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THE p-CENTER LOCATION PROBLEM IN AN AREA

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Abstract—The p-center problem seeks the location of p facilities. Each demand point receives its service from the closest facility. The objective is to minimize the maximal distance for all demand points. In this paper, the p-center location problem for demand originating in an area is investigated. This problem is equivalent to covering every point in the area by p circles with the smallest possible radius. Heuristic procedures are proposed and upper bounds on the optimal solution in a square are given. Computational results for the special case of a square area are reported. Some cases such as $p=9$ centers in a square yield unexpected and interesting results. Copyright © 1996 Elsevier Science Ltd

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1. INTRODUCTION

The p-center location problem over a continuous area of demand is an interesting practical problem with various applications. The problem is equivalent to covering a given area in the plane with p identical circles which have the smallest possible radius (the facilities are located at the centers of these circles). In the literature (e.g. Handler, 1990; Tansel *et al.*, 1990), it is usually assumed that demand for the required service originates from a finite set of "demand points". In many cases, assuming that an area represents the set of demand points is more realistic. Demand originating in an area rather than in a finite set of demand points applies to location of mobile demand. For example, the location of stations for cellular telephones requires a continuous approach.

We consider the unweighted version of the problem. Thus, as long as demand exists at a point, its magnitude is irrelevant to the problem. It therefore suffices to have the collection of points where demand exists. A continuous representation means that the area where demand exists is defined. The heuristic solution approach described in this paper addresses any area on the plane, although areas with a convex polygon shape are easiest to handle. We particularly concentrate on a square area for which we provide an upper bound for the solution.

In many applications, the radius of the covering circles is given and the problem is to find the minimum number of identical circles with a given radius that cover the area. This requires the solution of various p-center problems and the smallest p that covers the area within the prespecified radius is selected. Furthermore, covering an area such as a square with p identical circles with minimal radius is equivalent to covering the maximal area of the same shape with identical circles of a given radius. This is the appropriate objective for a problem if the maximal area of a given shape such as a square or a circle needs to be covered by p circles of a given radius.

The location of emergency facilities such as fire stations or hospitals is frequently modeled by the p-center problem. Each customer patronizes the closest facility and the objective is to minimize the distance for the farthest customer. Other examples include covering an area with p television transmitters, warning sirens, or sprinkler systems. Such an application for warning sirens is discussed in Current and O'Kelly (1992). Each of these facilities covers a circular area and p locations need to be found such that the maximal distance to a facility is minimized. Another interesting application is a radar coverage problem. An area in the sky contains objects to be detected by radar. The radar beam covers a circular area of a given radius and the desired area needs to be covered by the minimum number of identical circles.

While Current and O'Kelly's (1992) siren problem is similar to the problem presented in this paper, its solution approach is inherently different. They assume that a set of potential sites is available and the selection of sites from this set is modeled as a set covering problem. In our formulation, the facilities can be located anywhere in the area.

In this paper, for the case where the area to be covered is a square, we find an upper bound to the optimal value of the radius R. The problem in a general area is heuristically solved by applying the error-free Voronoi diagram method by Sugihara and Iri (1992, 1994). Our Voronoi-based solution procedure is an extension of Cooper's ideas (Cooper, 1963, 1964), extended in Drezner (1984) for the discrete case. Since convergence near the desired solution is slow, a special "finishing up" algorithm is devised. Computational experiments are reported for a square area.

2. RELATED PROBLEMS AND LITERATURE REVIEW

An excellent review of the p-center problem on trees and graphs can be found in Handler (1990) and Tansel *et al.* (1990). The p-center problem in the plane is discussed in Chen and Handler (1987), Drezner (1984), and Vijay (1985). The rectilinear distance version of the problem is discussed in Drezner (1987).

The p-dispersion problem is closely related to the p-center problem. The p-dispersion problem seeks to locate p facilities in an area or a graph such that the minimal distance between two facilities is maximized. For a review, see Erkut (1990) and Kuby (1987). Shier (1977) showed that on a graph, the $p+1$-dispersion problem is a dual problem of the p-center problem. This result was also discussed in Chandresekaran and Daughety (1981) and Chandrasekaran and Tamir (1982). The p-dispersion problem in a square, which is equivalent to packing p circles with maximal radius in a square, is discussed in Drezner and Erkut (1995).

Location problems with area demand are discussed in Suzuki and Okabe (1995). The p-median problem in a square was heuristically solved in Iri *et al.* (1984), and for competitive location in a square in Okabe and Suzuki (1987). In both cases, it was shown that the solutions are very close to the beehive hexagonal pattern. Such problems are usually solved heuristically by application of the Voronoi diagram; see Okabe *et al.* (1992). It was shown in Coxeter *et al.* (1959), Papadimitriou (1981) and Zemel (1984) that for a large p, the best solution to the p-center problem in an area is close to the beehive hexagonal pattern. In Drezner and Zemel (1992), it was shown that the beehive hexagonal pattern is best for the location of competitive facilities in the plane such that a future competitor will be able to capture the least market share.

3. PROBLEM STATEMENT

Consider a demand area A in the plane. Locations for p facilities at (x_i,y_i) for $i=1,\ldots,p$ are to be found such that the maximum distance from any point in the area A to its closest facility is minimized. The distance between a point (x,y) in the plane and facility i is the Euclidean distance, denoted by $d_i(x,y)$. Our objective is to:

$$\min_{(x_i,y_i),i=1,\ldots,p} \left\{ \max_{(x,y)\varepsilon A} \left\{ \min_i \{d_i(x,y)\} \right\} \right\} \tag{1}$$

The problem is equivalent to covering all the points in area A by p circles with the smallest possible radius.

