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Introduction to Dynamic Programming

Macroeconomics Theory - EAE320B

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Índice general

Dynamic Programming: Piece of Cake **2**

The dynamic programming approach with T = ∞ 6

Bellman equation 6

Dynamic Programming Theory **10**

Asumptions 10

Proof of existence 11

Stochastic Dynamic Programming **11**

The Markow Property 11

Assumptions on P and /pi_o 12

Transition over two periods 12

Conditional Expectation from Markov Chain 12

Stochastic Dinamic Programming 13

References **14**

Dynamic Programming: Piece of Cake

We have a cake of size W_1 and we need to decide how much of it consume in each period $t = 1, 2, 3, ...$ to maximize consumption utility.

Eventualmente pensaremos que la gente decide consumir todo el trozo de torta, pero más bien plantearemos que la gente piensa que maximizar su consumo para todos los periodos en un horizonte de tiempo definido o infinito.

Step 1. Assumptions

1. INADA: Cake consumption value as $u(c)$, u is increasing $u'(c) > 0$, concave $(u''(c) < 0)$, differentiable and $\lim_{c \to 0} u'(c) = \infty$
2. Lifetime utility is
$$u = \sum_{t=1}^{T} \beta^{t-1} u(c_t),\ \beta\ \epsilon[0, 1] \tag{1}$$
3. Constraints: the cake does not depreciate

$$W_{t+1} = W_t - c_t, \quad t = 1, 2, ..., T \tag{2}$$

Notice 2 the law of motion (or **transition equation**) implies

$$
\begin{aligned}
W_1 = W_2 + c_1 \\
&= (W_3 + c_2) + c_1 \\
&= ... \\
&= W_{t+1} + \sum_{t=1}^{T} c_t
\end{aligned} \tag{3}
$$

Step 2. Decide the optimal consumption sequence $\{\ c\ \}_{t=1}^{T}$

The problem can be written

$$v(W_1) = \max_{\{W_{t+1}, C_t\}\ _{t=1}^{T}} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{4}$$

subject to:

$$W_1 = W_{t+1} + \sum_{t=1}^{T} c_t \tag{5}$$

$$c_t, W_{t+1} \geq 0$$
$$W_1 \quad given$$

$v(W_1)$ Represents value function of state

Step 3. Formulate and solve Langrangian for 3 and

$$L = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \lambda[W_1 - W_{t+1} - \sum_{t=1}^{T} c_t] + \phi[W_{t+1}] \tag{6}$$

Step 4. First Order Conditions

$$\frac{\partial L}{\partial c_t} = 0 \Longrightarrow \beta^{t-1} u'(c_t) = \lambda \forall t \tag{7}$$

$$\frac{\partial L}{\partial W_{t+1}} = 0 \Longrightarrow \lambda = \phi \tag{8}$$

- ϕ is lagrange multiplier on non-negativity constraint for W_{t+1}
- I ignore the constraint $c_t \geq 0$ because the INADA assumption

Step 5. Interpreting sequential solution

If we take from 7 for *t+1*, therefore

$$\beta^{t-1} u'(c_t) = \lambda = \beta^{t} u'(c_t + 1) \tag{9}$$

Along an optimal sequence $\{c_t\}_{t+1}^{T}$ each adjacent period t and t+1 must satisfy 10, ie, utility in both periods is maximum.

$$u'(c_t) = \beta u'(c_t + 1) \tag{10}$$

Step 6. Generalization t from $t+n$ periods

$$u'(c_t) = \beta^2 u'(c_{t+2}) \tag{11}$$

The Euler Equation isn't sufficient for optimality. We could statisfy 10, but have some cake left ($W_t > c_t$).

We need to be ensure to given initial condition (W_1), terminal condition must be $W_{t+1} = 0$.

This form of solution is called **value function** ($v(W_1)$), were is the maximal utility flow over **T periods** given initial cake W_1 (Adda and Cooper, p. 13)

$$V'(W_1) = \lambda = \beta^{t-1} u'(c_t), t = 1, 2, ..., T \tag{12}$$

Example. Power Utility Functions

We will look at specific class of U functions: Power Utility, or, *isoelastic* utility functions.

This class includes the **hyperbolic** or **constant** relative **risk of aversion functions**.

