

Model Equations - Axisymmetric Atmosphere Simulation

Valentina Schüller

September 23, 2021

1 Original Equations from [1]

Equations of motion for zonal, meridional, and vertical winds:

$$\begin{aligned}\partial_t u &= -\nabla \cdot \begin{pmatrix} vu \\ wu \end{pmatrix} + fv + \frac{uv \tan \varphi}{a} + \partial_z (\nu \partial_z u) \\ \partial_t v &= -\nabla \cdot \begin{pmatrix} vv \\ wv \end{pmatrix} - fu - \frac{u^2 \tan \varphi}{a} - \frac{1}{a} \partial_\varphi \Phi + \partial_z (\nu \partial_z v) \\ \partial_t w &= -\nabla \cdot \begin{pmatrix} vw \\ ww \end{pmatrix} - \partial_z \Phi + \frac{g\Theta}{\Theta_0}\end{aligned}$$

First law of thermodynamics

$$\partial_t \Theta = -\nabla \cdot \begin{pmatrix} v\Theta \\ w\Theta \end{pmatrix} - (\Theta - \Theta_E) \tau^{-1} + \partial_z (\nu \partial_z \Theta)$$

Continuity equation:

$$0 = -\nabla \cdot \begin{pmatrix} v \\ w \end{pmatrix}$$

where u, v, w denote the zonally averaged winds in zonal, meridional, and vertical directions; z, φ denote the spherical coordinates; $\nabla = \begin{pmatrix} (a \cos \varphi)^{-1} \partial_\varphi (\cos \varphi) \\ \partial_z \end{pmatrix}$ is the gradient operator in the meridional-vertical plane; $f = 2\Omega \sin(\varphi)$ is the Coriolis parameter, a is Earth's radius, $\Phi = gz$ denotes the geopotential, ν viscosity, and g the gravitational acceleration. Θ is the potential temperature field, τ is a constant radiative damping time. Heating/cooling is implemented as a relaxation towards the equilibrium temperature field Θ_E , defined as:

$$\frac{\Theta_E}{\Theta_0} = 1 - \frac{2}{3} \Delta_H P_2(\sin \varphi) + \Delta_v \left(\frac{z}{H} - \frac{1}{2} \right)$$

Θ_0 is the average of Θ_E ; Δ_H is a nondimensional parameter representing the fractional change in potential temperature from equator to pole in radiative equilibrium; Δ_v is a nondimensional parameter representing the fractional change in potential temperature from top to bottom in radiative equilibrium; $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second Legendre polynomial.

1.1 Boundary Conditions

- at $z = H$:
 - $w = 0$
 - $\partial_z u = \partial_z v = \partial_z \Theta = 0$
- at $z = 0$:
 - $w = 0$
 - $\partial_z \Theta = 0$
 - $v \partial_z u = C u$
 - $v \partial_z v = C v$; C is a constant drag coefficient.

1.2 Model Parameters

- $\nu = 25, 10, 5, 2.5, 1, 0.5 \frac{m^2}{s}$
- $\Omega = \frac{2\pi}{8.64 \times 10^6 s}$
- $a = 6.4 \times 10^6 m$
- $g = 9.8 m s^{-2}$
- $H = 8.0 \times 10^3 m$
- $\Delta_H = \frac{1}{3}$
- $\Delta_v = \frac{1}{8}$
- $C = 0.005 m s^{-1}$
- $\tau = 20$ days

2 Simplified Equations

By plugging the definition of ∇ , the continuity equation becomes:

$$0 = \frac{1}{a} \tan \varphi \cdot v - \frac{1}{a} \partial_\varphi v - \partial_z w$$

Doing the same for the equations of motion and the first law of thermodynamics, assuming constant viscosity ν and plugging in the continuity equation results in the following reformulations:

$$\begin{aligned}
\partial_t u &= -\frac{1}{a} v \partial_\varphi u + f v + \frac{u v \tan \varphi}{a} - w \partial_z u + \nu \partial_z^2 u \\
\partial_t v &= -\frac{1}{a} v \partial_\varphi v + f v - \frac{u^2 \tan \varphi}{a} - w \partial_z v - \frac{1}{a} \partial_\varphi \Phi + \nu \partial_z^2 v \\
\partial_t w &= -\frac{1}{a} v \partial_\varphi w - w \partial_z w - \partial_z \Phi + g \frac{\Theta}{\Theta_0} \\
\partial_t \Theta &= -\frac{1}{a} v \partial_\varphi \Theta - w \partial_z \Theta - (\Theta - \Theta_E) \tau^{-1} + \nu \partial_z^2 \Theta
\end{aligned}$$