There are no restrictions on the shape of area A. However, for tractability of the solution procedures proposed below, it is suggested that area A is a convex polygon defined by its sides as constraints. In the computational results, we çonsider only the unit square for which we also derive an upper bound.

4. BOUNDS ON THE OPTIMAL SOLUTION

4.1. Lower bounds

Some trivial lower bounds can be derived. Let A be the region to be covered whose area is S. By adding up the areas of p identical circles of radius R, each of which circumscribes a region with area S/p, we get the inequality

$$p\pi R^2 \geqslant S \text{ or } R \geqslant \sqrt{S/\pi p}=0.56419\sqrt{S/p}$$

If we assume that covering the area with hexagons is best, then since the ara of a hexagon is $3\sqrt{3}/2R^2$, we get the more realistic bound of

$$R \geqslant \sqrt{\frac{2S}{3\sqrt{3}p}}=0.62040\sqrt{S/p}.$$

4.2. An upper bound for a square area

Any feasible solution to the p-center problem can serve as an upper bound. In the following, we solve the p-center problem in a unit square area, assuming that the centers are located at the centers of rectangles. Furthermore, we assume that the rectangles are arranged in "strips". Assume that there are N strips, where strip k contains n_k identical rectangles, $k=1,\ldots,N$. We also assume that all the rectangles are circumscribed by circles of the same radius. In what follows, we call this procedure the rectangle heuristic. In Appendix B, it is proved that the rectangle heuristic is optimal for $p \leqslant 4$.

Before the general problem is formulated, we present the rectangle heuristic for $p=5$. We consider the case of two strips: one containing 3 rectangles and the other containing 2 rectangles [See Fig. 1(a)]. In the figure, the widths of the two strips are variables with dimension x and $1-x$, respectively.

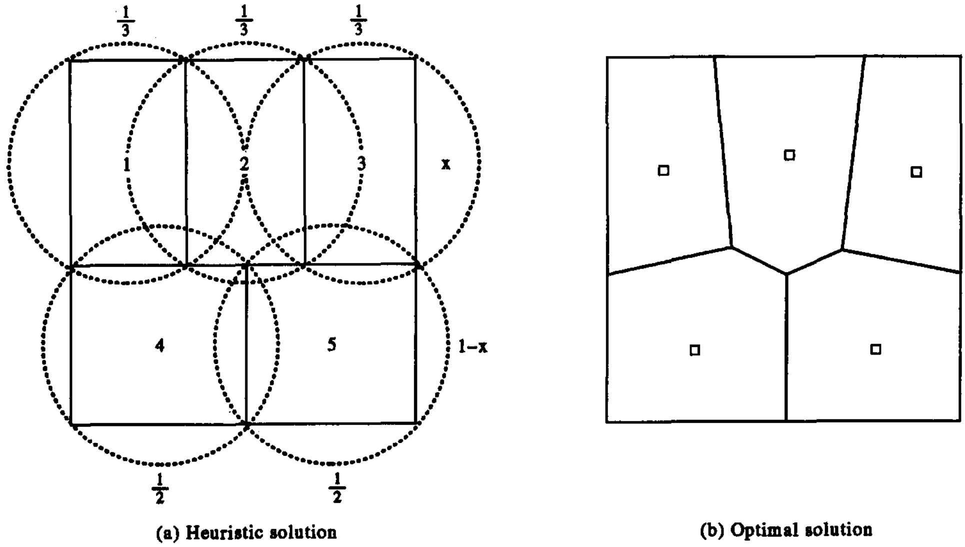

Fig. 1. Heuristic and optimal configurations for $p=5$.

Since the radii of the circles circumscribing these rectangles must be equal to each other, and therefore the diagonals of each of the rectangles must be identical, we get the equation $1/9+x^2=1/4+(1-x)^2$, which leads to $x=41/72$. The common radius of the circumscribing circles is therefore

$$R=1/2\sqrt{1/9+x^2}=\sqrt{2257/144}=0.32992$$

The five centers are located at the centers of the five rectangles depicted in Fig. 1(a). The area attracted by each center, which is determined by the Voronoi diagram, is not a rectangle because, for example, the perpendicular bisectors between point #2 and point #4 and #5 are not parallel to the x-axis. Consequently, the circle centered at rectangle "2" covers more than is really necessary. The circle centered at point #2 can be shrunk and its center moved up so that it passes through the top two corners of rectangle #2 and the center of its lower side, retaining coverage of the whole square. The radius R can be reduced to a lower new value for all five circles by moving center 2 upward and simultaneously moving the other centers downward, and moving centers "1" and "3" away from the center of the square. By executing these moves, the value of R is reduced to 0.32616. This better solution is depicted in Fig. 1(b). It can be shown (the length of the derivations prohibits us from showing the details) that R is a solution to the sixth order polynomial: $65536R^6+8192R^5+256R^4-10240R^3+352R^2-672R+425=0$. The polynomial has two real roots: 0.32616 and 0.46162.

The rectangle heuristic solution to the problem for a given N is detailed in Appendix A. Strips of identical rectangles are assumed, with a facility located at the center of each rectangle. Each rectangle is circumscribed in a circle of the same radius. In order to obtain the best rectangle heuristic solution, we need to find the best value of N.

When N and the number of rectangles in each strip are assumed to be real numbers and not necessarily integers, the optimal value for N is $N=\sqrt{p}$ (see Theorem 1 in Appendix A). Therefore, it is reasonable to assume that the optimal integer value for N will be close to this value. In the following, we limit the set of possible values for the optimal integer N.

