Optimal Parallel Randomized Algorithms for Sparse Addition and Identification

Advanced algorithms and parallel programming

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General problem

- *Parallel addition of n numbers in shared memory models*

Standard algorithms are not sensitive to properties of the input, thus randomized algorithms can beat standard parallel algorithms in specific cases:

e.g. let's consider the case when there is **a large amount of zero operands** among the numbers to be added

Proposed Solution

Randomized parallel algorithm:

- the number of non-zero operands must be much smaller than the number of zeros among the numbers to be added
- we suppose we are using a CRCW PRAM with unlimited memory and unlimited number of processors
- time complexity $O(\log m)$, space complexity $O(m)$ w.h.p.

The Algorithm

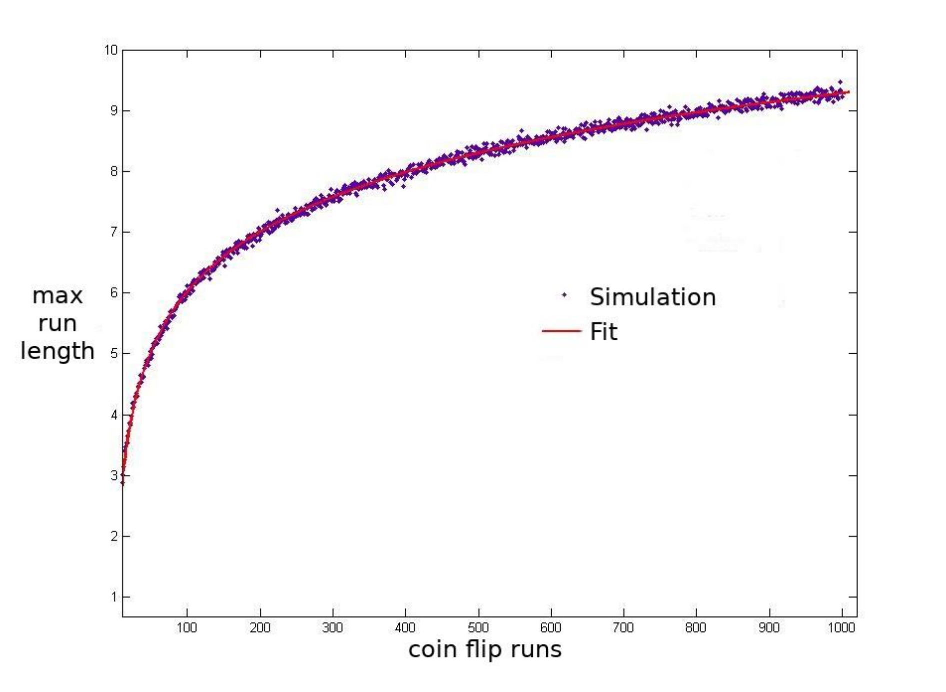
It consists of two phases:

- **Estimation** of the number m of non-zero terms
- **Parallel addition** on a smaller set of inputs

Estimation - *First phase*

1. Each processor P_i has a number to add.

2. **PRODUCE-AN-ESTIMATE**: every processor with $x_i \neq 0$ flips a coin until a head is found. A shared memory location will contain the *highest number of coin flips reached*.



Estimation - *First phase*

3. PRODUCE-AN-ESTIMATE is executed ***k* times.**

4. Then each P_i computes an estimation m' for the number of non-zero operands m using the following formula:

$$(1) \quad E \leftarrow (\log 2)\frac{E_1 + \cdots + E_k}{k}$$

$$(2) \quad m' \leftarrow \exp(2) \cdot \exp(E) + d$$

where $d \geqslant 1$ is a constant.

Code (Estimation)

INITIALIZATION

Processor P_1 initializes a special shared memory location (CLOCK) to zero.
Then, each P_i executes $TIME_i \leftarrow 0$.