3 Discretizing the Equations

3.1 Space

Making use of the staggered grid finite difference approaches used in [2, 3], I suggest the following discretizations for the spatial derivatives:

Equations of Motion:

$$\begin{aligned}
\partial_t u_{j+\frac{1}{2}}^k &= -\frac{1}{a} v_{j+\frac{1}{2}}^k \frac{u_{j+\frac{3}{2}}^k - u_{j-\frac{1}{2}}^k}{2\Delta\varphi_j} + f_j v_{j+\frac{1}{2}}^k + \frac{u_{j+\frac{1}{2}}^k v_{j+\frac{1}{2}}^k \tan \varphi_j}{a} \\
&\quad - w_{j+\frac{1}{2}}^k \frac{u_{j+\frac{1}{2}}^{k+1} - u_{j+\frac{1}{2}}^{k-1}}{2\Delta z_k} + \nu \frac{u_{j+\frac{1}{2}}^{k+1} - 2u_{j+\frac{1}{2}}^k + u_{j+\frac{1}{2}}^{k-1}}{4\Delta z_k^2} \\
\partial_t v_{j+\frac{1}{2}}^k &= -\frac{1}{a} v_{j+\frac{1}{2}}^k \frac{v_{j+\frac{3}{2}}^k - v_{j-\frac{1}{2}}^k}{2\Delta\varphi_j} + f_j v_{j+\frac{1}{2}}^k - \frac{\left(u_{j+\frac{1}{2}}^k\right)^2 \tan \varphi_j}{a} - w_{j+\frac{1}{2}}^k \frac{v_{j+\frac{3}{2}}^{k+1} - v_{j-\frac{1}{2}}^{k-1}}{2\Delta z_k} \\
&\quad - \frac{1}{a} \frac{\Phi_{j+1}^k - \Phi_{j-1}^k}{2\Delta\varphi_j} + \nu \frac{v_{j+\frac{1}{2}}^{k+1} - 2v_{j+\frac{1}{2}}^k + v_{j+\frac{1}{2}}^{k-1}}{4\Delta z_k^2}
\end{aligned}$$

First law of thermodynamics

$$\partial_t \Theta_j^k = -\frac{1}{a} v_{j+\frac{1}{2}}^k \frac{\Theta_{j+1}^k - \Theta_{j-1}^k}{2\Delta\varphi_j} - w_{j+\frac{1}{2}}^k \frac{\Theta_j^{k+1} - \Theta_j^{k-1}}{2\Delta z_k} - (\Theta_j^k - \Theta_E) \tau^{-1} + \nu \frac{\Theta_j^{k+1} - 2\Theta_j^k + \Theta_j^{k-1}}{4\Delta z_k^2}$$

Vertical winds:

$$\partial_t w_{j+\frac{1}{2}}^k = -\frac{1}{a} v_j^k \frac{w_{j+\frac{3}{2}}^k - w_{j-\frac{1}{2}}^k}{2\Delta\varphi_j} - w_{j+\frac{1}{2}}^k \frac{w_{j+\frac{1}{2}}^{k+1} - w_{j+\frac{1}{2}}^{k-1}}{2\Delta z_k} - \frac{\Phi_j^{k+1} - \Phi_j^{k-1}}{2\Delta z_k} + \frac{g\Theta_j^k}{\Theta_0}$$

where

$$f_j = 2\Omega \sin(\varphi_j)$$

3.2 Time

[2] use a backward difference formula, same as [3], who do backward Euler steps with a Δt of 15 minutes.

4 Resolution

I see no reason why we should use a non-equidistant grid. [1] use 90 grid-points in the vertical direction and 50 from equator to pole. [3] use a way coarser grid, 7.826° latitude and 9 vertical levels.