Let's defined as

$$u(c) = \begin{cases} \frac{c^1-\gamma}{1-\gamma} & \text{if} \quad \gamma \neq 1 \\ ln(c) & \text{if} \quad \gamma = 1 \end{cases}$$

The coefficient of **relative risk aversion** is a *constant*, γ, i.e, risk aversion does not depend on level of wealth. Also, $u'(c_t) = c_t^- $

```
# this is julia
library(JuliaCall)
function u(c,gamma)
if gamma==1
return log(c)
else
return (1/(1-gamma)) * c^(1-gamma)
end
end
```

```
using PGFPlots
using LaTeXStrings
p=Axis([
Plots.Linear(x->u(x,0),(0.5,2),legendentry=L"$\gamma=0$"),
Plots.Linear(x->u(x,1),(0.5,2),legendentry=L"$\gamma=1$"),
Plots.Linear(x->u(x,2),(0.5,2),legendentry=L"$\gamma=2$"),
Plots.Linear(x->u(x,5),(0.5,2),legendentry=L"$\gamma=5$")
```

```
],xlabel=L"$c$",ylabel=L"$u(c)$",style="grid=both")
p.legendStyle = "{at={(1.05,1.0)},anchor=north west}"
save("images/dp/CRRA.tex",p,include_preamble=false)
# then, next slide just has \input{images/dp/CRRA}
```

CRRA functions image

CRRA utility properties

We had:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma \neq 1 \\ ln(c) & \text{if} \quad \gamma = 1 \end{cases}$$

were γ^-1 is the elastic of intertermporal substitution (IES). **IES** is defined as the **percent change in consumption growth percent increase in the net interest rate.** [1]

Generally, it is accepted that $\gamma \geq 1$, in which case for

$$c \; \epsilon \; \mathbb{R}^+ \tag{13}$$

- $u(c) < 0, \lim_{c \to 0} u(c) = -\infty, \quad \lim_{c \to \infty} u(c) = 0$
- $u'(c) > 0, \lim_{c \to 0} u'(c) = \infty, \quad \lim_{c \to \infty} u'(c) = 0$

CRRA utility: solution 1

- Let's modify our cake eating problem
- $W_t \implies a_t$, and we introduce gross interest $R = 1 + r$ (for non-growing cake just take r = 0)

$$\max_{\{(c_1,\ldots,C_t)\epsilon(\mathbb{R}^+)^T\}} \sum_{t=1}^{T} \beta^{t-1} \frac{c^1 - \gamma}{1-\gamma} \tag{14}$$

subject to:

$$\sum_{t=1}^{T} R^{1-t} c_t \leq a_1 \tag{15}$$

- **Euler equations** are necessary for interior solutions. Rememver $u'(c_t) = c_t{-}^{\gamma}$

[1] medida de la capacidad de respuesta de la tasa de crecimiento del consumo a la tasa de interés real, es decir como cambia el consumo presente ante cambios en el consumo futuro. Si las subidas de los tipos reales, el consumo futuro puede aumentar debido a la mayor rentabilidad de los ahorros, pero el futuro consumo también puede disminuir a medida que el ahorrador decide consumir menos teniendo en cuenta que puede conseguir un mayor retorno de lo que ahorra (es decir, no consume). El efecto neto sobre el consumo futuro es la elasticidad de sustitución intertemporal.

$$c_t^{-}\gamma = \beta T c_{t+1}^{-}\gamma \Longrightarrow c_t = (R\beta)^{\frac{1}{\gamma}} c_{t+1} \quad \text{for t = 1, ..., T-1} \tag{16}$$

By sucessive substitution

$$c_t = (R\beta)^{\frac{t-1}{\gamma}} c_1 \tag{17}$$

The budget constraint and optimality condition imply

$$\begin{aligned} a_1 = \sum_{t=1,\dots,T} R^{1-t} c_t \\ &= \sum_{t=1,\dots,T} (R^{\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}})^{t-1} \\ &= \sum_{t=1,\dots,T} \alpha^{t-1} \end{aligned} \tag{18}$$

The solution for t = 1, ..., T:

$c_1 = \frac{1-\alpha}{1-\alpha^T} \cdot a_1$ and $ct = \frac{1-\alpha}{1-\alpha^T} \cdot (R\beta)^{\frac{t-1}{\gamma}} \cdot a_1$

In summary, **the consumption function** a *linear function of assets if utility is CRRA.

$$ct = \frac{1-\alpha}{1-\alpha^{T-t+1}} \cdot a_t$$

here, image of the profile of solution

The dynamic programming approach with T = ∞

Step 1. Assumptions

- $T = \infty$

$$\max_{\{(W_{t+1}, C_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \tag{19}$$

s.t

$$W_{t+1} = W_t - c_t \tag{20}$$

When we consider adding a period 0 to our original problem, we can take advantage of the information provided in $V_t(W_1)$, the solution of the T period problem given W_1 from and 19. Now we can write

$$v(W_t) = \max_{\{c_t \epsilon [0, W_t]\}} u(c_t) + \beta v(W_t - c_t) \tag{21}$$