ESTIMATE

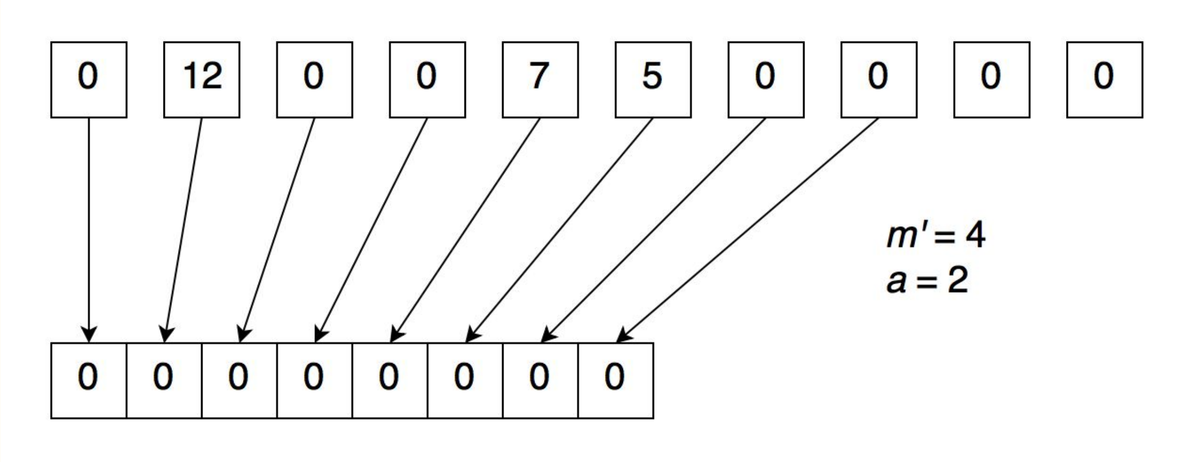
```
Processor Pi
        if x ≠ 0
                then begin (1) Flip a fair coin (two-sided)
                           (2 ) If the outcome is 'tail' then
                                  begin (2a) TIMEi = TIMEi + 1
                                             (2b) CLOCK ←TIMEi;
                                             (2c) goto(l)
                                  end
                      end
```

#all processors wait for everyone to be finished

Each P_i with $x_i \neq 0$ reads CLOCK and makes its value to be the current estimate.

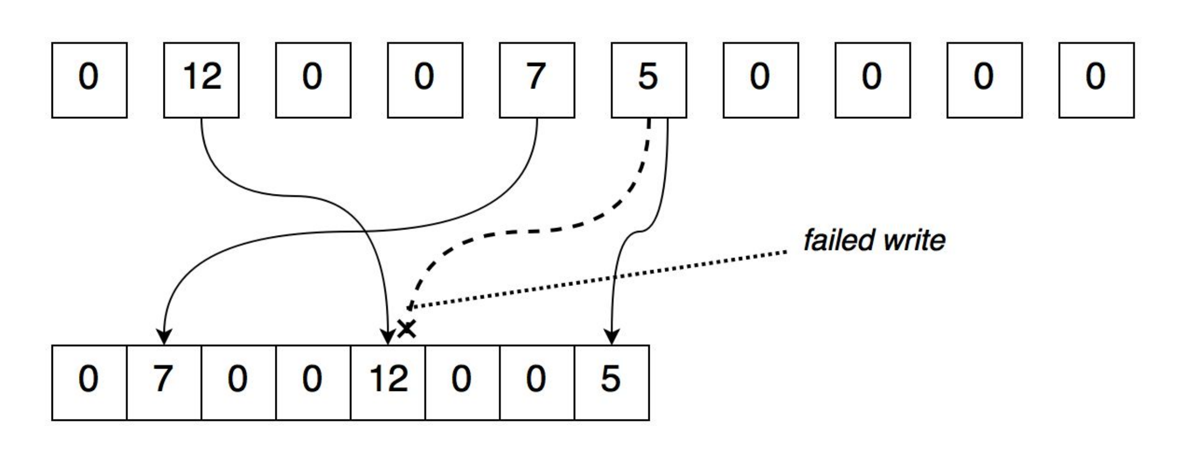
Addition - *Second phase*

1. **Initialization:**
   $a \cdot m'$ memory locations are set to zero. ($a \geq 4$ constant)



