Spectral Conformal Parameterization

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1 Introduction

Mesh parameterization plays one of the most important roles in geometry processing field. It has been researched for a long time and different methods and approaches were developed. Original task of mesh parameterization is to find the piecewise bijective mapping of a $3D$ model onto the 2-dimensional space. During this procedure every vertex of a given mesh gets a $(u,v)\epsilon\mathbb{R}^2$ to represent its location. This transformation opens large horizons for further manipulations. For example in $\mathbb{R}^2$ it is much easier to implement remeshing, surface fitting or texture mapping. However due to mesh complexity it is not always possible to generate perfect $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ correspondence without distortion, that's why new enhanced methods of parameterization are actively researched.

1.1 Previous research

The deformation of a 3D object will probably give us some sort of distortion in the resulting surface. This distortion can be minimized by using big variety of methods which will give (u,v) parameter values. In general, all these methods can be divided into 2 different groups:

1.1.1 ***Linear methods.***

These methods are significantly faster and easy to implement where only one linear system to be solved. However, the result is more distorted relative to non-linear methods, which will be described later. Linear methods maps the boundary vertices to the boundary of a convex area in 2D. Then the positions of the interior vertices are calculated by solving the following linear system:

$$Lu = Lv = 0$$

$$L_{i,j} = \begin{cases} -\sum_{k\neq i} L_{i,k} & i = j \\ \omega_{i,j} & i,j\epsilon E \\ 0 & otherwise \end{cases}$$

where ω is the edge weight, E is set of edges. The weights w_{ij} are defined for each edge of the mesh. For example the weights can be set to $w_{ij} = 1$ iff (i,j) is an edge of the mesh, called uniform weights

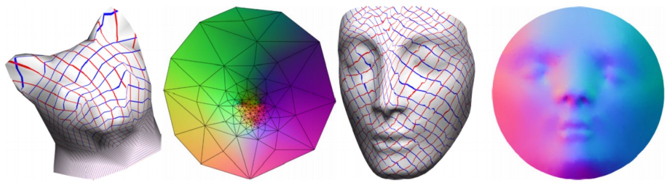

Figure 1. *Parameterization with uniform weights on a circular domain.*

The system above must be solved for the u and v coordinates independently. The disadvantage of this method is the absence of linear prescision.
Another linear method is "Harmonic parameterization". It aims to minimize the angular distortion. Here weights w_{ij} assigned as

$$w_{ij} = \frac{1}{2}(cot\alpha_{ij} + cot\beta_{ij}),$$

where α_{ij}, β_{ij} are opposite angles of two adjacent triangles with a common edge $(i,j)\epsilon E$. The weakness of this method the violation of positivity property, which requires $w_{ij} \geq 0$.

The "Mean-value weights" are calculated as $w_{ij} = \frac{tan\frac{\theta^1_{ij}}{2} + tan\frac{\theta^1_{ij}}{2}}{||c(i)+c(j)||}$, where $c(i)$ $c(j)$ - positions of vertices v_i,v_j in 3D.

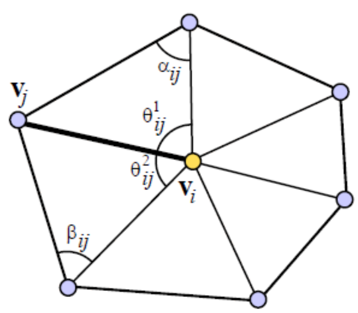

Figure 2. *Angles used for harmonic and mean-value weights*

But "Mean value weights" are not symmetric ($w_{ij} \neq w_{ji}$).
Thus it can be concluded that the linear methods have their own advantages and limitations.

1.1.2 *Nonlinear methods*

Most Isometric Parametrizations(MIPS)
MIPS method optimizes a nonlinear functional that measures mesh angle preservation of a mesh(mapped to $\mathbb{R}^2$). It begins with a harmonic fixed-boundary parameterization as an initial guess. Then the reduction of the distortion metric is

done by moving the vertices one by one. Vertices can only be moved inside the kernel of neighboring vertices. The method also prevents boundary overlaps. So the resulting parameterization remains globally bijective during the algorithm.

