Discrete Valuation Rings and Nonsingular Points

Valentin Boboc

University of Nottingham

pmxvb2@nottingham.ac.uk

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Overview

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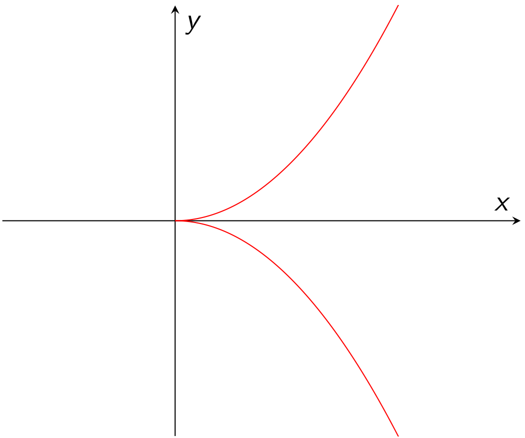
Motivation

When considering a geometric object, such as a curve, a conic or, more generally, any variety or scheme, one of the problems we are interested in is how well-behaved the points on these objects are.

A singular point is a badly behaved point. A precise definition of singularity depends heavily on the type of curve or objects we study.

A Classical Example

A typical example of a singular point is a cusp, shown in the figure below:



A Criterion for Singularity

The Jacobian Criterion

Let $X \subset \mathbb{A}^n$ be an affine algebraic variety which is generated by an ideal $I = (f_1, f_2, \cdots, f_s)$. Let $p \in X$ a point on X. Then X is nonsingular at the point p if the rank of the Jacobian matrix, $(\frac{\partial f_i}{\partial x_j}(p))_{i,j}$ has rank equal to $n - dimX$.

This is very useful for practical computations, but it is heavily dependant on choice of embedding. The main aim of this talk is to find an algebraic criterion for nonsingularity.

Discrete Valuations

Definition

We define **a discrete valuation** on a field F to be a function $\nu : F \backslash \{0\} \rightarrow \mathbb{Z}$ which satisfies the following properties for all $a, b \in F$

1. ν is surjective
2. $\nu(ab) = \nu(a) + \nu(b)$
3. $\nu(a+b) \geq min\{\nu(a), \nu(b)\}$

Example (The p-adic valuation)

Consider the valuation $\nu_p : \mathbb{Q}^x \rightarrow \mathbb{Z}$ for some prime integer p. We can write all rational numbers in the form $p^j \frac{a}{b}$ for $a, b \in \mathbb{Z}$ not divisible by p. Then the p-adic valuation is given by:

$$\nu_p(p^j \tfrac{a}{b}) = j$$

Valuation Rings

Definition

We call $R = \{0\} \cup \{r \in F \backslash \{0\} : \nu(r) \geq 0\}$ **the valuation ring of** ν.

Definition

An integral domain is a **discrete valuation ring** (DVR) if it is the valution ring of a discrete valuation of the domain's quotient field.

Example (The p-adic valuation)

We can associate to the p-adic valuation the following DVR:

$$\mathbb{Z}_{(p)} = \{\tfrac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z},\ p \nmid b\}$$

Valuations on $\mathbb{Q}$

Proposition

All discrete valuations on $\mathbb{Q}$ are p-adic valuations.

Proof. Let $val : \mathbb{Q}^x \to \mathbb{Z}$ be any discrete valuation.
Using the first property of the discrete valuation, we get that:

$$val(1) = val(1 \cdot 1) = val(1) + val(1)$$
$$So\ val(1) = 0$$

In a similar fashion, we note that $val(-1) = 0$.

Valuations on $\mathbb{Q}$ (cont)

Proof (cont).
Now consider the DVR associated to *val*:

$$P = \{n \in \mathbb{Z} : val(n) \geq 0\}$$

Let $n \in \mathbb{Z}$. Then $val(n) \geq 0$ for all n since $val(n) = val(1 + 1 + ... + 1) \geq 0$ using another property of discrete valuations.
Notice that $P = (p)$ is in fact a prime ideal of $\mathbb{Z}$ for some prime integer p.
Then consider an element in P of the form $p^j k$ such that $(p, k) = 1$.
Then $val(p^j k) = val(p) + ... + val(p) + val(k) = j \; val(p)$.
This is a multiple of j. We can extend this to fractions and obtain that *val* is just a multiple of the p-adic valuation. □

Some preliminary definitions

Definition

Let A be a Noetherian local ring and let $\mathfrak{m}$ be its unique maximal ideal. Let $k = A/\mathfrak{m}$ be the residue field of A. Then we say that A is **a regular local ring** if

$$dim_k \mathfrak{m}/\mathfrak{m}^2 = dim A$$

where the first is the dimension of $\mathfrak{m}/\mathfrak{m}^2$ (*the Zariski cotangent space*) as a k-vector space and the second is the Krull dimension of the ring A.

Definition

A discrete valuation ring is a regular local ring of dimension one.

The Main Theorem

Here we state the main result of this presentation, an algebraic criterion for being nonsingular.

Theorem

Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $p \in X$ be a point on this variety. Then X is nonsingular at p if and only if $\mathcal{O}_{X,p}$ is a regular local ring.

NB $\mathcal{O}_{X,p}$ is called the **stalk** of the **structure sheaf** of X at the point p. We will provide a definition later in the proof and in the subsequent examples.