Define $L = \lfloor\sqrt{p}\rfloor$ where $\lfloor z \rfloor$ is the largest integer not greater than z (z rounded down). Since the optimal value of R is nonincreasing in p, the value of R for p is no more than the value of R for L^2, which by Theorem 1 in Appendix A is $1/\sqrt{2}L$. Consider a given value N. If the number of rectangles in each strip is assumed to be a real number, the best radius, by Lemma 1 in Appendix A, is: $1/2\sqrt{1/N^2+N^2/p^2}$. This is defined as the "continuous solution". When the number of rectangles in each strip is restricted to integer values, then the radius for N strips cannot be smaller than the continuous solution. It therefore suffices to check only N's whose continuous solution is less than the solution to the problem with L^2 centers. This leads to $1/2\sqrt{1/N^2+N^2/p^2} \leqslant 1/\sqrt{2}L$. This condition leads to the inequality $L^2N^4 - 2p^2N^2 + L^2p^2 \leqslant 0$. This is a quadratic inequality in N^2 which is satisfied between the two positive roots of this quadratic equation. Some algebraic manipulations on the expressions for the two positive roots (solving the equation for $1/N$) lead to the following bounds for N:

$$\frac{L}{\sqrt{1+\sqrt{1-L^4/p^2}}} \leqslant N \leqslant \frac{L}{\sqrt{1-\sqrt{1-L^4/p^2}}} \tag{2}$$

This condition can be approximated for large p (i.e. assuming that $L^2 \approx p$).

$$\frac{L}{\sqrt{1+\sqrt{1-L^4/p^2}}} \approx L\left[1-\frac{1}{2}\sqrt{1-\frac{L^4}{p^2}}\right] \approx L\left[1-\frac{1}{2p}\sqrt{(p-L^2)(p+L^2)}\right]$$

and since $(L\sqrt{p+L^2}2p \approx 1/\sqrt{2})$, (and we get a similar expression for the right hand side of the inequality), we get:

$$L-\sqrt{\frac{p-L^2}{2}} \leqslant N \leqslant L+\sqrt{\frac{p-L^2}{2}} \tag{3}$$

Since $p-L^2 \leqslant 2L$ there are at most $2\sqrt{L}$ N's to check. This number is proportional to $\sqrt[4]{p}$ which is a fairly small number even for large p's.

We tested all p's up to 100,000 and in all cases the optimal N was either L or $L+1$. This can probably be proved as a theorem, but such a result will not improve the heuristic significantly and no attempt at such a proof was made.

The value of R, for selected values of p, calculated by the rectangle heuristic, is given in Table 1. Values for $p > 1000$ up to 100,000 were also calculated. It was found that $R = 1/\sqrt{2p}$ to five digits after the decimal point for these values. For a large p the upper bound is approximately $0.70711\sqrt{S/p}$ as compared with the hexagonal lower bound of $0.62040\sqrt{S/p}$.

5. THE VORONOI HEURISTIC

The p-center problem is a non-convex optimization problem and finding its global optimum is a difficult problem. We therefore constructed heuristic solution methods that provide good solutions.

We performed various numerical experiments with the Voronoi program developed by Ohya *et al.* (1984). The program is very fast, constructing the Voronoi diagram in 25 ms per

Table 1. Upper bounds for a square area

p	Upper bound	p	Upper bound
1	0.70711	60	0.09183
2	0.55902	70	0.08489
3	0.50389	80	0.07918
4	0.35355	90	0.07474
5	0.32992	100	0.07071
6	0.30046	200	0.05005
7	0.28410	300	0.04087
8	0.26501	400	0.03536
9	0.23570	500	0.03164
10	0.22755	600	0.02888
20	0.16008	700	0.02674
30	0.13017	800	0.02501
40	0.11265	900	0.02357
50	0.10021	1000	0.02237

one generator point with a 17 MIPS computer. It was improved by Sugihara and Iri (1992, 1994) so that it is robust against numerical errors and degeneration of the generator points of the Voronoi diagram.

Our heuristic approach to the p-center problem is as follows. A tolerance of ε (we used $\varepsilon=10^{-5}$), and a maximum allowable number of iterations m (we never achieved this number so it was practically infinite) is established.

5.1. *The Voronoi heuristic*

1. Generate p centers randomly in the area A.
2. Construct the Voronoi diagram based on the p centers.
3. Relocate the p centers to be the 1-center solution of their Voronoi polygon.
4. If the centers move from iteration to iteration by less than ε, or the maximum number iterations m is exceeded, stop.
5. Otherwise go to Step 2.

Notes:

1. Calculating the Voronoi diagram in Step 2 is done by the code given in Sugihara and Iri (1992, 1994). In such a Voronoi diagram, each center is dominating a polygon (called the Voronoi polygon) whose vertices are calculated by the program. This Voronoi polygon may be further reduced by intersecting it with the polygon defining the area A.
2. In Step 3, the 1-center solution to a continuous demand uniformly distributed in a convex polygon is identical to the discrete 1-center problem when demand is assumed to exist only at the vertices of the polygon. The number of vertices for such polygons is about six on average, which makes it a very fast and efficient procedure.

5.2. *A finishing-up algorithm*

Once the radii of the various polygons are close to one another, the convergence to a final configuration is very slow. Let the radius of the smallest circle covering polygon i be r_i for $i=1,\ldots,p$. Define $r_{\max}=\max\{r_i\}$ and $r_{\min}=\min\{r_i\}$. Our objective is to minimize $r_{\max}$. We conjecture that for optimal configurations $r_{\max}=r_{\min}$. Intuitively, if there exists a circle with a radius smaller than $r_{\max}$, then it is plausible that a change can be made in the configuration by increasing this radius a bit and simultaneously reducing the other radii while maintaining coverage of all points. A "finishing-up" algorithm can replace the iterative Voronoi approach once the improvement in $r_{\max}$ in one iteration is small. We applied the finishing-up procedure once the Voronoi heuristic terminated.

