Hilbert's 16th Problem

Valente Ramírez

December 3 2012

The International Congress of Mathematicians

Paris 1900

Background

In 1876 C. G. A. Harnack proved the following theorem:

Harnack's curve theorem

The number of connected components an algebraic curve of degree m has in the real projective plane is bounded above by

$$\frac{(m-1)(m-2)}{2}+1.$$

Problem 16

Problem of the topology of algebraic curves and surfaces

First part

"Investigate the relative position of these ovals and investigate the number and position of the sheets of an algebraic surface in three dimensional space".

Second part

"In connection with this purely algebraic problem, I wish to bring forward the question [...] *of the maximum number and positions of Poincaré's boundary cycles (cycles limites)..."*

In modern language...

Consider a system of polynomial differential equations on the plane

$$\frac{dx}{dt} = Q(x,y), \quad \frac{dy}{dt} = P(x,y),$$

where $(x,y) \in \mathbb{R}^2$ and $P, Q \in \mathbb{R}[x,y]$ are polynomials of degree n.

- *What is the maximum number and relative posion of limit cycles that the above system may have?*

A limit cycles is a non–trivial periodic solution isolated from other periodic solutions.

The 16th Problem

Problem 1

Is it true that a planar polynomial system has but a finite number of limit cycles?

Problem 2

Is it true that the number of limit cycles of a planar polynomial system is bounded by a constant depending only on the degree of the polynomials?

Such a bound is denoted by $H(n)$ and called the Hilbert number.

Problem 3

Give an upper bound for the number $H(n)$.

The birth of differential equations

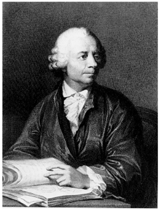
"Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa"

It is usefull to solve differential equations!

The laws of nature are expressed by differential equations.

Besides Newton and Leibniz the following names stand out:

- Euler
- Lagrange
- Laplace
- Gauss




Liouville's theorem

In 1832 Galois proved the impossibility of solving a general algebraic equation by radicals.

Liouville developed (1830's–1840's) an analogous theory for differential equations establishing the impossibility of solving most non–linear differential equations in quadratures.

We cannot find explicit expressions for the solutions!

Before Poincaré

Until 1880 differential equations were a purely analytic object.
Thus a plannar system

$$\frac{dx}{dt} = F_1(x, y), \qquad \frac{dy}{dt} = F_2(x, y),$$

was studied by means of finding its solutions

$$\varphi\colon (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^2, \quad \frac{d\varphi}{dt} = (F_1(\varphi(t)), F_2(\varphi(t))).$$

Henri Poincaré

Mémoire sur les courbes définies par une équation différentielle, 1881

COURBES DÉFINIES PAR UNE ÉQUATION DIFFÉRENTIELLE. 375

Mémoire sur les courbes définies par une équation différentielle;

PAR M. H. POINCARÉ,
Ingénieur des Mines.

Une théorie complète des fonctions définies par les équations différentielles serait d'une grande utilité dans un grand nombre de questions de Mathématiques pures ou de Mécanique. Malheureusement, il est évident que dans la grande généralité des cas qui se présentent on ne peut intégrer ces équations à l'aide des fonctions déjà connues, par exemple à l'aide des fonctions définies par les quadratures. Si l'on voulait donc se restreindre aux cas que l'on peut étudier avec des intégrales définies ou indéfinies, le champ de nos recherches serait singulièrement diminué, et l'immense majorité des questions qui se présentent dans les applications demeureraient insolubles.

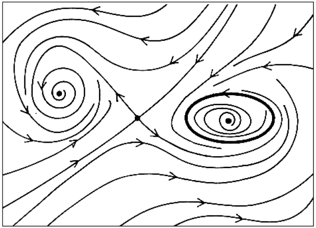
Il est donc nécessaire d'étudier les fonctions définies par des équations différentielles en elles-mêmes et sans chercher à les ramener à des fonctions plus simples, ainsi qu'on a fait pour les fonctions algébriques, qu'on avait cherché à ramener à des radicaux et qu'on étudie maintenant directement, ainsi qu'on a fait pour les intégrales de différentielles algébriques, qu'on s'est efforcé longtemps d'exprimer en termes finis.