The proof will aim to show that the condition of this theorem is equivalent to the Jacobian criterion.

Proof of the Main Theorem (1)

Proof. We will sketch the proof in 4 steps.

Step 1: A useful isomorphism
Let $p \in \mathbb{A}^n$ be the following point: $p = (a_1, a_2, \cdots, a_n)$.
The maximal ideal that corresponds to this point is:

$\mathfrak{a} = (x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$ in the ring $k[x_1, ..., x_n]$ (k a closed field)

The map $\phi : k[x_1, ..., x_n] \to k^n$ with $\phi(f) = (\frac{\partial f}{\partial x_1}(p), ..., \frac{\partial f}{\partial x_n}(p))$ will induce an isomorphism:

$$\mathfrak{a}/\mathfrak{a}^2 \simeq k^n$$

since the set $\{\phi(x_i - a_i) : i = 1, ..., n\}$ is a basis for k^n.

Proof of the Main Theorem (2)

Step 2: Look at the Jacobian criterion

Let $\mathfrak{b}$ be the ideal of X in $k[x_1, ..., x_n]$.

$$\mathfrak{b} = (f_1, f_2, ..., f_t) \text{ for } t \text{ polynomials } f_i \in k[x_1, ..., x_n].$$

The Jacobian matrix will have the form $J = \frac{\partial f_i}{\partial x_j}$ will be

$$\begin{aligned} rank(J) &= dim\ \phi(\mathfrak{b})\ with\ \phi(\mathfrak{b}) \subset k^n \\ &= dim\ (\mathfrak{b} + \mathfrak{a}^2)/\mathfrak{a}^2 \end{aligned}$$

where we read $\phi(\mathfrak{b})$ in $\mathfrak{a}/\mathfrak{a}^2$ using the isomorphism on the previous page.

Proof of the Main Theorem (3)

Step 3: Look at the structure sheaf

The stalk of the structure sheaf at p is

$$\mathcal{O}_{X,p} = (k[x_1, ..., x_n]/\mathfrak{b})_{(\mathfrak{a})}$$

As this is a localisation, $\mathcal{O}_{X,p}$ is a **local ring** and we can compute its unique maximal ideal:

$$\mathfrak{m} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \text{ and we have } \mathfrak{m}^2 = (\mathfrak{a}^2 + \mathfrak{b})/\mathfrak{b}$$

Now we can compute the **Zariski cotangent space**:

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{a}/(\mathfrak{a}^2 + \mathfrak{b})$$

Proof of the Main Theorem (4)

Step 4: Putting it all together

We notice the following fact:

$$\mathit{dim}\ (\mathfrak{b}+\mathfrak{a}^2)/\mathfrak{a}^2 + \mathit{dim}\ (\mathfrak{a}/(\mathfrak{a}^2+\mathfrak{b})) = \mathit{dim}\ (\mathfrak{a}/\mathfrak{a}^2)$$
$$\Leftrightarrow \mathit{rank}\ (J) + \mathit{dim}\ (\mathfrak{m}/\mathfrak{m}^2) = n$$

Let $\mathit{dim}(X) = r$.

Then p is nonsingular $\Longleftrightarrow \mathit{rank}(J) = n - r$ (Jacobian criterion)
$\Longleftrightarrow \mathit{dim}_k(\mathfrak{m}/\mathfrak{m}^2) = r$ (by relation above)
$\Longleftrightarrow \mathcal{O}_{X,p}$ is a regular local ring. $\square$

Worked Example 1

Example (Parabola)

Let $X = \mathbb{V}(y - x^2) \subset \mathbb{A}^2_{\mathbb{C}}$ be a parabola. We can show that all points on X are nonsingular.

Let p be any point on X, that is $p = (a, a^2)$.
Look at the structure sheaf:

$$\begin{aligned}\mathcal{O}_{X,p} &\simeq (\mathbb{C}[x,y]/(y-x^2))_{(x-a,y-a^2)}\\ &\simeq \mathbb{C}[x]_{(x-a,x^2-a^2)}\\ &\simeq \mathbb{C}[x]_{(x-a)}\end{aligned}$$

This is a regular local ring (i.e. a DVR), so every point $p \in X$ is nonsingular.

Worked Example 2

Example

Let $Y = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2_{\mathbb{C}}$ be an affine algebraic variety. We can show that the point $p = (0, 0) \in Y$ is a **singular point**.

We compute $\mathcal{O}_{Y,p} = (\mathbb{C}[x, y]/(y^2 - x^3))_{(x,y)}$.

Since $y^2 - x^3$ is not a zero divisor, we can use the following result from Atiyah's *Introduction to Commutative Algebra*:

$$\begin{aligned} dim\ \mathbb{C}[x, y]/(y^2 - x^3)_{(x,y)} &= dim\ \mathbb{C}[x, y]_{(x,y)} - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

The cotangent space $\mathfrak{m}/\mathfrak{m}^2$ has basis $\{x, y\}$ as a $\mathbb{C}$-vector space, and hence has dimension 2 as a $\mathbb{C}$-vector space. As these dimensions don't agree, $\mathcal{O}_{Y,p}$ is a local ring which is not regular, therefore $p \in Y$ is a singular point.

References

- Ravi Vakil's "Introduction to Algebraic Geometry" Notes
- M.F. Atiyah I.G. MacDonald, "Introduction to Commutative Algebra"
- Stacks Project

The End