5 Reformulation

$$\begin{aligned} \frac{u_{k,j+\frac{1}{2}}^{n+1} - u_{k,j+\frac{1}{2}}^n}{\Delta t} = & -\frac{1}{a} v_{k,j+\frac{1}{2}}^n \frac{u_{k,j+\frac{3}{2}}^n - u_{k,j-\frac{1}{2}}^n}{2\Delta\varphi_j} + f_{j+\frac{1}{2}} v_{k,j+\frac{1}{2}}^n + \frac{u_{k,j+\frac{1}{2}}^n v_{k,j+\frac{1}{2}}^n \tan\varphi_{j+\frac{1}{2}}}{a} \\ & - w_{k,j+\frac{1}{2}}^n \frac{u_{k+1,j+\frac{1}{2}}^n - u_{k-1,j+\frac{1}{2}}^n}{2\Delta z_k} + \nu \frac{u_{k+1,j+\frac{1}{2}}^n - 2u_{k,j+\frac{1}{2}}^n + u_{k-1,j+\frac{1}{2}}^n}{4\Delta z_k^2} \end{aligned}$$

$$\begin{aligned} \frac{v_{k,j+\frac{1}{2}}^{n+1} - v_{k,j+\frac{1}{2}}^n}{\Delta t} = & -\frac{1}{a} v_{k,j+\frac{1}{2}}^n \frac{v_{k,j+\frac{3}{2}}^n - v_{k,j-\frac{1}{2}}^n}{2\Delta\varphi_j} + f_{j+\frac{1}{2}} v_{k,j+\frac{1}{2}}^n - \frac{\left(u_{k,j+\frac{1}{2}}^n\right)^2 \tan\varphi_{j+\frac{1}{2}}}{a} \\ & - w_{k,j+\frac{1}{2}}^n \frac{v_{k+1,j+\frac{3}{2}}^n - v_{k-1,j-\frac{1}{2}}^n}{2\Delta z_k} - \frac{1}{a} \frac{\Phi_{k,j+1}^n - \Phi_{k,j-1}^n}{2\Delta\varphi_j} + \nu \frac{v_{k+1,j+\frac{1}{2}}^n - 2v_{k,j+\frac{1}{2}}^n + v_{k-1,j+\frac{1}{2}}^n}{4\Delta z_k^2} \end{aligned}$$

$$\frac{w_{k,j+\frac{1}{2}}^{n+1} - w_{k,j+\frac{1}{2}}^n}{\Delta t} = -\frac{1}{a} v_{k,j}^n \frac{w_{k,j+\frac{3}{2}}^n - w_{k,j-\frac{1}{2}}^n}{2\Delta\varphi_j} - w_{k,j+\frac{1}{2}}^n \frac{w_{k+1,j+\frac{1}{2}}^n - w_{k-1,j+\frac{1}{2}}^n}{2\Delta z_k} - \frac{\Phi_{k+1,j} - \Phi_{k-1,j}}{2\Delta z_k} + \frac{g\Theta_{k,j}^n}{\Theta_0}$$

$$\begin{aligned} \frac{\Theta_{k,j+\frac{1}{2}}^{n+1} - \Theta_{k,j+\frac{1}{2}}^n}{\Delta t} = & -\frac{1}{a} v_{k,j+\frac{1}{2}}^n \frac{\Theta_{k,j+1}^n - \Theta_{k,j-1}^n}{2\Delta\varphi_j} - w_{k,j+\frac{1}{2}}^n \frac{\Theta_{k+1,j}^n - \Theta_{k-1,j}^n}{2\Delta z_k} \\ & - (\Theta_{k,j}^n - \Theta_E) \tau^{-1} + \nu \frac{\Theta_{k+1,j}^n - 2\Theta_{k,j}^n + \Theta_{k-1,j}^n}{4\Delta z_k^2} \end{aligned}$$

References

- [1] Isaac M. Held and Arthur Y. Hou. Nonlinear Axially Symmetric Circulations in a Nearly Inviscid Atmosphere. *Journal of the Atmospheric Sciences*, 37(3):515–533, March 1980. Publisher: American Meteorological Society Section: Journal of the Atmospheric Sciences.

- [2] Isacc M. Held and Max J. Suarez. A Two-Level Primitive Equation Atmospheric Model Designed for Climatic Sensitivity Experiments. *Journal of Atmospheric Sciences*, 35(2):206 – 229, 1978. Place: Boston MA, USA
Publisher: American Meteorological Society.
- [3] Mao-Sung Yao and Peter H. Stone. Development of a Two-Dimensional Zonally Averaged Statistical-Dynamical Model. Part I The Parameterization of Moist Convection and its Role in the General Circulation. *Journal of Atmospheric Sciences*, 44(1):65–82, January 1987. Place: Boston MA, USA
Publisher: American Meteorological Society.