Notice in 21

- W is **state variable**
- c is **control variable**
- 20 is the **transition equation** or law of motion
- t is irrelevant, only state variable matters

Bellman equation

Considering this points we can substituting $c = W - W'$, where W' is the next period's value of W

$$v(W) = \max_{\{W' \epsilon [0,W]\}} u(W - W') + \beta v(W') \tag{22}$$

Notice in 23 equation

- v is in both sides
- The problems is more simple: before we need to find $\{(W_{t+1}, C_t)\}_{t=1}^{\infty}$ and now we need to find v (value function)
- This is called a **fixed point problem**: Find a function v such that plugging in W on the *RHS* and doing the maximization, we end up with the v on the *LHS*

Value function and policy function

We have reduced an infinite-length sequential problem to a one-dimensional maximization problem. But we have to find two unknown functions:

- The maximazer of the RHS of 23 is the **policy function** $g(W) = c^*$. This function gives the optimal value of the control variable, given the state. Then,

$$v(W) = u(g(W)) + \beta v(g(W)) \tag{23}$$

Now, the max operator vanished, because $g(W)$ is the optimal choice. In practice, finding value and policy function is the one operator.

Example

- Let's pretend that we knew v for now

$$v(W) = \max_{\{W' \epsilon [0,W]\}} u(W - W') + \beta v(W') \tag{24}$$

- Assuming v is differentiable, the FOC w.r.t W'

$$u'(c) = \beta v'(W') \tag{25}$$

Taking the partial derivative w.r.t the state W, we get the envolope condition

$$v'(W) = u'(c) \tag{26}$$

This needs to hold in each. Therefore

$$v'(W') = u'(c') \tag{27}$$

Combing 25 and 26 we obtain the usual Euler equation, and any solution v will satisfy this necessary condition as in the sequential case.

$$u'(c) = \beta v'(W') = \beta u'(c') \tag{28}$$

Finding v

Finding the Bellman equation v and associated policyfunction g is not easy. In general, it is impossible to find an analytic expression, ie, *to do it by hand*. We will see thar under some conditions we can always find an fixed point.

Find v: an example with closed form solution Now I'm going to find v by hand. Let's assume that u(c) = ln(c) and supusse T = 1. Then $u'(c) = \frac{1}{c}$, and $V_1(W_1) = ln(W_1)$

Step 1 Start with Euler equations For T = 2

$$u'(c_t) = \beta u'(c_{t+1})$$
$$\frac{1}{c_1} = \frac{\beta}{c_2}$$

And the constraint is

$$W_1 = c_1 + c_2$$

Working with these conditions

$$c_1 = \frac{W_1}{(1+\beta)} \quad \wedge \quad c_2 = \frac{\beta W_1}{(1+\beta)}$$

Step 2 Formulation of policy function bases on value function

$$v(W) = \max_{\{W' \epsilon [0,W]\}} u(W - W') + \beta v(W') \tag{29}$$

Now we can solve for the value of 2 period problem

$$v_2(W_1) = ln(c_1) + \beta ln(c_2) = A_2 + B_2 ln(W_1) \text{in general,} v(W) = A + Bln(W_1) \tag{30}$$

where A and B are constants associated with two period problem

$$A + Bln(W) = \max W' ln(W - W') + \beta(A + BlnW') \tag{31}$$

Step 3. Obtain W' = g(w)

First Order Condition for W'

$$\begin{aligned} -\frac{1}{(W-W')} + \frac{\beta B}{W'} &= 0 \\ W' = (\beta BW) - (\beta BW') \\ W' &= \frac{\beta BW}{1+\beta B} \\ W' &= g(W) \end{aligned} \tag{32}$$

i.e

$$g(w) = \frac{\beta BW}{1+\beta B}$$

Step 4 Replace g(w) in policy function

$$A + Bln(W) = \max W' ln(W - W') + \beta(A + BlnW') \tag{33}$$

Let's start with $LHS(guess)$

$$\begin{aligned} v(w) = ln(W - g(w)) + \beta(A + Blng(w)) \\ &= ln(W - \frac{\beta BW}{1+\beta B} + \beta(A + Bln[\frac{\beta BW}{1+\beta B}]) \\ &= ln(\frac{W}{1+\beta B} + \beta(A + Bln[\frac{\beta BW}{1+\beta B}]) \\ &= \beta A + ln(\frac{W}{1+\beta B} + \beta Bln[\frac{\beta BW}{1+\beta B}] \end{aligned} \tag{34}$$

