Smoothed particle hydrodynamics method (SPH)

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Abstract

The SPH method is a meshless numerical method used mostly in the astrophysics field. These researches have been done in order to apply the SPH method to other cases that can't be modelled through a regular mesh method.

Keywords : Smoothed particle hydrodynamics, Ghost particle, Cylindrical system, Kernel function

Introduction

Smoothed particle hydrodynamics is a method created for obtaining numerical solutions of equations by using a set of particles which are interpolation points. For the physicist or the engineer, the SPH particles are material particles which can be treated like any other particles system. The SPH method has many advantages. Problems with more than one material are often trivial for the SPH method because each material is described by its own set of particles. In addition to that, SPH can be used for many types of simulation such as huge deformation simulation, penetration simulation, machining simulation because, unlike the Finite Element method, the elements (particles) of the SPH method can be separated. This method is also used for fluid and astrophysics simulations. Unlike the Eulerian method, SPH allows having mass conservation and pressures without extra computation. The masses are contained in each particle and the pressures are given by the mathematical link between neighboring particles instead of solving linear equations. Besides, for Eulerian simulation method, we need boundary conditions which may be unknown (like stars), a problem that the SPH method does not have. However, the perfect simulation method does not exist. The SPH method still have some drawbacks, the computing time, the consistence at the boundaries (one of the issues we solved in this paper), the tensile instability and the spurious energy mode. This project is about performing researches about the SPH method, learning how it works mathematically speaking and give the mathematical parameters expressions in order to be able to run a simulation afterwards.

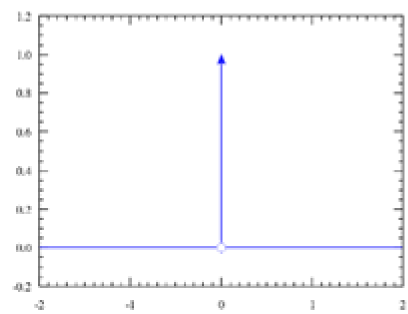
1 Explanation of the SPH method

1.1 Researches about the SPH method and its implementation

1.1.1 Researches about the SPH method

Dirac function

The Dirac function δ is equal to zero everywhere except at zero where the function is infinitely high at the spike's location, infinitely thin and an integral under the spike equal to one.



$$\int_{-\infty}^{+\infty} \delta(t)dt = 1 \tag{1}$$

Convolution product [6]

If we define a function f which is continuous (or not) we can write that :

$$f_g(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt \tag{2}$$

According to the convolution product properties :

Commutativity

$$(f * g)(x) \stackrel{def}{=} (g * f)(x) \tag{3}$$

Distribuvity

$$(f*(g+h))(x) \overset{def}{=} (f*g)(x)+(f*h)(x) \tag{4}$$

Associativity

$$((f*g)*h)(x) \overset{def}{=} (f*(g*h))(x) \tag{5}$$

If we set the function g as the Dirac function δ. According to the commutativity property of the convolution product ,if f is continuous (or not), knowing that the Dirac function is continuous, the convolution product will be continuous :

$$\int_{-\infty}^{+\infty} f(t)\delta(x-t)dt = \int_{-\infty}^{+\infty} \delta(t)f(x-t)dt \tag{6}$$

If the functions δ and f are continuous, we can apply the Mean Value Theorem which implies the existance of c, with $c \in [-\epsilon\ ;\ +\epsilon]$ such that :

$$\int_{a}^{b} f(t)dt = f(c)(b-a) \tag{7}$$

The point $f(c)$ is called the average value of $f(x)$ on $[a,b]$. Second Mean Value Theorem for Integrals: Let $f(x)$ and $g(x)$ be continuous on $[a,b]$. Assuming that $g(x)$ is positive, that is to say $g(x) \geq 0$ for any $x \in [a,b]$. Then it exists c $\in [a,b]$ such that [5]:

$$\int_{a}^{b} f(t)g(t)dt = f(c)\int_{a}^{b} g(t)dt \tag{8}$$

Thus in our case,

$$\int_{-\infty}^{+\infty} \delta(t)f(x-t)dt = f(x-c)\int_{-\infty}^{+\infty} \delta(t)dt \tag{9}$$

We set -ϵ and ϵ two values around the spike of the Dirac function

$$f(x-c)\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\infty}^{-\epsilon} \delta(t)f(x-t)dt = 0 \tag{10}$$

Thus, knowing that

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} \delta(t)dt = 1 \tag{11}$$

We have,

$$f(x-c) = \int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\epsilon}^{+\epsilon} \delta(t)f(x-t)dt \tag{12}$$

with c $\in (-\epsilon\ ;\ +\epsilon)$

Because the Dirac function is defined as infinitly thin we can write :

$$\lim_{\epsilon\to 0} f(x-c)\int_{-\epsilon}^{+\epsilon} \delta(t)dt = f(x) \approx \int_{-\infty}^{+\infty} \delta(t)f(x-t)dt \tag{13}$$

$$= \int_{-\infty}^{+\infty} f(t)\delta(x-t)dt \tag{14}$$

$$\Rightarrow f(x) = \int_{-\infty}^{+\infty} f(t)\delta(x-t)dt \tag{15}$$

Then each function $f(x)$ can be written as an integral. Even if the function f is not derivable we can calculate the derivative of the integral form.
However, the Dirac function is not a function because any function that is equal to zero everywhere but has a single point must have total integral zero. That is why we have to approximate the Dirac function by another function. The Gauss function has a shape which is quite similar to the Dirac function. However, we have to create an approximation of the Dirac function such that the function is very thin and its integral equal to 1. That is why we choose to create a polynomial function, the cubic spline function W(r,h).

Definition of the polynomial function W(r,h)

The shape function which is used is the cubic spline kernel's function. It is given by the formulation [2]:

$$W(r,h) = \frac{\alpha}{h^2}\begin{cases} 1-\frac{3}{2}q^2+\frac{3}{4}q^3 & \text{if } 0 \leqslant q < 1, \\ \frac{1}{4}(2-q)^3 & \text{if } 1 \leqslant q \leqslant 2, \\ 0 & \text{if } q > 2 \end{cases} \tag{16}$$

This kernel function must reach the condition of convergence, that is to say :

$$\int W(\vec{x})d\vec{x} = 1 \tag{17}$$

q is given by the formulation :

$$q = \frac{|r_j - r_i|}{h} = -\frac{\sqrt{r_{ij}}^2}{h^2} \tag{18}$$

The variable r represents the position of the particules and h represents the distance between two particules (the step). i and j are two neighboring interpolation points in a cartesian coordinate system that is mean :

All along this paper, whatever the variable, we set :

$$\alpha_{ij} = \alpha_i - \alpha_j \tag{19}$$

$$r_{ij} = \sqrt{(x_{ij})^2 + (y_{ij})^2 + (z_{ij})^2} \tag{20}$$

Consequently, we can write :
$\forall x \in \mathbb{R}$,

$$\lim_{h\to 0} W(r,h) = \delta(x) \Rightarrow W(r,h) \approx \delta(x) \tag{21}$$

Thus,

$$f(x) \approx \int f(\tau)W(x-\tau)d\tau \tag{22}$$

In order to code this approximation, we need a discrete system instead of a continuous system. To do that, we use the Monte Carlo method [8].

Thus,

$$\int f(\tau)W(x-\tau)d\tau \approx \sum_f f_j W_{ij} \frac{m_j}{\rho_j} \tag{23}$$

With ρ_j the density and m_j the mass of the particle j. We now have the approximation of any function $f(x)$. In fact, the approximation of a function is less important than its first derivative. To find the approximation of the first derivative, we use the Green's theorem [5].

$$\mathbf{d}\omega = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial \omega_i}{\partial q^i} dq^1 \wedge ... \wedge dq^i \wedge ... \wedge dq^n \tag{24}$$

If the distance between the interpolation point r_i and the border of the domain is greater than the "radius" of the kernel function, it is accurate to write that:

$$\nabla f_i(r) = -\int_\Omega f(r')\nabla_i W(r-r'),h)dr' \tag{25}$$

Then,

$$\nabla f_i(r) = \sum_{j=1}^{n} f_j \frac{m_j}{\rho_j} \nabla_j W(r_{ij},h) \tag{26}$$

1.2 Boundary conditions

1.2.1 The existing methods to simulate a boundary condition

As we said in the introduction, the SPH method has problems at the boundaries. At each time step, the particles at the boundaries accelerate and generate an unstable system. At each time step, this error which is due to the numerical method, will propagate through the system from the boundaries to the center of the system. That is the reason why, it is necessary to implement boundary conditions on particules which are located near the boundaries. When a particle is located next to the limit of the domain, the kernel function W is not complete because the set of interpolation points is not. Two cases are possible [6]:

The domain's limit is a wall (fixed or mobile) with free or imposed movements.

