Analysis of Bulk Interface Conditions in Atmosphere-Ocean-Sea Ice Coupling

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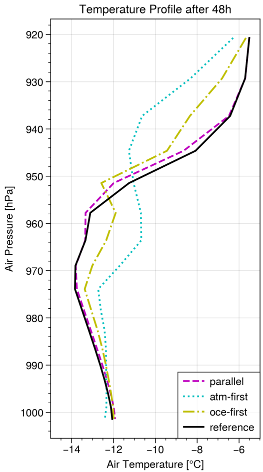
Coupling Errors in Climate Models

- Coupling methods in climate models: Schwarz Waveform Relaxation (SWR), stopped after one step
- Error w.r.t. converged SWR solution can be large[1]
- How can we reduce this error in practice?

⇒ Concrete advice to model developers!



[1] Marti et al. [1], S. et al. [2]

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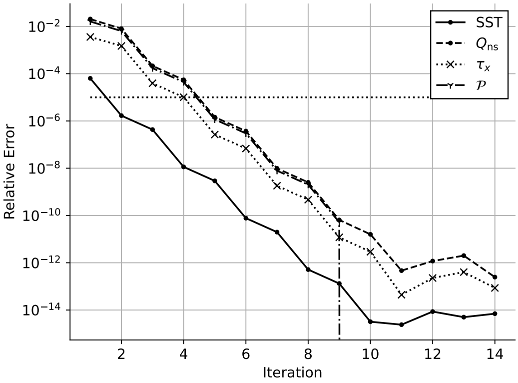
Convergence Rates Matter

Question: How much error reduction is possible after two iterations?

- Study SWR convergence factors ϱ
- Explain observed ϱ with idealized model
- Existing work: Blayo, Lemarié
  - Clément [3]: semi-discrete in space, for A-O coupling
  - Lozano [4]: continuous, for A-SI coupling

Our work: model + analysis + code for A-O-SI coupling.

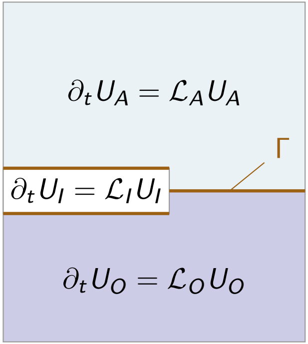


Parallel SWR in the EC-Earth SCM, $\mathcal{T} = 1\,\mathrm{h}$

A Case of Domain Decomposition

- Atmosphere-ocean-sea ice coupling:
  DD problem, three subdomains
- Sea ice covers fraction a_I of interface Γ
  $\Rightarrow$ affects interface boundary conditions (IBC)
- We focus on vertical turbulent thermodynamics: $U = T$
- Modeled as a diffusive process:

$$\partial_t T_j = \cdots + \alpha_j \partial_z^2 T_j, \quad j \in \{A, O\}$$



Schwarz Waveform Relaxation

Compute for $k = 1, 2, \ldots$

$$\left.\begin{aligned} \partial_t T_A^k - \alpha_A \partial_z^2 T_A^k &= f_A \quad \text{on } (0, \mathcal{T}] \times (0, H_A), \\ T_A^k(0, z) &= \vartheta_A(z), \\ \partial_z T_A^k(t, H_A) &= g_A(t), \\ \text{IBC } 1 \text{ on } &(0, \mathcal{T}] \times \Gamma, \end{aligned}\right. \tag{1a}$$

and

$$\left.\begin{aligned} \partial_t T_O^k - \alpha_O \partial_z^2 T_O^k &= f_O \quad \text{on } (0, T] \times (-H_O, 0), \\ T_O^k(0, z) &= \vartheta_O(z), \\ \partial_z T_O^k(t, -H_O) &= g_O(t), \\ \text{IBC } 2 \text{ on } &(0, \mathcal{T}] \times \Gamma. \end{aligned}\right. \tag{1b}$$

Comparing Interface Boundary Conditions ($a_I = 0$)

