Parameter Estimation of Diffusion Processes on the Space of 2D Shapes.

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Introduction

Goals:

- Construct diffusion processes for modeling the diffeomorphic evolution of 2D shapes.
- Estimate the missing parameters based on a sequence of observations.

Approach:

- Model the deformation of the boundary as induced by the flow of a finitely generated vector field defined over the plane.
- Use likelihood ratio [1] to estimate the unknown drift parameters.

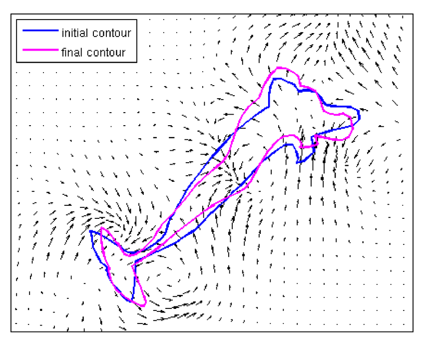
The Geometry of Shape Space

Shape space:
$\mathcal{S}$ = the space of curves obtained through the action of a group of diffeomorphisms on a fixed 2D curve:

$$S = \{\gamma | \gamma = \varphi(\gamma_0), \varphi \in Diff\}$$

V - a reproducing kernel Hilbert space of vector fields $v : \mathbb{R}^2 \to \mathbb{R}^2$ with kernel $K(\cdot,\cdot)\mathbb{I}_2$ such that

$$\frac{\partial \Phi(\cdot,\tau)}{\partial \tau} = v(\Phi(\cdot,\tau),\tau), \qquad \Phi(\cdot,0) = \mathrm{id}(\cdot)$$
$$\Phi(\cdot,1) = \varphi(\cdot).$$



Properties:

- $\mathcal{S}$ - smooth manifold
- tangent vectors on $\mathcal{S}$ are restrictions of v to γ
- the inner product on V induces a Riemannian metric on $\mathcal{S}$
- geodesics and the Riemannian exponential map (and its inverse - the logarithm) on $\mathcal{S}$ are computable.

Diffusions in Shape Space

Given an initial contour γ_0, we want to generate a random trajectory of curves γ_t following a diffusion equation

$$d\gamma_t = a(\gamma_t,\theta)dt + dW_t.$$

Q: What is $d\gamma_t$ on a nonlinear manifold?
A: $\gamma_{t+dt} - \gamma_t$ is not defined, but we can interpret it as an infinitesimal step along the exponential map:

$$\gamma_{t+dt} = \exp_{\gamma_t}(a(\gamma_t,\theta)dt + dW_t).$$

Q: How can we construct dW_t defined over γ_t?
A: Let χ_t be a finite set of n points on γ_t. Consider a subspace $\bar{V}(\chi_t) = span\{K(\cdot, x_i(t))\}$. Set

$$dW_t = \sum_{i=1}^{n} dw_i(t) K(\cdot, x_i(t)),$$

where dw_i - 2D (colored by K) Brownian motion.

Q: What are the possible choices for $a(\gamma_t,\theta)$?
A: We restrict the drift to be an element of $\bar{V}(\chi_t)$.

Examples:

1. Constant Drift

$$d\gamma_t = (K(\gamma_t,\chi_t)^{1/2})^T\theta dt + dW_t$$

2. Mean-Reverting Drift

$$d\gamma_t = -\frac{\theta}{2}\nabla dist(\gamma_t,\mu)^2 dt + dW_t$$

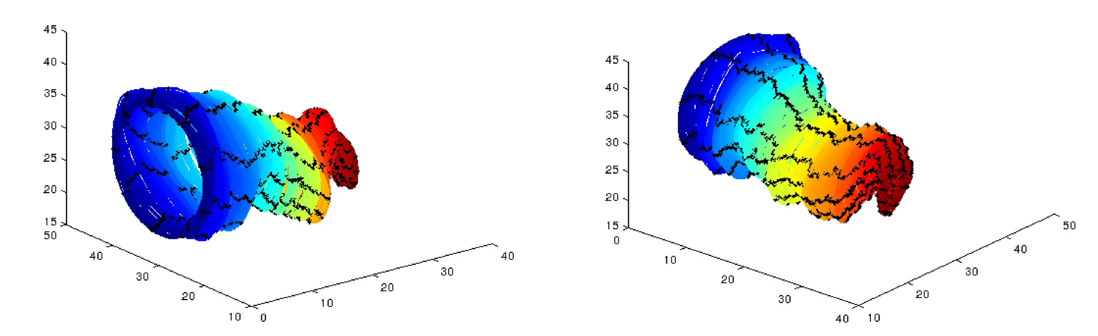
3. Shape-Gradient Drift

$$d\gamma_t = -\frac{\theta_1}{2}\nabla|length(\gamma_t) - L|^2 dt - \frac{\theta_2}{2}\nabla|area(\gamma_t) - A|^2 dt + dW_t$$

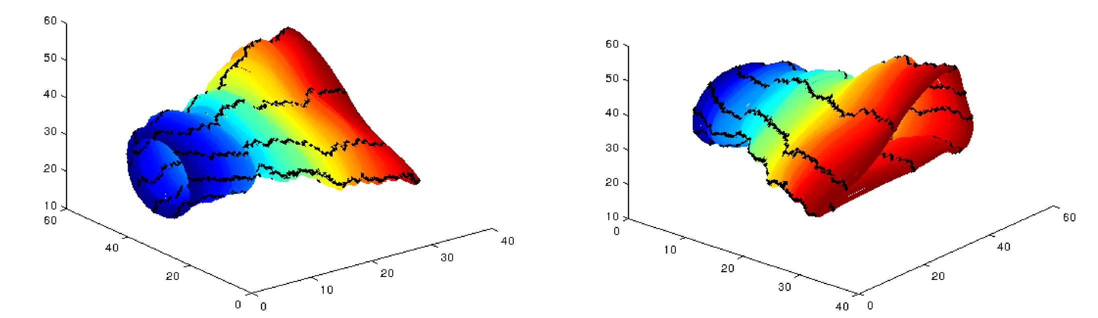
Notes:

- we set $dist(\gamma_t,\mu_t)$ equal to the area of mismatch between the two curves (instead of using the Riemannian distance)
- the gradients with respect to the shape can be calculated by evaluating the action of the differential on the basis functions of $\bar{V}$ [3].