In this case, one condition of non-penetration and one friction condition are imposed. There are some solutions to impose these conditions, for example:
- The Monaghan's one which is based on the Lennard's model
- The Morris border particles
- The Takeda border particles

These techniques are more or less efficient and the wall boundaries are generally not that developed for the SPH method. We will develop this kind of boundary condition in order to implement it, more precisely the technique of the Ghost particle.

The domains limit is a free surface boundary condition.

Dynamics and kinematics conditions are imposed.

1.2.2 The equation of conservation

Continuous form

In this paper we assume that the fluid's viscosity that we consider is negligible compared to the pressures. The Navier-Stokes equations are [11]:

$$\frac{d\vec{x}}{dt} = \vec{v} \tag{27}$$

$$\frac{\rho}{dt} = -\rho\nabla\vec{v} \tag{28}$$

$$\frac{d\vec{v}}{dt} = \vec{g} + \frac{\nabla\sigma}{\rho} \tag{29}$$

avec

$$\sigma_{ij} = \left(-P - \frac{2\mu}{3} d_{dkk}\right)\delta_{ij} + 2\mu d_{ij} \tag{30}$$

et

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{31}$$

As we said, the viscosity is neglected, thus the Navier-Stokes equations become the Euler equations :

$$\frac{d\vec{x}}{dt} = \vec{v} \tag{32}$$

$$\frac{\vec{v}}{dt} = \vec{g} - \frac{\nabla P}{\rho} \tag{33}$$

$$\frac{\rho}{dt} = -\rho\nabla\vec{v} \tag{34}$$

Because we suppose that the fluid is compressible we have to add an equation that takes into account this compressibility. Many state laws exist in the SPH method, there are chosen depending on the case which is studied (astrophysic, gas dynamic, ...). The equation of state can be as complicated as desired, as an example: Benz et al (1986, 1987) who used equations of state for metals and minerals, including phase changes (formation of the Moon). These equations link the density (sometimes the internal energy) with the pressure of the system which is studied. Below, an example of the state law equation for a free surface flow [5]:

$$P - P_0 = \frac{\rho_0 c_0^2}{\gamma}\left[\left(\frac{\gamma}{\gamma_0}\right)^{\gamma} - 1\right] \tag{35}$$

with γ : the polytrophic index, ρ_0 : the nominal density of the fluid, P_0 : the referential pressure and c_0 : the nominal sound's velocity within the fluid.

As far as the polytrophic index's concerned, it is equal to 1,4 for a perfect gas and 7 for water (incompressible but assumed slightly compressible). SPH method is explicit. When we use the equation of state, we avoid solving the Poisson equation (linear system).

The value of the nominal sounds velocity, is not the same that the real one (we generally take lower values). It is a particularity of the SPH method since we assumed that the system is a compressible problem with a fluid which apparently is not compressible. The compressibility of the numerical simulation would not be accurate for an acoustic study. However, as long as the number of Mach $Ma < 0,1$ the numerical solution is accurate if we study the dynamic of the fluid. We generally take lower values of the sounds velocity because the time step is linked to the CFL stability condition (Courant-Friedrich-Levy). If we study the acoustic of a system we have to follow the rule: The information transmission velocity is equal to the sounds velocity. However, if the velocity is too high, it could create instabilities.

It is important to underline the fact that for the modelling of structures, the sounds velocity cannot be arbitrarily chosen since it is linked to the elastic properties of the material (Young modulus). However, this paper is more about the method which is sometimes called CSPH, short for Compressible Smoothed Particle Hydrodynamic. Due to the low convergence problem of the ISPH, short for Incompressible Smoothed Particle Hydrodynamic, many researches are currently being worked on. An inversion of linear system occurs in the ISPH method, so the analogy is more difficult to make than for the CSPH method and its analogy with the Finite Volume method.

Until now, we considered that the particles are interpolation points, however each particle has a weight and a volume. This volume is not a constant and has to follow the equation

$$\frac{d\omega}{dt} = \omega\nabla.\vec{v} \tag{36}$$

Since we have the continuity equation in our system, we have by combination

$$\frac{d\omega\rho}{dt} = 0 \tag{37}$$

We have the notion of interpolation points but also the notion of particles method since each interpolation point has a weight and a volume. Then each particle has discrete values such as speed, volume and pressure. These values are supposed homogenously distributed in the discrete volume of the particle.

Discrete form [2]

In order to keep a symmetry between the neighbor particles i and j in the momentum conservation equation and the continuity equation we have to express the equations in a different way. Otherwise, we would have an inequality problem. Normally, we should have:

$$\vec{F_{i\to j}} = -\vec{F_{j\to i}} \tag{38}$$

with

$$\vec{F_{j\to i}} = -m_i\frac{1}{\rho_i}P_j\nabla W(\vec{x_{ij}})\omega_j \tag{39}$$

But if we directly do the discretization we wouldnt have the correct equalities. We would have:

$$-m_i\frac{1}{\rho_i}P_j\nabla W(\vec{x_{ij}})\omega_j \neq -m_j\frac{1}{\rho_i}P_j\nabla W(\vec{x_{ij}})\omega_i \tag{40}$$

$$\vec{v_i}\nabla W(\vec{x_{ji}})\omega_i\omega_j \neq \vec{v_j}.\nabla W(\vec{x_{ij}})\omega_j\omega_i \tag{41}$$

Many ways exist to keep this symmetry if we use the way to write the equation:

$$\frac{\vec{\nabla}P}{\rho} = \frac{P}{\rho^{\sigma}}\vec{\nabla}\left(\frac{1}{\rho^{1-\sigma}}\right) + \rho^{\sigma-2}\vec{\nabla}\left(\frac{P}{\rho^{\sigma-1}}\right) \tag{42}$$

$$\rho\vec{\nabla}\vec{v} = \frac{\vec{\nabla}(\rho^{\sigma-1}\vec{v}) - \vec{v}\vec{\nabla}\rho^{\sigma-1}}{\rho^{\sigma-2}} \tag{43}$$

σ can be chosen such as $\sigma\epsilon\mathbb{R}$.

Then, we can deduce the discrete form of the Euler equations

$$\frac{d\vec{x_i}}{dt} = \vec{v_i} \tag{44}$$

$$\frac{d\rho_i}{dt} = \sum_j m_j \left(\frac{\vec{v_{ij}}}{\rho_i^{\sigma-2}\rho_j^{2-\sigma}} \right) \nabla W(\vec{x_{ij}}) \tag{45}$$

$$\frac{d\vec{v_i}}{dt} = \vec{g} - \sum_j m_j \left(\frac{P_i}{\rho - i^{\sigma}\rho_j^{2-\sigma}} + \frac{P_j}{\rho_i^{2-\sigma}\rho_j^{\sigma}} \right) \nabla W(\vec{x_{ij}}) \tag{46}$$

Thanks to these modifications in the way to write the equations we now have symmetries and we respect the third Newton's law. Generally, the value of σ is equal to 2 [1], we then obtain :

$$\frac{d\rho_i}{dt} = \sum_j m_j(\vec{v_i} - \vec{v_j})\nabla W(\vec{x_i} - \vec{x_j}) \tag{47}$$

$$\frac{d\vec{v_i}}{dt} = \vec{g} - \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla W(\vec{x_{ij}}) \tag{48}$$

