Numerical Methods for Coupled Environmental Problems

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Introduction

Many research problems in environmental and climate science are solved with the *partitioned approach*: Specialized models for subprocesses are developed independently and coupled for the full numerical simulation.

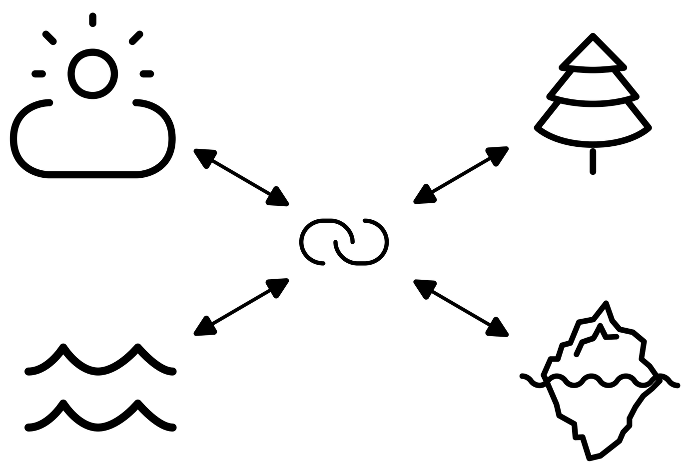

Figure 1: For example, Earth system models consist of various submodels for individual processes/subsystems.

The better we model the subprocesses, the more we have to consider numerical errors developing at the interface between components.

$\Rightarrow$ We study the *coupling error in time*.

Iterative Coupling Algorithms

Simple coupling algorithms:

- severely restrict the convergence order
- introduce instabilities

Iterative coupling algorithms

- enable high-order solutions
- pose few requirements on subsolvers (black box assumption)
- confirm validity of interface boundary conditions in highly nonlinear settings

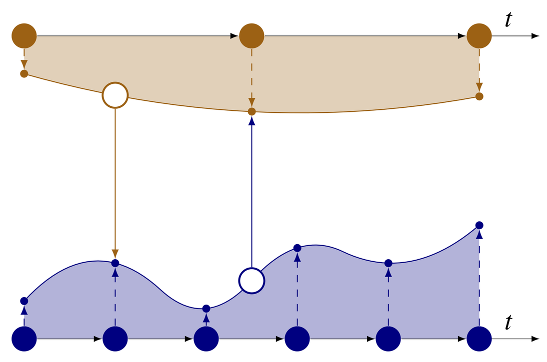

Figure 2: The most generic, high-order coupling algorithm are waveform iterations, where the solvers exchange interpolants of their data. *Adapted with permission from Benjamin Rodenberg.*

Outlook

Focus: atmosphere-ocean coupling & groundwater-surface flows.

- *fully discrete analysis* in idealized models
- verify results & test new algorithms in full complexity models

$\Rightarrow$ develop **high-order, energy efficient, black-box** coupling algorithms

Selected References

[1] P. Bastian, H. Berninger, A. Dedner *et al.*, 'Adaptive Modelling of Coupled Hydrological Processes with Application in Water Management,' in *Progress in Industrial Mathematics at ECMI 2010*, M. Günther, A. Bartel, M. Brunk, S. Schöps and M. Striebel, Eds., vol. 17, Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 561–567, ISBN: 978-3-642-25100-9. DOI: 10.1007/978-3-642-25100-9_65.

[2] A. Monge and P. Birken, 'On the convergence rate of the Dirichlet–Neumann iteration for unsteady thermal fluid–structure interaction,' *Computational Mechanics*, vol. 62, no. 3, pp. 525–541, 1st Sep. 2018, ISSN: 1432-0924. DOI: 10.1007/s00466-017-1511-3.

[3] P. Bastian, M. Blatt, A. Dedner *et al.*, 'The Dune framework: Basic concepts and recent developments,' *Computers & Mathematics with Applications*, Development and Application of Open-source Software for Problems with Numerical PDEs, vol. 81, pp. 75–112, 1st Jan. 2021, ISSN: 0898-1221. DOI: 10.1016/j.camwa.2020.06.007.

[4] G. Chourdakis, K. Davis, B. Rodenberg *et al.*, 'preCICE v2: A sustainable and user-friendly coupling library,' *Open Research Europe*, vol. 2, no. 51, 2022. DOI: 10.12688/openreseurope.14445.2.

Code available at: `github.com/valentinaschueller/richards-swe-coupling/`.

Example: Coupled Groundwater and Surface Flows

Water management projects such as stream bed re-naturalization affect the water table. We can use numerical models to predict how the aquifer is affected by river systems, e.g., in case of floods.

Collaboration with Andreas Dedner and Robert Klöfkorn.

Modeling Approach [1]

We restrict ourselves to a simplified 1D model.

Groundwater: Richards equation in 1D

$$c(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\underbrace{\left(K(\psi)\left(\frac{\partial \psi}{\partial z}+1\right)\right)}_{=:-v(\psi)} \tag{1}$$

Surface: shallow water equations in 0D

$$\frac{dh}{dt} = v_s(t) \tag{2}$$

Coupling: mass + pressure conservation

$$v_s(t) = v(\psi(t,0)) \tag{3a}$$
$$\psi(t,0) = h(t) \tag{3b}$$

Figure 3: Öre river in Västerbotten. (Credit: SiberianJay, CC BY-SA 4.0)

Coupling Algorithm

Dirichlet-Neumann Iteration with Relaxation

$k \leftarrow 1$, h^0 given

repeat

  solve (1) using (3b) $\rightarrow \psi^k$ given h^{k-1}

  solve (2) using (3a) $\rightarrow \tilde{h}^k$ given ψ^k

  relax: $h^k = \omega\tilde{h}^k + (1-\omega)h^{k-1}$

  $k \leftarrow k+1$

until termination

Linear Analysis [2]

- **linearization:** assume K, c constant
- **space discretization:** linear finite elements
- **time discretization:** implicit Euler method

We can now express the water height at $t = t^n$ and in iteration k in terms of old data:

$$\boxed{h^{n,k} = (\omega S + (1-\omega))h^{n,k-1} + \omega b^{n-1},}$$

where $S = S(c, K, \Delta t, \Delta z)$.

