Appendix : *Mathematica* script

As sketched in Section 3.3: “The Elimination lemma”, in order to prove that the system $F_3 =. .. = F_6 = 0$ has a unique solution it is enough to compute the resultant $\mathrm{Res}^3{}_6$ and show that it is not identically zero as a function on (λ,α).

```
<< Notation`

Symbolize[ p̃ ]

Symbolize[ η̃ ]

Symbolize[ S̃ ]

Symbolize[ c̃ ]

Symbolize[ q̃ ]

Symbolize[ λ1 ]

Symbolize[ λ2 ]

Symbolize[ α0 ]

Symbolize[ α1 ]

Symbolize[ α2 ]
```

We first introduce the basic objects defining the normal forms for foliations F and $\tilde{F}$ (cf. the begining of Section 4: “Definitions and normalizations”).

```
r := w^2 - 1
s := λ1 * (w - 1) + λ2 * (w + 1)
σ := λ1 + λ2
p := α1 * (w - 1) + α2 (w + 1)
p̃ :=  β1 * (w - 1) + β2 * (w + 1)
η := α1 + α2
η̃ := β1 + β2
```

We now proceed to define the polynomials $S_d(w)$ and the coefficients c_d and their tilde-analogs which are defined in Definition 4.1 and Proposition 4.2. Explicit formulas for these objects appear at the begining of Section 7: “Proof of Elimination lemma”.

```
S2 := r
S̃2 := r
c2 := α0 * (1 - σ)
c̃2 := β0 * (1 - σ)

S3 := -s * p * r + (η - α0 * σ) * r^2
S̃3 := -s * p̃ * r + (η̃ - β0 * σ) * r^2
c3 := -α0^2 * σ * (1 - σ)
c̃3 := -β0^2 * σ * (1 - σ)

S4 := -p * r^2 + α0 * (2 * σ - 1) * s * p * r^2 + α0 * σ * (α0 * σ - η) * r^3
S̃4 := -p̃ * r^2 + β0 * (2 * σ - 1) * s * p̃ * r^2 + β0 * σ * (β0 * σ - η̃) * r^3
c4 := α0^3 * σ^2 * (1 - σ)
c̃4 := β0^3 * σ^2 * (1 - σ)
```

```
S5 := s*p^2*r^2 + (2*α0*σ - η)*p*r^3 + α0^2*σ*(2 - 3*σ)*s*p*r^3 + α0^2*σ^2*(η - α0*σ)*r^4
S̃5 := s*p̃^2*r^2 + (2*β0*σ - η̃)*p̃*r^3 + β0^2*σ*(2 - 3*σ)*s*p̃*r^3 + β0^2*σ^2*(η̃ - β0*σ)*r^4
c5 := -α0^4*σ^3*(1 - σ)
c̃5 := -β0^4*σ^3*(1 - σ)

S6 := p^2*r^3 + α0*(1 - 3*σ)*s*p^2*r^3 + (2*α0*σ*η - 3*α0^2*σ^2)*p*r^4 -
   α0^3*σ^2*(3 - 4*σ)*s*p*r^4 + α0^3*σ^3*(α0*σ - η)*r^5
S̃6 := p̃^2*r^3 + β0*(1 - 3*σ)*s*p̃^2*r^3 + (2*β0*σ*η̃ - 3*β0^2*σ^2)*p̃*r^4 -
   β0^3*σ^2*(3 - 4*σ)*s*p̃*r^4 + β0^3*σ^3*(β0*σ - η̃)*r^5
```

Now that above objects have been defined we proceed to making some linear computations. We first compute the polynomials $P_d(w)$ that appear in the statement of the Key lemma and the polynomials $F_d(\beta)$ that appear in the statement of the Main lemma. In order to compute the polynomials P_d(w) we use the formulas found in Propositions 6.4, 6.6, 6.8 and 6.10. In order to obtain the $F_d(\beta)$ we shall need first the polynomials R_d(w) defined in Section 3.2: "Deducing Main lemma from Key lemma". Once we have computed the R_d(w) we compute $F_d(\beta)$ using the strategy outlined in Section 7.2: "Computing the polynomials F_d ".

