 21–60, Swiss Finance Institute.

Polbennikov, S., A. Desclée, and M. Dubois. 2020. "Implementing Value and Momentum Strategies in Credit Portfolios." *The Journal of Portfolio Management* 47 (2): 82–98.

Puerto, J., M. Rodríguez-Madrena, and A. Scozzari. 2020. "Clustering and Portfolio Selection Problems: A Unified Framework." *Computers & Operations Research* 117: 104891.

Raffinot, T. 2018. "Hierarchical Clustering-Based Asset Allocation." *The Journal of Portfolio Management* 44 (2): 89–99.

Rokach, L., and O. Maimon. 2005. "Clustering Methods." In *Data Mining and Knowledge Discovery Handbook*, pp. 321–352. Boston: Springer.

Rouwenhorst, K. G. 1998. "International Momentum Strategies." *The Journal of Finance* 53 (1): 267–284.

Schmid, O., and P. Wirth. 2021. "Optimal Allocation to Time-Series and Cross-Sectional Momentum." *The Journal of Portfolio Management* 47 (4): 160–179.

Sharpe, W. F. 1963. "A Simplified Model for Portfolio Analysis." *Management Science* 9 (2): 277–293.

Tola, V., F. Lillo, M. Gallegati, and R. N. Mantegna. 2008. "Cluster Analysis for Portfolio Optimization." *Journal of Economic Dynamics and Control* 32 (1): 235–258.

Tumminello, M., F. Lillo, and R. N. Mantegna. 2010. "Correlation, Hierarchies, and Networks in Financial Markets." *Journal of Economic Behavior & Organization* 75 (1): 40–58.

Ulrych, U., and N. Vasiljević. 2020. "Global Currency Hedging with Ambiguity." Research Paper No. 20–73, Swiss Finance Institute.

Výrost, T., S. Lyócsa, and E. Baumöhl. 2019. "Network-Based Asset Allocation Strategies." *The North American Journal of Economics and Finance* 47: 516–536.

Wang, C. D., Z. Chen, Y. Lian, and M. Chen. 2022. "Asset Selection Based on High Frequency Sharpe Ratio." *Journal of Econometrics* 227 (1): 168–188.

Zhu, Z., and K. Yung. 2016. "The Interaction of Short-Term Reversal and Momentum Strategies." *The Journal of Portfolio Management* 42 (4): 96–107.