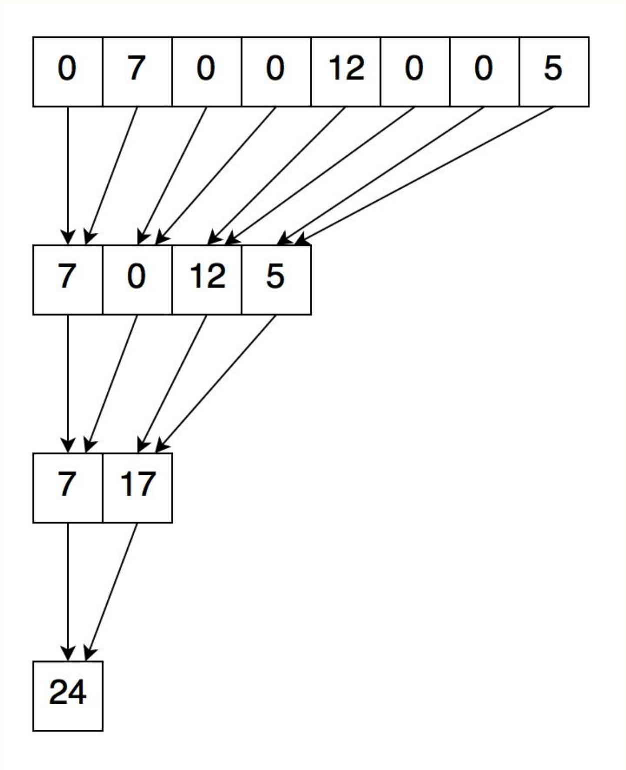
Addition - *Second phase*

2. **Memory Marking:**
   each processor with a non-zero number tries to write it to a random empty location in the zeroed space, until they all succeed.



Addition - *Second phase*

3. **Addition:**
   standard parallel addition of all the $a \cdot m'$ numbers (zero and non-zero). The result is of course the sum of all the non-zero numbers.



Code (Addition)

INITIALIZATION

In one parallel step, processors initialize $a \cdot m' + 2$ shared memory locations to zero, by executing: "Processor Pi writes a zero to M(j), if $j < am' + 2$". Then, they all execute TIMEj ← 0.

MEMORY MARKING

```
Processor Pj
IF xj≠0 then
    BEGIN (1) Select y equiprobably at random from { 1, 2, . . . . am' }
          (2) TIMEj ←TIMEj + 1
          (3) Read M(y); TIMEj ← TIMEj ← 1
          (4) If M(y) = 0 then write xj into M(y). Also, TIMEj ← TIMEj + 1
          (5) If the "write" failed then
                BEGIN (5a) write TIME, into M(am' + 1)
                      (5b) go to (1)
                END
    END
```

Code (Addition)

The following part is executed by P_j with $x_j = 0$ and by "successful" P_j with $x_j \neq 0$.

(6) Read M(am' + 1) into a local variable R1

(7) Wait for 8 steps

(8) Read M(am' + 1) into a local variable R2

(9) If R1 $\neq$ R2, then go to (6)

ADDITION

(Processor P, is assigned to location M(j), $1 < j < am'$) From this point on, processors P_j (where $1 < j < am'$) perform a standard parallel addition of the numbers M(l), ... M(am'). In the i-th parallel step of the addition, processor P_j adds M(j) and M(j+2^i) into M(j), for $j = k \cdot 2^i + 1$, $k = 0, 1, 2, \ldots, am'/2^i$.

Properties

- ➢ The sum is always correct
- ➢ The time complexity is in O(log m) and the space complexity in O(m) (where m is the number of non-zero entries among the terms we need to sum) with high probability
- ➢ We can control the precision of the estimate, and thus the variance of the running time, by changing the amount of estimations of m to run (represented by the constant k)

Running time for Estimation

Given any $\delta > 0$, if we choose $k \geq 4/\delta$, then with probability at least $1 - \delta$ we have that:

the total running time of estimation is $O(4/\delta \cdot \log m)$, *or* $O(k \cdot \log m)$

Brief explanation:

1. **a single iteration** takes time equal to the maximum of m independent geometric random variables with $p = 1/2$, which **can be approximated as log m/log 2 time** using the harmonic series
2. with Chebyshev's inequality **we can predict how often the sum** of the times for each iteration **will exceed the logarithmic expected time**, thus the relationship with δ

Estimation accuracy

The estimation is computed using the harmonic series as an approximation ($E \approx \log m / \log 2$), but the approximation **does not converge** for big ks.