Bulk interface conditions — standard in climate models

1. Flux $\propto$ temperature **jump** (law of the wall):

$$
\begin{aligned}
-k_A \partial_z T_A^k \Big|_\Gamma &= C_{AO} \left(T_A^k \Big|_\Gamma - T_O^{k-1} \Big|_\Gamma \right) \\
\Leftrightarrow \left(k_A \partial_z T_A^k + C_{AO} T_A^k \right) \Big|_\Gamma &= C_{AO}\ T_O^{k-1} \Big|_\Gamma
\end{aligned}
$$

2. Flux continuous across Γ:

$$
k_O \partial_z T_O^k \Big|_\Gamma = k_A \partial_z T_A^k \Big|_\Gamma
$$

For comparison: Dirichlet-Neumann (DN)

$$
1.\ \ T_A^k \Big|_\Gamma = T_O^{k-1} \Big|_\Gamma \qquad 2.\ \ k_O \partial_z T_O^k \Big|_\Gamma = k_A \partial_z T_A^k \Big|_\Gamma
$$

Continuous Analysis Steps

1. Write coupling iteration (1) for the error in iteration k:

$$e^k(t, z) := T^k - T.$$

2. Apply Fourier transform in time, assuming $e^k = 0$ for $t \leq 0$:

$$\hat{e}^k(\omega, z) := \mathcal{F}\{e^k(t, z)\}.$$

3. Solve differential equations to obtain convergence factor:

$$\varrho(\omega) := \frac{\hat{e}^k\big|_\Gamma}{\hat{e}^{k-1}\big|_\Gamma}.$$

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Continuous Analysis Results ($a_I = 0$)

We obtain the convergence factor:

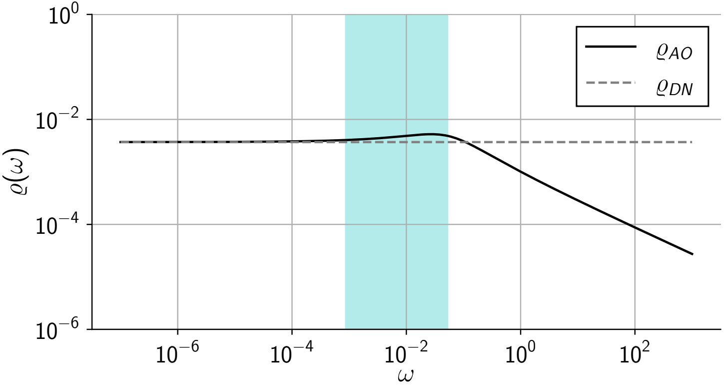
$$\varrho_{AO}(\omega) = \frac{k_A}{k_O}\sqrt{\frac{\alpha_O}{\alpha_A}}\left|\frac{\chi_A(\omega)}{\chi_O(\omega)}\right|\left|\frac{1}{1-\nu_A\sqrt{i\omega}\chi_A(\omega))}\right|,$$

$$\varrho_{DN}(\omega) = \frac{k_A}{k_O}\sqrt{\frac{\alpha_O}{\alpha_A}}\left|\frac{\chi_A(\omega)}{\chi_O(\omega)}\right|,$$

with $\chi_j(\omega) = \tanh\left(H_j\sqrt{\frac{i\omega}{\alpha_j}}\right) \approx 1$, $\nu_A = k_A/(C_{AO}\sqrt{\alpha_A})$.

Analysis Results: Continued ($a_l = 0$)

Assuming realistic material parameters for the atmosphere-ocean setting:



Shaded frequency band: $[\omega_{\min}, \omega_{\max}] = \left[\frac{\pi}{\Delta t_{cpl}}, \frac{\pi}{\Delta t_A}\right]$ (cf. Gander and Halpern [5]).

Introducing Sea Ice ($a_I = 1$)

Toy model for sea ice: 0-layer model from Semtner [6]

- ODE for ice thickness $h(t)$
- Ocean sees constant ice bottom temperature $T_{I,b}$
- Atmosphere sees ice surface temperature, computed from energy balance

$$T_{I,s}^k = \min\left\{ \frac{SW_{net} + LW_{net} + C_{AI}\; T_A^k\big|_\Gamma + \frac{k_I}{h} T_{I,b}}{\frac{k_I}{h} + B + C_{AI}}, 0\,^\circ\mathrm{C} \right\}.$$

Continuous Analysis: Ice-Covered Ocean ($a_I = 1$)