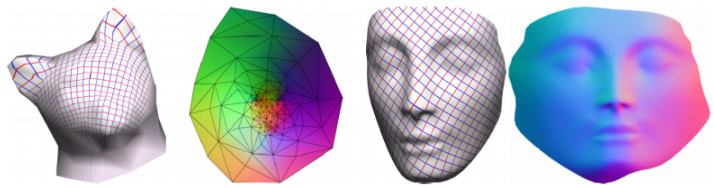

Figure 3. *Parameterization with MIPS*

1.2 Contributions

Spectral Conformal Parameterization(2008) paper presents conformal parameterization. The idea is to eliminate the boundary constraints like it is done in LSCM/DCP algorithm. This feature allows to succeed in getting low distorted parameterization by solving an eigenvalue problem of a linear operator.

1.3 Notations

The following notation is used to formulate the problem:
V denotes set of vertices, E is number of edges, $V_b$is number of vertices which belong to the boundary of the mesh.
$X_i = (x_i, y_i, y_i)$ - position of i-th vertex of the mesh χin 3D. $U_i = (u_i, v_i)$ - parameter value of the corresponding node in the 2D mesh U.
The vector $\mathbf{u}$ will denote the column vector $(u_1, v_1, u_2, v_2, ..., u_v, v_v)^T$, e_{ij}defines the edge in mesh U between points u_i and u_j .

2 Background on Descrete Conformal Maps

2.1 Definition for Continuous setting

First we describe several geometric notions relevant to conformal parameterization.

Dirichlet Energy

For given a surface patch χ and map $\mathbf{u}$ Dirichlet energy E_D can be written in the form

$$E_D = \frac{1}{2} \int_\chi |\nabla u|^2 dA,$$

Where ∇u is the gradient vector field of the $\mathbf{u}$. The Dirichlet Energy and its value determines the amount of distortion performed by map $\mathbf{u}$. If the area of $\mathbf{u}$ is given by$A(u) = \int_\chi det(\mathbf{u}) dA$, then for any map $\mathbf{u}$, the Dirichlet energy has a lower limit related to the map area $E_D \geq A(u)$.

Harmonic and Conformal maps

A map is **u** called **harmonic** if it minimizes the Dirichlet energy given fixed boundary conditions, since its Laplacian $\Delta u = 0$.
If the minimal value of the Dirichlet energy is achievable, the resulting map is **conformal** (angle-preserving).
Therefore it is convenient to define the conformal energy E_C as:

$$E_C(u) = E_D(u) - A(u)$$

Thus a map **u** is conformal if and only if $E_C = 0$. In the continuous setting, the Riemann mapping theorem states that for any given shape of the boundary of the image U, a conformal parameterization always exists. Nevertheless, if we fix the map **u** along the boundary, then conformality is usually impossible. Minimizing the conformal energy will reduce the angle distortion of the map given the boundary, but will not not get rid of the distortion completely.

2.2 Discretization

Since geometry processing is working with meshes, which are discrete objects, the above definitions should be transferred to a discrete setting.
Actually, only the map **u** needs to be discretized. It can be considered as piecewise-linear.

