Model Equations - Axisymmetric Atmosphere Simulation

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1 Original Equations from [1]

Equations of motion for zonal, meridional, and vertical winds:

$$\partial_t u = -\nabla \cdot \begin{pmatrix} vu \\ wu \end{pmatrix} + fv + \frac{uv \tan\varphi}{a} + \partial_z \left(\nu \partial_z u\right)$$

$$\partial_t v = -\nabla \cdot \begin{pmatrix} vv \\ wv \end{pmatrix} - fu - \frac{u^2 \tan\varphi}{a} - \frac{1}{a}\partial_\varphi \Phi + \partial_z \left(\nu \partial_z v\right)$$

$$\partial_t w = -\nabla \cdot \begin{pmatrix} vw \\ ww \end{pmatrix} - \partial_z \Phi + \frac{g\Theta}{\Theta_0}$$

First law of thermodynamics

$$\partial_t \Theta = -\nabla \cdot \begin{pmatrix} v\Theta \\ w\Theta \end{pmatrix} - (\Theta - \Theta_E)\tau^{-1} + \partial_z \left(\nu \partial_z \Theta\right)$$

Continuity equation:

$$0 = -\nabla \cdot \begin{pmatrix} v \\ w \end{pmatrix}$$

where u, v, w denote the zonally averaged winds in zonal, meridional, and vertical directions; z, φ denote the spherical coordinates; $\nabla = \begin{pmatrix} (a\cos\varphi)^{-1}\partial_\varphi(\cos\varphi) \\ \partial_z \end{pmatrix}$ is the gradient operator in the meridional-vertical plane; $f = 2\Omega\sin(\varphi)$ is the Coriolis parameter, a is Earth's radius, $\Phi = gz$ denotes the geopotential, ν viscosity, and g the gravitational acceleration. Θ is the potential temperature field, τ is a constant radiative damping time. Heating/cooling is implemented as a relaxation towards the equilibrium temperature field Θ_E, defined as:

$$\frac{\Theta_E}{\Theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\varphi) + \Delta_v \left(\frac{z}{H} - \frac{1}{2}\right)$$

Θ_0 is the average of Θ_E; Δ_H is a nondimensional parameter representing the fractional change in potential temperature from equator to pole in radiative equilibrium; Δ_v is a nondimensional parameter representing the fractional change in potential temperature from top to bottom in radiative equilibrium; $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second Legendre polynomial.

1.1 Boundary Conditions

- at $z = H$:
  - $w = 0$
  - $\partial_z u = \partial_z v = \partial_z \Theta = 0$
- at $z = 0$:
  - $w = 0$
  - $\partial_z \Theta = 0$
  - $v \partial_z u = Cu$
  - $v \partial_z v = Cv$; C is a constant drag coefficient.

1.2 Model Parameters

- $\nu = 25, 10, 5, 2.5, 1, 0.5 \frac{m^2}{s}$
- $\Omega = \frac{2\pi}{8.64 \times 10^6 s}$
- $a = 6.4 \times 10^6 m$
- $g = 9.8 m s^{-2}$
- $H = 8.0 \times 10^3 m$
- $\Delta_H = \frac{1}{3}$
- $\Delta_v = \frac{1}{8}$
- $C = 0.005 m s^{-1}$
- $\tau = 20$ days

2 Simplified Equations

By plugging the definition of ∇, the continuity equation becomes:

$$0 = \frac{1}{a} \tan \varphi \cdot v - \frac{1}{a} \partial_\varphi v - \partial_z w$$

Doing the same for the equations of motion and the first law of thermodynamics, assuming constant viscosity ν and plugging in the continuity equation results in the following reformulations:

$$\partial_t u = -\frac{1}{a} v \partial_\varphi u + f v + \frac{uv\tan\varphi}{a} - w\partial_z u + \nu \partial_z^2 u$$

$$\partial_t v = -\frac{1}{a} v \partial_\varphi v - f u - \frac{u^2\tan\varphi}{a} - w\partial_z v - \frac{1}{a}\partial_\varphi \Phi + \nu \partial_z^2 v$$

$$\partial_t w = -\frac{1}{a} v \partial_\varphi w - w\partial_z w - \partial_z \Phi + g\frac{\Theta}{\Theta_0}$$

$$\partial_t \Theta = -\frac{1}{a} v \partial_\varphi \Theta - w\partial_z \Theta - (\Theta - \Theta_E)\tau^{-1} + \nu \partial_z^2 \Theta$$

