Curve-constrained Gaussian Random Fields

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Introduction

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- have natural geometric properties
- allow for easy approximation

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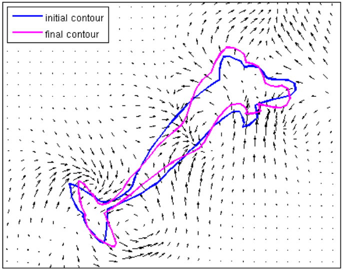
1. select a vector field defined over $\mathbb{R}^2$
2. deform an initial curve along the flow of this vector field



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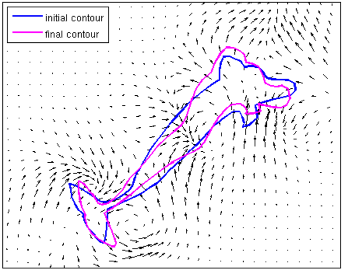

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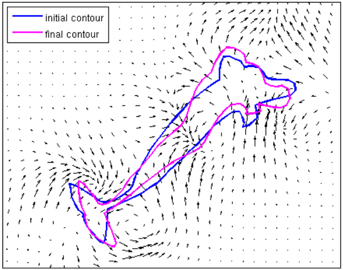

- have natural geometric properties
  - select random vector fields which drive neighboring points to move together
- allow for easy approximation
  - use the structure of reproducing kernel Hilbert spaces (RKHS)

Motivating example: GRF on $\mathbb{R}^2$

Let $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ be a centered Gaussian random vector field with covariance

$$C(x, y) = \frac{1}{2\pi\sigma_1^2} e^{-\frac{\|x-y\|_2^2}{2\sigma_1^2}} \mathbb{I}_2, \tag{1}$$

and let $V(C)$ be the associated RKHS with reproducing kernel C.

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A: No.

Conditions[1]

Conditions for the realizations to belong to an RKHS (I)

$$P(\xi \in V(K)) = 1 \quad \text{if} \quad \sup_{\chi_n} \mathrm{tr}(C(\chi_n)K(\chi_n)^{-1}) < \infty \tag{3}$$

$$P(\xi \in V(K)) = 0 \quad \text{if} \quad \sup_{\chi_n} \mathrm{tr}(C(\chi_n)K(\chi_n)^{-1}) = \infty \tag{4}$$

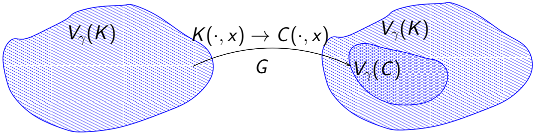
Note: when $K = C$, then $tr(CK^{-1}) = n$, hence the realizations never belong to the RKHS of the covariance.

[1] M. Driscoll, The Reproducing Kernel Hilbert Space Structure of the Sample Paths of Gaussian Processes, 1973

Alternative conditions

If $V_\gamma(C) \subset V_\gamma(K)$, then there exists a self-adjoint bounded linear operator G, s.t.

$$GK(\cdot, x) = C(\cdot, x), \quad \forall x \in \gamma. \tag{5}$$



Conditions for the realizations to belong to an RKHS (II)

1. $V_\gamma(C) \subset V_\gamma(K)$
2.

$$P(\xi \in V(K)) = 1 \quad \text{if} \quad \mathrm{tr}(G) < \infty \tag{6}$$
$$P(\xi \in V(K)) = 0 \quad \text{if} \quad \mathrm{tr}(G) = \infty \tag{7}$$

Gaussian random fields over $\mathbb{R}^2$ - revisited

1. if $\sigma_1 \geq \sigma_0$, then $V_{\mathbb{R}^2}(C) \subset V_{\mathbb{R}^2}(K)$
2. the action of the G operator is

$$G[f] = \text{const} \int_{\mathbb{R}^2} e^{-\frac{\|x-y\|_2^2}{2\sigma_1^2 - 2\sigma_0^2}} f(x)dx \tag{8}$$

$tr(G) = \infty$, therefore the realizations of ξ are not in $V_{\mathbb{R}^2}(K)$

Curve-constrained Gaussian random vector fields

Idea:

- define a Gaussian random field over a curve γ
- still need a vector field defined over $\mathbb{R}^2$
- extend to $\mathbb{R}^2$ by interpolation

RKHS - review

Restriction of an RKHS to a set

$V_\gamma(K)$ consists of functions in $V_{\mathbb{R}^2}(K)$ resricted to γ with a norm

$$\|f\|_{V_\gamma(K)} = \inf_{g \in V(K),\ \text{s.t.}\ g_{|\gamma}=f} \|g\|_{V(K)}. \tag{9}$$

Note: there is an isometry between functions in $V_\gamma(K)$ and the subspace of functions in $V_{\mathbb{R}^2}(K)$ which minimize the above norm.