But we know that if $\sigma = 1$ [13] and [14]. We have :

$$\frac{d\rho_i}{dt} = \rho_i \sum_j \frac{m_j}{\rho_j}(\vec{v_{ij}})\nabla W(\vec{x_{ij}}) \tag{49}$$

$$\frac{d\vec{v_i}}{dt} = \vec{g} - \sum_j m_j \left(\frac{P_i + P_j}{\rho_i\rho_j} \right) \nabla W(\vec{x_{ij}}) \tag{50}$$

Thanks to these last two equations, we analyze the symmetric discretized equations. We set $\nabla\rho$ equal to $\nabla 1$.We then have the following symmetric equations:

$$\nabla\vec{v} = \nabla\vec{v} - \vec{v}1 \tag{51}$$

$$\nabla P = \nabla P + P\nabla 1 \tag{52}$$

According to some papers, the choice of the value $\sigma = 1$ offers the possibility to improve the accuracy of the method.
It is possible to add an artificial viscosity. This artificial viscosity is generally used to allow shock phenomena to be simulated. It is also a way to stabilize an algorithm (numerically speaking). In the Navier-Stocks equations for viscous flow, there are the shear viscosity coefficient and the bulk viscosity. These coefficients are functions of temperature and density of the particles. Monaghan and Gingold set an artificial viscosity related to gas viscosity [1]. This viscosity is denoted by:

$$\Pi_{ij} = \frac{-\alpha\mu_{ij}\bar{c_{ij}} + \beta\mu_{ij}^2}{\bar{\rho_{ij}}} \tag{53}$$

$$\mu_{ij} = \begin{cases} h\frac{\vec{v_{ij}}.\vec{r_{ij}}}{r_{ij}^2+h^2} \text{ if } v_{ij}.\vec{r_{ij}} < 0 \\ 0 \text{ elseif} \end{cases}$$

Knowing that the meaning of $\bar{f_{ij}}$; $\bar{f_{ij}} = \frac{f_i+f_j}{2}$

Then the term Π_{ij} is added to the pressure in the previous equations, we finally have the following equations:

$$\frac{d\vec{x_i}}{dt} = \vec{v_i} \tag{55}$$

$$\frac{d\rho_i}{dt} = \rho_i \sum_j \frac{m_j}{\rho_j}(\vec{v_{ij}})\nabla W(\vec{x_{ij}}) \tag{56}$$

$$\frac{d\vec{v_i}}{dt} = \vec{g} - \sum_j m_j \left(\frac{P_i + P_j}{\rho_i\rho_j} + \Pi_{ij} \right) \nabla W(\vec{x_{ij}}) \tag{57}$$

The term Π_{ij} which is added to the pressure is quite interesting because the artificial viscosity produces a force between interacting particles and create a stabilization of the system. If the particles get closer and closer, a repulsive force is created. Besides, if the particles are receding, an attractive force is created.

1.2.3 Ghost Particles explanation

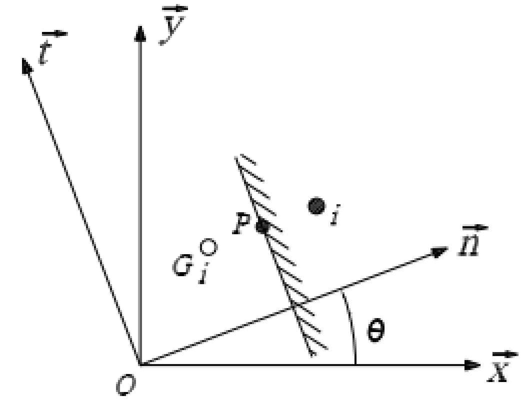
Currently, the "Ghost Particle" technique seems to provide the best results and, unlike other methods, the mathematical formulation offers the possibility to add many kind of improvements.

Creation of the Ghost Particles

If we consider the particle i, next to the border, and p the point which is the projection of this particle on the wall along the normal vector $\vec{n}$, the position of the ghost particle G_i which corresponds to the particle i is [13]:

$$\overrightarrow{\mathrm{OG_i}} = 2\overrightarrow{\mathrm{OP}} - \overrightarrow{\mathrm{O_i}} = \begin{pmatrix} 2x_{pn} - x_{in} \\ 2x_{pt} - x_{it} \end{pmatrix} \tag{58}$$

O the origin, this vector is expressed in the basis $\vec{n}$(O,$\vec{n}$,$\vec{t}$) linked to the wall



For a general case, if the wall has a movement of translation and rotation, the ghost particle speed is :

$$\frac{d_0\overrightarrow{\mathrm{OG_i}}}{dt} = \frac{d_n\overrightarrow{\mathrm{OG_i}}}{dt} + \vec{\Omega_{0n}} \wedge \overrightarrow{\mathrm{OG_i}} \tag{59}$$

$$\vec{v_{Gi}} = \frac{d_0\overrightarrow{\mathrm{OG_i}}}{dt} \tag{60}$$

Ω_0 is the rotational vector of $\vec{n}$(O,$\vec{n}$,$\vec{t}$)relative to the absolute basis. If we assume that the problem is a 2D problem where :

$$\vec{\Omega_{0n}} = \dot{\theta}\vec{z_1} \tag{61}$$

and assuming that the tangential velocity of the wall is equal to the tangential velocity of the particle i which has to be symmetrizes $v_{pt} = v_{it}$ because $x_{pt} = x_{it}$, the ghost particle's speed is :

$$\vec{v_{Gi}} = (2v_{pn} - \dot{\theta}x_{pt} - v_{in})\vec{n} + (v_{it} + \dot{\theta}(2x_{pn} - x_{in}))\vec{t} \tag{62}$$

In the case of a flat wall, we impose the same volume for the ghost particle and the particle i. So $\omega_{Gi} = \omega_i$. It is possible to study the problem for a three dimensions problem by using the equation [1]:

$$\frac{d_0\overrightarrow{\mathrm{OG_i}}}{dt} = \frac{d_n\overrightarrow{\mathrm{OG_i}}}{dt} + \vec{\Omega_{0n}} \wedge \overrightarrow{\mathrm{OG_i}} \tag{63}$$

When a particle is near the border, it generates a gradient of pressure which affects the value of the pressure of the Ghost Particle i,P_{Gi}. In order to take this pressure gradient into account, several entities must be mastered like the gravity and the wall's kinematic.

The condition of non-penetration is given by the formulation [2]:

$$(\vec{v} - \vec{v_p}).\vec{n} = 0 \tag{64}$$

In order to know the Ghost particle's pressure, we derivate the formulation below :

$$\frac{d\vec{v}}{dt}.\vec{n} = (\vec{v_p} - \vec{v})\frac{d\vec{n}}{dt} + \frac{dv_p}{dt}.\vec{n} \tag{65}$$

$$= (\vec{v_p} - \vec{v})(\vec{\Omega_{0n}} \wedge \vec{n}) + \frac{d\vec{v_p}}{dt}.\vec{n} \tag{66}$$

As we said, the velocity is about the pressure and the gravity, that's why we have :

$$-\frac{1}{\rho}\frac{\partial P}{\partial n} + \vec{g}.\vec{n} = (\vec{v_p} - \vec{v}).(\vec{\Omega_{0n}} \wedge \vec{n}) + \frac{d\vec{v_p}}{dt}.\vec{n} \tag{67}$$

$$\frac{\partial P}{\partial n} = -\rho((\vec{v_p} - \vec{v}).(\vec{\Omega_{0n}} \wedge \vec{n}) + \frac{d\vec{v_p}}{dt}.\vec{n} - \vec{g}.\vec{n}) \tag{68}$$

Knowing that

$$\frac{\partial P}{\partial n} = \frac{\partial P}{\partial p}\frac{\partial p}{\partial n} = c_0^2(\frac{\rho}{\rho_0})^{\gamma-1}\frac{\partial p}{\partial n} \tag{69}$$