We proceed doing this degree by degree, from $d = 3$ to $d = 6$. We will explain in detail the process for $d = 3$. The following cases are completely analogous.

We first imput the expression for $P_3(w)$ found in Proposition 6.4

```
P3 := S̃3 - S3
```

Second, we define the linear map L_3 which appears on Section 3.2 and compute

```
L3[f_] := D[f, w]*r + 2*(s - 2*w)*f
```

The next step is to compute $R_3(w)$ by finding an inverse image of $P_3(w)$ under L_3; that is, we define $R_3(w) = L_3^{-1}(P_3(w))$.

```
A3 := Transpose[{Coefficient[L3[1], w, #] & /@ Range[0, 4], Coefficient[L3[w], w, #] & /@ Range[0, 4],
    Coefficient[L3[w^2], w, #] & /@ Range[0, 4], Coefficient[L3[w^3], w, #] & /@ Range[0, 4]}]
B3 := Drop[A3, 1]
r3 := LinearSolve[B3, Coefficient[P3, w, #] & /@ Range[1, 4]]
R3 := r3.{1, w, w^2, w^3}
```

We finally obtain the polynomial $F_3(\beta)$ by the formula $F_3 = L_3(R_3)\,(0) - P_3(0)$ (cf. Section 7.2).

```
v3 := Take[A3, 1]
F3 = Simplify[v3.r3 - Coefficient[P3, w, 0]][[1]];
```

We repeat the process for d=4,5,6. For these degrees we will also need the auxiliary polynomials $q_d(w)$ in order to define $P_d(w)$ (cf. Propositions 5.3, 5.4 and 5.5).

```
q4 := S4 + c2*S3*r - (c3 / 2)*S2*r^2
q̃4 := S̃4 + c̃2*S̃3*r - (c̃3 / 2)*S̃2*r^2
P4 := q̃4 - q4 - S2*R3
L4[f_] := D[f, w]*r + 3*(s - 2*w)*f

A4 := Transpose[{Coefficient[L4[1], w, #] & /@ Range[0, 6], Coefficient[L4[w], w, #] & /@ Range[0, 6],
    Coefficient[L4[w^2], w, #] & /@ Range[0, 6], Coefficient[L4[w^3], w, #] & /@ Range[0, 6],
    Coefficient[L4[w^4], w, #] & /@ Range[0, 6], Coefficient[L4[w^5], w, #] & /@ Range[0, 6]}]
B4 := Drop[A4, 1]
v4 := Take[A4, 1]
r4 := LinearSolve[B4, Coefficient[P4, w, #] & /@ Range[1, 6]]
R4 := r4.{1, w, w^2, w^3, w^4, w^5}

F4 = Simplify[v4.r4 - Coefficient[P4, w, 0]][[1]];


q5 := S5 + 2*c2*S4*r + c2^2*S3*r^2 - (2 / 3)*(c4 + c3*c2)*S2*r^3
q̃5 := S̃5 + 2*c̃2*S̃4*r + c̃2^2*S̃3*r^2 - (2 / 3)*(c̃4 + c̃3*c̃2)*S̃2*r^3
P5 := q̃5 - q5 - 2*S2*R4
L5[f_] := D[f, w]*r + 4*(s - 2*w)*f
```

```
A5 := Transpose[{Coefficient[L5[1], w, #] &/@Range[0, 8], Coefficient[L5[w], w, #] &/@Range[0, 8],
    Coefficient[L5[w^2], w, #] &/@Range[0, 8], Coefficient[L5[w^3], w, #] &/@Range[0, 8],
    Coefficient[L5[w^4], w, #] &/@Range[0, 8], Coefficient[L5[w^5], w, #] &/@Range[0, 8],
    Coefficient[L5[w^6], w, #] &/@Range[0, 8], Coefficient[L5[w^7], w, #] &/@Range[0, 8]}]
B5 := Drop[A5, 1]
v5 := Take[A5, 1]
r5 := LinearSolve[B5, Coefficient[P5, w, #] &/@Range[1, 8]]
R5 := r5.{1, w, w^2, w^3, w^4, w^5, w^6, w^7}

F5 := Simplify[v5.r5 - Coefficient[P5, w, 0]][[1]]


q6 := S6 + 3 * c2 * S5 * r + (c3 / 2 + 3 * c2^2) * S4 * r^2 +
   (-c4 / 3 + c3 * c2 / 6 + c2^3) * S3 * r^3 + (-3 * c5 / 4 - 3 * c4 * c2 / 2 - c3^2 / 8 - 3 * c3 * c2^2 / 4) * S2 * r^4
q̃6 := S̃6 + 3 * c̃2 * S̃5 * r + (c̃3 / 2 + 3 * c̃2^2) * S̃4 * r^2 + (-c̃4 / 3 + c̃3 * c̃2 / 6 + c̃2^3) * S̃3 * r^3 +
   (-3 * c̃5 / 4 - 3 * c̃4 * c̃2 / 2 - c̃3^2 / 8 - 3 * c̃3 * c̃2^2 / 4) * S̃2 * r^4
P6 := q̃6 - q6 + q̃4 * R3 - (1 / 2) * S2 * R3^2 - S3 * R4 - 3 * S2 * R5
L6[f_] := D[f, w] * r + 5 * (s - 2 * w) * f

A6 := Transpose[{Coefficient[L6[1], w, #] &/@Range[0, 10], Coefficient[L6[w], w, #] &/@Range[0, 10],
    Coefficient[L6[w^2], w, #] &/@Range[0, 10], Coefficient[L6[w^3], w, #] &/@Range[0, 10],
    Coefficient[L6[w^4], w, #] &/@Range[0, 10], Coefficient[L6[w^5], w, #] &/@Range[0, 10],
    Coefficient[L6[w^6], w, #] &/@Range[0, 10], Coefficient[L6[w^7], w, #] &/@Range[0, 10],
    Coefficient[L6[w^8], w, #] &/@Range[0, 10], Coefficient[L6[w^9], w, #] &/@Range[0, 10]}]
B6 := Drop[A6, 1]
v6 := Take[A6, 1]
r6 := LinearSolve[B6, Coefficient[P6, w, #] &/@Range[1, 10]]

F6 := Simplify[v6.r6 - Coefficient[P6, w, 0]][[1]]
```

