Model Equations - Axisymmetric Atmosphere Simulation

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September 29, 2021

1 Original Equations from [2]

1.1 Stationary Case

Equations of motion for zonal, meridional, and vertical winds

$$0 = -\nabla \cdot \begin{pmatrix} vu \\ wu \end{pmatrix} + fv + \frac{uv \tan\varphi}{a} + \partial_z \left(\nu \partial_z u\right)$$

$$0 = -\nabla \cdot \begin{pmatrix} vv \\ wv \end{pmatrix} - fu - \frac{u^2 \tan\varphi}{a} - \frac{1}{a}\partial_\varphi \Phi + \partial_z \left(\nu \partial_z v\right)$$

$$0 = -\partial_z \Phi + \frac{g\Theta}{\Theta_0}$$

First law of thermodynamics

$$0 = -\nabla \cdot \begin{pmatrix} v\Theta \\ w\Theta \end{pmatrix} - (\Theta - \Theta_E)\tau^{-1} + \partial_z \left(\nu \partial_z \Theta\right)$$

Continuity equation

$$0 = -\nabla \cdot \begin{pmatrix} v \\ w \end{pmatrix}$$

where u, v, w denote the zonally averaged winds in zonal, meridional, and vertical directions; z, φ denote the spherical coordinates; $\nabla = \begin{pmatrix} (a\cos\varphi)^{-1}\partial_\varphi(\cos\varphi) \\ \partial_z \end{pmatrix}$ is the gradient operator in the meridional-vertical plane; $f = 2\Omega\sin(\varphi)$ is the Coriolis parameter, a is Earth's radius, $\Phi = gz + \delta$ denotes the geopotential with a pressure variation term δ, ν viscosity, and g the gravitational acceleration. Θ is the potential temperature field, τ is a constant radiative damping time. Heating/cooling is implemented as a relaxation towards the equilibrium temperature field Θ_E, defined as:

$$\frac{\Theta_E}{\Theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\varphi) + \Delta_v \left(\frac{z}{H} - \frac{1}{2}\right)$$

Θ_0 is the average of Θ_E; Δ_H is a nondimensional parameter representing the fractional change in potential temperature from equator to pole in *radiative* equilibrium; Δ_v is a nondimensional parameter representing the fractional change in potential temperature from top to bottom in radiative equilibrium; $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second Legendre polynomial.

1.2 Time-Dependent Equations

For the numerical solution, [2] use the "time-dependendent version" of these equations, i.e.:

Equations of motion for zonal, meridional, and vertical winds

$$\partial_t u = -\nabla \cdot \begin{pmatrix} vu \\ wu \end{pmatrix} + fv + \frac{uv \tan\varphi}{a} + \partial_z (\nu \partial_z u)$$

$$\partial_t v = -\nabla \cdot \begin{pmatrix} vv \\ wv \end{pmatrix} - fu - \frac{u^2 \tan\varphi}{a} - \frac{1}{a}\partial_\varphi \Phi + \partial_z (\nu \partial_z v)$$

$$\partial_t w = -\partial_z \Phi + \frac{g\Theta}{\Theta_0}$$

First law of Thermodynamics

$$\partial_t \Theta = -\nabla \cdot \begin{pmatrix} v\Theta \\ w\Theta \end{pmatrix} - (\Theta - \Theta_E)\tau^{-1} + \partial_z (\nu \partial_z \Theta)$$

Continuity equation

$$0 = -\nabla \cdot \begin{pmatrix} v \\ w \end{pmatrix}$$

1.3 Boundary Conditions

- at $z = H$:
  - $w = 0$
  - $\partial_z u = \partial_z v = \partial_z \Theta = 0$
- at $z = 0$:
  - $w = 0$
  - $\partial_z \Theta = 0$
  - $\nu \partial_z u = Cu$
  - $\nu \partial_z v = Cv$; C is a constant drag coefficient.
- at $\varphi = -\frac{\pi}{2}, \varphi = \frac{\pi}{2}$:

  - $u = v = w = 0$
  - $\Theta = \Theta_{eq}$
- von Neumann boundary conditions for δ on all boundaries

1.4 Model Parameters

- $\nu = 25, 10, 5, 2.5, 1, 0.5 \frac{m^2}{s}$
- $\Omega = \frac{2\pi}{8.64 \times 10^6 s}$
- $a = 6.4 \times 10^6 m$
- $g = 9.8 m s^{-2}$
- $H = 8.0 \times 10^3 m$
- $\Delta_H = \frac{1}{3}$
- $\Delta_v = \frac{1}{8}$
- $C = 0.005 m s^{-1}$
- $\tau = 20$ days

2 Simplified Equations

By plugging the definition of ∇, the continuity equation becomes:

$$0 = \frac{1}{a} \tan\varphi \cdot v - \frac{1}{a} \partial_\varphi v - \partial_z w$$

Doing the same for the equations of motion and the first law of thermodynamics, assuming constant viscosity ν and plugging in the continuity equation results in the following reformulations:

$$\partial_t u = -\frac{1}{a} v \partial_\varphi u + f v + \frac{u v \tan\varphi}{a} - w \partial_z u + \nu \partial_z^2 u$$

$$\partial_t v = -\frac{1}{a} v \partial_\varphi v - f u - \frac{u^2 \tan\varphi}{a} - w \partial_z v - \frac{1}{a} \partial_\varphi \Phi + \nu \partial_z^2 v$$

$$\partial_t w = -\frac{1}{a} v \partial_\varphi w - w \partial_z w - \partial_z \Phi + g \frac{\Theta}{\Theta_0}$$

$$\partial_t \Theta = -\frac{1}{a} v \partial_\varphi \Theta - w \partial_z \Theta - (\Theta - \Theta_E) \tau^{-1} + \nu \partial_z^2 \Theta$$

Inserting our definition $\Phi := gz + \delta$, the equations for v and w further simplify to:

$$\partial_t v = -\frac{1}{a} v \partial_\varphi v - fu - \frac{u^2 \tan\varphi}{a} - w\partial_z v - \frac{1}{a}\partial_\varphi \delta + \nu \partial_z^2 v$$

$$\partial_t w = -\frac{1}{a} v \partial_\varphi w - w\partial_z w - \partial_z \delta + g\left(\frac{\Theta}{\Theta_0} - 1\right)$$

3 Discretizing the Equations

We start off by treating our velocity and temperature derivatives explicitly in time:

$$u^{n+1} = u^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi u - w\partial_z u + fv + \frac{uv\tan\varphi}{a} + \nu\partial_z^2 u\right)$$

$$v^{n+1} = v^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi v - fu - \frac{u^2\tan\varphi}{a} - w\partial_z v - \frac{1}{a}\partial_\varphi\delta + \nu\partial_z^2 v\right)$$

$$w^{n+1} = w^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi w - w\partial_z w - \partial_z\delta + g\left(\frac{\Theta}{\Theta_0} - 1\right)\right)$$

$$\Theta^{n+1} = \Theta^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi \Theta - w\partial_z \Theta - (\Theta - \Theta_E)\tau^{-1} + \nu\partial_z^2 \Theta\right)$$

Then we use the helpful descriptions in [1] and abbreviate:

$$F \coloneqq v^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi v - fu - \frac{u^2\tan\varphi}{a} - w\partial_z v + \nu\partial_z^2 v\right)$$

$$G \coloneqq w^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi w - w\partial_z w + g\left(\frac{\Theta}{\Theta_0} - 1\right)\right)$$

to obtain:

$$u^{n+1} = u^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi u - w\partial_z u + fv + \frac{uv\tan\varphi}{a} + \nu\partial_z^2 u\right)$$

$$v^{n+1} = F - \frac{\Delta t}{a}\partial_\varphi \delta$$

$$w^{n+1} = G - \Delta t \partial_z \delta$$

$$\Theta^{n+1} = \Theta^n + \Delta t\left(-\frac{1}{a} v\partial_\varphi \Theta - w\partial_z \Theta - (\Theta - \Theta_E)\tau^{-1} + \nu\partial_z^2 \Theta\right)$$

We evaluate all derivatives, velocities, and potential temperatures on the right hand sides of the equations explicitly at time t^n, **except** for the derivatives of our pressure/geopotential variation δ, which we evaluate at t^{n+1}, i.e., implicitly.