If we choose $k \geq 4/\delta$, with probability $1 - \delta$ we have:

$$|E - \log m| \leq 2 , \qquad \text{i.e.} \quad m \leq m' \leq m \cdot e^4$$

(note: $e^4 \approx 55$)

We have no guarantees except for these bounds.

Running time for Addition

Memory Marking

With $a \cdot m'$ free memory, each processor will have chance at least $(a - 1)/a$ to succeed in writing to an empty memory space.

The running time is the maximum of m geometric random variables, the same as in the estimation step, thus the time **can be approximated as logarithmic**. The memory marking step is therefore in O(log m) on average.

Parallel Addition

The parallel addition algorithm's running time is trivially in O(log m').

Total running time

The total average running time is O(log m) + O(log m) + O(log m) => **O(log *m*)**

Worst case:

- *Estimation*: each iteration has standard deviation 2, very improbable to take much longer than log m
- *Memory marking*: each processor's probability to need a repeated number of tries decreases geometrically (again, improbable)
- *Parallel addition*: can't take more than log m' steps

Considerations

Compared to the standard parallel addition, we need to remember that:

- the estimation step is expensive (uses rand()) and grows with k
- the memory marking step usually takes less than log m' since the estimation is in excess
- the parallel addition takes log m' time, which can be up to $\log(e^4) = 4$ times more than log m

Thus we expect the algorithm to be better than the simple parallel addition o**nly on extreme cases**, when the array of number to sum is **very sparse** (a very small fraction of non-zero numbers).

Problems encountered

- rand() returns 30-bit numbers, but we needed 64-bit random numbers
  - solution: generate a number bit by bit

- difficulties in integrating the project on both Windows and Linux environments

- measuring the performance of the algorithm is not easy, because of the noise created by thread scheduling
  - solution: taking the average time of many runs; running when the machine is idle
  - consequence: hard to measure worst times

- the paper is not so clear and glosses over a few things, like the proofs for the algorithm complexity

Testing and Performance

Datasets for testing are easy to generate (random numbers).

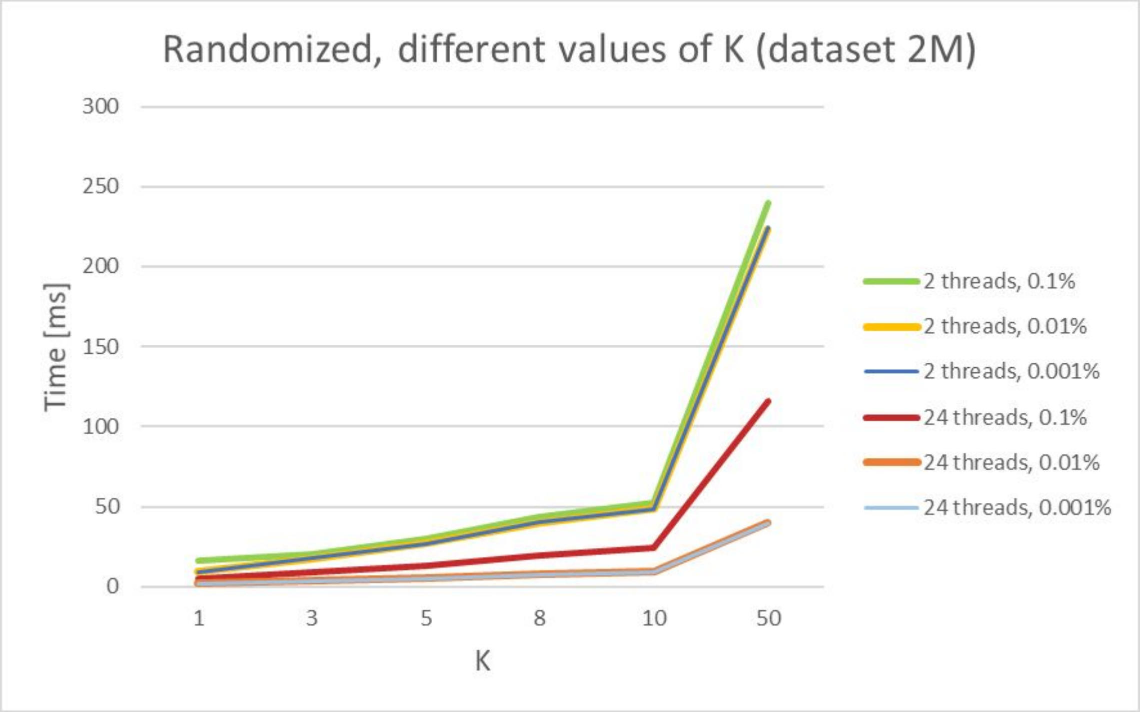
Tests to perform:

