$\langle \mathrm{true}, \mathrm{true} \rangle$

$\pi_1(\mathrm{false})$

$\mathrm{curry}_{\sigma,\tau,\delta}$

$:= \lambda f : \sigma \times \tau \to \delta . \lambda x : \sigma . \lambda y : \tau .$

$f \langle x, y \rangle$

- $M, N ::= \cdots \mid \langle M, N \rangle \mid \pi_1(M) \mid \pi_2(M)$
- $\sigma, \tau ::= \cdots \mid \sigma \times \tau$

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau}\ \text{pair}$$

$$\frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \pi_1(M) : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \pi_2(M) : \tau}$$

$V ::= \dots \mid \langle V, V \rangle$

$\langle M, N \rangle$

$$\frac{M \to M'}{\langle M, N \rangle \to \langle M', N \rangle}\ pair_{c1}$$

$$\frac{M \to M'}{\langle V, M \rangle \to \langle V, M' \rangle}\ pair_{c}$$

$$\frac{M \to M'}{\pi_1(M) \to \pi_1(M')}\ \pi_{1c}$$

$$\frac{M \to M'}{\pi_2(M) \to \pi_2(M')}\ \pi_{2c}$$

$$\frac{}{\pi_1(\langle V_1, V_2 \rangle) \to V_1}\ \pi_1$$

$$\frac{}{\pi_2(\langle V_1, V_2 \rangle) \to V_2}\ \pi_2$$

$$\pi_1(\langle M, N\rangle) \to M$$

$$\pi_1(\langle \text{if true then zero else zero}, \text{false}\rangle)$$

$$\to \pi_1(\langle \text{zero}, \text{false}\rangle) \qquad \to \text{if true} \ldots$$

- Verificar el siguiente juicio de tipado:
  $\emptyset \vdash \pi_1((\lambda x : Nat.\langle x, \text{True}\rangle)\ 0) : Nat$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{x:Nat\} \vdash x : Nat}\ ax_v \qquad \dfrac{}{\{x:Nat\} \vdash \text{True} : Bool}\ ax_{true}}{\{x:Nat\} \vdash \langle x, \text{True}\rangle : Nat \times Bool}}{\emptyset \vdash \lambda x : Nat.\langle x, \text{True}\rangle : Nat \to Nat \times Bool} \qquad \dfrac{}{\emptyset \vdash 0 : Nat}\ ax_{zero}}{\dfrac{\emptyset \vdash (\lambda x : Nat.\langle x, \text{True}\rangle)\ 0 : \langle Nat, Bool\rangle}{\emptyset \vdash \pi_1((\lambda x : Nat.\langle x, \text{True}\rangle)\ 0) : Nat}\ \pi_1}\ \to_e$$

$$\dfrac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N\rangle : \sigma \times \tau}\ \text{par}$$

$$\dfrac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \pi_1(M) : \sigma} \qquad \dfrac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \pi_2(M) : \tau}$$

$$\pi_1((\lambda x : Nat.\langle x, \mathsf{True}\rangle)\ 0)$$

$$\xrightarrow[\pi_{1c}]{\beta} \pi_1(\langle 0, true\rangle) \xrightarrow{\pi_1} 0$$

$$\frac{M \to M'}{\langle M, N\rangle \to \langle M', N\rangle}\ pair_{c1}$$

$$\frac{M \to M'}{\pi_2(M) \to \pi_2(M')}\ \pi_{2c}$$

$$\frac{M \to M'}{\langle V, M\rangle \to \langle V, M'\rangle}\ pair_c$$

$$\frac{}{\pi_1(\langle V_1, V_2\rangle) \to V_1}\ \pi_1$$

$$\frac{M \to M'}{\pi_1(M) \to \pi_1(M')}\ \pi_{1c}$$

case Left($\underline{0}$) of left(x) $\rightsquigarrow$ iszero(x)
right(y) $\rightsquigarrow$ False

$\longrightarrow$ iszero($\underline{0}$)

left(zero) : Nat + Bool

$$\begin{array}{lll} \sigma & ::= & \ldots \mid \sigma + \sigma \\ M & ::= & \ldots \mid \mathsf{left}_\tau(M) \mid \mathsf{right}_\tau(M) \mid \\ & & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{left}(x) \rightsquigarrow M \,\|\, \mathsf{right}(y) \rightsquigarrow M \end{array}$$

$\Gamma \vdash \mathsf{left}(\mathsf{zero}) : \mathsf{Nat} + \tau$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathsf{left}_\tau(M) : \sigma + \tau}\ \mathsf{left}$$