To make this clear, we write this once more:

$$v^{n+1} = F^n - \frac{\Delta t}{a}\partial_\varphi \delta^{n+1}$$

$$w^{n+1} = G^n - \Delta t \partial_z \delta^{n+1}$$

We determine δ^{n+1} by solving the continuity equation at t^{n+1}:

$$0 = \frac{1}{a}\tan\varphi \cdot v^{n+1} - \frac{1}{a}\partial_\varphi v^{n+1} - \partial_z w^{n+1}$$

$$0 = \frac{1}{a}\tan\varphi \cdot \left(F^n - \frac{\Delta t}{a}\partial_\varphi \delta^{n+1}\right) - \frac{1}{a}\partial_\varphi \left(F^n - \frac{\Delta t}{a}\partial_\varphi \delta^{n+1}\right) - \partial_z \left(G^n - \Delta t \partial_z \delta^{n+1}\right)$$

$$0 = \frac{1}{a}F^n \tan\varphi - \frac{\Delta t}{a}\tan\varphi \partial_\varphi \delta^{n+1} - \frac{1}{a}\partial_\varphi F^n - \frac{\Delta t}{a^2}\partial_\varphi^2 \delta^{n+1} - \partial_z G^n - \Delta t \partial_z^2 \delta^{n+1}$$

$$\frac{\Delta t}{a}\tan\varphi \cdot \partial_\varphi \delta^{n+1} + \frac{\Delta t}{a^2}\partial_\varphi^2 \delta^{n+1} + \Delta t \partial_z^2 \delta^{n+1} = \frac{1}{a}F^n \tan\varphi - \frac{1}{a}\partial_\varphi F^n - \partial_z G^n$$

TODO: Discretize this equation for δ^{n+1}! (von Neumann at all boundaries, stable discretization? discretization of F^n, G^n derivatives?)

Using these new values for δ, we can solve for v^{n+1}, w^{n+1}. (TODO: Discretize $\partial_\varphi \delta^{n+1}, \partial_z \delta^{n+1}$)

Question: Should we use new values of v, w for computation of u, Θ?

3.1 Space

Making use of the staggered grid finite difference approaches used in [3, 4], I suggest the following discretizations for the spatial derivatives:

Equations of Motion:

$$\begin{aligned}\partial_t u^k_{j+\frac{1}{2}} = &-\frac{1}{a}v^k_{j+\frac{1}{2}}\frac{u^k_{j+\frac{3}{2}} - u^k_{j-\frac{1}{2}}}{2\Delta\varphi_j} + f_j v^k_{j+\frac{1}{2}} + \frac{u^k_{j+\frac{1}{2}} v^k_{j+\frac{1}{2}}\tan\varphi_j}{a} \\ &- w^k_{j+\frac{1}{2}}\frac{u^{k+1}_{j+\frac{1}{2}} - u^{k-1}_{j+\frac{1}{2}}}{2\Delta z_k} + \nu\frac{u^{k+1}_{j+\frac{1}{2}} - 2u^k_{j+\frac{1}{2}} + u^{k-1}_{j+\frac{1}{2}}}{4\Delta z_k^2}\end{aligned}$$