- different amounts of numbers to sum
- different ratios of zero and non-zero terms
- different values of k

Comparison with other algorithms:

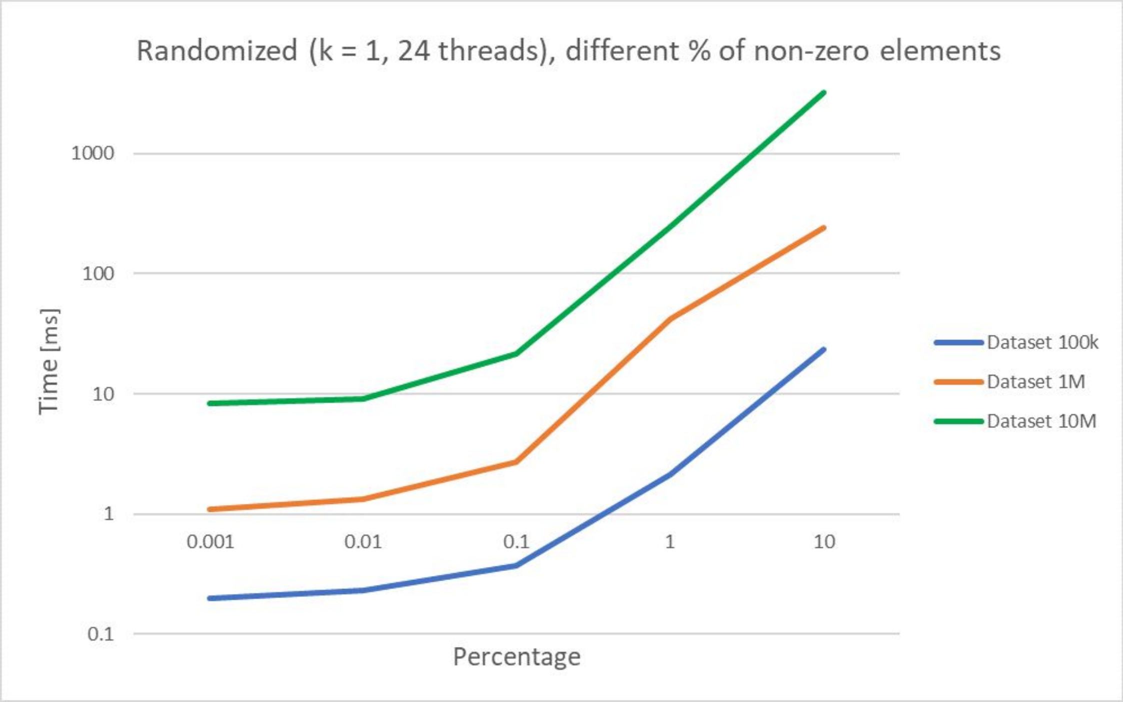
- performance gain w.r.t. a serial algorithm
- performance gain w.r.t. the standard parallel sum (parallel “lower bound”)