$\mathsf{left}_{\mathsf{Bool}}(\mathsf{zero}) : \mathsf{Nat} + \mathsf{Bool}$

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash \mathsf{right}_\sigma(M) : \sigma + \tau}\ \mathsf{right}$$

$\mathsf{right}_{\mathsf{Nat}}(\mathsf{True}) : \mathsf{Nat} + \mathsf{Bool}$

$$\frac{\Gamma \vdash M : \sigma + \tau \quad \Gamma, x : \sigma \vdash N_1 : \rho \quad \Gamma, y : \tau \vdash N_2 : \rho}{\Gamma \vdash \mathsf{case}\ M\ \mathsf{of}\ \mathsf{left}(x) \rightsquigarrow N_1 \,\|\, \mathsf{right}(y) \rightsquigarrow N_2 : \rho}\ \mathsf{case}$$

$$V ::= \ldots \mid \mathsf{left}_\sigma(V) \mid \mathsf{right}_\sigma(V)$$

$$\frac{M \to M'}{\mathsf{left}_\sigma(M) \to \mathsf{left}_\sigma(M')}\ \mathsf{left}_c$$

$$\frac{M \to M'}{\mathrm{right}_\sigma(M) \to \mathrm{right}_\sigma(M')}\ \mathrm{right}_C$$

$$\frac{M \to M'}{\begin{array}{l}\mathrm{case}\ M\ \mathrm{of}\ \mathrm{left}(x) \rightsquigarrow N_1 \,\|\, \mathrm{right}(y) \to N_2 \\ \to \mathrm{case}\ M'\ \mathrm{of}\ \mathrm{left}(x) \rightsquigarrow N_1 \,\|\, \mathrm{right}(y) \to N_2\end{array}}\ \mathrm{case}_C$$

$$\frac{}{\begin{array}{l}\mathrm{case}\ \mathrm{left}_\sigma(V)\ \mathrm{of}\ \mathrm{left}(x) \rightsquigarrow N_1 \,\|\, \mathrm{right}(y) \rightsquigarrow N_2 \\ \to N_1\{x := V\}\end{array}}\ \mathrm{case}_{\mathrm{left}}$$

$$\frac{}{\begin{array}{l}\mathrm{case}\ \mathrm{right}_\sigma(V)\ \mathrm{of}\ \mathrm{left}(x) \rightsquigarrow N_1 \,\|\, \mathrm{right}(y) \rightsquigarrow N_2 \\ \to N_2\{y := V\}\end{array}}\ \mathrm{case}_{\mathrm{right}}$$

- $M, N, O ::= \cdots \mid \mathsf{Nil}_\sigma \mid \mathsf{Bin}(M, N, O) \mid \mathsf{raíz}(M) \mid \mathsf{der}(M) \mid \mathsf{izq}(M) \mid \mathsf{esNil}(M)$
- $\sigma ::= \cdots \mid \mathsf{AB}_\sigma$

$\mathrm{Nil}_{Bool \to Bool}$

$A = \mathrm{Bin}(\mathrm{Nil}_{B \to B}, \lambda x : B . x, \mathrm{Nil}_{B \to B})$

$\mathrm{raíz}(A) \to \ldots \to \lambda x : B . x$

$\mathrm{esNil}(A) \to \ldots \to \mathrm{False}$

$$\frac{}{\Gamma \vdash \mathrm{Nil}_\sigma : AB_\sigma} \text{T-Nil}$$

$$\frac{\Gamma \vdash M : AB_\sigma \qquad \Gamma \vdash N : \sigma \qquad \Gamma \vdash O : AB_\sigma}{\Gamma \vdash \mathrm{Bin}(M, N, O) : AB_\sigma} \text{T-Bin}$$

$$\frac{\Gamma \vdash M : AB_\sigma}{\Gamma \vdash \mathrm{raíz}(M) : \sigma} \text{T-raíz}$$

$$\frac{\Gamma \vdash M : AB_\sigma}{\Gamma \vdash \mathrm{izq}(M) : AB_\sigma} \text{T-izq}$$