We can define that

$$\rho_{Gi} = \left[\rho_i^{\gamma-1} + A\left(B - \frac{d\vec{v_p}}{dt}.\vec{n} + \vec{g}.\vec{n}\right)(x_{Gin} - x_{in})\right]^{\frac{1}{\gamma-1}} \tag{70}$$

with

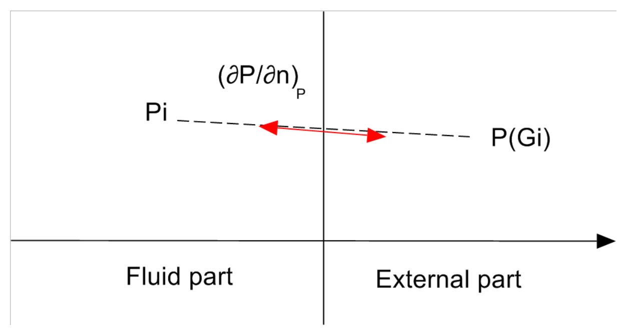
$$A = \frac{\rho_0^{\gamma-1}(\gamma-1)}{c_0^2} \tag{71}$$

$$B = (\vec{v} - \vec{v_p}).(\vec{\Omega_{0n}} \wedge \vec{n}) \tag{72}$$

In order to access the value of the pressure, we use the state law equation for a free surface flow. It exists many laws, we decided to present this one :

$$P - P_0 = \frac{\rho_0 c_0^2}{\gamma}\left[\left(\frac{\gamma}{\gamma_0}\right)^{\gamma} - 1\right] \tag{73}$$

Now, it is possible to find the Ghost particle's pressure P_{Gi}



2 Results of 2D SPH function

2.1 The partial derivative of W

We now have to calculate the partial derivative of W(r,h)in order to validate the approximation of the function f. We used the chain rule to make the calculation easier.

$$\frac{\partial W(q(X,Y,Z))}{\partial X} = \frac{\partial W}{\partial q}.\frac{\partial q}{\partial X} \tag{74}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Y} = \frac{\partial W}{\partial q}.\frac{\partial q}{\partial Y} \tag{75}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Z} = \frac{\partial W}{\partial q}.\frac{\partial q}{\partial Z} \tag{76}$$

First of all, we calculated the partial derivative of q by X, Y and Z.

$$\frac{\partial q}{\partial x_j} = \frac{x_{ji}}{h}.\frac{1}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{77}$$

$$\frac{\partial q}{\partial y_j} = \frac{y_{ji}}{h}.\frac{1}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{78}$$

$$\frac{\partial q}{\partial z_j} = \frac{z_{ji}}{h}.\frac{1}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{79}$$

Once these calculations are done, we have to determine the partial derivative of the shape function W (kernel function).

For $0 < q \leqslant 1$:

$$\frac{\partial W}{\partial q} = \frac{\alpha}{h^2}(\frac{9}{4}.q^2 - 3q) \tag{80}$$

For $1 \leqslant q \leqslant 2$:

$$\frac{\partial W}{\partial q} = \frac{3\alpha}{h^2}(-\frac{1}{4}q^2 + q - 1) \tag{81}$$

For $q > 2$:

$$\frac{\partial W}{\partial q} = 0 \tag{82}$$

Finally, we have all the elements in order to write the expression of the partial derivative of W(r,h).

For $0 < q \leqslant 1$:

$$\frac{\partial W(q(X,Y,Z))}{\partial X} = \frac{\frac{\alpha}{h^3}(\frac{9}{4}.q^2 - 3q)(x_j - x_i)}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{83}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Y} = \frac{\frac{\alpha}{h^3}(\frac{9}{4}.q^2 - 3q)(y_j - y_i)}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{84}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Z} = \frac{\frac{\alpha}{h^3}(\frac{9}{4}.q^2 - 3q)(z_j - z_i)}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{85}$$

For $1 \leqslant q \leqslant 2$:

$$\frac{\partial W(q(X,Y,Z))}{\partial X} = \frac{\frac{3\alpha}{h^3}(-\frac{1}{4}q^2 + q - 1)(x_{ij})}{\sqrt{(x_{ji})^2+(y_{ij})^2+(z_{ij})^2}} \tag{86}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Y} = \frac{\frac{3\alpha}{h^3}(-\frac{1}{4}q^2 + q - 1)(y_j - y_i)}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{87}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Z} = \frac{\frac{3\alpha}{h^3}(-\frac{1}{4}q^2 + q - 1)(z_{ji})}{\sqrt{(x_{ij})^2+(y_{ij})^2+(z_{ij})^2}} \tag{88}$$

For $q > 2$:

$$\frac{\partial W(q(X,Y,Z))}{\partial X} = 0 \tag{89}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Y} = 0 \tag{90}$$

$$\frac{\partial W(q(X,Y,Z))}{\partial Z} = 0 \tag{91}$$

2.2 The calculation of the acceleration

The calculation of the acceleration is done in several steps. First of all we need to express the velocity in a three dimensional base.

$$\underset{-}{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{92}$$

Then, we express the velocity's gradient.

$$\underline{\underline{\nabla u}} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{93}$$

Then, we have to express the divergence of the stress matrix. This stress matrix is a function of the strain's matrix :

$$\underline{\underline{\sigma}} = f(\underline{\underline{\epsilon}}) \tag{94}$$

And the strain's matrix is given by the equation:

$$\underline{\underline{\epsilon}} = \frac{1}{2}(\underline{\underline{\nabla u}} + {}^t\underline{\underline{\nabla u}}) \tag{95}$$

Thus, we obtain the divergence of the stress' matrix.

$$\mathrm{div}\sigma_{\mathrm{xx}} = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial y} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{bmatrix} \tag{96}$$

With all the equations described above, we can calculate the acceleration by using this equation :

$$\frac{d^2u}{dt^2} = -\frac{1}{\rho}div\,\underline{\underline{\sigma}} + f_v \tag{97}$$

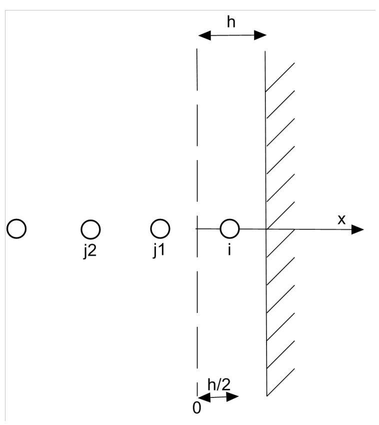
2.3 The interface between the fluid and the boundaries

In many papers, we found the expressions of the parameters of the ghost particles (speed, density, pressure, and mass). In order to show the effects of the ghost particle, we compare two similar systems. The first system does not have ghost particles and the second one has it. In order to highlight the effects of the ghost particles on the SPH method, we chose to use a very simple system. Below, the assumptions of the case we study: The system studied is a one dimension system. The particles are gas particles. At the time t=0, we assume that the particles are evenly distributed. The gravity field is negligible (g=0). The wall is fixed. We consider that there is no artificial viscosity. We set h=dx, so for this special case, the approximation of a partial derivative is:

$$A\prime = \frac{1}{2h}(A_{i+1} - A_{i-1}) \tag{98}$$

Without Ghost particle

All the particles have an initial velocity v_0.



The initial conditions are:

$$v_i = v_{j1} = v_0 \tag{99}$$

$$x_i = \frac{h}{2} \tag{100}$$

Without using the approximation of the SPH method and according to the Euler equations we should have, at the instant t:

$$\frac{d\rho_i}{dt} = -\rho.\frac{\partial v}{\partial x} = 0 \tag{101}$$

It means that normally, there is no variation of the density through time.

Method SPH with h=dx

If we calculate the approximation of the Euler equation with the SPH method, knowing that h=dx, we obtain,

$$\frac{d\rho_i}{dt} = -\rho.\frac{1}{2h}(v_{j1}) > 0 \tag{102}$$

It means that the particle's density changes. This inequality is incompatible with what happen when we do not ue the SPH approximation. We then have instabilities on the particle which is at the limit of the domain. This is due to the fact that the set of points is not complete.