Now see RHS (*verify*)

$$v(w) = A + ln(W) + \beta BlnW v(w) = A + (1+\beta B)lnW \tag{35}$$

Then,

$$B = (1 + \beta B)B = \frac{1}{1-\beta} \tag{36}$$

$$\therefore \; g(w) = \beta W$$

Solving the Cake problem with T $< \infty$

- When time is finite, solving this DP is fairly simple.
- If we knoe the value in the final period, we can simply go backwards in time.
- In period T there is no point setting W' > 0. Therefore

$$v_t(W) = u(W)$$

Notice that we index the value function with time in this case. It's not the same to have W in period 1 as it is to have W in period T. Right? But if we know v_t for all values of W, we can construct v_{T-1}

We know that

$$\begin{aligned} v_{t-1}(W_{t-1}) = \max W_t \epsilon [0, W_{T-1}] u(W_{T-1} - W_T) + \beta(v_t(W_t)) &= \max W_t \epsilon [0, W_{T-1}] u(W_{T-1} - W_T) + \beta(u(W_t)) \\ &= \max W_t \epsilon [0, W_{T-1}] ln(W_{T-1} - W_T) + \beta(ln(W_t)) \end{aligned} \tag{37}$$

FOC for W_t

$$\frac{1}{WT - 1 - WT} = \frac{\beta}{W_t} W_T = \frac{\beta}{1+\beta} W_{T-1} \tag{38}$$

Thus, the value function in T - 1 is

$$V_{T-1}(W_{T-1}) = ln(\frac{W_{T-1}}{\beta}) + \beta ln(\frac{\beta}{1+\beta} \cdot W_{T-1}) \tag{39}$$

Thus, the value function in T - 2 is

$$v_{T-2}(W_{T-2}) = u(W_{T-2} - W_{T-1}) + \beta(v_{T-1}(W_{T-1})) v_{T-2}(W_{T-2}) = u(W_{T-2} - W_{T-1}) + \beta[ln(\frac{W_{T-1}}{\beta} + \beta ln(\frac{\beta}{1+\beta} \cdot W_{T-1})] \tag{40}$$

Notice that with T finite, there is no fixed point problem if we do backwards induction

Dynamic Programming Theory

$$u = \sum_{t=1}^{T} \beta^{t-1} \tilde{u}(s_t, c_t) \tag{41}$$

- $\beta < 1$: discount factor
- s_{t+1}: state vector evolves as $s_{t+1} = h(s_t, c_t)$
  - s_t: state, and all past decisions are contained in s_t
- c_t: control

Asumptions

- Let $c_t \epsilon c(s_t), s_t \epsilon S$ and assume $\tilde{u}$ is bounded in $(c, s) \epsilon cxs$
- Stationarity: neither payoff $\tilde{u}$ nor transition h depend on time
- Modify $\tilde{u}$ to u subject to in terms of s' (as in cake: $c = W - W'$):

$$v(s) = \max s' \epsilon \Gamma(s) u(s, s') + \beta v(s') \tag{42}$$

- $\Gamma(s)$ is the contraint set (or feasible set) for s' when the current state is s:
  - before that was $\Gamma(W) = [0, W]$
- We will work towards one possible set of sufficient conditions for the existence to the functional equation (see Stokey and Lucas (1989))

Proof of existence

Theorem Assume that $u(s, s')$ is real-valued, continuos, and bounded, that $\beta \epsilon (0, 1)$, and te constraint set $\Gamma(s)$ is *nonempty, compact and continuos.*Then there exists a unique function $v(s)$ that solves 42 (Stokey and Lucas, 1989, theorem 4.6 + Theorem of the Maximum)

Stochastic Dynamic Programming

There are several ways to include uncertainty into this framework. Let's assume the existence of a variable ϵ, representing a *shock*.

Assumptions:

1. ϵ_t affects the agent's payoff in period t
2. ϵ_t is exogenous: the agent cannot influence it.

3. ϵ_t depends only on ϵ_{t-1} (and not on ϵ_{t-2} although we could add ϵ_{t-1} as a state variable)
4. The distribution of $ '| $ is time-invariant.

Defined this assumptions in this way, we call ϵ a **first order Markow process**

The Markow Property

Definition A stochastic process $\{x_t\}$ is sait to have *Markov property* if for all $k \geq 1$ and all t, que have this property to $\{\epsilon_t\}$:

$$Pr(x_{t+1}|x_t, xt-1, ..., x_{t-k}) = Pr(x_{t+1}|x_t)$$

This property is characterize by a **Markov Chain**

A **Markov Chain** consist in a time-invariant state. More formally a **n-state Markov Chain** consist of:

1. n vector of size $(n,1)$: $e_i, i = 1, ..., n$ such that the $i-th$ entry of e_i is one and all others zero,
2. one (n,n) **transition matrix** P, giving the probability of probability of moving from state i to state j, and
3. a vector $\pi_{0i} = Pr(x = e_i)$ holding the probability of being in state i at time 0.

