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Analysis of Bulk Interface Conditions in Atmosphere-Ocean-Sea Ice Coupling

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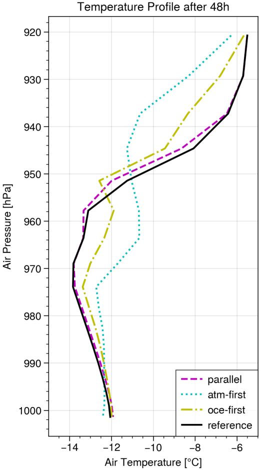
Coupling Errors in Climate Models

- Coupling methods in climate models: Schwarz Waveform Relaxation (SWR), stopped after one step
- Error w.r.t. converged SWR solution can be large[1]
- How can we reduce this error in practice?

⇒ Concrete advice to model developers!

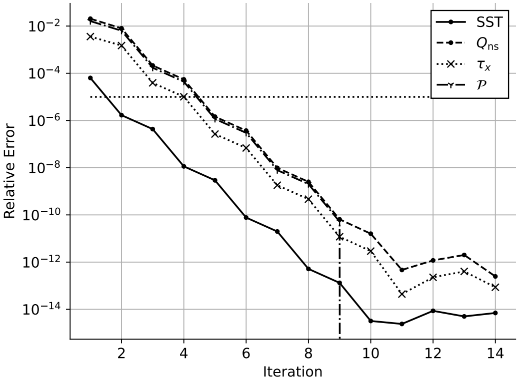


[1] Marti et al. [1], S. et al. [2]

Convergence Rates Matter

Question: How much error reduction is possible after two iterations?

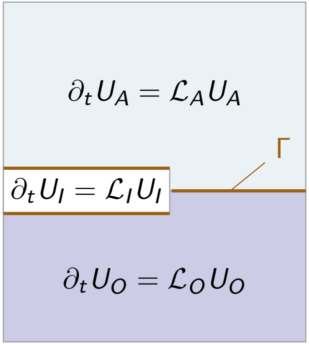
- Study SWR convergence factors ϱ
- Explain observed ϱ with idealized model
- Existing work: Blayo, Lemarié
  - Clément [3]: semi-discrete in space, for A-O coupling
  - Lozano [4]: continuous, for A-SI coupling



Parallel SWR in the EC-Earth SCM, $\mathcal{T} = 1\,\mathrm{h}$

A Case of Domain Decomposition

- Atmosphere-ocean-sea ice coupling: DD problem, three subdomains
- Ice covers a fraction of the interface Γ
- Sea ice presence affects interface boundary conditions
- We focus on thermodynamics: $U = T$



Bulk Interface Conditions: Motivation (Atmosphere)

- Vertical turbulence assumed to be a diffusive process:

$$\partial_t T_A = \cdots + \partial_z \left(k_A \partial_z T_A\right)$$

- Boundary condition on Γ necessary!
- If surface layer resolved: $T_A|_\Gamma = T_{sfc}$
- But: We only have T_A at first grid level $z_1 \approx 10\,\mathrm{m}$!

$$T_A(z_1) \neq T_{sfc}$$

- We need a different boundary condition!
  $\Rightarrow$ bulk interface conditions: based on law of the wall.

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Bulk Interface Conditions: Ice-Free Ocean

1. Flux proportional to temperature **jump**:

$$
\begin{aligned}
-k_A \partial_z T_A|_\Gamma &= C_{AO} \left(T_A|_\Gamma - T_O|_\Gamma \right) \\
\Leftrightarrow (k_A \partial_z T_A + C_{AO} T_A)|_\Gamma &= C_{AO} \; T_O|_\Gamma
\end{aligned}
$$