3 Discretizing the Equations

3.1 Space

Making use of the staggered grid finite difference approaches used in [2, 3], I suggest the following discretizations for the spatial derivatives:

Equations of Motion:

$$\begin{aligned}\partial_t u^k_{j+\frac{1}{2}} = {} & -\frac{1}{a} v^k_{j+\frac{1}{2}} \frac{u^k_{j+\frac{3}{2}} - u^k_{j-\frac{1}{2}}}{2\Delta\varphi_j} + f_j v^k_{j+\frac{1}{2}} + \frac{u^k_{j+\frac{1}{2}} v^k_{j+\frac{1}{2}} \tan\varphi_j}{a} \\ & - w^k_{j+\frac{1}{2}} \frac{u^{k+1}_{j+\frac{1}{2}} - u^{k-1}_{j+\frac{1}{2}}}{2\Delta z_k} + \nu \frac{u^{k+1}_{j+\frac{1}{2}} - 2u^k_{j+\frac{1}{2}} + u^{k-1}_{j+\frac{1}{2}}}{4\Delta z_k^2}\end{aligned}$$

$$\begin{aligned}\partial_t v^k_{j+\frac{1}{2}} = {} & -\frac{1}{a} v^k_{j+\frac{1}{2}} \frac{v^k_{j+\frac{3}{2}} - v^k_{j-\frac{1}{2}}}{2\Delta\varphi_j} - f_j u^k_{j+\frac{1}{2}} - \frac{\left(u^k_{j+\frac{1}{2}}\right)^2 \tan\varphi_j}{a} - w^k_{j+\frac{1}{2}} \frac{v^{k+1}_{j+\frac{3}{2}} - v^{k-1}_{j-\frac{1}{2}}}{2\Delta z_k} \\ & - \frac{1}{a}\frac{\Phi^k_{j+1} - \Phi^k_{j-1}}{2\Delta\varphi_j} + \nu \frac{v^{k+1}_{j+\frac{1}{2}} - 2v^k_{j+\frac{1}{2}} + v^{k-1}_{j+\frac{1}{2}}}{4\Delta z_k^2}\end{aligned}$$

First law of thermodynamics

$$\partial_t \Theta^k_j = -\frac{1}{a} v^k_{j+\frac{1}{2}} \frac{\Theta^k_{j+1} - \Theta^k_{j-1}}{2\Delta\varphi_j} - w^k_{j+\frac{1}{2}} \frac{\Theta^{k+1}_j - \Theta^{k-1}_j}{2\Delta z_k} - (\Theta^k_j - \Theta_E)\tau^{-1} + \nu \frac{\Theta^{k+1}_j - 2\Theta^k_j + \Theta^{k-1}_j}{4\Delta z_k^2}$$

Vertical winds:

$$\partial_t w^k_{j+\frac{1}{2}} = -\frac{1}{a} v^k_j \frac{w^k_{j+\frac{3}{2}} - w^k_{j-\frac{1}{2}}}{2\Delta\varphi_j} - w^k_{j+\frac{1}{2}} \frac{w^{k+1}_{j+\frac{1}{2}} - w^{k-1}_{j+\frac{1}{2}}}{2\Delta z_k} - \frac{\Phi^{k+1}_j - \Phi^{k-1}_j}{2\Delta z_k} + \frac{g\Theta^k_j}{\Theta_0}$$

where

$$f_j = 2\Omega\sin(\varphi_j)$$

3.2 Time

[2]use a backward difference formula, same as [3], who do backward Euler steps with a Δt of 15 minutes.

4 Resolution

I see no reason why we should use a non-equidistant grid. [1] use 90 grid-points in the vertical direction and 50 from equator to pole. [3] use a way coarser grid, 7.826° latitude and 9 vertical levels.