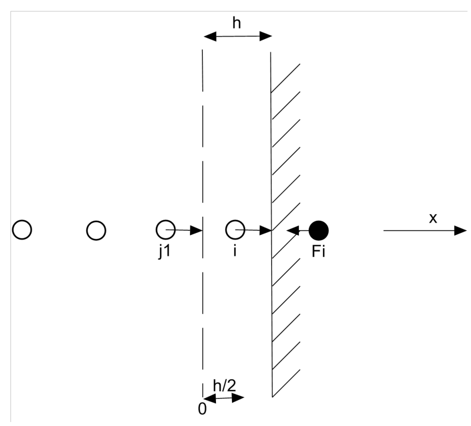
At the instant t+Δt :

$$\rho^{t+\Delta t} = \rho^t + \Delta t(\frac{d\rho}{dt} = \rho^t + \Delta t(-\rho.\frac{1}{2h}(v_{j1})) > \rho^t \tag{103}$$

The density will increase at each time step. That is why, in order to complete the set of interpolation points, we have to add an extra particle one the right.

With Ghost particle

SPH with h=dx



We assume that the wall does not have any rotation or translation movement. Then, according to the literature:[3] and [2], the expressions of the parameters of the ghost particle are:

$$\begin{gathered}
\mathrm{v}_{Gi} = 2v_w - v_i = -v_i = -v_0 \\
\mathrm{p}_{Gi} = 2p_w - p_i \\
\rho_{Gi} = \rho_i \\
\mathrm{x}_{Gi} = 2x_w - x_i \\
\mathrm{m}_{Gi} = m_i \\
\mathrm{v}_{Gi} = m_i \rho_{Gi}
\end{gathered}$$

(104)

Then, if we calculate the approximation of the density's derivative we obtain :

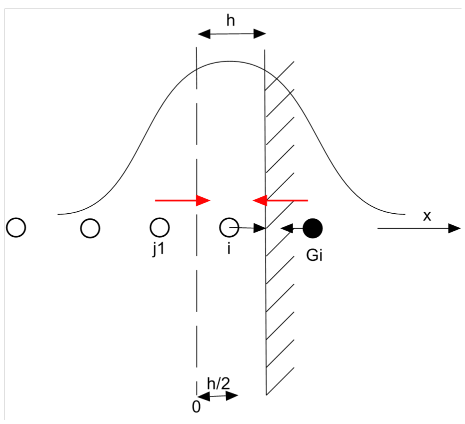
$$\frac{d\rho_i}{dt} = -\rho\frac{1}{2h}(v_{Gi} - v_{j1}) = 0 \tag{105}$$

because of the equation (2.3)

At the time $t + \Delta t$:

$$\rho_i^{t+\Delta t} = \rho_i^t + \Delta t\frac{d\rho_i}{dt} = \rho_i^t \Rightarrow \rho_i = cste \tag{106}$$

It means that there is no variation of the density through time. Then, the method seems to be stabilized because we added a point to the set of points. Then the Kernel function is complete.



This approach produces the desired repulsion mechanism. However, if the time step is not small enough to keep all the fluid particules inside the domain, an extra force is added to avoid a penetration.

3 SPH function in cylindrical coordinate

At first, we will use a given function in a cartesian coordinate that we will transform in cylindrical coordinate. The point of this research is determining the Gaussian function that will be used to find the Kernel function. It will depend on the Bessel function.

3.1 Calculous of the Gaussian and Kernel function

As said before, we change the cartesian coordinates into cylindrical coordinates:

$$\begin{cases} x &= rcos(\theta) \\ y &= rsin(\theta) \\ z &= z \end{cases}$$

With i and j two points, the difference of the coordinates of those two points is:

$$\begin{cases} (x_j - x_i)^2 &= (r_j cos(\theta_j))^2 + (r_i cos(\theta_i))^2 - 2r_i r_j cos(\theta_i)cos(\theta_j) \\ (y_j - y_i)^2 &= (r_j sin(\theta_j))^2 + (r_i sin(\theta_i))^2 - 2r_i r_j sin(\theta_i)sin(\theta_j) \end{cases}$$

We do the sum of the two previous factors with $a = (x_j - x_i)^2$ and $b = (y_j - y_i)^2$:

$$\begin{aligned} a + b = {} & r_j^2(cos(\theta_j)^2 + sin(\theta_j)^2) + r_i^2(cos(\theta_j)^2 + sin(\theta_j)^2) - \\ & 2r_i r_j cos(\theta_j - \theta_i) \\ & = (r_j - r_i)^2 + 2r_i r_j(1 - cos(\theta_{ij})) \end{aligned}$$

We will be able to implement this result in the Gaussian function:

$$\begin{aligned} W &= exp(-\frac{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}{h^2}) \\ &= exp(-\frac{(r_j - r_i)^2 + 2r_i r_j(1 - cos(\theta_{ij})) + (z_j - z_i)^2}{h^2}) \end{aligned}$$

To simplify our equation, we write $q = r_{ij}^2 + z_{ij}^2$ and $G = exp(-\frac{q}{h^2})$ in order to have:

$$W = Gexp(\frac{-2r_i r_j(1 - cos(\theta_{ij}))}{h^2})$$

The reason why the factors of the function have been separated is because the Kernel function (called W_c which represents a statistical method for estimating a real valued function) needs to be integrated in function of θ (G is not a function of θ). Moreover, we multiply W_c with a constant $\alpha = \frac{1}{h\sqrt{\pi}}$. The purpose of this constant is to adjust correctly the Kernel function with the right result. It gives:

$$W_c = \alpha G \int_0^{2\pi} exp(\frac{-2r_i r_j}{h^2}(1 - cos(\theta_{ij})))d\theta$$

In order to simplify the equation, we will call $\beta = \frac{2r_i r_j}{h^2}$:

$$W_c = \alpha G \int_0^{2\pi} exp(-\beta)exp(\beta cos(\theta_{ij}))d\theta$$

We will call $I_0 = \int_0^{2\pi} exp(-\beta)exp(\beta cos(\theta_{ij}))d\theta$.

Finally,

$$Wc = \alpha G(q)I_0(\beta)$$

I_0 is a modified Bessel's function of the first kind.

To explain this demonstration, the following picture are an example of the using of this theorem. We can take a function $F = (r - \dfrac{ra+rb}{2})^2(z+1)^3 + rz^5$ where r and z are variables, ra and rb are the boundaries of the set of r.

This picture represents the values of F depending on the variation of r and z.

Figure 1: Analytical function

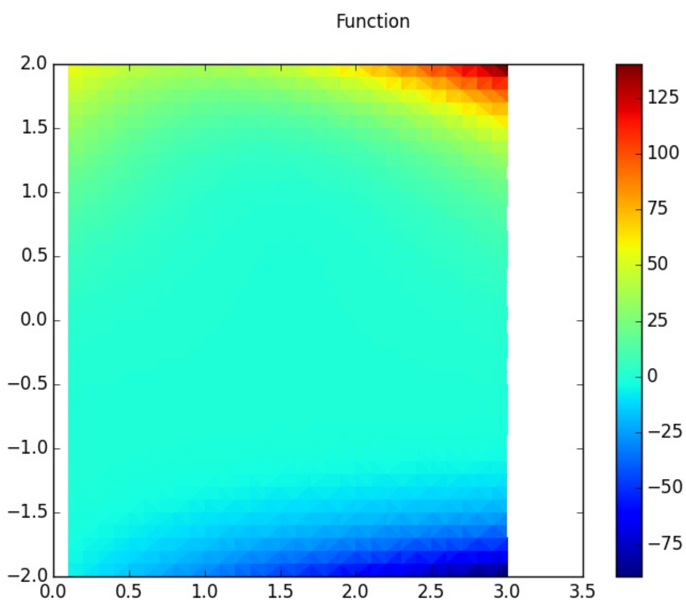
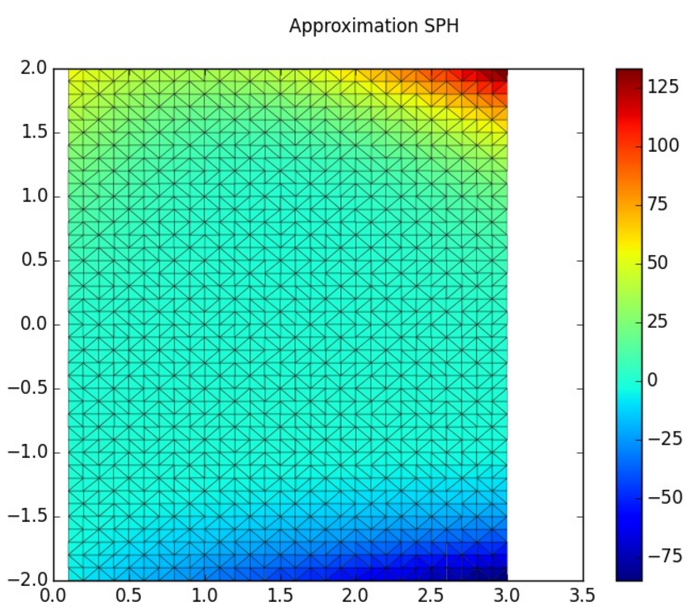


Figure 2: Analytical function - Triangular



We notice that despite the great precision of the representation, we cannot completely follow those graphics because there are errors at the edges of the graphics.