Optimal Relaxation Parameter ω_{opt}

Convergence is obtained when $h^{n,k}$ does not change with increasing k. We can find an optimal value of ω which ensures convergence of the coupling algorithm in two iterations:

$$\omega_{opt} = \frac{1}{1-S}, \quad S = b^2\alpha - \frac{a}{2}$$

with

$$a = \frac{2}{3}c\Delta z + 2\frac{K\Delta t}{\Delta z}, \quad b = \frac{1}{6}c\Delta z - \frac{K\Delta t}{\Delta z},$$

and

$$\alpha = \Delta z \sum_{j=1}^{M-1} \sin^2(j\pi\Delta z)\left(a - 2b\cos(j\pi\Delta z)\right)^{-1}.$$

Analysis Verification

We verify our analysis with an example code, using DUNE [3] for the Richards equation and preCICE [4] for equation coupling. The experimental results are in agreement with the derived formula. ω_{opt} is strongly affected by the material parameters K and c which can span various orders of magnitude in the Richards equation.

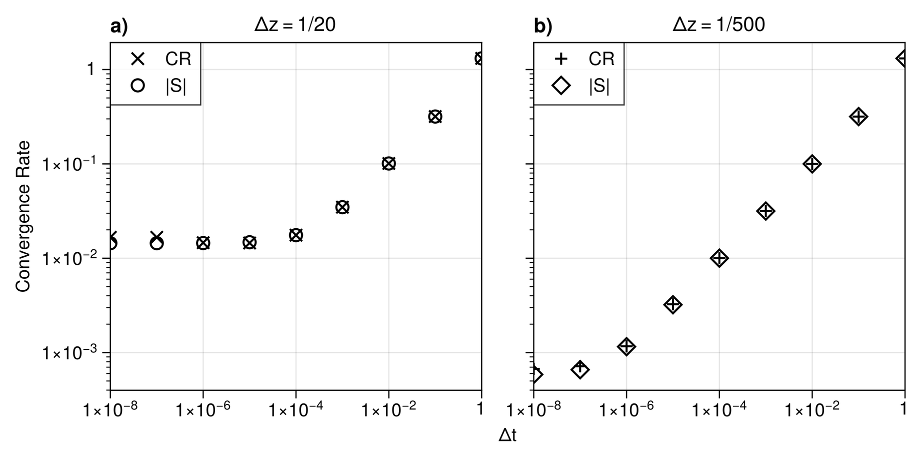

Figure 4: Experimental and theoretical convergence rates, CR and |S|, for varying grid resolutions ($K = c = 1$).

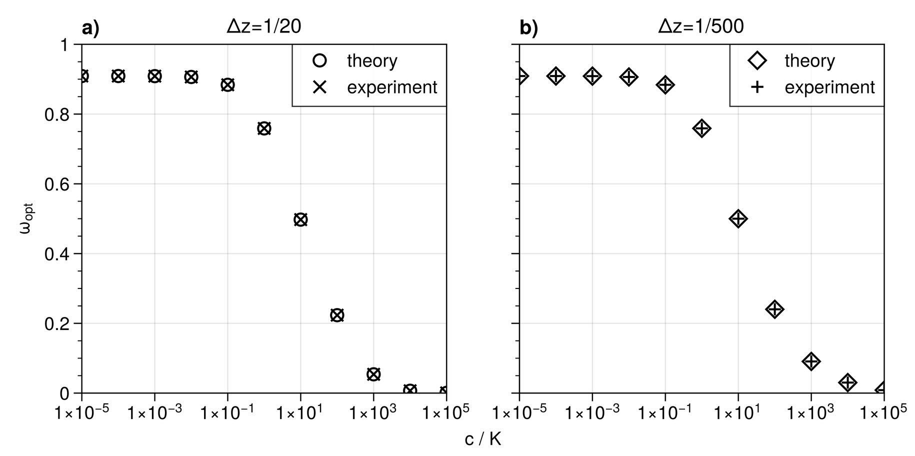

Figure 5: Optimal relaxation parameter ω_{opt} for varying material parameters ($\Delta t = 1/10$).

Work in Progress

How well does our 1D-0D analysis translate to the nonlinear 2D-1D case?

Test case: Lake at rest, varying soil types.

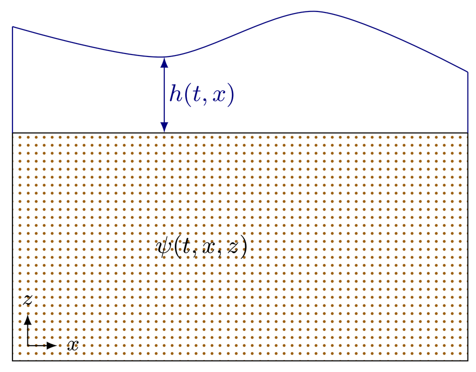

Figure 6: 2D setup of the coupled surface-groundwater-flow system. We obtain the 1D-0D case by omitting the x-axis.

The analysis indicates that the robustness of the coupling algorithm is very sensitive to the nonlinearities in the Richards equation.

$\Rightarrow$ need to track K and c at runtime!

Future work: Study advanced coupling setups (waveform iterations, Quasi-Newton acceleration) using preCICE.