Consider p centers in the area. For simplicity, we assume that the area is a convex polygon defined by its sides as linear constraints. A feasible point is a point fulfilling all the constraints. υ vertices of the Voronoi diagram (called also Voronoi points) are calculated. For each center, there is a well-defined list of vertices that form the influence of polygon of that center. Let $V(i)$ be the set of vertices associated with center i, $i=1,\ldots,p$. Conversely, for each vertex j, let $C(j)$ be the set of centers where center i is in $C(j)$ if $j\in V(i)$. We also define the set CV of center–vertex pairs:

$$CV=\{(i,j)|j\in V(i)\}=\{(i,j)|i\in C(j)\}$$

We find the optimal solution (i.e. the best R) for a *given* set CV. (Such a set CV may be obtained by the terminal Voronoi heuristic solution.) The distance between a center and a vertex in CV cannot exceed R, and vertices located on the boundary of the area A are forced to remain on the same side (or sides) on which they are located. This leads to the following non-linear programming formulation for a *given* set CV:

$$\min\{R^2\} \tag{4}$$

$$\text{Subject to: } (x_i-u_j)^2+(y_i-\upsilon_j)^2\leqslant R^2 \quad \forall (i,j)\in CV$$

where the centers (x_i,y_i) for $i=1,\ldots,p$, are variables. The vertices (u_j,υ_j) for $j=1,\ldots,\upsilon$ are also variables *except* when the point (u_j,υ_j) lies on a side of the area, in which case they are forced to remain on that side. If the vertex (u_j,υ_j) lies on a vertex of the area, it is held in its place and u_j and υ_j are no longer variables.

This non-linear formulation is a convex programming problem with one local optimum, which is the global optimum. It can be solved by standard mathematical programming methods.

6. NUMERICAL EXPERIMENTS WITH THE VORONOI PROGRAM IN A SQUARE

In order to solve the finishing-up algorithm, we used the student's version of AMPL (Fourer *et al.*, 1993) which uses MINOS 5.4 as the optimization package for non-linear programming. The AMPL Program and sample data for a square area are given in Appendix C. The finishing-up algorithm was solved by MINOS in less than a minute on a PC-486, 33 MHz for each of the test problems. The student version of AMPL limits the finishing-up solution to $p\leqslant 53$. It is interesting that a square root of the constraints in (4) was required for efficient solution.

For $p=n^2$ (and to a lesser extent for $p=n(n+1)$) we noticed the following pattern: we observe an $n\times n$ hexagonal pattern where each column consists of n hexagons but the hexagons are staggered, alternating between being shifted up and shifted down. Such patterns can be generated for AMPL for finishing-up. The case $p=49$ depicted in Fig. 2 was obtained this way. The pattern in Fig. 2 is quite similar to the pattern obtained by Iri *et al.* (1984). It is interesting that the p-center solution obtained here and the p-median solution in Iri *et al.*

(1984) are both quite similar beehive patterns. This approach cannot be used for p's that are not complete squares or of the form $n(n+1)$.

Some computational results are summarized in Table 2. The Voronoi heuristic was run once for each p. The values $r_{\min}$ and $r_{\max}$ are the minimal and maximal radii, respectively, obtained in the final configuration of the Voronoi heuristic. In Fig. 2, some solutions obtained for a square area are depicted. In our opinion, the results for $p=9$ are counter-