We can extend functions in $V_\gamma(K)$ to be defined over $\mathbb{R}^2$.

From finite dimension to infinite dimension

Finite dimensional random fields:

$$\xi_n = \sum_{k=1}^{n} K(\cdot, x_k)\alpha_k, \quad x_k \in \gamma, \qquad \alpha_k \sim \mathcal{N}(0, \Sigma) \tag{10}$$

Covariance:

$$\begin{aligned} C_n(x,y) &= \mathbb{E}[\xi_n(x)\xi_n(y)^T] = \mathbb{E}[K(x,\chi_n)\alpha(K(y,\chi_n)\alpha)^T] = \\ &= K(x,\chi_n)\mathbb{E}[\alpha\alpha^T]K(\chi_n,y) = K(x,\chi_n)\Sigma K(\chi_n,y) \end{aligned} \tag{11}$$

How do we extend to infinite dimensions?

Consistency

Need to select Σ so that ξ_n is consistent with projections on lower dimensional subspaces.

A random field $\xi_m \in V_{\chi_m}(K)$ is *consistent* with the random field $\xi_n \in V\chi_n(K)$, where $\chi_n \subset \chi_m$, if its orthogonal projection $\bar{\xi}_m$ onto $V(\chi_n)$ satisfies

$$Cov(\bar{\xi}_m) = Cov(\xi_n). \tag{12}$$

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Covariance of the coefficients:

$$\Sigma(\chi_n) = K(\chi_n)^{-1} C(\chi_n) K(\chi_n)^{-1}. \tag{13}$$

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Covariance of the infinite dimensional random field (when $\chi_n \to \gamma$):

$$C_\gamma(x,y) = \langle \pi_{V_\gamma(K)}(K(\cdot,x)), G\pi_{V_\gamma(K)}(K(\cdot,y))\rangle_{V_\gamma(K)}. \tag{15}$$

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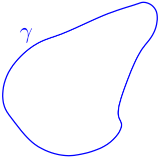
Note: when $x, y \in \gamma$, $C_\gamma(x, y) = C(x, y)$.

Question

Let ξ_γ be a centered GRF with covariance $C_\gamma(x, y)$.



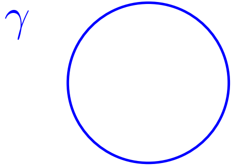
Q: Do the realizations of ξ_γ belong to $V_\gamma(K)$?

Example: circle

Let γ be a circle.

Steps to calculate the trace:

1. pick an orthonormal basis for $V_\gamma(K)$
   - eigenfunctions of $K[f] = \int_\gamma K(x,y)f(x)dx$
2. calculate explicitly: spherical harmonics!
3. sum the ratio of eigenvalues $\lambda_k(K)$, $\lambda_k(C)$



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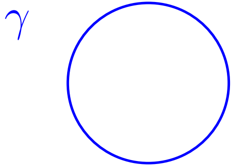

Trace formula

$$tr(G) = \sum_{k=0}^{\infty} \frac{\lambda_k(C)}{\lambda_k(K)} = (e^{2/\sigma_1^2 - 2/\sigma_0^2}) \sum_{k=0}^{\infty} \underbrace{\left(\frac{\sigma_0}{\sigma_1}\right)}_{<1}^{2k} \underbrace{\frac{\sum_{l=0}^{\infty} \frac{(1/\sigma_1)^{2l}}{l!(k+l)!}}{\sum_{l=0}^{\infty} \frac{(1/\sigma_0)^{2l}}{l!(k+l)!}}}_{<1} \leq \infty \qquad (16)$$

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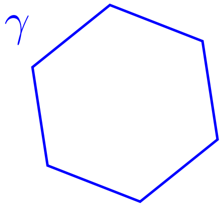
A: Yes.

Example: polygonal curve

Let γ be a polygonal curve.



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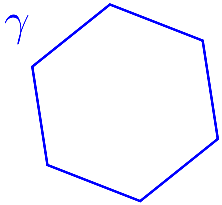

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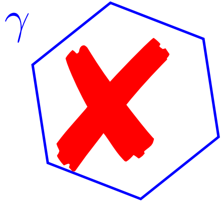

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A: No.

Example: disc

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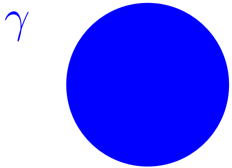

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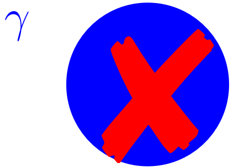

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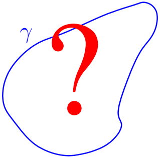

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Conclusion

- We have defined a model for curve-constrained Gaussian random vector fields.
- We need to establish for what curves the realizations of those random fields belong to the RKHS of interest.
- We can explore other kernels and covariances.