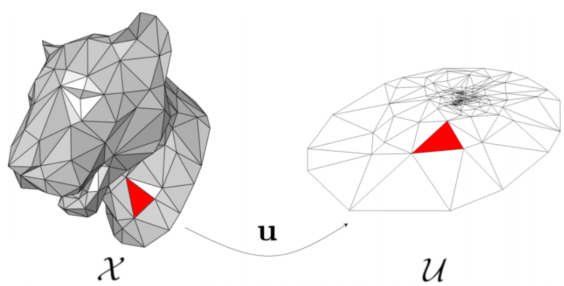

Figure 4. Piecewise-linear mapping

In such a case Dirichlet energy of a map **u** between two discrete surfaces can be defined as the sum of all energies on triangles:
$E_D(u) = \sum_{e_{ij}} \frac{1}{4}(cot\theta_{ij} + cot\theta_{ji})(\mathbf{u}_i - \mathbf{u}_j)^2$, where the angles θ_i and θ_i are the angles opposite to common edge in the two adjacent triangles.
Matrix form representation of Dirichlet energy:

$$E_D(\mathbf{u}) = \frac{1}{2}\mathbf{u}^t L_D \mathbf{u} \ ,$$

where L_D- $2V \times 2V$ sparse, symmetric matrix containing only the cotangent coefficients computed on χ.
The total area of parameterization can be computed using only the coordinates of the boundary vertices

$$A(u) = \sum_{e_{ij} \epsilon \partial U} \frac{1}{2}(u_i v_j - u_j v_i),$$

where e_{ij} is along the boundary ∂U: Therefore, we can define a matrix A such that $A(\mathbf{u}) = \frac{1}{2}\mathbf{u}^t A \mathbf{u}$
Conformal energy E_C which is the difference between $E_D$and the area of the parameterization can be written in quadratic form:

$$E_C(\mathbf{u}) = \frac{1}{2}\mathbf{u}^t L_C \mathbf{u}.$$

Minimization of Conformal Energy

There are two well-known approaches of conformal enegry minimization: "Discrete Conformal Mapping" and "Least Squares Conformal Mapping"
If the piecewise-linear mapping minimizes the discrete quadratic energy $E_{\text{C}}(u)$ then it is called "Discrete Conformal Mapping". In this method only one linear system has to be solved.
On the other hand the condition of LSCM is that gradients of u and v coordinates in the parameterization should be as orthogonal as possible per triangle: $E_{LSCM}(u) = \int_{\chi} \frac{1}{2}|\nabla u^{\llcorner} - \nabla v|^2$, where $u^{\llcorner}$ is $\frac{\pi}{2}$ degrees counterclockwise rotation in χ.
Both DCP and LSCM provide the result which looks identical to the human eye. In practice the difference is only in cotangent weights approximations.

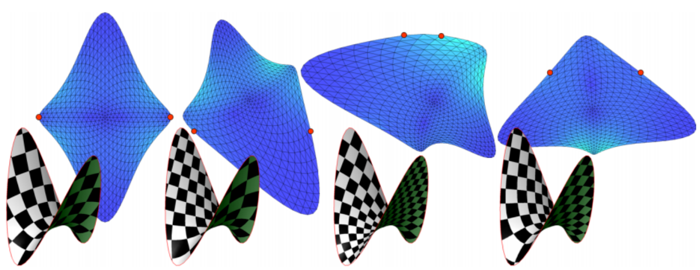

Figure 5. DCP/LSCM with different boundary vertices choice

3 Spectral Parameterizations

3.1 Fiedler vector and free boundary

As was seen earlier, the previous methods require a fixed of at least two boundary vertices. Nevertheless, the choice of these vertices significantly affect the results, as shown in the picture above. In general, the best result is obtained by taking the most distant vertices. However, this choice does not guarantee the best results in the processing of complicated models. Consequently the relatively high distortion may appear near those fixed points.
Now we're going to get rid of these constraints by presenting so-called "Fiedler verctor" $u*$ of the system L_C matrix.

$$\mathbf{u}^* = \underset{\mathbf{u}}{argmin}\ u^t L u$$

$$\mathbf{u}^t e = 0 \text{ (barycenter of solution must be 0)}$$

$$\mathbf{u}^t \mathbf{u} = 1 \text{ (sum of squared distences to barycenter must be unit),}$$