Benchmark results - selection of K



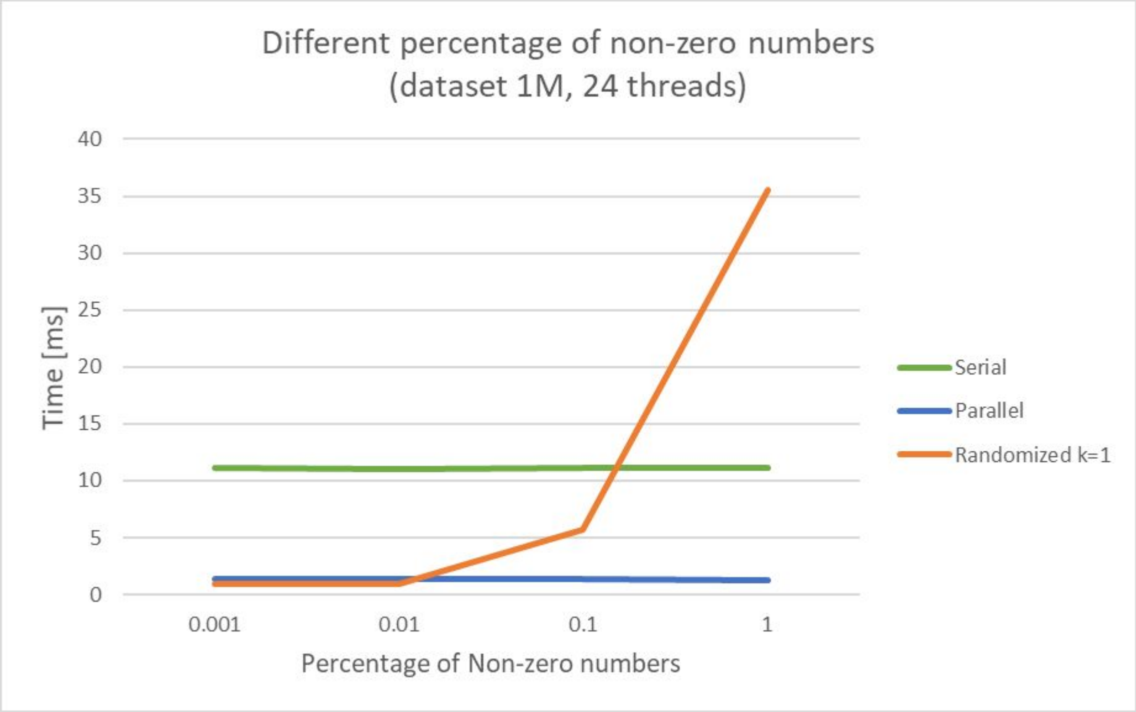
- The time to execute the algorithm always increases with K
- We choose a low K for our tests (K=1)

Benchmark results - percentage of nonzero numbers



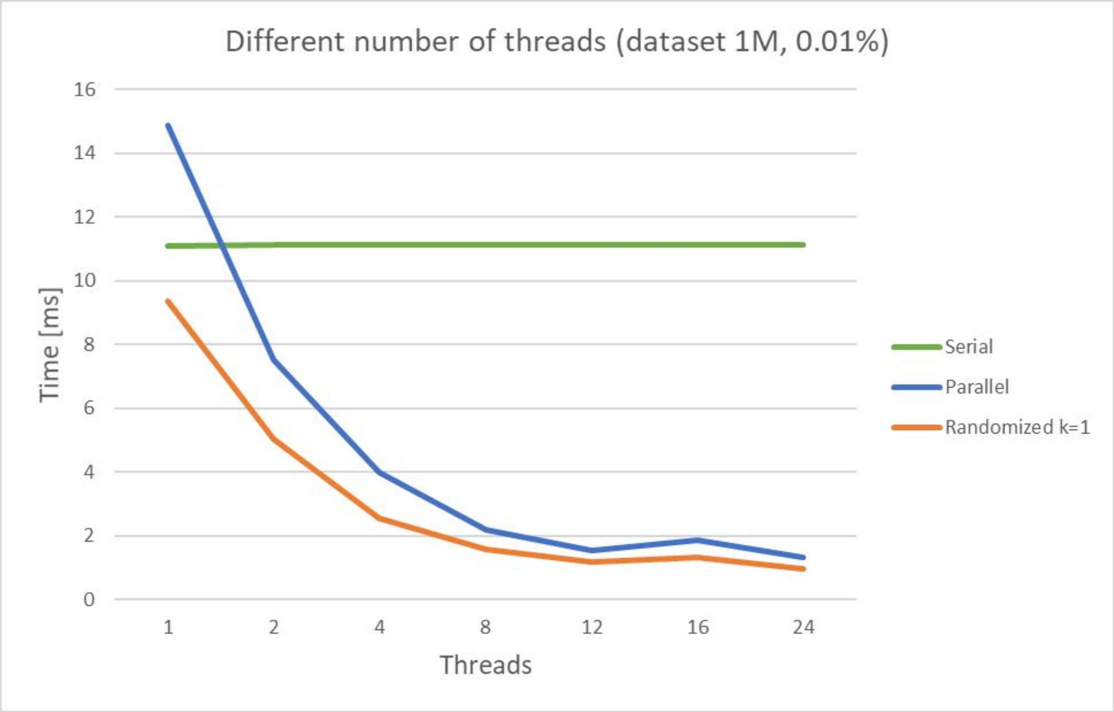
- The execution time for the randomized algorithm increases a lot when the number of non-zero elements is great
- We should use this algorithm only with a small percentage of non-zero elements