Rechercher quelles sont les propriétés des équations différentielles est donc une question du plus haut intérêt. On a déjà fait un premier pas dans cette voie en étudiant la fonction proposée *dans le voisinage d'un des points du plan*. Il s'agit aujourd'hui d'aller plus loin et d'étu-

Poincaré's program

Poincaré proposes the following program to study planar systems.

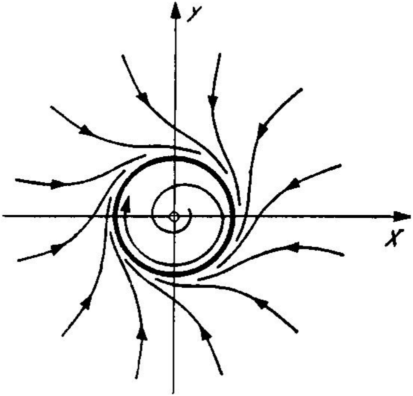
> "A complete study of a system consists on two parts:
>
> 1.- *Qualitative study* (so to speak), or *geometric study* of the curves defined by the solutions,
>
> 2.- Quantitative study, or numerical compuation of the values of the solutions."

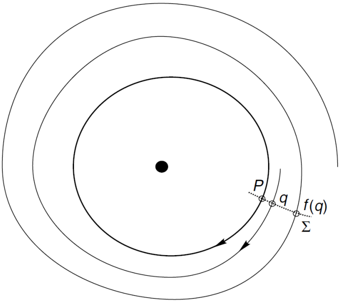


- Differential equations
- Vector fields
- Differential 1–forms

Poincaré's program

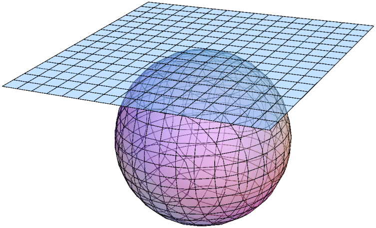
Poincaré defined the concept of a limit cycle and studied them via the so–called Poincaré map.

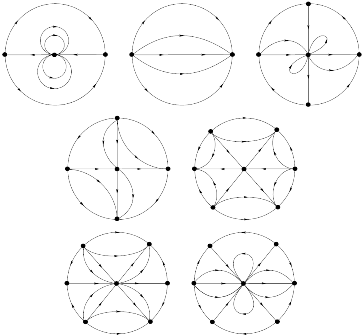




Poincaré's program

Poincaré studied *polynomial systems* by compactifying the phase space $\mathbb{R}^2$.





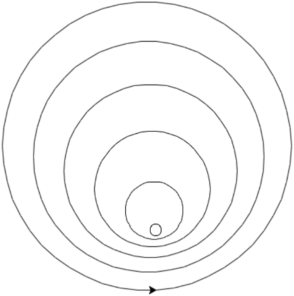
Limit cycles
Why do we care?

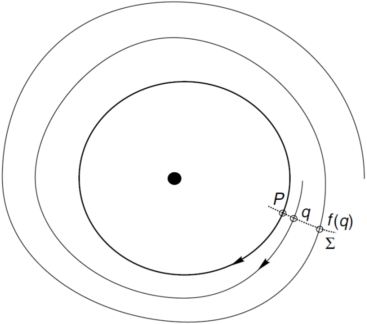
- They are an escential part of the system!
- ω–limit sets
- Stability of the system

The first finiteness theorem

Theorem (Poincaré 1883)

Any polynomial system with no saddle–connections and no degenerate singular points has finitely many limit cycles.





Dulac and the general finiteness theorem

In 1921 Henri Dulac published the following result:

Theorem

Any individual polinomial system has but finitely many limit cycles.

Proof of Dulac's theorem

If a planar system has infinitely many limit cycles then there is a sequence of them accumulating to a polycycle.

Definition

A germ $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is called semiregular if it is smooth outside zero and admits the following asymptotic expanssion:

$$\widehat{f}(x) = cx^{\nu_0} + \sum_j P_j(\ln x)\, x^{\nu_j},$$

where $c, \nu_j > 0$, $\nu_j \nearrow \infty$ and $P_j \in \mathbb{R}[x]$.