3.2 Calculous of the divergence with the SPH method

The SPH method allows to determine SPH's approximation of the divergence of a function thanks to the exact function. If we have a function f and a Gaussian function W, for all x and y in an Ω set, we can write:

$$\forall(x,y) \in \Omega, < f(x) > \approx \iiint_{\Omega} f(y)\, W(|x-y|, h) d\Omega \tag{107}$$

Then, we can assume:

$$< \nabla f(x) > \approx \iiint_{\Omega} \nabla f(y)\, W(|x-y|, h) d\Omega \tag{108}$$

We have the simplified expression of $\nabla f(y)\, W$, where $W = W(|x-y|, h)$. Using an integration by parts:

$$\nabla f(y)\, W \approx \nabla(f(y)\, W) - f(y) \nabla W \tag{109}$$

We implement (108) in (107):

$$< \nabla f(x) > \approx \iiint_{\Omega} [(\nabla(f(y)\, W) - f(y)\, \nabla W] d\Omega \tag{110}$$

We can simplify (109) by using the divergence theorem. We set $\Gamma = \partial\Omega$ and the surface normal n:

$$\iiint_{\Omega} (\nabla(f(y)\, W)) d\Omega = \oiint_{\Gamma} (f(y)\, W\, \vec{n}) d\Gamma \tag{111}$$

We use (110) in (109):

$$< \nabla f(x) > \approx \oiint_{\Gamma} (f(y)\, W\, \vec{n}) d\Gamma - \iiint_{\Omega} \nabla(f(y)\, \nabla W\, d\Omega$$

In this case, the values outside of Γ are equaled to 0. Then:

$$\oiint_{\Gamma} (f(y)\, W\, \vec{n}) d\Gamma = 0$$

Finally, it comes:

$$< \nabla f(x) > \approx - \iiint_{\Omega} f(y)\, \nabla W\, d\Omega \tag{112}$$

We can use (22) to transform the continuous system into a discrete system with D_j being the volume of the particle j:

$$< \nabla f(x) >_i \approx -\sum f_j \nabla_j W_{ij} D_j \tag{113}$$

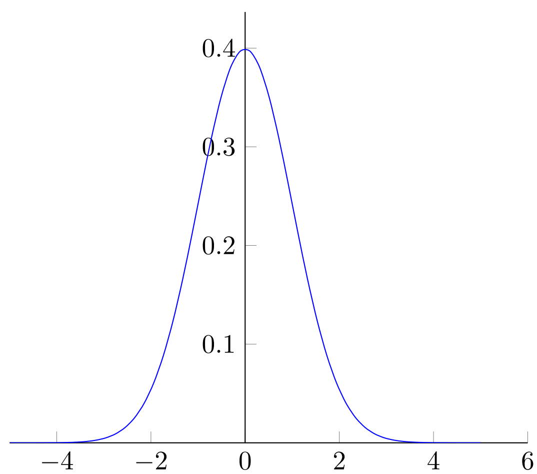
With m and ρ respectively the mass and the density of a particle, we can define $D = \frac{m}{\rho}$. Then:

$$< \nabla f(x) >_i \approx -\sum f_j \nabla_j W_{ij} \frac{m_j}{\rho_j} \tag{114}$$

3.3 Representation of W and its divergence

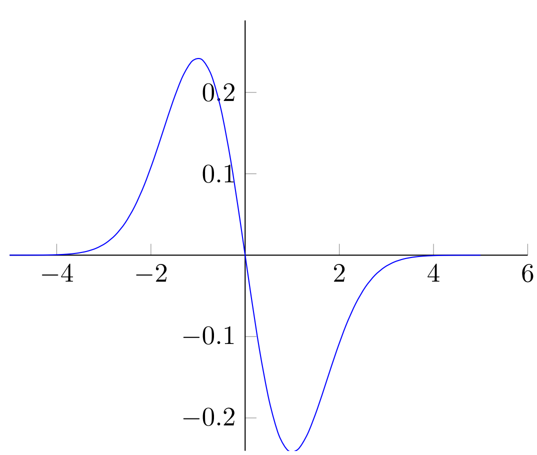
W is a gaussian function which is used in every calculous of SPH. It determines the effect of the particles j applied on the particle i. As we saw in (22) and (114), W or its derivative will be used in the equation depending of the result requested.
As W_{ij} is a Gaussian function, W_{ij} is symmetric. Therefore:

$$W_{ij} = W_{ji}$$



The divergence of a symmetric function is odd, it comes:

$$\nabla_j W_{ij} = -\nabla_i W_{ij}$$



3.4 Example of the use of the SPH method

The SPH method can be very useful to solve some complicated physic equations like the continuity equation or the acceleration depending on the strain. We call (114) with $f := A$,ρ_j and m_j are respectively the density and the mass of the particle. The set contains n particles.

$$< \nabla A >_i \approx -\sum_{j=0}^{n} \frac{m_j}{\rho_j} A_j \nabla_j W_{ij} \tag{115}$$

3.4.1 SPH method applied on the continuity equation

The continuity equation can be solved using the SPH method:

$$\frac{d\rho}{dt} = -\rho.div(v) = -\rho \nabla v \tag{116}$$

We can use the laws of derivation based of the derivation of a product of factors:

$$\nabla(\rho v) = \rho \nabla v + \nabla \rho.v \tag{117}$$
$$\Leftrightarrow \rho \nabla v = \nabla(\rho v) - \nabla \rho.v \tag{118}$$
$$\Leftrightarrow -\rho \nabla v = -\nabla(\rho v) + \nabla \rho.v \tag{119}$$

We take the factors of (119) and we implant them in (115). We set $A := \rho v$:

$$\nabla(\rho v) = -\sum \frac{m_j}{\rho_j} (\rho_j v_j) \nabla_j W_{ij} = -\sum m_j v_j \nabla_j W_{ij} \tag{120}$$

$$\nabla \rho.v_i = -v_i \sum \frac{m_j}{\rho_j} \rho_j \nabla_j W_{ij} = -v_i \sum m_j \nabla_j W_{ij} \tag{121}$$

We implement (120) and (121) in (119):

$$-\rho \nabla v = \sum m_j v_j \nabla_j W_{ij} - v_i \sum m_j \nabla_j W_{ij} \tag{122}$$
$$\Leftrightarrow -\rho \nabla v = \sum (v_j - v_i) m_j \nabla_j W_{ij} \tag{123}$$
$$\Leftrightarrow -\rho \nabla v = -\sum (v_i - v_j) m_j \nabla_j W_{ij} \tag{124}$$

The continuity equation can be solved using the difference of velocity between particles. This method of calculation is very useful because it improves the accuracy of the results compared to the regulard method. A more general method can be used in order to get a better approximation of the result based on the calculous of the continuity equation. We take (124) and we consider $\rho := 1$. It comes:

$$\nabla(1v) = \rho \nabla 1 + \nabla.v \tag{125}$$
$$\Leftrightarrow 1 \nabla v = \nabla(1v) - \nabla 1.v \tag{126}$$

We take the factors of (126) and we put them in (115). It comes:

$$\nabla(1v) = -\sum \frac{m_j}{\rho_j} v_j \nabla_j W_{ij} \tag{127}$$

$$\nabla 1.v = -v_i \sum \frac{m_j}{\rho_j} \nabla_j W_{ij} \tag{128}$$

Finally, we put (127) and (128) in (126). It comes:

$$1\nabla v = \sum (v_i - v_j) \frac{m_j}{\rho_j} \nabla_j W_{ij} \tag{129}$$

This method is very powerful since it improves the accuracy of the results for every equations.