$$\begin{aligned}\partial_t v^k_{j+\frac{1}{2}} = &-\frac{1}{a}v^k_{j+\frac{1}{2}}\frac{v^k_{j+\frac{3}{2}} - v^k_{j-\frac{1}{2}}}{2\Delta\varphi_j} - f_j u^k_{j+\frac{1}{2}} - \frac{\left(u^k_{j+\frac{1}{2}}\right)^2\tan\varphi_j}{a} - w^k_{j+\frac{1}{2}}\frac{v^{k+1}_{j+\frac{3}{2}} - v^{k-1}_{j-\frac{1}{2}}}{2\Delta z_k} \\ &- \frac{1}{a}\frac{\Phi^k_{j+1} - \Phi^k_{j-1}}{2\Delta\varphi_j} + \nu\frac{v^{k+1}_{j+\frac{1}{2}} - 2v^k_{j+\frac{1}{2}} + v^{k-1}_{j+\frac{1}{2}}}{4\Delta z_k^2}\end{aligned}$$

First law of thermodynamics

$$\partial_t \Theta_j^k = -\frac{1}{a} v_{j+\frac{1}{2}}^k \frac{\Theta_{j+1}^k - \Theta_{j-1}^k}{2\Delta\varphi_j} - w_{j+\frac{1}{2}}^k \frac{\Theta_j^{k+1} - \Theta_j^{k-1}}{2\Delta z_k} - (\Theta_j^k - \Theta_E)\tau^{-1} + \nu \frac{\Theta_j^{k+1} - 2\Theta_j^k + \Theta_j^{k-1}}{4\Delta z_k^2}$$

Vertical winds:

$$\partial_t w_{j+\frac{1}{2}}^k = -\frac{1}{a} v_j^k \frac{w_{j+\frac{3}{2}}^k - w_{j-\frac{1}{2}}^k}{2\Delta\varphi_j} - w_{j+\frac{1}{2}}^k \frac{w_{j+\frac{1}{2}}^{k+1} - w_{j+\frac{1}{2}}^{k-1}}{2\Delta z_k} - \frac{\Phi_j^{k+1} - \Phi_j^{k-1}}{2\Delta z_k} + \frac{g\Theta_j^k}{\Theta_0}$$

where

$$f_j = 2\Omega \sin(\varphi_j)$$

3.2 Time

[3]use a backward difference formula, same as [4], who do backward Euler steps with a Δt of 15 minutes.

4 Resolution

I see no reason why we should use a non-equidistant grid. [2] use 90 grid-points in the vertical direction and 50 from equator to pole. [4] use a way coarser grid, 7.826° latitude and 9 vertical levels.

5 Reformulation

$$v_{k+\frac{1}{2},j+\frac{1}{2}}^n = \frac{v_{k,j}^n + v_{k,j+1}^n}{2}$$

$$w_{k+\frac{1}{2},j+\frac{1}{2}}^n = \frac{w_{k+1,j+\frac{1}{2}}^n + w_{k,j+\frac{1}{2}}^n}{2}$$

$$w_{k+\frac{1}{2},j}^n = \frac{w_{k+\frac{1}{2},j+\frac{1}{2}}^n + w_{k+\frac{1}{2},j-\frac{1}{2}}^n}{2}$$

$$\begin{aligned}\frac{u_{k+\frac{1}{2},j+\frac{1}{2}}^{n+1} - u_{k+\frac{1}{2},j+\frac{1}{2}}^n}{\Delta t} = &-\frac{1}{a} v_{k+\frac{1}{2},j+\frac{1}{2}}^n \hat{\partial_\varphi} u_{k+\frac{1}{2},j+\frac{1}{2}}^n + f_{j+\frac{1}{2}} v_{k+\frac{1}{2},j+\frac{1}{2}}^n + \frac{u_{k+\frac{1}{2},j+\frac{1}{2}}^n v_{k+\frac{1}{2},j+\frac{1}{2}}^n \tan\varphi_{j+\frac{1}{2}}}{a} \\ &- w_{k+\frac{1}{2},j+\frac{1}{2}}^n \hat{\partial_z} u_{k+\frac{1}{2},j+\frac{1}{2}}^n + \nu \frac{u_{k+\frac{3}{2},j+\frac{1}{2}}^n - 2u_{k+\frac{1}{2},j+\frac{1}{2}}^n + u_{k-\frac{1}{2},j+\frac{1}{2}}^n}{\Delta z^2}\end{aligned}$$