2. Flux continuous across the interface:

$$
k_O \partial_z T_O|_\Gamma = k_A \partial_z T_A|_\Gamma
$$

Ice-Free Setting: Schwarz Waveform Relaxation

Compute for $k = 1, 2, \ldots$

$$\begin{aligned}
\partial_t T_A^k - \alpha_A \partial_z^2 T_A^k &= f_A \quad \text{on } (0, \mathcal{T}] \times (0, H_A), \\
T_A^k(0, z) &= \vartheta_A(z), \\
\partial_z T_A^k(t, H_A) &= g_A(t), \\
\text{IBC } 1 &\text{ on } (0, \mathcal{T}] \times \Gamma,
\end{aligned} \tag{1a}$$

and

$$\begin{aligned}
\partial_t T_O^k - \alpha_O \partial_z^2 T_O^k &= f_O \quad \text{on } (0, T] \times (-H_O, 0), \\
T_O^k(0, z) &= \vartheta_O(z), \\
\partial_z T_O^k(t, -H_O) &= g_O(t), \\
\text{IBC } 2 &\text{ on } (0, \mathcal{T}] \times \Gamma.
\end{aligned} \tag{1b}$$

Comparing IBCs

I: Bulk interface conditions

$$
\begin{aligned}
&1.\quad \left(k_A \partial_z T_A^k + C_{AO} T_A^k\right)\Big|_\Gamma &&= C_{AO} T_O^{k-1}\Big|_\Gamma \\
&2.\quad k_O \partial_z T_O^k\Big|_\Gamma &&= k_A \partial_z T_A^k\Big|_\Gamma
\end{aligned}
$$

II: Dirichlet-Neumann (for comparison)

$$
\begin{aligned}
&1.\quad T_A^k\Big|_\Gamma &&= T_O^{k-1}\Big|_\Gamma \\
&2.\quad k_O \partial_z T_O^k\Big|_\Gamma &&= k_A \partial_z T_A^k\Big|_\Gamma
\end{aligned}
$$

Continuous Analysis Steps

1. Write (1) for the error in iteration k:

$$e^k(t, z) := T^k - T.$$

2. Apply Fourier transform in time, assuming $e^k = 0$ for $t \leq 0$:

$$\hat{e}^k(\omega, z) := \mathcal{F}\{e^k(t, z)\}.$$

3. Solve differential equations to obtain convergence factor:

$$\varrho(\omega) := \frac{\hat{e}^k\big|_\Gamma}{\hat{e}^{k-1}\big|_\Gamma}.$$

Continuous Analysis Results

We obtain the convergence factors:

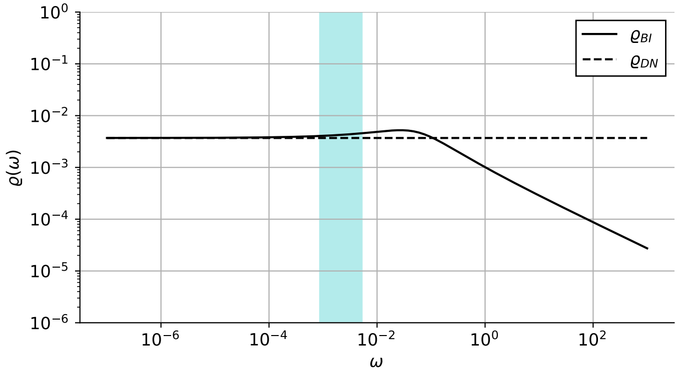
$$\varrho_{BI}(\omega) = \frac{k_A}{k_O}\sqrt{\frac{\alpha_O}{\alpha_A}}\left|\frac{\chi_A(\omega)}{\chi_O(\omega)}\right|\left|\frac{1}{1-\nu_A\sqrt{i\omega}\chi_A(\omega))}\right|,$$

$$\varrho_{DN}(\omega) = \frac{k_A}{k_O}\sqrt{\frac{\alpha_O}{\alpha_A}}\left|\frac{\chi_A(\omega)}{\chi_O(\omega)}\right|,$$

with $\chi_j(\omega) = \tanh\left(H_j\sqrt{\frac{i\omega}{\alpha_j}}\right) \approx 1$, $\nu_A = k_A/(C_{AO}\sqrt{\alpha_A})$.