3.4.2 SPH method applied on the acceleration's equation

Like the contuity equation, the acceleration can be determined thanks to laws of derivation.

$$\nabla(\frac{\sigma}{\rho}) = \frac{\rho\nabla\sigma - \nabla\rho.\sigma}{\rho^2} \tag{130}$$

$$\Leftrightarrow \rho^2\nabla(\frac{\sigma}{\rho}) = \rho\nabla\sigma - \nabla\rho.\sigma \tag{131}$$

$$\Leftrightarrow \rho^2\nabla(\frac{\sigma}{\rho}) + \nabla\rho.\sigma = \rho\nabla\sigma \tag{132}$$

$$\Leftrightarrow \nabla(\frac{\sigma}{\rho}) + \frac{\nabla\rho.\sigma}{\rho^2} = \frac{\nabla\sigma}{\rho} \tag{133}$$

We take the factors of (133) and we implement them in (115):

$$\nabla(\frac{\sigma}{\rho})_i = -\sum \frac{m_j}{\rho_j}(\frac{\sigma_j}{\rho_j})\nabla_j W_{ij} = -\sum \frac{m_j}{\rho_j^2}\sigma_j \nabla_j W_{ij} \tag{134}$$

$$(\frac{\nabla\rho.\sigma}{\rho^2})_i = -\sum \rho_j \frac{m_j}{\rho_j}\frac{\sigma_i}{\rho_i^2}\nabla_j W_{ij} = -\sum \frac{m_j}{\rho_i^2}\sigma_i \nabla_j W_{ij} \tag{135}$$

Finally, we implement (134) and (135) in (133):

$$\langle\frac{\nabla\sigma}{\rho}\rangle_i \approx -\sum (\frac{\sigma_i}{\rho_i^2} + \frac{\sigma_j}{\rho_j^2}) m_j \nabla_j W_{ij} \tag{136}$$

3.4.3 Calculous of the density

The SPH method allows to determine the density of a particle depending on its mass. We use the equation (26), we call $f := \rho$. It comes:

$$< \rho >_i \approx \sum \frac{m_j}{\rho_j}\rho_j W_{ij}$$

$$< \rho >_i \approx \sum m_j W_{ij}$$

This equation is very useful beacause it gives the opportunity of determining the density of each particles depending on its mass.

3.4.4 Calculous of the revision matrix

In this part, we will deduce the maxtrix B that will revise our function. There is the vector A:

$$A_{ij} := A_i - A_j$$

We call $f(x) := A$ in (112):

$$\nabla A = -\iiint_\Omega (A_{ij}\, \nabla_j W_{ij}) d\Omega \tag{137}$$

If we call a function $v(x) = ax + b$, then $\nabla v = a$. If we put $A := v$ in (111):

$$\widehat{\nabla v_i} = -\iiint_\Omega (v_{ij}\, \nabla_j W_{ij} B) d\Omega$$

Basically, $v_{ij} = ax_{ij} = a(x_i - x_j)$. Then:

$$\widehat{\nabla v_i} \; - \iiint_\Omega (ax_{ij}\, \nabla_j W_{ij} B) d\Omega$$

$$\Longleftrightarrow a = -a \iiint_\Omega x_{ij}\, \nabla_j W_{ij} B d\Omega$$

$$\Longleftrightarrow 1 = -\iiint_\Omega x_{ij}\, \nabla_j W_{ij} B d\Omega$$

$$\Longleftrightarrow B = -\frac{1}{\iiint_\Omega x_{ij}\, \nabla_j W_{ij} B d\Omega}$$

3.5 Application of the SPH method

The SPH method are very useful in dynamic systems. It means that the particles have a movement which imply that they have a velocity. We will use the velicity vector $\underline{U}$. In the case of a cylinder, U^θ is equal to 0. It comes:

$$\underline{U} = \begin{bmatrix} U^{r*} \\ 0 \\ U^{z*} \end{bmatrix}$$

This is the reason why only $\frac{\partial U^r}{\partial t}$ and $\frac{\partial U^z}{\partial t}$ will be determined for the acceleration.

3.5.1 Calculous of the acceleration of the particles through the divergence of the tensor stress in a cylindric domain

In order to dertemine the acceleration of the particles with the SPH method, we have to set the divergence of the tensor stress. The idea is that the method will use the stress applied on the particles to determine their acceleration on every axis ($\frac{\partial U}{\partial t} =$

$\frac{\nabla\sigma}{\rho}$). By definition, the equation of divergence of a tensor in an cylindric domain is:

$$div(\sigma) = \begin{cases} \frac{\partial\sigma^{rr*}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^{r\theta*}}{\partial\theta} + \frac{\partial\sigma^{rz*}}{\partial z} + \frac{\sigma^{rr*}-\sigma^{\theta\theta*}}{r} \\ \frac{\partial\sigma^{\theta r*}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^{\theta\theta*}}{\partial\theta} + \frac{\partial\sigma^{\theta z*}}{\partial z} + \frac{\sigma^{r\theta*}+\sigma^{\theta r*}}{r} \\ \frac{\partial\sigma^{zr*}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^{z\theta*}}{\partial\theta} + \frac{\partial\sigma^{zz*}}{\partial z} + \frac{\sigma^{zr*}}{r} \end{cases}$$

In order to calculate this divergence with the SPH approximation, we have to apply to the tensor σ the contravariant tensor transformation P:

$$P = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The contravariant is applied in our case because of the change of basis of every particle.

To simplify the equations, we will write $c := cos(\theta)$ and $s := sin(\theta)$. Since the equation of the contravariant is $\sigma^* = P^t\sigma P$, it comes:

$$\sigma^* = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^{rr} & 0 & \sigma^{rz} \\ 0 & \sigma^{\theta\theta} & 0 \\ \sigma^{rz} & 0 & \sigma^{zz} \end{bmatrix} \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{138}$$

$$= \begin{bmatrix} c^2\sigma^{rr} + s^2\sigma^{\theta\theta} & sc\sigma^{rr} - cs\sigma^{\theta\theta} & c\sigma^{rz} \\ sc\sigma^{rr} - cs\sigma^{\theta\theta} & s^2\sigma^{rr} + c^2\sigma^{\theta\theta} & s\sigma^{rz} \\ c\sigma^{rz} & s\sigma^{rz} & \sigma^{zz} \end{bmatrix} \tag{139}$$

Calculous of $\frac{\partial U^r}{\partial t}$

In order to determine $\frac{\partial U_r}{\partial t}$, the divergence tensor of the r axis will be taken in account in the calculous.

Basically,

$$\nabla\sigma^r = \frac{\partial\sigma^{rr*}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^{r\theta*}}{\partial\theta} + \frac{\partial\sigma^{rz*}}{\partial z} + \frac{\sigma^{rr*}-\sigma^{\theta\theta*}}{r}$$

We remind that $\frac{\partial U_r}{\partial t} = \frac{\nabla\sigma^r}{\rho}$. It comes:

$$\frac{\partial U^t}{\partial t} = \frac{1}{\rho}\frac{\partial\sigma^{rr*}}{\partial r} + \frac{1}{\rho r}\frac{\partial\sigma^{r\theta*}}{\partial\theta} + \frac{1}{\rho}\frac{\partial\sigma^{rz*}}{\partial z} + \frac{\sigma^{rr*}-\sigma^{\theta\theta*}}{\rho r} \tag{140}$$

In order to simplify the calculous, we will set:

$$\begin{cases} a &= \frac{1}{\rho}\frac{\partial\sigma^{rr*}}{\partial r} \\ b &= \frac{1}{\rho r}\frac{\partial\sigma^{r\theta*}}{\partial\theta} + \frac{\sigma^{rr*}-\sigma^{\theta\theta*}}{\rho r} \\ c &= \frac{1}{\rho}\frac{\partial\sigma^{rz*}}{\partial z} \end{cases} \tag{141}$$