So this vector satisfies to $L\mathbf{u}^* = \lambda \mathbf{u}^*$where λ represents the smallest non-zero eigenvalue.
The Fiedler vector minimizes the Rayleigh quotient: $u^* = \underset{\mathbf{u}}{argmin} |\frac{\mathbf{u}^t L_C \mathbf{u}}{\mathbf{u}^t \mathbf{u}}|$, however minimization of this value does nod guarantee good parameterization and requires some modifications, as can be seen in following figure(top right)

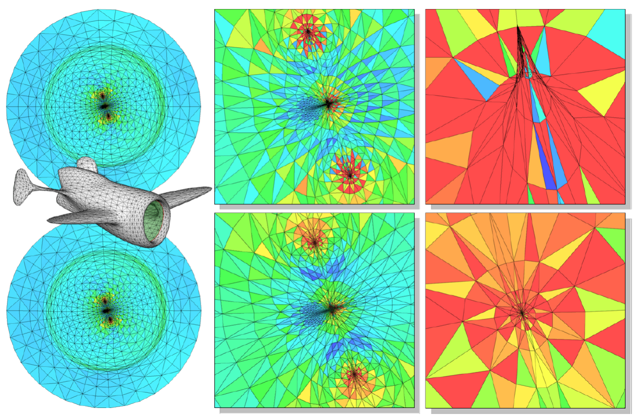

Figure 6. Fiedler vector parameterization(top) vs Spectral Conformal Parameterization(bottom)

3.2 Spectral Conformal Map

To improve the situation define spectral conformal parameterization u^* as a generalized eigenvector satisfying which satisfies to:
$L\mathbf{u}^* = \lambda B \mathbf{u}^*$, where B is a $2V \times 2V$ diagonal matrix with 1 at each diagonal element corresponding to boundary vertices and 0 everywhere else.
This matrix is used for setting the constraints only on the boundary of a mesh.
This addition tends to modified minimization of a Fiedler vector:
This provides the parameterization where the constraints are smoothly spread in the resulting mesh:

$$\mathbf{u}^* = \underset{\mathbf{u}}{argmin}\ \mathbf{u}^t L_C \mathbf{u}$$

$$\mathbf{u}^t B e = 0$$

$$\mathbf{u}^t B \mathbf{u} = 1$$

In this way the Rayleigh quotient will be $\frac{\mathbf{u}^t L_C \mathbf{u}}{\mathbf{u}^t B \mathbf{u}}$. So we can see that now the denominator depends only on the boundary vertices. Therefore, the optimal eigenvector balances conformality at the boundary, eliminating the distortion for inner vertices which appear when standard Fiedler vector is used.
From the geometrical point of view new spectral parameterization maximizes the squared distance from boundary vertices to their barycenter over the unit ball defined by the conformal energy.

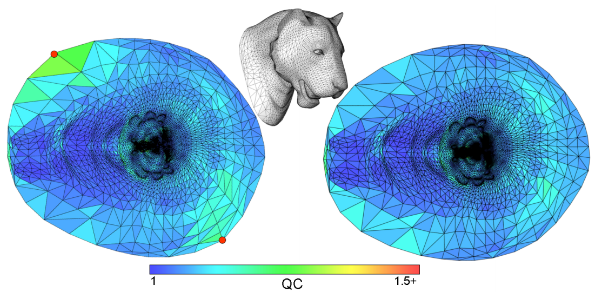

Figure 7. LSCM/DCP with two constrained vertices vs. Spectral conformal parameterization, with no visual bias