Now that we have expresions for the polynomials F_d, we proceed to compute the resultants defined in Section 3.3.

```
R14 := Resultant[F3, F4, β2]
R15 := Resultant[F3, F5, β2]
R16 := Resultant[F3, F6, β2]
R25 := Resultant[R14, R15, β1]
R26 := Resultant[R14, R16, β1]
R36 := Resultant[PolynomialQuotient[R25, β0 - α0, β0], R26, β0]
```

In order to prove that R_{36} is not identically zero as a function of (λ,α) we evaluate it at the following specific values to obtain a non-zero complex number:

```
λ1 = 2 - i;
λ2 = 2 * i;
α0 = 1;
α1 = 0;
α2 = 0;
```

$\mathbf{R_{36}}$

1 139 132 346 851 261 972 844 438 537 600 339 344 017 050 909 765 016 162 665 597 454 661 514 992 363 984 953 671 805 073 \
303 589 256 278 329 063 734 447 264 797 494 684 006 274 034 986 417 982 381 876 948 845 277 434 723 489 378 025 830 554
019 719 482 402 795 748 291 332 712 495 243 410 005 291 278 511 103 697 343 423 866 393 743 897 177 899 776 910 159 236
355 325 952 /
382 782 474 751 194 656 935 499 150 514 101 659 066 355 261 955 881 356 551 638 008 175 442 412 332 756 259 019 106 \
437 335 722 148 418 426 513 671 875 +
(316 571 246 264 261 451 513 223 958 287 182 218 066 698 186 828 724 890 033 909 068 686 721 275 482 267 295 563 143 \
690 921 432 494 134 514 538 725 814 359 513 133 358 239 154 093 234 034 868 353 864 202 336 721 532 865 359 711 273 \
364 434 527 303 029 712 283 367 265 224 827 939 621 980 392 569 302 532 557 669 597 795 542 855 514 867 957 249 581 \
115 963 414 688 038 912 ⅈ) /
127 594 158 250 398 218 978 499 716 838 033 886 355 451 753 985 293 785 517 212 669 391 814 137 444 252 086 339 702 \
145 778 574 049 472 808 837 890 625

This is a non-zero complex number and so R_{36} is not identically zero.