Analysis Results: Continued

Assuming realistic material parameters for the atmosphere-ocean setting:



Shaded frequency band: $[\omega_{\min}, \omega_{\max}] = \left[\frac{\pi}{\Delta t_{cpl}}, \frac{\pi}{\Delta t_A}\right]$ (cf. Gander and Halpern [5]).

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Introducing Sea Ice

Toy model for sea ice: 0-layer model from Semtner [6]

- ODE for ice thickness $h(t)$
- Ocean sees constant ice bottom temperature $T_{I,b}$
- Atmosphere sees ice surface temperature, computed from energy balance

$$T_{I,s}^k = \min\left\{ \frac{SW_{net} + LW_{net} + C_{AI}\ T_A^k\big|_\Gamma + \frac{k_I}{h} T_{I,b}}{\frac{k_I}{h} + \epsilon B + C_{AI}}, 0\,^\circ\mathrm{C} \right\}.$$

Continuous Analysis: Ice-Covered Ocean

- Assume full ice cover:

$$-k_A \partial_z T_A^k \Big|_\Gamma = C_{AI} \left(T_A^k \Big|_\Gamma - T_{I,s}^{k-1} \right)$$

- Linearize: $\dot{h} \ll h, \; T_{I,s} < 0\,^\circ\mathrm{C}$
- Resulting convergence factor:

$$\varrho_{AI}(\omega) = \frac{C_{AI}}{k_I/h + \epsilon B + C_{AI}} \frac{1}{\left|1 - \nu_{AI}\sqrt{i\omega}\chi_A(\omega)\right|}$$

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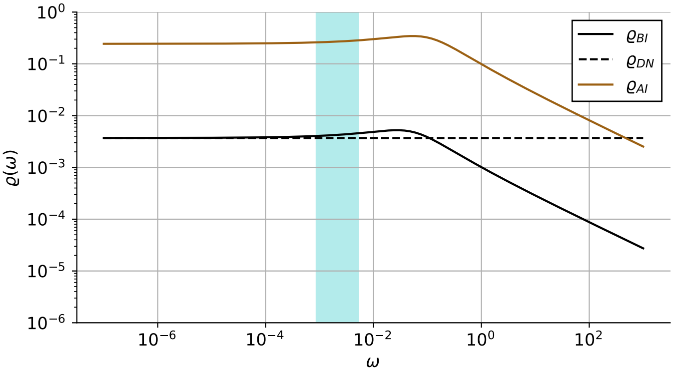
Analysis Results: Ice-Covered Ocean

Assuming realistic material parameters for the atmosphere-sea ice setting, $h = 1\,\mathrm{m}$:



Shaded frequency band: $[\omega_{\min}, \omega_{\max}] = \left[\frac{\pi}{\Delta t_{cpl}}, \frac{\pi}{\Delta t_A}\right]$.

Numerical Results

- Implementation “as we would do it in a climate model”
- Combined model:

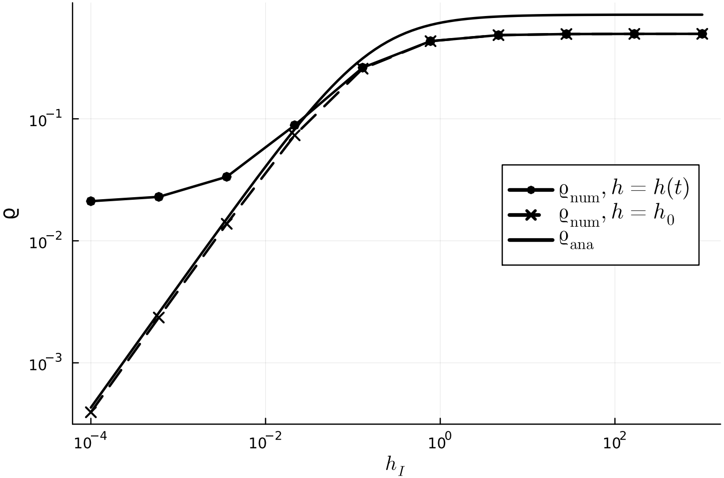
$$k_A \partial_z T_A|_\Gamma = a_I C_{AI}(T_A|_\Gamma - T_{I,s}) + (1 - a_I) C_{AO}(T_A|_\Gamma - T_O|_\Gamma))$$