- Assume full ice cover:

$$-k_A \partial_z T_A^k \Big|_\Gamma = C_{AI} \left(T_A^k \Big|_\Gamma - T_{I,s}^{k-1} \right)$$

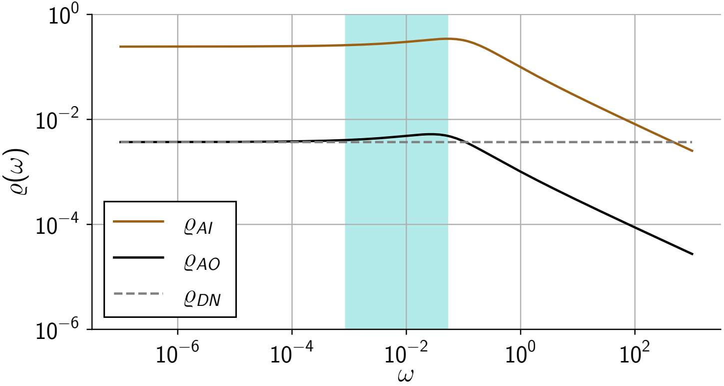
- Linearize: $\dot{h} \ll h,\ T_{I,s} < 0\,^\circ\mathrm{C}$
- Resulting convergence factor:

$$\varrho_{AI}(\omega) = \frac{\zeta}{\left|1 - \nu_{AI}\sqrt{i\omega}\chi_A(\omega)\right|},$$

with $\zeta := C_{AI}/\left(k_I/h + B + C_{AI}\right)$.

Analysis Results: Ice-Covered Ocean ($a_I = 1$)

Assuming realistic material parameters for the atmosphere-sea ice setting, $h = 1\,\mathrm{m}$:



Shaded frequency band: $[\omega_{\min}, \omega_{\max}] = \left[\frac{\pi}{\Delta t_{cpl}}, \frac{\pi}{\Delta t_A}\right]$.

Analysis Results: Combined Model, $a_I \in [0, 1]$

Linear combination of boundary conditions

$$k_A \partial_z T_A|_\Gamma = a_I C_{AI}(T_A|_\Gamma - T_{I,s}) + (1 - a_I) C_{AO}(T_A|_\Gamma - T_O|_\Gamma))$$

yields nonlinear combination of convergence factors:

$$\varrho(\omega) = \left| \frac{a_I C_{AI} \zeta + (1 - a_I)^2 C_{AO} \frac{\frac{k_A}{\sqrt{\alpha_A}} \sqrt{i\omega} \chi_A(\omega)}{a_I C_{IO} - \frac{k_O}{\sqrt{\alpha_O}} \sqrt{i\omega} \chi_O(\omega)}}{\frac{k_A}{\sqrt{\alpha_A}} \sqrt{i\omega} \chi_A(\omega) - a_I C_{AI} - (1 - a_I) C_{AO}} \right|, \tag{2}$$

with

$$\lim_{a_I \to 0} \varrho = \varrho_{AO}, \quad \lim_{a_I \to 1} \varrho = \varrho_{AI}.$$

Numerical Results

- Compare continuous analysis with discretized model
- Implementation "as we would do it in a climate model"
- Code[2] based on `CliMA`[3] & `ClimaCoupler.jl`
  - Implicit Euler in time
  - Centered FD in space
  - Multirate: e.g., $\Delta t_A < \Delta t_{cpl}$
  - Coupling variables time-averaged over Δt_{cpl}

Discretization parameters

Δt_A	$60\,\mathrm{s}$
Δt_O	$600\,\mathrm{s}$
Δt_I	$600\,\mathrm{s}$
Δt_{cpl}	$3600\,\mathrm{s}$
$\mathcal{T}$	$3600\,\mathrm{s}$
Δz_A	$1\,\mathrm{m}$
Δz_O	$1\,\mathrm{m}$