Proof of Dulac's theorem

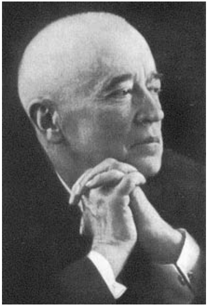
The steps in Dulac's finiteness theorem are the following:

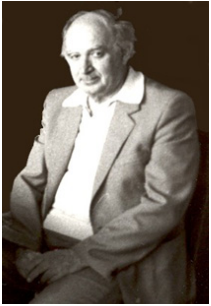
- The Poincaré map $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ of such a polycycle is either *semiregular*, *flat* or *inverse to flat*.
- (i) Flat and inverse to flat germs cannot have infinitely many fixed points,
  (ii) If f is a semiregular germ with infinitely many fixed points then $f(x) \equiv x$.

This provides an affirmative answer to Problem 1.

The Petrovskii–Landis theorems

In the middle of the 50s Petrovskii and Landis published the following results:





Theorem (1955)

Any quadratic system has at most 3 limit cycles.

Theorem (1957)

There exists a polynomial P_3 of degree 3 such that $H(n) \leq P_3(n)$.

The Petrovskii–Landis strategy

The strategy: Complexify the problem

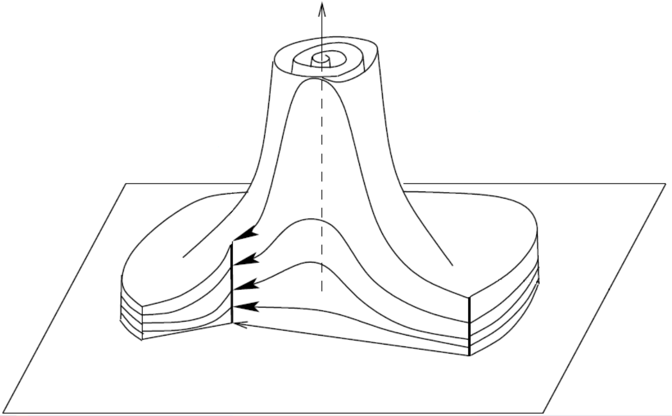
Consider:

$$\frac{dx}{dt} = Q(x,y), \qquad \frac{dy}{dt} = P(x,y),$$

but now $(x,y) \in \mathbb{C}^2$, $P, Q \in \mathbb{R}[x,y] \subset \mathbb{C}[x,y]$ and $t \in \mathbb{C}$.

The Petrovskii–Landis strategy

Now our solutions are *complex curves* on the *complex plane*, that is, Riemann surfaces on the four–dimensional space $\mathbb{C}^2$,



The Petrovskii–Landis strategy

The following concepts extend naturally to the complex domain:

- Complex cycle,
- Complex limit cycle,
- Poincaré map.

The Petrovskii–Landis strategy

The proof of the Petrovskii–Landis result goes as follows:

- Prove that the number of real limit cycles is bounded by the number of complex limit cycles,
- Prove that complex systems admit a uniform bound on the number of limit cycles depending on the degree of the system only.

This provides a final answer to the 16th problem.

The above claims are wrong!!

- A crutial mistake in the proof was found by Ilyashenko and Novikov in 1963.
- Later quadratic systems with 4 limit cycles were constructed.
- In fact, Ilyashenko proved that *generic* complex systems have infinitely many limit cycles!!

Parametric versions of the problem

Let us consider families of vector fields depending on finitely many parameters such that the parameter space is compact.

Problem 4

Is it true that for any analytic finite parameter family of vector fields on the 2–sphere the number of limit cycles of the equations in the family is uniformly bounded?

Parametric versions of the problem

Hilbert–Arnold problem

The previous problem has a $\mathcal{C}^\infty$ counterpart:

Problem 5

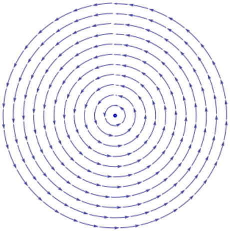
Is it true that for a *generic* finite parameter family of smooth vector fields on the 2–sphere the number of limit cycles of the equations in the family is uniformly bounded?

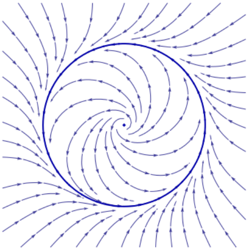
Limit cycles via bifurcations

Consider the function which assigns to each polynomial system its number of limit cycles.

This function is discontinuous at those systems whose perturbations generate limit cycles via bifurcations.

Of particular interest is the study of perturbation of Hamiltonian systems.