$$\frac{v^{n+1}_{k+\frac{1}{2},j}-v^n_{k+\frac{1}{2},j}}{\Delta t}=-\frac{1}{a}v^n_{k+\frac{1}{2},j}\hat{\partial_\varphi}v^n_{k+\frac{1}{2},j}-f_j u^n_{k+\frac{1}{2},j}-\frac{\left(u^n_{k+\frac{1}{2},j}\right)^2\tan\varphi_j}{a}$$
$$-w^n_{k+\frac{1}{2},j}\hat{\partial_z}v^n_{k+\frac{1}{2},j}-\frac{1}{a}\frac{\Phi^n_{k,j+1}-\Phi^n_{k,j-1}}{2\Delta\varphi}+\nu\frac{v^n_{k+1,j+\frac{1}{2}}-2v^n_{k,j+\frac{1}{2}}+v^n_{k-1,j+\frac{1}{2}}}{4\Delta z^2}$$

$$\frac{w^{n+1}_{k,j+\frac{1}{2}}-w^n_{k,j+\frac{1}{2}}}{\Delta t}=-\frac{1}{a}v^n_{k,j}\frac{w^n_{k,j+\frac{3}{2}}-w^n_{k,j-\frac{1}{2}}}{2\Delta\varphi}-w^n_{k,j+\frac{1}{2}}\frac{w^n_{k+1,j+\frac{1}{2}}-w^n_{k-1,j+\frac{1}{2}}}{2\Delta z}-\frac{\Phi_{k+1,j}-\Phi_{k-1,j}}{2\Delta z}+\frac{g\Theta^n_{k,j}}{\Theta_0}$$

$$\frac{\Theta^{n+1}_{k,j+\frac{1}{2}}-\Theta^n_{k,j+\frac{1}{2}}}{\Delta t}=-\frac{1}{a}v^n_{k,j+\frac{1}{2}}\frac{\Theta^n_{k,j+1}-\Theta^n_{k,j-1}}{2\Delta\varphi}-w^n_{k,j+\frac{1}{2}}\frac{\Theta^n_{k+1,j}-\Theta^n_{k-1,j}}{2\Delta z}$$
$$-(\Theta^n_{k,j}-\Theta_E)\tau^{-1}+\nu\frac{\Theta^n_{k+1,j}-2\Theta^n_{k,j}+\Theta^n_{k-1,j}}{4\Delta z^2}$$

$$\Phi = gz + \delta^{(n+1)}$$

where $\hat{\partial_\varphi}$, $\hat{\partial_z}$ are upwind discretizations of the first partial derivative in φ and z directions, respectively: t.b.c.

References

[1] Michael Griebel, Thomas Dornseifer, and Tilman Neunhoeffer. *Numerical Simulation in Fluid Dynamics*. Mathematical Modeling and Computation. Society for Industrial and Applied Mathematics, January 1998.

[2] Isaac M. Held and Arthur Y. Hou. Nonlinear Axially Symmetric Circulations in a Nearly Inviscid Atmosphere. *Journal of the Atmospheric Sciences*, 37(3):515–533, March 1980. Publisher: American Meteorological Society Section: Journal of the Atmospheric Sciences.

[3] Isacc M. Held and Max J. Suarez. A Two-Level Primitive Equation Atmospheric Model Designed for Climatic Sensitivity Experiments. *Journal of Atmospheric Sciences*, 35(2):206 – 229, 1978. Place: Boston MA, USA Publisher: American Meteorological Society.

[4] Mao-Sung Yao and Peter H. Stone. Development of a Two-Dimensional Zonally Averaged Statistical-Dynamical Model. Part I The Parameterization of Moist Convection and its Role in the General Circulation. *Journal of Atmospheric Sciences*, 44(1):65–82, January 1987. Place: Boston MA, USA Publisher: American Meteorological Society.