$$\frac{\Gamma \vdash M : AB_\sigma}{\Gamma \vdash \mathrm{esNil}(M) : Bool}$$

$$V ::= \ldots \mid Nil_\sigma \mid Bin(V, V, V)$$

$$\frac{M \to M'}{Bin(M, N, O) \to Bin(M', N, O)}$$

$$\frac{N \to N'}{Bin(V, N, O) \to Bin(V, N', O)}$$

$$\frac{O \to O'}{Bin(V_1, V_2, O) \to Bin(V_1, V_2, O')}$$

$$\frac{}{esNil(Nil_\sigma) \to True} \text{ E-esNil-Nil}$$

$$\frac{}{esNil(Bin(V_1, V_2, V_3)) \to False} \text{ E-esNil-Bin}$$

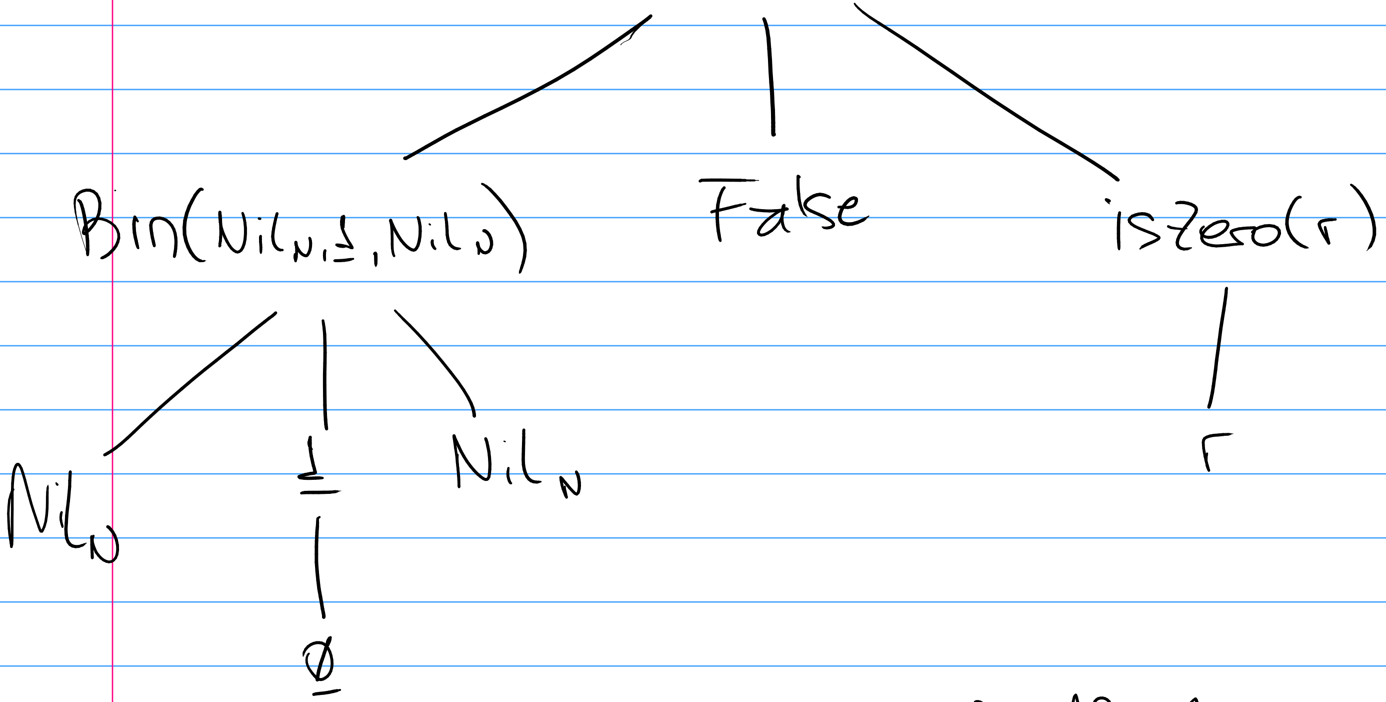
$$\frac{}{raiz(Bin(V_1, V_2, V_3)) \to V_2}$$

$$\frac{}{izq(Bin(V_1, V_2, V_3)) \to V_1}$$

$$\frac{\Gamma \vdash r : Nat}{\Gamma \vdash isZero(r) : Bool}$$

- $M, N, O ::= \cdots \mid Nil_\sigma \mid \mathrm{Bin}(M, N, O) \mid \mathrm{case}_{AB_\sigma}\ M\ of\ Nil \rightsquigarrow N\ ; \mathrm{Bin}(i, r, d) \rightsquigarrow O$

$$\mathrm{case}_{AB_N}\ \mathrm{Bin}(Nil_N, \perp, Nil_N)\ \text{of}\ Nil \rightsquigarrow \mathrm{False}\ ;\ \mathrm{Bin}(i,r,d) \rightsquigarrow \mathrm{isZero}(r) \twoheadrightarrow \ldots \twoheadrightarrow \mathrm{False}$$