5 Reformulation

$$\frac{u^{n+1}_{k,j+\frac{1}{2}} - u^n_{k,j+\frac{1}{2}}}{\Delta t} = -\frac{1}{a} v^n_{k,j+\frac{1}{2}} \frac{u^n_{k,j+\frac{3}{2}} - u^n_{k,j-\frac{1}{2}}}{2\Delta\varphi} + f_{j+\frac{1}{2}} v^n_{k,j+\frac{1}{2}} + \frac{u^n_{k,j+\frac{1}{2}} v^n_{k,j+\frac{1}{2}} \tan\varphi_{j+\frac{1}{2}}}{a}$$
$$- w^n_{k,j+\frac{1}{2}} \frac{u^n_{k+1,j+\frac{1}{2}} - u^n_{k-1,j+\frac{1}{2}}}{2\Delta z} + \nu \frac{u^n_{k+1,j+\frac{1}{2}} - 2u^n_{k,j+\frac{1}{2}} + u^n_{k-1,j+\frac{1}{2}}}{4\Delta z^2}$$

$$\frac{v^{n+1}_{k,j+\frac{1}{2}} - v^n_{k,j+\frac{1}{2}}}{\Delta t} = -\frac{1}{a} v^n_{k,j+\frac{1}{2}} \frac{v^n_{k,j+\frac{3}{2}} - v^n_{k,j-\frac{1}{2}}}{2\Delta\varphi} - f_{j+\frac{1}{2}} u^n_{k,j+\frac{1}{2}} - \frac{\left(u^n_{k,j+\frac{1}{2}}\right)^2 \tan\varphi_{j+\frac{1}{2}}}{a}$$
$$-w^n_{k,j+\frac{1}{2}} \frac{v^n_{k+1,j+\frac{3}{2}} - v^n_{k-1,j-\frac{1}{2}}}{2\Delta z} - \frac{1}{a} \frac{\Phi^n_{k,j+1} - \Phi^n_{k,j-1}}{2\Delta\varphi} + \nu \frac{v^n_{k+1,j+\frac{1}{2}} - 2v^n_{k,j+\frac{1}{2}} + v^n_{k-1,j+\frac{1}{2}}}{4\Delta z^2}$$

$$\frac{w^{n+1}_{k,j+\frac{1}{2}} - w^n_{k,j+\frac{1}{2}}}{\Delta t} = -\frac{1}{a} v^n_{k,j} \frac{w^n_{k,j+\frac{3}{2}} - w^n_{k,j-\frac{1}{2}}}{2\Delta\varphi} - w^n_{k,j+\frac{1}{2}} \frac{w^n_{k+1,j+\frac{1}{2}} - w^n_{k-1,j+\frac{1}{2}}}{2\Delta z} - \frac{\Phi_{k+1,j} - \Phi_{k-1,j}}{2\Delta z} + \frac{g\Theta^n_{k,j}}{\Theta_0}$$

$$\frac{\Theta^{n+1}_{k,j+\frac{1}{2}} - \Theta^n_{k,j+\frac{1}{2}}}{\Delta t} = -\frac{1}{a} v^n_{k,j+\frac{1}{2}} \frac{\Theta^n_{k,j+1} - \Theta^n_{k,j-1}}{2\Delta\varphi} - w^n_{k,j+\frac{1}{2}} \frac{\Theta^n_{k+1,j} - \Theta^n_{k-1,j}}{2\Delta z}$$
$$- (\Theta^n_{k,j} - \Theta_E)\tau^{-1} + \nu \frac{\Theta^n_{k+1,j} - 2\Theta^n_{k,j} + \Theta^n_{k-1,j}}{4\Delta z^2}$$

References

[1] Isaac M. Held and Arthur Y. Hou. Nonlinear Axially Symmetric Circulations in a Nearly Inviscid Atmosphere. *Journal of the Atmospheric Sciences*, 37(3):515–533, March 1980. Publisher: American Meteorological Society Section: Journal of the Atmospheric Sciences.

[2] Isacc M. Held and Max J. Suarez. A Two-Level Primitive Equation Atmospheric Model Designed for Climatic Sensitivity Experiments. *Journal of Atmospheric Sciences*, 35(2):206 – 229, 1978. Place: Boston MA, USA Publisher: American Meteorological Society.

[3] Mao-Sung Yao and Peter H. Stone. Development of a Two-Dimensional Zonally Averaged Statistical-Dynamical Model. Part I The Parameterization of Moist Convection and its Role in the General Circulation. *Journal of Atmospheric Sciences*, 44(1):65–82, January 1987. Place: Boston MA, USA Publisher: American Meteorological Society.