In order to prove the Elimination Lemma we still need to prove Propositin 3.3.
We have computed the polynomials F_3 and F_4 and we now declare $\beta_0 = \alpha_0$. We can now see that $F_3 = 0$, $F_4 = 0$ forms a linear inhomogeneous system on β_1 and β_2, and we can easily verify that this system has non-zero determinant.

$\mathbf{Clear}[\lambda_1, \lambda_2, \alpha_0, \alpha_1, \alpha_2]$

$\beta_0 := \alpha_0$

The following is the explicit expressions for F_3. It is clear from it that F_3 is linear on β_1, β_2.

$\mathbf{Collect}[F_3, \{\beta_1, \beta_2\}, \mathbf{Simplify}]$

$$\frac{16 (-1+\lambda_1)(-1+\lambda_2)(\alpha_1 (-1+2\lambda_1+\lambda_2)(-1+2\lambda_2)+\alpha_2(-1+2\lambda_1)(-1+\lambda_1+2\lambda_2))}{(-2+\lambda_1+\lambda_2)(-1+\lambda_1+\lambda_2)(-3+2\lambda_1+2\lambda_2)(-1+2\lambda_1+2\lambda_2)} - \frac{16(-1+\lambda_1)(-1+\lambda_2)(-1+2\lambda_1+\lambda_2)(-1+2\lambda_2)\beta_1}{(-2+\lambda_1+\lambda_2)(-1+\lambda_1+\lambda_2)(-3+2\lambda_1+2\lambda_2)(-1+2\lambda_1+2\lambda_2)} - \frac{16(-1+\lambda_1)(-1+2\lambda_1)(-1+\lambda_2)(-1+\lambda_1+2\lambda_2)\beta_2}{(-2+\lambda_1+\lambda_2)(-1+\lambda_1+\lambda_2)(-3+2\lambda_1+2\lambda_2)(-1+2\lambda_1+2\lambda_2)}$$

The expression for F_4 is a bit more complicated but still linear on β_1, β_2. To see this we proceed as follows: consider the power series expansion $F_4 = c_{00} + c_{10}\,\beta_1 + c_{01}\,\beta_2 + \ldots$. We will prove that in fact F_4 coincides with its 1-jet.