$$\frac{\Gamma \vdash M : AB_\sigma \qquad \Gamma \vdash N : \tau \qquad \Gamma \cup \{i : AB_\sigma,\ r : \sigma,\ d : AB_\sigma\} \vdash O : \tau}{\Gamma \vdash \mathrm{case}_{AB_\sigma}\ M\ \text{of}\ Nil \rightsquigarrow N\ ;\ \mathrm{Bin}(i,r,d) \rightsquigarrow O : \tau}$$

$$\frac{M \longrightarrow M'}{\mathrm{case}_{AB_\sigma}\ M\ \text{of}\ Nil \rightsquigarrow N\ ;\ \mathrm{Bin}(i,r,d) \rightsquigarrow O \longrightarrow \mathrm{case}_{AB_\sigma}\ M'\ \text{of}\ Nil \rightsquigarrow N\ ;\ \mathrm{Bin}(i,r,d) \rightsquigarrow O}\ \text{E-CASE}$$

$$\frac{}{\mathrm{case}_{AB_\sigma}\ Nil_\sigma\ \text{of}\ Nil \rightsquigarrow N\ ;\ \mathrm{Bin}(i,r,d) \rightsquigarrow O \longrightarrow N}\ \text{E-CASE-NIL}$$

E-CASE-BIN

$\text{case}_{AB_\sigma}\ \text{Bin}(V_1, V_2, V_3)\ \text{of}\ \text{Nil} \leadsto N\ ;\ \text{Bin}(i,r,d) \leadsto O$

$\longrightarrow O\{i := V_1,\ r := V_2,\ d := V_3\}$

$\text{esNil}_\sigma \stackrel{\text{def}}{=} \lambda a : AB_\sigma .\ \text{case}_{AB_\sigma}\ a\ \text{of}\ \text{Nil} \leadsto \text{true}\ ;\ \text{Bin}(i,r,d) \leadsto \text{false}$

$\text{raíz}_\sigma \stackrel{\text{def}}{=} \lambda a : AB_\sigma .\ \text{case}_{AB_\sigma}\ a\ \text{of}\ \text{Nil} \leadsto \bot_\sigma\ ;\ \text{Bin}(i,r,d) \leadsto r$

$\cancel{\mu x : \sigma . M \longrightarrow M\{x := \mu x : \sigma . M\}}$

$\text{fix}\ \lambda x : \sigma . M \longrightarrow M\{x := \text{fix}\ \lambda x : \sigma . M\}$

$\bot_\sigma \stackrel{\text{def}}{=} \text{fix}\ \lambda x : \sigma . x$

$\text{case}_{AB_{Nat}}$ if $(\lambda x : Bool.x)$ True then Bin(Nil$_{Nat}$, $\underline{1}$, Nil$_{Nat}$) else Nil$_{Nat}$
of Nil $\rightsquigarrow$ False ; Bin$(i, r, d) \rightsquigarrow$ iszero(r)

$$\xrightarrow[\substack{\text{E-CASE} \\ \text{E-IF} \\ \beta}]{} \text{case}_{AB_N} \text{ if true then Bin}(\text{Nil}_N, \underline{1}, \text{Nil}_N) \text{ else Nil}_N \text{ of Nil} \rightsquigarrow \text{False}; \text{Bin}(i,r,d) \rightsquigarrow \text{iszero}(r)$$

$$\xrightarrow[\substack{\text{E-CASE} \\ \text{E-IF-TRUE}}]{} \text{case}_{AB_N} \text{ Bin}(\text{Nil}_N, \underline{1}, \text{Nil}_N) \text{ of Nil} \rightsquigarrow \text{False}; \text{Bin}(i,r,d) \rightsquigarrow \text{iszero}(r)$$

$$\xrightarrow[\text{E-CASE-BIN}]{} \text{iszero}(\underline{1})$$

$$\xrightarrow[\text{iszero-succ}]{} \text{False}$$

$$M ::= \ldots \mid map(M, M)$$

$$map(\lambda x:N.\ isZero(x),\ Bin(Nil_N, 1, Nil_N)) \to \ldots \to Bin(Nil_B, false, Nil_B)$$

$$\frac{\Gamma \vdash N : AB_\gamma \qquad \Gamma \vdash M : \gamma \to \sigma}{\Gamma \vdash map(M, N) : AB_\sigma}\ \text{T-MAP}$$

Faltan reglas de congruencia

$$\frac{\emptyset \vdash V : \sigma \to \gamma}{map(V, Nil_\sigma) \to Nil_\gamma}$$

$$map(\lambda x:N.\ isZero(x),\ Nil_N) \to Nil_{???}$$

$$\frac{}{map(V, Bin(V_1, V_2, V_3)) \to Bin(map(V, V_1), V\ V_2, map(V, V_3))}$$