There are no new index theorems for quadratic vector fields on $\mathbb{C}^2$

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Abstract

Consider a polynomial vector field of degree $n \geq 2$ on $\mathbb{C}^2$. In the generic case, it has n^2 isolated singularities, and the foliation it defines on $\mathbb{P}^2$ has an invariant line at infinity with $n+1$ singular points.

Each equilibrium carries two analytic invariants: the spectrum of its linearization matrix. Each singular point at infinity carries one analytic invariant: its Camacho-Sad index. Define the *extended spectra of singularities* to be the collection of these $2n^2+n+1$ numbers.

These numbers are constrained by classical *index theorems*: the Euler-Jacobi relations, the Camacho-Sad theorem and the Baum-Bott theorem. A simple dimensional argument shows that, for each fixed degree n, there must exist yet more algebraic relations among these numbers. Not one of these *hidden relations* were, until very recently, known.

In the quadratic case, there is only one such hidden relation. In this poster we will exhibit and explain this last relation. Moreover, we will show that it does not come from an index theorem. In fact, we show that any possible "index-theorem-like equation" can be deduced from the classical index theorems, hence concluding the lack of existence of new index theorems. These results can be found in [1].

Introduction

Any polynomial vector field v on $\mathbb{C}^2$ defines a singular holomorphic foliation $\mathcal{F}_v$ on $\mathbb{P}^2$. We will only consider here those vector fields whose induced foliations have isolated non-degenerate singularities and an invariant line at infinity; these assumptions are generic.

Definition. Denote by $\mathcal{V}_2$ the space of all quadratic vector fields v on $\mathbb{C}^2$ having exactly 4 isolated singularities, and such that the foliation $\mathcal{F}_v$ has an invariant line at infinity carrying exactly 3 singular points.

The extended spectra of singularities

Let $p \in \mathbb{C}^2$ be a singular point of v. We define the *spectrum* of v at p as the ordered pair

$$\operatorname{Spec}(v,p) = (\operatorname{tr} Dv(p), \det Dv(p)).$$

Denote by $\mathcal{L}$ the invariant line at infinity: $\mathcal{L} = \mathbb{P}^2 \setminus \mathbb{C}^2$, and let $p \in \mathcal{L}$ be a singular point of $\mathcal{F}_v$. The *characteristic number* of p is defined to be the Camacho-Sad index

$$\lambda(v,p) = \operatorname{CS}(\mathcal{F}_v, \mathcal{L}, p).$$

Definition. The *extended spectra of singularities* of a vector field $v \in \mathcal{V}_2$ is the collection of the spectra of v at each of its singular points on $\mathbb{C}^2$, together with the Camacho-Sad indices of the singularities of $\mathcal{F}_v$ at infinity.

Remark. Affine equivalent vector fields have the same extended spectra. In fact, two generic quadratic vector fields are affine equivalent *if and only if* they have the same spectra [2].

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Figure 1: Hello world :)

The extended spectra consists of 11 complex numbers: 8 coming from the spectra and 3 characteristic numbers at infinity. The aim of this poster is to give a complete description of *all the algebraic relations* among these 11 numbers.

The classical relations

The extended spectra are related by the following four classical index theorems:

$$\sum_{v(p)=0} \frac{1}{\det Dv(p)} = 0, \tag{EJ1}$$

$$\sum_{v(p)=0} \frac{\operatorname{tr} Dv(p)}{\det Dv(p)} = 0, \tag{EJ2}$$

$$\sum_{p \in \operatorname{Sing} \mathcal{F}_v} \operatorname{BB}(\mathcal{F}_v, p) = 16, \tag{BB}$$

$$\sum_{p \in \mathcal{L} \cap \operatorname{Sing} \mathcal{F}_v} \operatorname{CS}(\mathcal{F}_v, \mathcal{L}, p) = 1. \tag{CS}$$

Dimension count

On one hand, the space $\mathcal{V}_2$ has dimension 12, and the group $\operatorname{Aff}(2,\mathbb{C})$ is 6-dimensional. Hence, the geometric quotient $\mathcal{V}_2 /\!\!/ \operatorname{Aff}(2,\mathbb{C})$ has dimension 6. On the other hand, the extended spectra, which consists of 11 numbers modulo 4 relations, is a space of dimension 7. The gap in the dimensions implies that there must exist at least one more algebraic relation among these numbers. We call this the *hidden relation.*

The hidden relation

Let us number the singularities of v as $p_1, \dots, p_4$. Denote $\operatorname{Spec}(v,p_k) = (t_k, d_k)$, and let $\lambda_1, \lambda_2, \lambda_3$ be the characteristic numbers at infinity. Let Λ be the product $\Lambda = \lambda_1\lambda_2\lambda_3$.

Definition. Let S be the graded polynomial ring $S = \mathbb{C}[t_1, t_2, t_3, d_1, d_2, d_3]$, where t_k are of degree 1 and d_k of degree 2. The variables of S do not include t_4 and d_4, because we can use (EJ1) and (EJ2) to solve for them.

Theorem 1. *There exist homogeneous polynomials $H_0, H_1, H_2 \in S$ of degree 14 such that every generic quadratic vector field $v \in \mathcal{V}_2$ satisfies the following equation*

$$\boldsymbol{H_2(t,d)\,\Lambda^2 + H_1(t,d)\,\Lambda + H_0(t,d) = 0.}$$

Moreover, the above relation is independent from the identities (EJ1), (EJ2), (BB), (CS).

Index Theory

An index theorem consists of two parts: an index, which is a number we assign to a singularity of a vector field, foliation, space or map; and a *Lefschetz number* that measures some global property of our geometric object. An index theorem should satisfy the following:

- The index depends only on the local behavior around the singularity,
- The index is invariant under analytic equivalence.
- The sum of the indices taken over all singularities equals the Lefschetz number.

Any index theorem for generic polynomial vector fields on $\mathbb{C}^2$ may be written as

$$\sum_{k=1}^{n^2} \operatorname{ind}_{\mathbb{C}^2}(v, p_k) + \sum_{j=1}^{n+1} \operatorname{ind}_{\mathcal{L}}(\mathcal{F}_v, w_j) = L(n).$$

Moreover, the indices may only depend on the extended spectra. Note that all of the classical relations on the extended spectra come from index theorems. The following result implies that there are no more index theorems for (generic) quadratic vector fields on $\mathbb{C}^2$.

Theorem 2. *There exists no pair (R, r) consisting of a rational function R on $\mathbb{C}^8$ and a symmetric rational function r on $\mathbb{C}^3$ with the property that every quadratic vector field with non-degenerate singularities satisfies the relation*

$$R(t_1, d_1, \dots, t_4, d_4) = r(\lambda_1, \lambda_2, \lambda_3),$$

except for those that can be derived from the previously known relations: Euler-Jacobi, Baum-Bott and Camacho-Sad.

The proof of Theorem 1 and Theorem 2 can be found in [1].

References

[1] Y. Kudryashov and V. Ramírez. There are no new index theorems for generic quadratic vector fields on $\mathbb{C}^2$. arXiv:1705.06340, 2017.

[2] V. Ramírez. Twin vector fields and independence of spectra for quadratic vector fields. *J. Dynam. Control Syst.*, 23(3):623–633, 2017. doi: 10.1007/s10883-016-9344-5.

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