The analytic classification of germs of maps $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ tangent to identity

Valente Ramírez

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Let X be a Riemann surface and $f\colon X \to X$ an analytic map with a fixed point $p \in X$.

Problem

Study they dynamics of f in a neighborhood of the fixed point p.

Strategy: Study maps $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and their *normal forms*.

$$\begin{tikzcd}
(X, p) \arrow[r, "f"] \arrow[d, "\varphi"'] & (X, p) \arrow[d, "\varphi"] \\
(\mathbb{C}, 0) \arrow[r, "\tilde{f}"'] & (\mathbb{C}, 0)
\end{tikzcd}$$

Definition

Two maps $f_1, f_2\colon (\mathbb{C},0) \to (\mathbb{C},0)$ are *analytically equivalent* whenever there exits an invertible analytic map $\varphi\colon (\mathbb{C},0) \to (\mathbb{C},0)$ such that $\varphi \circ f_1 = f_2 \circ \varphi$.

$$\begin{tikzcd}
(\mathbb{C},0) \arrow[r, "f_1"] \arrow[d, "\varphi"'] & (\mathbb{C},0) \arrow[d, "\varphi"] \\
(\mathbb{C},0) \arrow[r, "f_2"'] & (\mathbb{C},0)
\end{tikzcd}$$

We wish to:

- Find *particularly nice* elements in each conjugacy class,
- Be able to decide whether two given maps f_1 and f_2 are analytically equivalent or not.

Let $f\colon (\mathbb{C},0) \to (\mathbb{C},0)$ be given locally by the power series

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots.$$

The number $\lambda = f'(0)$ is called the *multiplier* of f and it plays a fundamental role in the dynamics of f.

Classification of fixed points

We say that the fixed point at the origin is

- **Attracting:** If $|\lambda| < 1$,
- **Superttracting:** If $|\lambda| = 0$,
- **Repelling:** If $|\lambda| > 1$,
- **Indifferent:** If $|\lambda| = 1$.

Definition

A map $f\colon (\mathbb{C},0) \to (\mathbb{C},0)$ is called *hyperbolic* if $|\lambda| \neq 0, 1$.

Heuristics: If $|z|$ is very small then $f(z)$ is approximately $f'(z) \cdot z$.

Is f equivalent to the linear map $z \mapsto \lambda z$?

i.e. does there exist h such that $h \circ f \circ h^{-1}(z) = \lambda\, z$?

Think about power series!

Definition

Two maps $f_1, f_2 \colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ are *formally equivalent* whenever there exits a formal power series $h(z) = \beta_1 z + \beta_2 z^2 + \ldots$ such that $h \circ f_1 = f_2 \circ h$ in $\mathbb{C}[[z]]$.

Theorem

Any hyperbolic germ is formally equivalent to its linear part.

Theorem (Schröeder–Kœnigs Theorem)

Any hyperbolic germ is analytically equivalent to its linear part.

Indeed, in a small enough neighborhood of the origin (and assuming $|\lambda| < 1$) the map

$$\phi(z) = \lim_{n\to\infty} \lambda^{-n} f^{\circ n}(z)$$

satisfies

$$\phi \circ f \circ \phi^{-1}(z) = \lambda z.$$

Hyperbolic fixed points are well understood!!

Some general conclusions:

- Formal equivalence is necessary, yet not always sufficient, for analytic equivalence.
- The formal classification problem is *easier* than its analytic counterpart!

Definition

A map $f\colon (\mathbb{C},0) \to (\mathbb{C},0)$ is *tangent to identity* if $\lambda = 1$, that is, if

$$f(z) = z + a_{p+1}z^{p+1} + \dots$$

The number $p+1 \in \mathbb{N}$ is the *multiplicity* of f, and the number p is called the *level* of tangency to the identity.