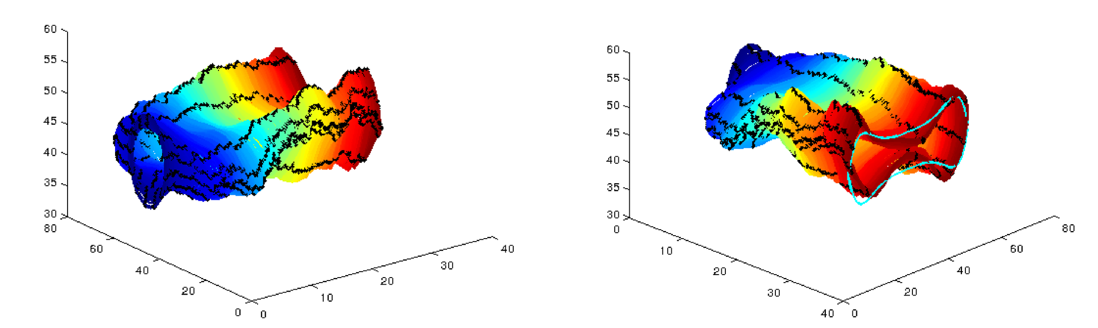
Simulated Diffusions



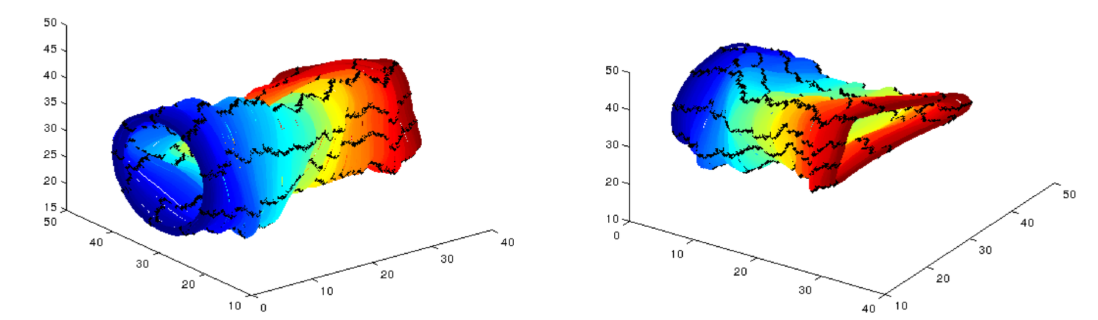
(a) driftless



(b) constant drift



(c) mean-reverting drift



(d) shape-gradient drift.

References

[1] R.S. Lipster and A.N. Shiryaev,"Statistics of Random Processes," *Springer-Verlag*, 1977.

[2] K.D. Elworthy, "Stochastic Differential Equations on Manifolds," *Cambridge University Press*, 1982.

[3] J. Sokolowski, J. P. Zolésio, Introduction to Shape Optimization",*Springer-Verlag*,1992.

Likelihood Ratio Estimation

Girsanov's Theorem [2] gives a closed form for the likelihood ratio:

$$\frac{dP_{\gamma_t}}{dP_{W_t}} = \exp\left\{\int_0^T \langle a(\gamma_t,\theta), d\gamma_t\rangle - \frac{1}{2}\int_0^T \langle a(\gamma_t,\theta), a(\gamma_t,\theta)\rangle dt\right\},$$

and we can maximize it to obtain an estimate for θ:

1. Constant Drift

$$\hat{\theta} = \frac{1}{N dt}\sum_{i=1}^{N-1} K_{\chi_i}^{-1/2}\log(\gamma_i,\gamma_{i+1})$$

2. Mean-Reverting Drift

$$\hat{\theta} = \frac{\sum_{i=1}^{N-1}\langle \nabla dist(\gamma_i,\mu), \log(\gamma_i,\gamma_{i+1})\rangle}{\sum_{i=1}^{N}\|\nabla dist(\gamma_i,\mu)\|^2 dt}$$

3. Shape-Gradient Drift

Let M_i - the Grammian matrix of $\nabla|L(\gamma_i) - L|^2$ and $\nabla|A(\gamma_i) - A|^2$,

$$b = \begin{bmatrix} \sum_{i=1}^{N-1}\langle \nabla|L(\gamma_i) - L|^2, \log(\gamma_i,\gamma_{i+1})\rangle \\ \sum_{i=1}^{N-1}\langle \nabla|A(\gamma_i) - A|^2, \log(\gamma_i,\gamma_{i+1})\rangle \end{bmatrix}.$$

Then,

$$\begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = \left(\sum_{i=1}^{N-1} M_i dt\right)^{-1} b,$$

Future Work

- Develop parameter estimation techniques suited for sparse observations.
- Consider drifts based on higher-order shape descriptors (e.g. curvature, torsion)
- Apply to curves extracted from images to train video tracking algorithms

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