- $e_1 = [10...0]'$, $e_2 = [01...0]'$, ... are just a way of a saying "x is in state i".
- The elements of P are

$$P_{ij} = Pr(x_{t+1} = e_j|x_t = e_i)$$

Assumptions on P and /pi_o

P is a stochastic matrix, where each row sums to one. Row i has probabilities to move any possible state j. Remember that a valid probability distribution must to one. Also, P defines the probabilities of moving from **current state** i **to future state** j. Finally π_o is a valid initial probability distribution

1. For $i = 1, ..., n$ the matrix P satisfies

$$\sum_{j=1}^{n} P_{ij} = 1$$

2. The vector π_o satisfies

$$\sum_{i=1}^{n} \pi_{0i} = 1$$

Transition over two periods

The probability to ove from i to j over two periods is given by P^2_{ij}. The reason is

$$Pr(x_{t+2} = e_j | x_t = e_i)$$
$$\sum_{h=1}^{n} Pr(x_{t+2} = e_j | x_{t+1} = e_h) \cdot Pr(x_{t+1} = e_h | x_{t+1} = e_i)$$
$$\sum_{h=1}^{n} P_{ih} \cdot P_{hj} = P^2_{ij}$$

Conditional Expectation from Markov Chain

Conditional expectation is an importan operator in our problem. The question is *Wht is the expected value of* x_{t+1} *given* $x_t = e_i$. The answer is the following

$$E[x_{t+1} | x_t = e_j] = values\ of\ x\ Prob\ of\ those\ values$$
$$= \sum_{j=1}^{n} e_j \cdot Pr(x_{t+1} = e_j | e_i)$$
$$[x_1 x_2 ... x_n](P_i)'$$

where P_i is the $i-th$ row of P, and $(P_i)'$ is the transpose of that row (i.e. a column vector)

In this problem we have the condition expectation of a function (**value function**). We can notice the result is the same.

Stochastic Dinamic Programming

Now we know something about Makov Chains, we can write

$$v_(s, \epsilon) = \max S' \epsilon [s, \epsilon] u(s, s', \epsilon) + \beta(v(s', \epsilon')|\epsilon) \tag{43}$$

If $u(s, s', \epsilon)$ is **real-valued, continuos,concave and bounded** and if $\beta \epsilon (0, 1)$ and constrain set is compact and convex, then:

1. there exist a **unique value function** $v(s, \epsilon)$ that solves the problem.
2. there exists a stationary **policy function** $\phi(s, \epsilon)$

Proof

This is a direct application of suddiciency conditions[2]

1. with $ < 1 $ discounting hold for the operator on the problem
2. Monotonicity can be established as before.

Now we can derive the first order conditions ($\frac{\partial}{\partial s}$)

$$u'(s, s', \epsilon) + \beta(V'(s', \epsilon')|\epsilon) = 0$$

The result to differentiating and find $V'(s', \epsilon')$

$$u'(s, s', \epsilon) + \beta(u'(s', s'', \epsilon')|\epsilon) = 0$$

We will now solve the deterministic growth model with dynamic programming.

Remember:

$$V(k) \max_{c=f(k)-k' \geq 0} u(c) + \beta V(k')$$

If we will assume $f(k))k^{\alpha}, u(c) = \ln(c)$, discrete state of dinamic programming to know V. This finite set of points are called **grid**. The steps are the following

$$V(k) \max_{k' \epsilon k^{\alpha}} ln(k^{\alpha} - k') + \beta V(k')$$

1. Discretize V onto a **grid** of n points $ K = {k_1, k_2, ..., k_n}$
2. Discretize variable control k': we change the control variable from $k' \epsilon [0, k^{\alpha}]$ *to* $[0, K]$, i.e choose k' from the discrete **grid**.
3. Guess an initial function of $V_o(k)$
4. Iterate on the equation **util** the distance $d(V' - V) < \epsilon$, where $\epsilon > 0$ is the tolerance choosen.

References

Adda and Cooper (2003): Dynamic Economics.

Ljungqvist and Sargent (2012) (LS): Recursive Macroeconomic Theory.

Lucas and Stokey (1989): Recursive Methods in Economics Dynamics

Stokey, N. L., Lucas, R. E., & Prescott, E. (1989). Recursive methods in dynamic economics. Cambridge, MA: Harvard University.

[2] Blackwell's suffiency conditions