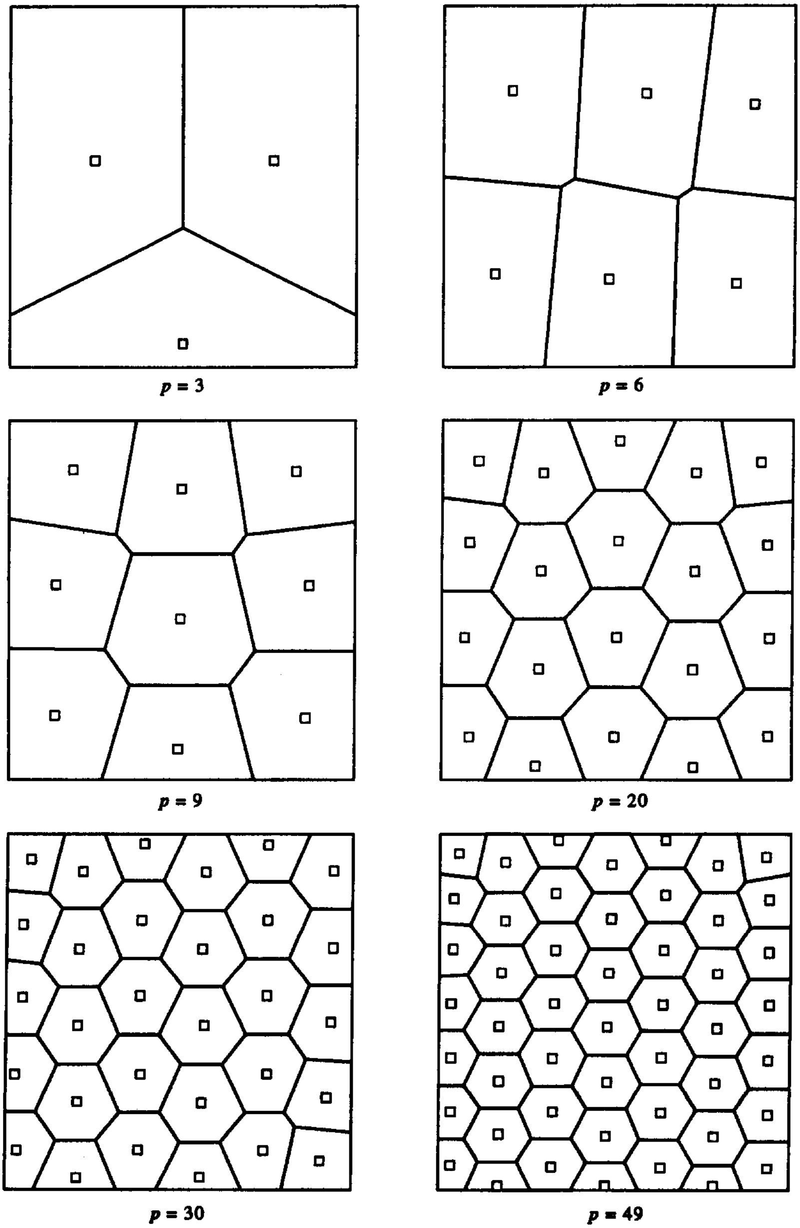

Fig. 2. Various p-center solutions in a square.

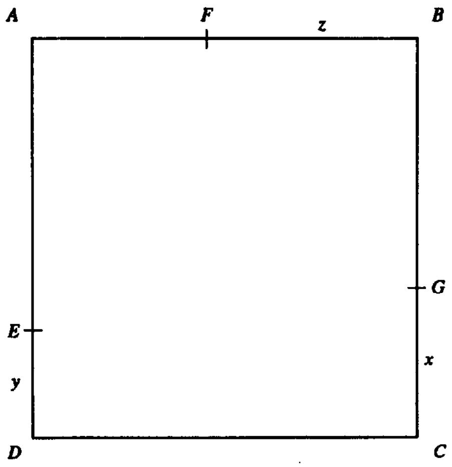

Fig. 3. Proof of $p=3$.

intuitive. It is intuitive that the 3×3 square configuration is the optimal one. This solution is by Table 1 $R=0.23570$. As can be seen from Fig. 2 and Table 2, the algorithm found a better solution with $R=0.23064$.

7. CONCLUSIONS

The p-center problem seeks to locate p facilities such that the maximum distance between a demand point and its closest facility is minimized. It is equivalent to covering all demand by p circles (defined by the norm used) of the smallest possible radius. This problem was extensively researched on graphs, trees and the Euclidean plane for a discrete set of demand points. In this paper, the problem is solved for demand which is continuously spread in an area.

Most location researchers formulate their models for a finite set of demand points because such formulations are more tractable; however, modeling demand as continuous is more

Table 2. Description of logistics costs modeled

p	Voronoi heuristic results		R (after finishing up)	Upper bound
	r_{max}	r_{min}		
3	0.50390	0.50387	0.50389	0.50389
5	0.32617	0.32615	0.32616	0.32992
6	0.29873	0.29871	0.29873	0.30046
9	0.23085	0.23083	0.23064	0.23570
16	0.17122	0.17119	0.16943	0.17678
20	0.15694	0.14756	0.15225	0.16008
30	0.12885	0.12715	0.12204	0.13017
256	0.04264	0.03961	not run	0.04419

realistic. It may be the only tractable model when the number of customers in the area is too large, yielding a problem that cannot be solved by a discrete model algorithm. Cases where demand is mobile (such as the location of stations for mobile telephones) are best modeled using continuous demand. Applications include those for the discrete p-center problems, such as building hospitals, schools, fire stations, as well as problems specific to area demand, such as the location of warning sirens, sprinkler systems, television transmitters, or area coverage by radar beams.

In this paper, we suggested a heuristic solution procedure based on Voronoi diagrams, followed by a "finishing-up" algorithm based on a non-linear programming formulation. We also provide lower bounds on the optimal solution as well as upper bounds for the p-center problem in a square.

It is interesting to note that some of the computational results are counter-intuitive; see Fig. 2. For example, covering a square with $p=9$ circles with the smallest possible radius is *not* achieved by locating nine centers in a grid, each covering a square of sides $\frac{1}{3}\times\frac{1}{3}$. The best configuration is not symmetric. Covering a square grassy area with nine sprinklers is more efficient by our configuration than placing the sprinklers in three straight rows.

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REFERENCES

Chandrasekaran, R. & Daughety, A. (1981) Location on tree networks: p-center and N-dispersion problems. *Mathematics of Operations Research*, **6**, 50–57.

Chandrasekaran, R. & Tamir, A. (1982) Polynomial bounded algorithms for locating p-centers on a tree. *Mathematical Programming*, **22**, 304–315.

Chen, R. & Handler, G. Y. (1987) A relaxation method for the solution of the minimax location–allocation problem in Euclidean space. *Naval Research Logistics Quarterly*, **34**, 775–788.

Cooper, L. (1963) Location–allocation problems. *Operations Research*, **11**, 331–343.

Cooper, L. (1964) Heuristic methods for location–allocation problems. *SIAM Review*, **6**, 37–53.

Coxeter, H. S. M., Few, L. & Rogers, C. A. (1959) Covering space with equal spheres. *Mathematica*, **6**, 247.

Current, J. & O'Kelly, M. (1992) Locating emergency warning sirens. *Decision Sciences*, **23**, 221–234.

Drezner, Z. (1984) The p-center problem—heuristic and optimal algorithms. *Journal of the Operational Research Society*, **35**, 741–748.

Drezner, Z. (1987) On the rectangular p-center problem. *Naval Research Logistics Quarterly*, **34**, 229–234.