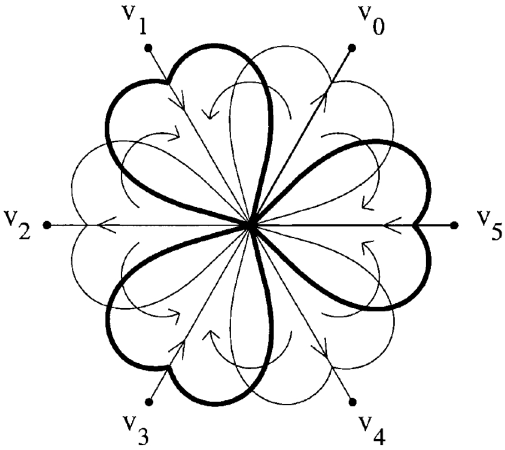
A map is called *parabolic* if $\lambda^q = 1$ for some integer q.

Remark

If $\lambda^q = 1$ then $f^{\circ q}$ is tangent to the identity.

Throughout this talk parabolic will be understood to mean tangent to the identity.

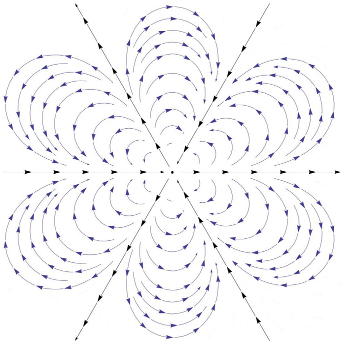
Dynamics of parabolic maps



Theorem (Camacho–Sad, 1982)

Any parabolic map of level p is topologically equivalent to the time–one map of the complex vector field

$$v(z) = z^{p+1}\frac{\partial}{\partial z}.$$



Formal classification of parabolic maps

A parabolic map $f(z) = z + a_{p+1}z^{p+1} + \dots$ is formally equivalent to a unique polynomial map of the form

$$z \mapsto z + z^{p+1} + \alpha z^{2p+1}.$$

The number $\alpha \in \mathbb{C}$ is called the *formal invariant* of f.

The formal invariant may be recovered from f by the formula

$$\alpha = \frac{1}{2\pi i} \int_{\partial D_\epsilon} \frac{d\zeta}{\zeta - f(\zeta)}.$$

Definition

The *iterative residue* of f is defined to be

$$\text{résit}(f) = -\frac{1}{4\pi i}\int_{\partial D_\epsilon} \frac{1+f'(\zeta)}{\zeta - f(\zeta)}\, d\zeta.$$

The iterative residue has the following nice properties:

- $\text{résit}(f) = \frac{p+1}{2} - \alpha$,
- $\text{résit}(f^{\circ k}) = \text{résit}(f)/k$,
- The map f is formally equivalent to the time–one map of the vector field
$$\frac{z^{p+1}}{1+\beta z^p}\frac{\partial}{\partial z},$$
if and only if $\beta = \text{résit}(f)$.

This map is a nice formal normal form!

Question

Does formal equivalence of parabolic maps imply their analytic equivalence?

Answer

No! Normalizing series $\widehat{H} \in \mathbb{C}[[z]]$ taking f to its *formal normal form* are almost always divergent.

New questions

What are the analytic invariants of parabolic maps?
How different is the analytic classification from the formal one?

Problem

Construct a minimal set of analytic invariants for parabolic maps such that a necessary and sufficient condition for the analytic equivalence of two maps is the coincidence of their respective invariants.

We will assign to each parabolic map f such a set of invariants $\mathcal{M}_f$, which we'll called the *modulus* of analytic classification of f. The space of all such moduli is called the *moduli space* of analytic classification of parabolic maps.

Example

The moduli space of *both* the analytic and formal classification of hyperbolic maps is $\mathbb{C}^* \setminus S^1$.

Example

The moduli space of formal classification of parabolic maps is $\mathbb{Z}_+ \times \mathbb{C}$.