$\mathbf{c_{00} = Simplify}[F_4 \;/.\; \{\beta_1 \to 0, \beta_2 \to 0\}]$

$$\Big(16(-1+\lambda_1)(-1+\lambda_2)\Big(\alpha_1\big(-48+266\lambda_2-629\lambda_2^2+793\lambda_2^3-535\lambda_2^4+171\lambda_2^5-18\lambda_2^6+6\lambda_1^5(25-54\lambda_2+18\lambda_2^2)+\lambda_1^4(-715+2031\lambda_2-1638\lambda_2^2+432\lambda_2^3)+\lambda_1^3(1262-4453\lambda_2+5481\lambda_2^2-2988\lambda_2^3+648\lambda_2^4)+\lambda_1^2(-1021+4295\lambda_2-6960\lambda_2^2+5640\lambda_2^3-2376\lambda_2^4+432\lambda_2^5)+\lambda_1(374-1818\lambda_2+3598\lambda_2^2-3757\lambda_2^3+2211\lambda_2^4-720\lambda_2^5+108\lambda_2^6)+4\alpha_0(-2+3\lambda_1)(2-9\lambda_2+9\lambda_2^2)(8\lambda_1^5+4\lambda_1^4(-11+9\lambda_2)+2\lambda_1^3(45-78\lambda_2+32\lambda_2^2)+(-1+\lambda_2)^2(-6+19\lambda_2-16\lambda_2^2+4\lambda_2^3)+\lambda_1^2(-85+235\lambda_2-204\lambda_2^2+56\lambda_2^3)+\lambda_1(37-145\lambda_2+200\lambda_2^2-116\lambda_2^3+24\lambda_2^4))\big)+\alpha_2\big(48-374\lambda_2+1021\lambda_2^2-1262\lambda_2^3+715\lambda_2^4-150\lambda_2^5-18\lambda_1^6(-1+6\lambda_2)-9\lambda_1^5(19-80\lambda_2+48\lambda_2^2)+\lambda_1^4(535-2211\lambda_2+2376\lambda_2^2-648\lambda_2^3)+\lambda_1^3(-793+3757\lambda_2-5640\lambda_2^2+2988\lambda_2^3-432\lambda_2^4)+\lambda_1^2(629-3598\lambda_2+6960\lambda_2^2-5481\lambda_2^3+1638\lambda_2^4-108\lambda_2^5)+\lambda_1(-266+1818\lambda_2-4295\lambda_2^2+4453\lambda_2^3-2031\lambda_2^4+324\lambda_2^5)+4\alpha_0(2-9\lambda_1+9\lambda_1^2)(-2+3\lambda_2)(4\lambda_1^5+24\lambda_1^4(-1+\lambda_2)+\lambda_1^3(55-116\lambda_2+56\lambda_2^2)+(1-2\lambda_2)^2(-6+13\lambda_2-9\lambda_2^2+2\lambda_2^3)+4\lambda_1^2(-15+50\lambda_2-51\lambda_2^2+16\lambda_2^3)+\lambda_1(31-145\lambda_2+235\lambda_2^2-156\lambda_2^3+36\lambda_2^4))\big)\Big)\Big) \Big/ \Big((-2+\lambda_1+\lambda_2)^2(-1+\lambda_1+\lambda_2)^2(-3+2\lambda_1+2\lambda_2)(-1+2\lambda_1+2\lambda_2)(-5+3\lambda_1+3\lambda_2)(-4+3\lambda_1+3\lambda_2)(-2+3\lambda_1+3\lambda_2)\Big)$$

```
c10 = Simplify[D[F4, β1] /. {β1 → 0, β2 → 0}]
```

$$-\Big(\big(16\,(-1+\lambda_1)\,(-1+\lambda_2)\,\big(-48+266\,\lambda_2-629\,\lambda_2^2+793\,\lambda_2^3-535\,\lambda_2^4+171\,\lambda_2^5-18\,\lambda_2^6+6\,\lambda_1^5\,(25-54\,\lambda_2+18\,\lambda_2^2)+\lambda_1^4\,(-715+2031\,\lambda_2-1638\,\lambda_2^2+432\,\lambda_2^3)+\lambda_1^3\,(1262-4453\,\lambda_2+5481\,\lambda_2^2-2988\,\lambda_2^3+648\,\lambda_2^4)+\lambda_1^2\,(-1021+4295\,\lambda_2-6960\,\lambda_2^2+5640\,\lambda_2^3-2376\,\lambda_2^4+432\,\lambda_2^5)+\lambda_1\,(374-1818\,\lambda_2+3598\,\lambda_2^2-3757\,\lambda_2^3+2211\,\lambda_2^4-720\,\lambda_2^5+108\,\lambda_2^6)+4\,\alpha_0\,(-2+3\,\lambda_1)\,(2-9\,\lambda_2+9\,\lambda_2^2)\,(8\,\lambda_1^5+4\,\lambda_1^4\,(-11+9\,\lambda_2)+2\,\lambda_1^3\,(45-78\,\lambda_2+32\,\lambda_2^2)+(-1+\lambda_2)^2\,(-6+19\,\lambda_2-16\,\lambda_2^2+4\,\lambda_2^3)+\lambda_1^2\,(-85+235\,\lambda_2-204\,\lambda_2^2+56\,\lambda_2^3)+\lambda_1\,(37-145\,\lambda_2+200\,\lambda_2^2-116\,\lambda_2^3+24\,\lambda_2^4))\big)\big)\Big/\big((-2+\lambda_1+\lambda_2)^2\,(-1+\lambda_1+\lambda_2)^2\,(-3+2\,\lambda_1+2\,\lambda_2)\,(-1+2\,\lambda_1+2\,\lambda_2)\,(-5+3\,\lambda_1+3\,\lambda_2)\,(-4+3\,\lambda_1+3\,\lambda_2)\,(-2+3\,\lambda_1+3\,\lambda_2)\big)\Big)$$

```
c01 = Simplify[D[F4, β2] /. {β1 → 0, β2 → 0}]
```