Drezner, Z. & Erkut, E. (1995) Solving the continuous p-dispersion problem using non-linear programming. *Journal of the Operational Research Society*, **36**, 516–520.

Drezner, Z. & Zemel, E. (1992) Competitive location in the plane. *Annals of Operations Research*, **40**, 173–193.

Erkut, E. (1990) The discrete p-dispersion problem. *European Journal of Operational Research*, **40**, 48–60.

Fourer, R., Gay, D. M. & Kernighan, B. W. (1993) *AMPL A Modeling Language for Mathematical Programming.* The Scientific Press, South San Francisco.

Handler, G. Y. (1990) "p-Center Problems", in *Discrete Location Theory*, P. Mirchandani & R. L. Francis (eds), New York, John Wiley & Sons.

Iri, M., Murota, K. & Ohya, T. (1984) A fast Voronoi diagram algorithm with applications to geographical optimization problems. *Proceedings of the IFIP Conference on System Modelling and Optimization*, 1983, Copenhagen, Denmark. Also in *System Modelling and Optimization, Lecture Notes in Control and Information Science*, Vol. 59, pp. 273–288, P. Thoft-Christensen (ed.) Berlin, Springer-Verlag.

Kuby, M. J. (1987) Programming models for facility dispersion: the p-dispersion and maximum dispersion problem. *Geographical Analysis*, **19**, 315–329.

Ohya, T., Iri, M. & Murota, T. (1984) Improvements of the incremental method of the Voronoi diagram with computational comparison of various algorithms. *Journal of the Operations Research Society of Japan*, **27**, 306–337.

Okabe, A., Boots, B. & Sugihara, K. (1992) *Spatial Tesselations—Concepts and Applications of Voronoi Diagrams.* Chichester, John Wiley & Sons.

Okabe, A. & Suzuki, A. (1987) Stability in spatial competition for a large number of firms on a bounded two-dimensional space. *Environment and Planning A*, **19**, 1067–1082.
Papadimitriou, C. H. (1981) Worst case and probabilistic analysis of a geometric location problem. *SIAM Journal on Computing*, **10**, 542.
Shier, D. R. (1977) A min-max theorem for p-center problems on a tree. *Transportation Science*, **11**, 243–252.
Sugihara, K. & Iri, M. (1992) Construction of the Voronoi diagram for "one million" generators in single-precision arithmetic. *Proceedings of the IEEE*, **80**, 1471–1484.
Sugihara, K. & Iri, M. (1994) A robust topology-oriented incremental algorithm for Voronoi diagram. *International Journal of Computational Geometry and Applications*, **4**, 179–228.
Suzuki, A. & Okabe, A. (1995) Using Voronoi diagrams. In *Facility Location: A Survey of Applications and Methods*, Z. Drezner (ed.). New York, Springer-Verlag.
Tansel, B. C., Francis, R. L. & Lowe, T. J. (1990) Duality: covering and constraining p center problems on trees. In *Discrete Location Theory*, P. Mirchandani & R. L. Francis (eds). New York, John Wiley & Sons.
Vijay, J. (1985) An algorithm for the p-center problem in the plane. *Transportation Science*, **19**, 235–245.
Zemel, E. (1984) Probabilistic analysis of geometric location problems. *Annals of Operations Research*, **1**, 215–238.

APPENDIX A

Calculation of the best radius for a given N

Consider a unit square divided into N strips, where strip i contains n_i identical rectangles, for $i=1,\dots,N$. The widths of the strips are $x_1,\dots,x_N$, respectively. Each rectangle is circumscribed in a circle of a common radius R. The following relationships must exist:

$$\frac{1}{n_i^2}+x_i^2=4R^2, \quad \forall i$$

$$\sum_{i=1}^{N} n_i=p$$

$$\sum_{i=1}^{N} x_i=1$$

These relationships lead to the following optimization problem:

$$\min_N \{ \min_{n_1,\dots,n_N} \{R\}\} \tag{A.1}$$

subject to:

$$\sum_{i=1}^{N} n_i=p \tag{A.2}$$

$$\sum_{i=1}^{N} \sqrt{4R^2-\frac{1}{n_i^2}}=1 \tag{A.3}$$

Although R is a function of N and $n_1,\dots,n_N$, in the mathematical programming (A.1–A.3), R is just a variable.

Lemma 1: For a given N, the radius R is bounded by $R\geqslant 1/2\sqrt{1/N^2+N^2/p^2}$.

Proof: Suppose that the n_i's are allowed to be real numbers and not necessarily integers. Let λ and μ be the Lagrange multipliers for (A.2) and (A.3), respectively. The Lagrange function is:

$$R+\lambda\left(p-\sum_{i=1}^{N} n_i\right)+\mu\left(1-\sum_{i=1}^{N}\sqrt{4R^2-\frac{1}{n_i^2}}\right)$$

The partial derivative of the Lagrangean function by n_i leads to the equation:

$$-\lambda-\mu\frac{1}{n_i^3\sqrt{4R^2-1/n_i^2}}=0$$

which is equivalent to:

$$4R^2n_i^6-n_i^4-\left(\frac{\mu}{\lambda}\right)^2=0 \tag{A.4}$$

Since the square root in equation (A.3) must be real, n_i^2 must be greater than $1/4R^2$. It can be shown that the derivative of the polynomial (A.4) is positive for $n_i^2 \geqslant 1/4R^2$ and therefore there is only one solution for n_i. Since all the n_i have the same value, then by equation (6) $n_i=p/N$ for all i. Equation (A.3) turns to $N\sqrt{4R^2-N^2/p^2}=1$, or $R=1/2\sqrt{1/N^2+N^2/p^2}$. The lemma follows because the integer solution to (A.1–A.3) cannot be lower than the solution for real n_i's. □

The expression for R in Lemma 2 is minimized for $N=\sqrt{p}$. This leads to the following Theorem.