Benchmark results - % nonzero, comparison



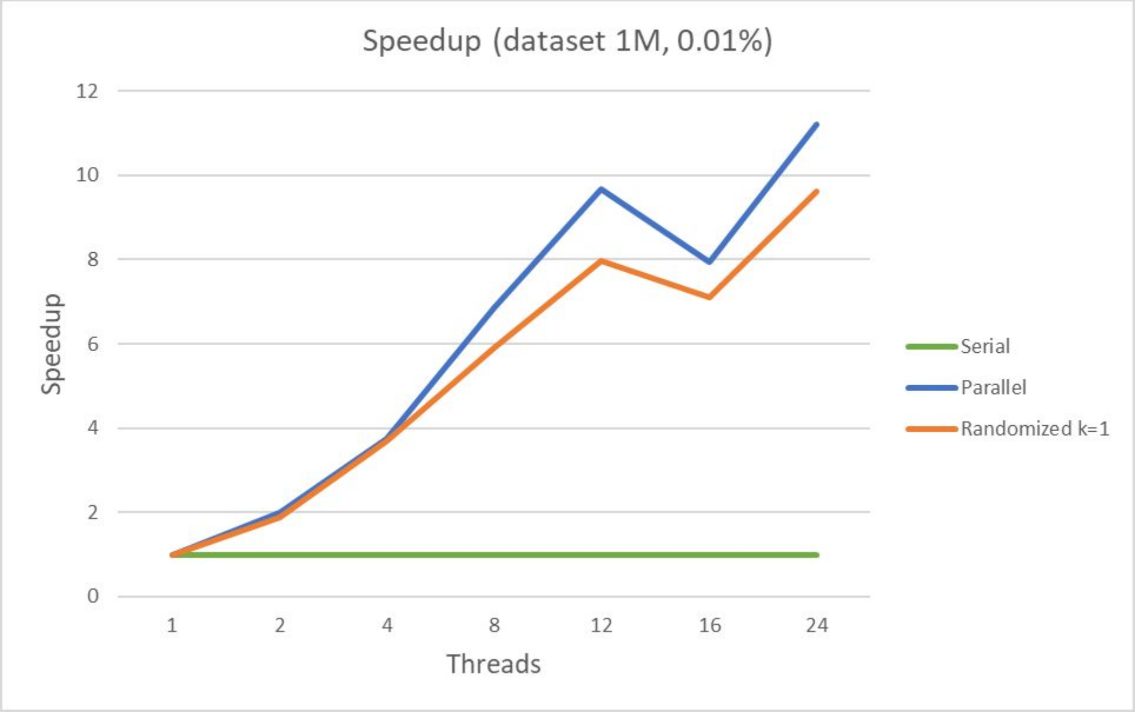
- The randomized algorithm is better than the parallel only when the % of nonzero numbers is under 0.01% (1 over 1000)

Benchmark results - threads



- Fixed K=1 and non-zero=0.01%
- The algorithm is well parallelized, the execution times easily improves when running with more threads

Benchmark results - threads (speedup)



- The speedup from the parallelization doesn't saturate (the dataset is much bigger than the number of threads)
- We have a dip at 16 threads because we actually have only 12 physical cores

Conclusions

- The randomized algorithm works and it is feasible, but it is better than the standard parallel sum only when the non-zero numbers in the dataset are **very sparse** ($\leq 0.01\%$ of all numbers).

- Even if having a bigger k gives more theoretical guarantees, this in practice makes the performance worse. The best results are reliably obtained with $\boldsymbol{k}$**=1** (only one estimation).