$$-\Big(\big(16\,(-1+\lambda_1)\,(-1+\lambda_2)\,\big(48-374\,\lambda_2+1021\,\lambda_2^2-1262\,\lambda_2^3+715\,\lambda_2^4-150\,\lambda_2^5-18\,\lambda_1^6\,(-1+6\,\lambda_2)-9\,\lambda_1^5\,(19-80\,\lambda_2+48\,\lambda_2^2)+\lambda_1^4\,(535-2211\,\lambda_2+2376\,\lambda_2^2-648\,\lambda_2^3)+\lambda_1^3\,(-793+3757\,\lambda_2-5640\,\lambda_2^2+2988\,\lambda_2^3-432\,\lambda_2^4)+\lambda_1^2\,(629-3598\,\lambda_2+6960\,\lambda_2^2-5481\,\lambda_2^3+1638\,\lambda_2^4-108\,\lambda_2^5)+\lambda_1\,(-266+1818\,\lambda_2-4295\,\lambda_2^2+4453\,\lambda_2^3-2031\,\lambda_2^4+324\,\lambda_2^5)+4\,\alpha_0\,(2-9\,\lambda_1+9\,\lambda_1^2)\,(-2+3\,\lambda_2)\,(4\,\lambda_1^5+24\,\lambda_1^4\,(-1+\lambda_2)+\lambda_1^3\,(55-116\,\lambda_2+56\,\lambda_2^2)+(1-2\,\lambda_2)^2\,(-6+13\,\lambda_2-9\,\lambda_2^2+2\,\lambda_2^3)+4\,\lambda_1^2\,(-15+50\,\lambda_2-51\,\lambda_2^2+16\,\lambda_2^3)+\lambda_1\,(31-145\,\lambda_2+235\,\lambda_2^2-156\,\lambda_2^3+36\,\lambda_2^4))\big)\big)\Big/\big((-2+\lambda_1+\lambda_2)^2\,(-1+\lambda_1+\lambda_2)^2\,(-3+2\,\lambda_1+2\,\lambda_2)\,(-1+2\,\lambda_1+2\,\lambda_2)\,(-5+3\,\lambda_1+3\,\lambda_2)\,(-4+3\,\lambda_1+3\,\lambda_2)\,(-2+3\,\lambda_1+3\,\lambda_2)\big)\Big)$$

```
F4 == c00 + c10 * β1 + c01 * β2 // FullSimplify
```

```
True
```

This shows that F_4 coincides with its 1-jet and so is linear on β_1, β_2.

Consider now the linear system $F_3 = 0,\ F_4 = 0$.

```
A := {{Coefficient[F3, β1, 1], Coefficient[F3, β2, 1]}, {Coefficient[F4, β1, 1], Coefficient[F4, β2, 1]}}
```

Again, to show that the determinant of the matrix A is not identically zero it is enough to evaluate it at concrete values of (λ, α) and verify that we obtain a non-zero complex number:

```
Det[A] /. {λ1 → 2 - ⅈ, λ2 → 2 * ⅈ, α0 → 1, α1 -> 0, α2 -> 0}
```

$$\frac{848\,896}{325} - \frac{5\,379\,072\,i}{325}$$

This proves that A is not identically zero with respect to (λ,α) and so proves Proposition 3.3.

This completes the proof of the Elimination Lemma.