Theorem (Écalle 1982, Voronin 1981)

The analytic classification of parabolic maps has functional moduli, that is, the moduli space is an infinite dimensional functional space.

Construction of the Écalle–Voronin modulus

The basic idea is the following: Restrict our attention to the class

$$\mathscr{A}_{p,\beta} = \{f \mid f(z) = z + z^{p+1} + o(z^{p+1}),\ \text{résit}(f) = \beta\}.$$

Among the elements of this class we distinguish the map

$$f_{p,\beta} = \exp\left(F_{p,\beta}\right), \qquad F_{p,\beta} = \frac{z^{p+1}}{1+\beta z^p}\,\frac{\partial}{\partial z}.$$

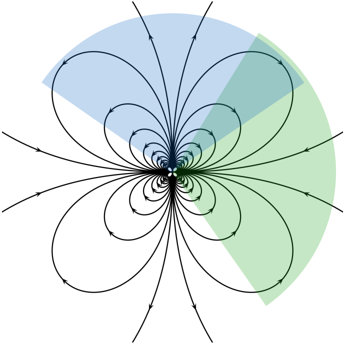
Given any $f \in \mathscr{A}_{p,\beta}$ we wil try to conjugate f to the map $f_{p,\beta}$ and somehow *measure* the extent to which the analytic class of f differs from the analytic class of $f_{p,\beta}$.

Definition

A *nice p–covering* of a punctured neighborhood of the origin is a covering $\mathcal{S} = \{S_1, \dots, S_{2p}\}$ by $2p$ sectors of the form

$$S_j = \left\{ z \;\middle|\; |\mathrm{Arg}\, z - \pi j/p| < \theta, \quad |z| < r \right\}, \qquad j = 1, \dots, 2p,$$

where the angle θ satisfies $\pi/2p < \theta < \pi/p$.



Sectorial Normalization Theorem

Given any $f \in \mathscr{A}_{p,\beta}$, any $\widehat{H} \in \mathbb{C}[[z]]$ tangent to identity transforming f into $f_{p,\beta}$, and a nice p–covering $\mathcal{S}$, there exits a unique *holomorphic cochain*

$$\mathcal{H} = \{H_1, \dots, H_{2p}\}, \qquad H_j \in \mathcal{O}(S_j)$$

such that

- H_j conjugates f to $f_{p,\beta}$ in S_j,
- $\widehat{H}$ is a common asymptotic series for each H_j on S_j. That is, if $\widehat{H}(z) = \sum_{k=1}^{\infty} a_k z^k$ then for any integer N,

$$\left| \sum_{k=1}^{N} a_k z^k - H_j(z) \right| \in o(|z|^N) \quad \text{as} \quad z \to 0 \quad \text{on } S_j.$$

Sketch of the proof

We introduce the change of coordinates

$$t = \varphi(z), \qquad \varphi(z) = -\frac{1}{pz^p} + \beta \ln z.$$

In these new coordinates the vector field $F_{p,\beta}$ is constant,

$$\varphi_*(F_{p,\beta}) = \frac{\partial}{\partial t}, \qquad \exp \frac{\partial}{\partial t} = t + 1.$$

An attracting (repelling) sector is transformed into a domain invariant by forward (backward) iteration of the shift

$$T\colon\, t \longmapsto t + 1.$$

In these coordinates the problem reduces to finding a conjugacy α between the map $\tilde{f} = \varphi \circ f \circ \varphi^{-1}$ and the shift map T. This is achieved by solving the *Abel equation*:

$$\alpha(\tilde{f}(t)) = 1 + \alpha(t).$$

The desired solution is given by

$$\alpha(t) = \lim_{n \to \infty} \left(\tilde{f}^{\circ n}(t) - n \right).$$

Taking $\alpha(t)$ back to z–coordinates gives us the conjugacy H_j between f and $f_{p,\beta}$. $\square$