Theorem 1: Treating the variables in problem (A.1–A.3) as real numbers and not necessarily integers, yields the continuous solution $n_i \approx N=\sqrt{p}$ with the optimal radius $R=1/\sqrt{2p}$.

Corollary 1: If $p=n^2$ for some integer n, then the solution to problem (A.1–A.3) is $n_i=N=n$.

Corollary 2: For a given N: if p/N is an integer, then the solution to problem (A.1–A.3) is $n_i=p/N$.

Lemma 2: The function $f(x)=\sqrt{4R^2-1/x^2}$ is concave for $x>1/(2R)$.

Proof: It can be verified that the second derivative of $f(x)$ is negative. □

Lemma 3: $f(x+2)+f(x) \leqslant 2f(x+1)$ for $x>1/2R$.

Proof: Follows from the concavity of $f(x)$ by Lemma 2. □

Lemma 4: For $x>1/2R$ and $m \geqslant 1$ (m integer): $f(x+m)-f(x+m-1) \leqslant f(x+1)-f(x)$.

Proof: We prove the lemma by mathematical induction. The lemma is trivially true for $m=1$. Assume that the lemma is true for m and we prove it for $m+1$. By Lemma 3 $f(x+2)-f(x+1) \leqslant f(x+1)-f(x)$. By substituting $x+m-1$ into this inequality we get $f(x+m+1)-f(x+m) \leqslant f(x+m)-f(x+m-1)$. But $f(x+m)-f(x+m-1) \leqslant f(x+1)-f(x)$ by the induction assumption and therefore $f(x+m+1)-f(x+m) \leqslant f(x+1)-f(x)$. □

Lemma 5: For $y>x>1/2R$ (x,y integers): $f(x)+f(y) \leqslant f(x+1)+f(y-1)$.

Proof: It is a reformulation of Lemma 4 by substituting $y=x+m$. □

Lemma 6: The optimal R is obtained when $\max\{n_i\}-\min\{n_i\} \leqslant 1$.

Proof: Consider a solution n_i for $i=1,\dots,N$ that yields a value of R. Assume that $\max\{n_i\}-\min\{n_i\}>1$. We construct a sequence of solutions for which the value of R decreases each step, until we get a solution for which $\max\{n_i\}-\min\{n_i\} \leqslant 1$. Let j, k be indices such that $n_j=\max\{n_i\}$; $n_k=\min\{n_i\}$. Changing n_j to n_j-1 and n_k to n_k+1 retains constraint (A.2). The sum in constraint (A.3) increases for the same R by substituting $x=n_k$ and $y=n_j$ in Lemma 5. Therefore, the value of the feasible R for the new set $\{n_i\}$ decreases because the sum in (A.3) is an increasing function of R and the solution R to (A.3) must be smaller. This is the next member in the sequence of solutions. The cardinalities of the sets $\{s|n_s=\max\{n_i\}\}$ and $\{s|n_s=\min\{n_i\}\}$ are reduced by 1. These cardinalities are reduced each step until one of them becomes zero, which means that either $\max\{n_i\}$ is reduced by 1 or $\min\{n_i\}$ is increased by 1 and the difference is decreased by at least 1. If $\max\{n_i\}-\min\{n_i\}>1$ in the new solution, we generate the next value in the sequence of solutions. The sequence is continued until $\max\{n_i\}-\min\{n_i\} \leqslant 1$. □

Define $\lfloor z \rfloor$ as the largest integer not greater than z (z rounded down). For a given N define $K=\lfloor p/N \rfloor$. Consider a feasible solution $n_1,\dots,n_N$ with a value of R.

Theorem 2: For a given N: if p/N is integer, then the solution is $n_i=p/N$. And if p/N is not an integer, then in the optimal integer solution to (A.1–A.3) n_i is either equal to K or is equal to $K+1$, where $K=\lfloor p/N \rfloor$.

Proof: If p/N is integer, the Theorem follows Corollary 2. Therefore, in the following we assume that p/N is not an integer. By Lemma 6 $\max\{n_i\}-\min\{n_i\} \leqslant 1$. We prove that $\min\{n_i\} \leqslant p/N$ and $\max\{n_i\} \geqslant p/N$. Indeed, if

$\min\{n_i\} > p/N$ then $\sum_{i=1}^{N} n_i > N \cdot p/N = p$ contradicting (A.2). A similar argument proves that $\max\{n_i\} \geqslant p/N$, and the Theorem follows. □

For a given N, and K defined as above ($K = \lfloor p/N \rfloor$), the number of $n_i = K$ and the number of $n_i = K+1$ can be determined. Let $\alpha(K)$ be the number of $n_i = K$. Then, $\alpha(K) + \alpha(K+1) = N$, and by Equation (A.2) $K\alpha(K) + (K+1)\alpha(K+1) = p$. Solving these equations yield: $\alpha(K) = (K+1)N - p$ and $\alpha(K+1) = p - KN$ respectively. This leads to the following equation for R:

$$\alpha(K)\sqrt{4R^2 - \frac{1}{K^2}} + \alpha(K+1)\sqrt{4R^2 - \frac{1}{(K+1)^2}} = 1 \tag{A.5}$$

The solution to (A.5) is as follows. Substitute $c^2 = 4R^2 - 1/K^2$. The first term of (A.5) is linear in c. By transferring it to the right hand side and squaring the equation, a quadratic equation in c is obtained. This leads to:

1. If $\alpha(K) = \alpha(K+1)$ calculate $c = 1/N - p/2K^2(K+1)^2$.
2. Otherwise calculate

$$c = \frac{1}{N} - \frac{\alpha(K+1)}{N[\alpha(K) - \alpha(K+1)]}\left[\sqrt{1 + \frac{[\alpha(K) - \alpha(K+1)]N(2K+1)}{K^2(K+1)^2}} - 1\right].$$

3. Then $R = 1/2\sqrt{c^2 + 1/K^2}$

APPENDIX B

In this Appendix, we prove that $p=5$ is the smallest p for which the solution from the rectangle heuristic is not optimal. The rectangular solution is clearly optimal for $p=1$ and $p=2$. It is not optimal for $p=5$, as illustrated in Fig. 1. To complete the proof, we show that the rectangular solutions for $p=3$ and $p=4$ are optimal.