- Code[2] based on `CliMA`[3] & `ClimaCoupler.jl`
  - ERK4 in time
  - Centered FD in space
  - Multirate: e.g., $\Delta t_A < \Delta t_{cpl}$
  - Coupling variables time-averaged over Δt_{cpl}

[2] Code available at: `https://github.com/valentinaschueller/clima-playground`

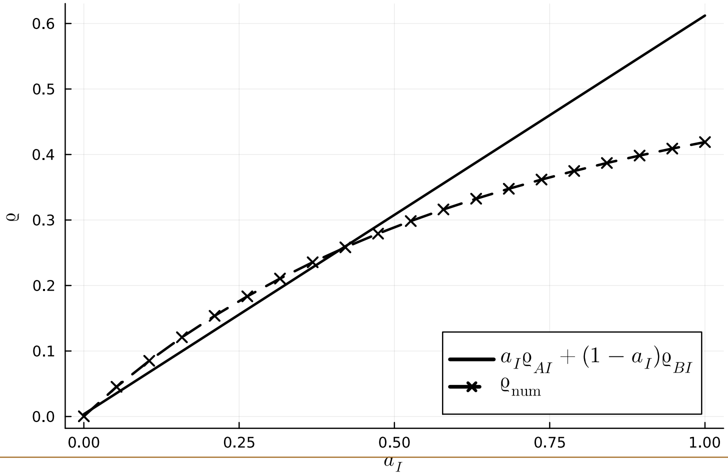
[3] `https://clima.caltech.edu/`

Numerical Results: Ice-Atmosphere Convergence



Numerical Results: Varying Sea Ice Cover

How good is the estimate $a_I \varrho_{AI} + (1 - a_I)\varrho_{BI}$?



Conclusion

- Idealized model combining work by Clément [3] and Lozano [4]
  - Fully continuous analysis for thermodynamic A-O-SI coupling
  - Allow for varying sea ice area fraction a_I
- Bulk interface conditions introduce an additional term in ϱ
  - $\Rightarrow$ Accelerates high frequency error decay
- Sea ice presence and thickness slow down convergence *nonlinearly*
- Continuous analysis overestimates numerical results, especially for large $\Delta t, \Delta z$
- But coarse grids are the norm in climate models!
  - $\Rightarrow$ Discrete analysis necessary here?

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Thank You!

References

[1] Olivier Marti et al. "A Schwarz Iterative Method to Evaluate Ocean–Atmosphere Coupling Schemes: Implementation and Diagnostics in IPSL-CM6-SW-VLR". In: *Geosci. Model Dev.* 14.5 (May 2021), pp. 2959–2975. DOI: 10.5194/gmd-14-2959-2021.

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[3] Simon Clément. "Numerical Analysis for a Combined Space-Time Discretization of Air-Sea Exchanges and Their Parameterizations". PhD thesis. Université Grenoble Alpes, 2022.

[4] Pierre Lozano. "Analysis and Optimization of Schwarz Algorithms for Ocean-Sea Ice-Atmosphere Coupling". MS Thesis. Université Grenoble Alpes, 2022.

[5] Martin J. Gander and Laurence Halpern. "Optimized Schwarz Waveform Relaxation Methods for Advection Reaction Diffusion Problems". In: *SIAM Journal on Numerical Analysis* 45.2 (2007), pp. 666–697. ISSN: 0036-1429.

[6] Albert J. Semtner. "A Model for the Thermodynamic Growth of Sea Ice in Numerical Investigations of Climate". In: *Journal of Physical Oceanography* (1976). ISSN: 1520-0485.