Given a cochain $\mathcal{H} = \{H_1, \dots, H_{2p}\}$, we define

$$H_{j,j+1} := H_j \circ H_{j+1}^{-1} \quad \text{on} \quad S_j \cap S_{j+1}$$

The *coboundary* of $\mathcal{H}$ is thus defined to be

$$\delta\mathcal{H} = \{H_{j,j+1} \,|\, j = 1, \dots, 2p\}.$$

Important remark

The coboundary maps in the Sectorial Normalization Theorem satisfy

- $|H_{j,j+1}(z) - z|$ decreases exponentially fast as $z \to 0$,
- $H_{j,j+1}$ commutes with $f_{p,\beta}$.

Definition

A *map–cochain* is a cochain $\mathcal{H} = \{H_1, \ldots, H_{2p}\}$ such that:

- All the H_j have a common asymptotic series

$$\widehat{H}(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

- The coboundary maps are exponentially flat.

Note that a map–cochain with trivial coboundary is in fact a regular map.

Definition

A *normalizing cochain* for f is a map–cochain $\mathcal{H} = \{H_1, \ldots, H_{2p}\}$ such that each H_j conjugates f to $f_{p,\beta}$ on S_j.

Remark

By the sectorial normalization theorem, normalizing cochains always exist. Moreover if $\mathcal{H}$ and $\mathcal{H}'$ are two normalizing cochains for f then there exists a number $t \in \mathbb{C}$ such that

$$\mathcal{H}' = g^t \circ \mathcal{H}, \qquad g^t = \exp(tF_{p,\beta}).$$

In particular their couboundry maps are all conjugate by g^t

$$H'_{j,k} = g^t \circ H_{j,k} \circ g^{-t}.$$

Definition

Let $M^j_{p,\beta}$ be the set of all maps ϕ_j that satisfy the following properties:

- ϕ_j is an analytic map defined on $S_j \cap S_{j+1}$,
- $|\phi_j(z) - z|$ decreases exponentially fast as $z \to 0$ on $S_j \cap S_{j+1}$,
- ϕ_j commutes with $f_{p,\beta}$.

Note that the space $\widetilde{M}^j_{p,\beta} = \{\phi_j - id \mid \phi_j \in M^j_{p,\beta}\}$ is an *infinite dimensional* complex vector space. Define

$$M_{p,\beta} = \prod_{j=1}^{2p} M^j_{p,\beta},$$

That is, $M_{p,\beta}$ is the space of all cochains $\Phi = (\phi_1, \ldots, \phi_{2p})$ such that each ϕ_j satisfies the conditions stated above.

The flow $\{g^t\}_{t\in\mathbb{C}}$ acts on $M_{p,\beta}$ by conjugacy,

$$g^t \cdot \Phi = g^t \circ \Phi \circ g^{-t}.$$

Define $\mathscr{M}_{p,\beta}$ to be the quotient of $M_{p,\beta}$ by this action.

The space $\mathscr{M}_{p,\beta}$ is the Ecalle–Voronin moduli space; it is the quotient of the infinite dimensional space $M_{p,\beta}$ by the action of the one–dimensional Lie group $\{g^t\}$. This tells us that the space $\mathscr{M}_{p,\beta}$ contains as much information as an infinite dimensional vector space over $\mathbb{C}$.

Definition

The *Ecalle–Voronin modulus* of a parabolic map $f \in \mathscr{A}_{p,\beta}$ is the equivalence class $\mathcal{M}_f \in \mathscr{M}_{p,\beta}$ of the coboundary of *any* normalizing cochain for f.

Theorem (Analytic classification of parabolic maps)

- Invariance. Two analytically equivalent parabolic maps have the same Ecalle–Voronin modulus,
- Equimodality. Two parabolic maps with the same Ecalle–Voronin modulus must be analytically equivalent,
- Realization. Any equivalence class $\mathcal{M} \in \mathscr{M}_{p,\beta}$ can be realized as the Ecalle–Voronin modulus of some parabolic map.