Limit cycles via bifurcations

Consider a system:

$$dH + \epsilon\omega = 0,$$

- $H \in \mathbb{R}[x, y]$ has degree $n + 1$,
- $\omega = P\,dx + Q\,dy$ with P, Q, polynomials of degree at most n,
- $\epsilon \in (\mathbb{R}, 0)$.

Pontryagin criterion

If an oval $\delta(t)$ generates a limit cycle of the above system then

$$I(t) := \int_{\delta(t)} \omega \;=\; 0.$$

The integral above is called an **Abelian integral**.

Limit cycles via bifurcations

Problem 6 (Infinitesimal Hilbert's 16th problem)

Find an upper bound $V(n)$ of the number of zeros of Abelian integrals as above. The bound should depend on the degree n only.

Problem 7

Is it true that from a polycycle ocurring in a finite parameter family of analytic vector fields there may bifurcate only finitely many limit cycles?

Problem 8

Is it true that from a polycycle ocurring in a *generic* finite parameter family of smooth vector fields there may bifurcate only finitely many limit cycles?

Ilyashenko 1981

A huge gap was found in Dulac's proof!!

The map $f(x) = x + (\sin \frac{1}{x})e^{-\frac{1}{x}}$ is semiregular but has infinitely many fixed points.

It is **not true** that $\hat{f} = id \implies f = id$.

After a hundred years from Poincaré we're back to where we started!

Some important developements

Even though all the stated problems remained open, several important developements took place in the 70's and 80's.

- Study of plannar analytic foliations.
- Theory of normal forms.
- Resurgent functions.
- Theory of fewnomials.
- Non–linear Stokes phenomena.

Definite finiteness theorem

Theorem (Ilyashenko '91, Ecalle '92)

Any polynomial vector field on the plane has finitely many limit cycles.

Proof of the finiteness theorem

Easy case: Non–acumulation theorem for hyperbolic polycycles

Definition

An exponential Dulac series is a formal series

$$\alpha\zeta + \beta + \sum p_j(\zeta) e^{-\nu_j \zeta},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$ and $0 < \nu_j \nearrow \infty$.

Definition

A germ $f\colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$ is called almost regular if in the logarithmic chart $\zeta = -\ln x$ it has a representative that can be extended as a biholomorphic map on a *standard domain* and is expandable by a Dulac exponential series there.

Proof of the finiteness theorem

Easy case: Non–acumulation theorem for hyperbolic polycycles

Note that in the logarithmic chart we obtain a germ

$$\zeta \mapsto -\log \circ f \circ \exp(-\zeta), \quad \zeta \in (\mathbb{C}, \infty).$$

Lemma 1

The Poincaré map of a hyperbolic polycycle is almost regular.

Lemma 1 is proved using the theory of normal forms.

Lemma 2

An almost regular germ is uniquely defined by its Dulac series.

Lemma 2 is proved using a refinement of the Phragmén–Lindelöf theorem.

So, what do we know?

Problem 1 (Ilyashenk, Ecalle 1991-1992)

A polynomial vector field may have finitely many limit cycles only.

Problem 2

Is it true that the number of limit cycles of a planar system of differential equations is bounded by a constant depending only on the degree of the polinomials?

Problem 3

Give an upper bound for the number $H(n)$.

So, what do we know?

Problem 4

Is it true that for any analytic finite parameter family of vector fields on the 2–sphere the number of limit cycles of the equations in the family is uniformly bounded?

Problem 5

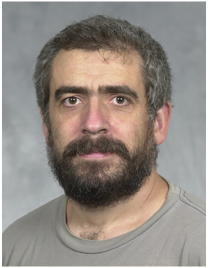
Is it true that for a *generic* finite parameter family of smooth vector fields on the 2–sphere the number of limit cycles of the equations in the family is uniformly bounded?

So, what do we know?

Problem 6 (Binyamini, Novikov, Yakovenko 2010)

The upper bound $V(n)$ of the number of real zeros of Abelian integrals is of the form
$$2^{2^{\mathrm{Poly}(n)}}.$$





So, what do we know?

Problem 7

Is it true that from a polycycle ocurring in a finite parameter family of analytic vector fields there may bifurcate only finitely many limit cycles?

Problem 8

Is it true that from a polycycle ocurring in a *generic* finite parameter family of smooth vector fields there may bifurcate only finitely many limit cycles?

Are we close to solving the problem?

Probably in a hundred years more...