A proof that $p=3$ rectangle heuristic solution is optimal

For $p=3$, one center must cover two corners of the square. It can be easily verified that having all corners covered by two centers is not optimal and leads to an R greater than the one obtained in the following. The boundary of the square must be covered in segments as depicted in Fig. 3.

One center covers the part EF of the perimeter, one covers FG, and one covers EG including the bottom side. The covering circle has a radius R. All the Euclidean distances between the points E and F, F and G, D and G, and E and C cannot exceed $2R$. This leads to the following equations with the objective of minimizing R:

$$(1-y)^2 + (1-z)^2 \leqslant 4R^2$$
$$(1-x)^2 + z^2 \leqslant 4R^2$$
$$1 + x^2 \leqslant 4R^2$$
$$1 + y^2 \leqslant 4R^2$$

We first show that the minimal R entails $x=y$. If $x>y$ (and it is similarly shown for $y>x$), then because of the third constraint, y can be increased to a value less than x without violating the fourth constraint. That will make the first constraint a strict inequality, which means that z can be decreased, keeping it strict and having the second inequality strict. That makes all inequalities except for the third one strict and therefore x can be reduced, yielding a smaller R. Therefore, $x=y$. Now, for a given x (equal to y), the smallest R that minimizes the first two equations is obtained when $z=1-z$ which means $z=0.5$. That leaves the equations $(1-x)^2 + 0.25 \leqslant 4R^2$ and $1 + x^2 \leqslant 4R^2$. R is minimized when both are equalities which leads to $x=0.125$ which is our solution.

A proof that p=4 *rectangle heuristic solution is optimal*

For $p=4$, each corner must be covered by one center; otherwise a center that covers two corners must have $R \geqslant 0.5$. Consider a center located at (x,y) that covers the lower left corner (0,0). We must have $x^2 + y^2 \leqslant R^2$. Such a center covers $2\sqrt{R^2 - y^2}$ of the positive side of the x-axis. The total part of the circumference of the square covered by the center is $2\sqrt{R^2 - y^2} + 2\sqrt{R^2 - x^2}$. Since all four sides of the square must be covered $2\sqrt{R^2 - y^2} + 2\sqrt{R^2 - x^2} \geqslant 1$. For the smallest R, these two conditions must be fulfilled as equalities. Since

$x^2+y^2=R^2$, there is an angle θ such that $x=R\cos\theta$, $y=R\sin\theta$. The second condition is $2R(\cos\theta+\sin\theta)=1$. The mimimal R is attained for $\theta=\pi/4$ which is the square solution for $p=4$.

APPENDIX C

The AMPL program for a square area

```
set CENTER;
set VERTEX;
param R:
param xx{i in CENTER};
param yy{i in CENTER};
param uu{j in VERTEX};
param vv{j in VERTEX};
set U1 := {j in VERTEX: uu[j]=0 or uu[j]=1};
set U2 := {j in VERTEX: uu[j]<>0 and uu[j]<>1};
set V1 := {j in VERTEX: vv[j]=1 or vv[j]=0};
set V2 := {j in VERTEX: vv[j]<>0 and vv[j]<>1};
set PAIRS := {i in CENTER, j in VERTEX:
(xx[i]-uu[j])U^2+(yy[i]-vv[j])^2<R^2};
var x {i in CENTER}>=0;
var y {i in CENTER}>=0;
var u {j in U2}>=0;
var v {j in V2}=0;
var L;
minimize objective:L;
subject to
Constraint1{j in U1 inter V1, (i,j) in PAIRS}:
sqrt((x[i]-uu[j]^2+(y[i]-vv[j])^2+1.e-16) <=L;
Constraint2{j in U1 inter V2, (i,j) in PAIRS}:
sqrt((x[i]-uu[j])^2+(y([i]-v[j])^2+1.e-16) <=L;
Constraint3{j in U2 inter V1, (i,j) in PAIRS}:
sqrt((x[i]-u[j])^2+(y[i]-vv[j])^2+1.e-16) <=L;
Constraint4{j in U2 inter V2, (i,j) in PAIRS}:
sqrt((x[i]-u[j])^2+(y[i]-v[j])^2+1.e-16) <=L;

Sample Data (for p=5)
set CENTER :=1 2 3 4 5;
set VERTEX :=1 2 3 4 5 6 7 8 9 10 11 12;
param R:=.3272
param: xx yy:=
1 .148141 .709531
2 .500000 .745115
3 .851859 .709531
4 .250000 .209477
5 .750000 .209477;
param: uu vv:=
 1 .000000 1.000000
 2 .296494 1.000000
 3 .703506 1.000000
 4 1.000000 1.000000
 5 .000000 .418954
 6 .351763 .454740
 7 .648237 .454740
 8 1.000000 .418954
 9 .500000 .418955
10 .000000 .000000
11 .500000 .000000
12 1.000000 .000000;
```