$\log n! = \Theta(n \log n)$

We need to show that log *n*! is less than one multiple of *n* log *n* and greater than another multiple of *n* log *n* for all values of *n* greater than some constant. That is

$$0 \le c_1 n\log n \le \log(n!) \le c_2 n\log n \quad \text{for all } n \ge n_0.$$

First, show log *n*! is less than or equal to *n* log *n*. This is true for all *n* > 0.

$$\begin{aligned}\log(n!) &= \log(1 \cdot 2 \cdot 3 \cdots n) \\ &= \log 1 + \log 2 + \log 3 + \cdots + \log n \\ &\le \log n + \log n + \log n + \cdots + \log n \\ &= n \log n\end{aligned}$$

So, log *n*! = O(*n* log *n*) with $c_2 = 1$.

Next, show log *n*! is greater than or equal to a constant multiple of *n* log *n*.

$$\begin{aligned}\log(n!) &= \log(1 \cdot 2 \cdot 3 \cdots n) \\ &= \log 1 + \log 2 + \log 3 + \cdots + \log n\end{aligned}$$

Deleting the first half of the terms gives

$$\log(n!) \ge \log \frac{n}{2} + \log\left(\frac{n}{2}+1\right) + \log\left(\frac{n}{2}+2\right) + \cdots + \log n \;.$$

Replacing all remaining terms by the smallest one gives

$$\log(n!) \ge \frac{n}{2} \log \frac{n}{2} \;.$$

$$\log(n!) \ge \frac{n}{2}\log\frac{n}{2} = \frac{n}{2}(\log n - 1) = \frac{n}{2}\log n - \frac{n}{2}$$

We want to show this is greater than a multiple of *n* log *n*.

To show $\frac{n}{2}\log n - \frac{n}{2} \ge \frac{n}{2}\log n$, we can use the following argument.

For $n \geq 100$,

$$\log n \geq 2$$

$$\frac{1}{4}\log n \geq \frac{1}{2}$$

$$\frac{1}{4}n\log n \geq \frac{1}{2}n$$

$$\frac{1}{4}n\log n - \frac{1}{2}n \geq 0$$

$$\frac{1}{2}n\log n - \frac{1}{2}n \geq \frac{1}{4}n\log n$$

Putting these together gives the following.

$$\begin{aligned}\log(n!) &\geq \frac{n}{2}\log\frac{n}{2} \\ &= \frac{n}{2}(\log n - 1) \\ &= \frac{n}{2}\log n - \frac{n}{2} \\ &\geq \frac{n}{4}\log n \\ &= \frac{1}{4}n\log n\end{aligned}$$

Therefore, $n_0 = 100$ and $c_1 = ¼$ and log $n! = \Omega(n \log n)$.

Since we have log $n! = \Omega(n \log n)$ and log $n! = O(n \log n)$ or since

$$\frac{1}{4}n\log n \leq \log(n!) \leq n\log n$$

we can conclude log $n! = \Theta(n \log n)$

$$
\begin{aligned}
\log(n!) &\geq \frac{n}{2}\log\frac{n}{2} \\
&= \frac{n}{2}(\log n - 1) \\
&= \frac{n}{2}\log n - \frac{n}{2} \\
&\geq \frac{n}{4}\log n \\
&= \frac{1}{4}n\log n
\end{aligned}
$$