3.3 Practical realization of the algorithm

We need to solve the sparse symmetric generalized eigenvalue problem by finding the smallest eigenvalues. Generally it can be efficiently done by using some well-known methods, like the Cholesky decomposition where the positive-defined matrix A is represented by the product of a lower triangular matrix and its conjugate transpose:
$A = LL^t$. Cholesky decomposition can be very useful in processing of the large meshes. Another way of solving the eigen problem is to use the Lanczos algorithm which is an adaptation of 'power methods' to find the most useful eigenvalues and eigenvectors of an n^{th} order linear system with a limited number of operations, m, where m is much smaller than n.
Particularly these methods are quite fast since we operate with sparse matrices. We can thus obtain a spectral free-boundary parameterization very efficiently. However, there is an improved solution method that allows us to reduce the computation time further.
Instead of minimizing the $L\mathbf{u}^* = \lambda B\mathbf{u}^*$we will search the largest eigenvalue $\mu = 1/\lambda$ of the eigenvalue problem where we switched the left and right side of the expression. To avoid problems in processing of thr large meshes with extremely degenerate triangles we can add $\varepsilon \cdot Id$ to L_C to guarantee that numerical inaccuracies in the coefficients of the Laplacian matrix will not alter the positive semi-definiteness of the matrix. Lastly, we remove the known kernel of $L_C$in computation:

$$[B - \tfrac{1}{V_b} e_b e_b^t]\mathbf{u} = \mu L_C \mathbf{u},$$

where e_b is the matrix derived from e by setting 'zero' to the values that correspond to internal vertices.
In fact, these improvements provide a performance boost by 5-15%.

4 Further improvements

Building on this Spectral Conformal Parameterizations algorithm some extenstions could be done making it more applicable to different purposes.

4.1 Sampling control

Since the original mesh can have samples with relatively different sizes. This irregularity can be preserved by apply some rather simple and efficient modification of conformal energy. Namely, we weight the area functional and Dirichlet energy from each triangle T by the inverse of its original area $|T|$ in χ.

$$A_{u_i,v_j} = \frac{1}{2}[\frac{1}{|T_{ijk}|} - \frac{1}{|T_{ijl}|}], A_{v_i,u_j} = \frac{1}{2}[-\frac{1}{|T_{ijk}|} + \frac{1}{|T_{ijl}|}].$$

Each cotangent coefficients in L_D are divided by the area of the triangle they were computed on. The resulting inverse-area-weighted L_C has the same sparsity as before. This modification does not require additional computational cost for our eigensolver.

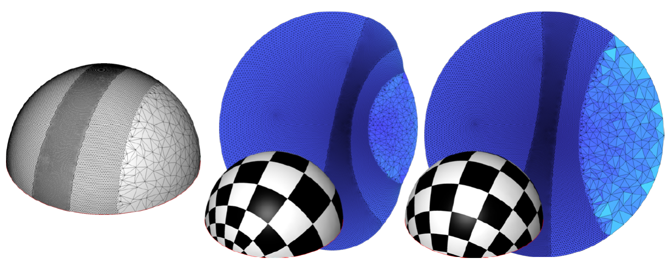

Figure 8. Area weighting

Weighting aslo can be applied to LSCM/DCP but since we have 2 point constraint, the result wouldn't be as effective as the spectral approach performs.

4.2 User control

Spectral method can be modified in a way that we can give more control to the user. We can let the user "interact" with the parameterization by adding "extenders" to move two vertices u_{i} and u_j away from each other. If we add a quadratic term $\alpha(\mathbf{u}_i - \mathbf{u}_j)^2$to the quadratic form represented by matrix B then a force will be added between these two vertices, where $|\alpha|$ responds for the amount of conformality that can be lost to satisfy this user-specified force. This idea can be used to prevent holes in a mesh from collapsing in the parameterization

5 Results

5.1 Conclusion

We presented an efficient spectral approach to discrete conformal parameterization which is robust to sampling irregularity and does not require positional constraints on vertices. To compute the parameterization we need to solve a symmetric generalized eigenvalue problem. It is not as simple as a single linear solve but still efficient and robust according to the tests. Solving the eigenvalue problem is only two to three times slower than the linear solve required by LSCM/DCP but it gives us flexibility of constraints.

5.2 Limitations

Unfortunately SCP method does not handle folds and triangle flips in general case. This happens if we have very degenerate triangles in the initial mesh. Also non-injectivity can be observed in cases where the boundary has large and sharp concavities.

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