[2] Code available at: `https://github.com/valentinaschueller/clima-playground`

[3] `https://clima.caltech.edu/`

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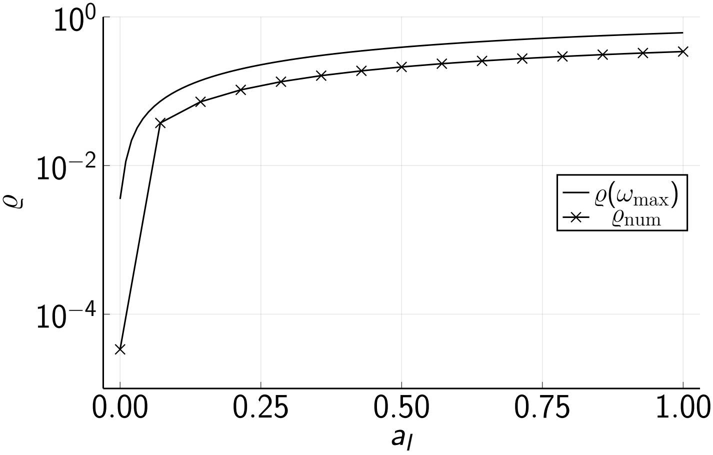
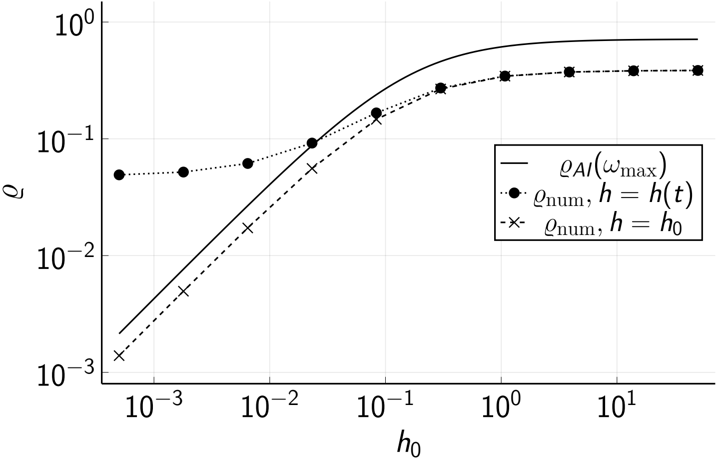
Numerical Results: Varying Sea Ice Cover



Numerical Results: Ice Thickness Dependence



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Summary & Conclusion

- Idealized model combining work by Clément [3] and Lozano [4]
  - Fully continuous analysis for thermodynamic A-O-SI coupling
  - Allow for varying sea ice area fraction a_I
- Bulk interface conditions introduce an additional term in ϱ

  $\Rightarrow$ Accelerates high frequency error decay
- Sea ice presence and thickness slow down convergence *nonlinearly*
- Continuous analysis overestimates numerical results, especially for large $\Delta t, \Delta z$
- But coarse grids are the norm in climate models!

  $\Rightarrow$ Discrete analysis necessary here?

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Thank You!

References

[1] Olivier Marti et al. “A Schwarz Iterative Method to Evaluate Ocean–Atmosphere Coupling Schemes: Implementation and Diagnostics in IPSL-CM6-SW-VLR”. In: *Geosci. Model Dev.* 14.5 (May 2021), pp. 2959–2975. DOI: 10.5194/gmd-14-2959-2021.

[2] V. S. et al. “Quantifying Coupling Errors in Atmosphere-Ocean-Sea Ice Models: A Study of Iterative and Non-Iterative Approaches in the EC-Earth AOSCM”. In: *EGUsphere* (2025), pp. 1–32. DOI: 10.5194/egusphere-2025-1342.

[3] Simon Clément. “Numerical Analysis for a Combined Space-Time Discretization of Air-Sea Exchanges and Their Parameterizations”. PhD thesis. Université Grenoble Alpes, 2022.

[4] Pierre Lozano. “Analysis and Optimization of Schwarz Algorithms for Ocean-Sea Ice-Atmosphere Coupling”. MS Thesis. Université Grenoble Alpes, 2022.

[5] Martin J. Gander and Laurence Halpern. “Optimized Schwarz Waveform Relaxation Methods for Advection Reaction Diffusion Problems”. In: *SIAM Journal on Numerical Analysis* 45.2 (2007), pp. 666–697. ISSN: 0036-1429.

[6] Albert J. Semtner. “A Model for the Thermodynamic Growth of Sea Ice in Numerical Investigations of Climate”. In: *Journal of Physical Oceanography* (1976). ISSN: 1520-0485.