We will transform the terms of (141) into a SPH calcoulous thanks to (136):

$$\begin{cases} a &= -\frac{\partial}{\partial r_i}\sum(\frac{\sigma_i^{rr*}}{\rho_i^2} + \frac{\sigma_j^{rr*}}{\rho_j^2})m_jW_{ij} \\ b &= \frac{1}{r_i}\sum\Big[-\frac{\partial}{\partial\theta_i}\Big(m_jW_{ij}\Big(\frac{\sigma_i^{r\theta*}}{\rho_i^2} + \frac{\sigma_j^{r\theta*}}{\rho_j^2}\Big)\Big) \\ & +(\sigma_j^{rr*} - \sigma_j^{\theta\theta*})\frac{m_jW_{ij}}{\rho_j^2}\Big] \\ c &= -\frac{\partial}{\partial z_i}\sum(\frac{\sigma_i^{rz*}}{\rho_i^2} + \frac{\sigma_j^{rz*}}{\rho_j^2})m_jW_{ij} \end{cases}$$

We will apply the contravariant tensor transformation:

$$\begin{cases} a &= -\frac{\partial}{\partial r_i}\sum\frac{m_j}{\rho_i^2}(\sigma_i^{rr}c_i + \sigma_i^{\theta\theta}s_i)W_{ij} \\ & -\frac{\partial}{\partial r_i}\sum\frac{m_j}{\rho_j^2}(\sigma_j^{rr}c_j + \sigma_j^{\theta\theta}s_j)W_{ij} \\ b &= \frac{1}{r_i}\sum\Big[-\frac{\partial}{\partial\theta_i}\Big(\Big(\frac{(c_js_j\sigma_j^{rr} - c_js_j\sigma_j^{\theta\theta})}{\rho_j^2} + \frac{(c_is_i\sigma_i^{rr} - c_is_i\sigma_i^{\theta\theta})}{\rho_i^2}\Big) \\ & m_jW_{ij}\Big) + (c_j^2\sigma_j^{rr} + s_j^2\sigma_j^{\theta\theta} - s_j^2\sigma_j^{rr} - c_j^2\sigma_j^{\theta\theta})\frac{m_jW_{ij}}{\rho_j^2}\Big] \\ &= -\frac{1}{r_i}\sum\Big[\frac{\partial}{\partial\theta_i}\Big(\frac{(c_is_i\sigma_i^{rr} - c_is_i\sigma_i^{\theta\theta})W_{ij}}{\rho_i^2}\Big) + \frac{(c_js_j\sigma_j^{rr} - c_js_j\sigma_j^{\theta\theta})}{\rho_j^2} \\ & \frac{\partial W_{ij}}{\partial\theta_i} + (-c_j^2\sigma_j^{rr} - s_j^2\sigma_j^{\theta\theta} + s_j^2\sigma_j^{rr} + c_j^2\sigma_j^{\theta\theta} \\ & +c_j^2\sigma_j^{rr} + s_j^2\sigma_j^{\theta\theta} - s_j^2\sigma_j^{rr} - c_j^2\sigma_j^{\theta\theta})\frac{W_{ij}}{\rho_j^2}\Big]m_j \\ c &= -\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial z_i}\sum c_im_jW_{ij} - \frac{\partial}{\partial z_i}\sum\frac{\sigma_j^{rz}}{\rho_j^2}c_jm_jW_{ij} \end{cases}$$

Since the contravariant changes the basis in function of the particles, we choose $\theta_i = 0$ which will simplify the calculous. Moreover, $\nabla_jW_{ij} = -\nabla_iW_{ij}$, then $-\frac{\partial W_{ij}}{\partial\theta_i} = \frac{\partial W_{ij}}{\partial\theta_j}$. It comes:

$$\begin{cases} a &= -\frac{\partial}{\partial r_i}\sum\frac{m_j}{\rho_i^2}\sigma_i^{rr}W_{ij} + \frac{m_j}{\rho_j^2}(\sigma_j^{rr}c_j^2 + \sigma_j^{\theta\theta}s_j^2)W_{ij} \\ b &= \frac{1}{r_i}\sum\frac{m_j}{\rho_j^2}c_js_j(\sigma_j^{rr} - \sigma_j^{\theta\theta})\frac{\partial W_{ij}}{\partial\theta_j} \\ c &= -\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial z_i}\sum m_jW_{ij} - \frac{\partial}{\partial z_i}\sum\frac{\sigma_j^{rz}}{\rho_j^2}c_jm_jW_{ij} \end{cases} \tag{142}$$

Some integrals will be needed to solve the equations, with $\beta = \frac{2r_ir_j}{h^2}$. Those integrals are used to take off the trigonometric functions.

$$\frac{1}{2\pi}\int W cos(\theta) d\theta = W_c \frac{I_1}{I_0} \tag{143}$$

$$\frac{1}{2\pi}\int W cos^2(\theta) d\theta = W_c \frac{I_0 + I_2}{2I_0} \tag{144}$$

$$\frac{1}{2\pi}\int W sin^2(\theta) d\theta = W_c \frac{I_0 - I_2}{2I_0} \tag{145}$$

$$\frac{1}{2\pi}\int \frac{dW}{d\theta_j} sin(\theta) d\theta = -\beta W_c \frac{I_0 - I_2}{2I_0} \tag{146}$$

$$\frac{1}{2\pi}\int \frac{dW}{d\theta_j} cos(\theta) sin(\theta) d\theta = -\beta W_c \frac{I_1 - I_3}{4I_0} \tag{147}$$

We implement (143), (144), (145) and (147) in (142). It comes:

$$\begin{cases} a & = -\dfrac{\sigma_i^{rr}}{\rho_i^2}\dfrac{\partial}{\partial r_i}\sum m_j W_c \\ & -\dfrac{\partial}{\partial r_i}\sum \dfrac{m_j}{\rho_j^2}\left(\sigma_j^{rr}\dfrac{I_0 + I_2}{2I_0} + \sigma_j^{\theta\theta}\dfrac{I_0 - I_2}{2I_0}\right) W_c \\ b & = -\dfrac{2}{h^2}\sum r_j \dfrac{m_j}{\rho_j^2}(\sigma_j^{rr} - \sigma_j^{\theta\theta})\dfrac{I_1 - I_3}{4I_0} W_c \\ c & = -\dfrac{\sigma_i^{rz}}{\rho_i^2}\dfrac{\partial}{\partial z_i}\sum m_j W_c - \dfrac{\partial}{\partial z_i}\sum \dfrac{\sigma_j^{rz}}{\rho_j^2} m_j \dfrac{I_1}{I_0} W_c \end{cases} \tag{148}$$

We import the results of the equation (148) in (140). It comes:

$$\frac{\partial U^r}{\partial t} = -\sum_j \frac{m_j}{\rho_j^2}\left(\sigma_j^{rr}\frac{I_0 + I_2}{2I_0} + \sigma_j^{\theta\theta}\frac{I_0 - I_2}{2I_0}\right)\frac{\partial W_c}{\partial r_i} \tag{149}$$

$$-\sum_j \frac{m_j}{\rho_j^2}\sigma_j^{rz}\frac{I_1}{I_0}\frac{\partial W_c}{\partial z_i} \tag{150}$$

$$-\frac{\sigma_i^{rr}}{\rho_i^2}\frac{\partial}{\partial r_i}\sum_j m_j W_c \tag{151}$$

$$-\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial z_i}\sum_j m_j W_c \tag{152}$$

$$-\frac{2}{h^2}\sum_j \frac{m_j}{\rho_j^2} r_j (\sigma_j^{rr} - \sigma_j^{\theta\theta})\left(\frac{I_1 - I_3}{4I_0}\right) W_c \tag{153}$$

The final step in the derivation of $\dfrac{\partial U^r}{\partial t}$ is the differentiation. The derivatives required are:

$$\frac{\partial W_c}{\partial r_i} = -\frac{2}{h^2}\left(r_i - r_j\frac{I_1}{I_0}\right) W_c \tag{154}$$

$$\frac{\partial W_c}{\partial z_i} = -\frac{2}{h^2}(z_i - z_j) W_c \tag{155}$$

$$\frac{\partial}{\partial r_i}\left(\frac{I_1}{I_0}W_c\right) = -\frac{2}{h^2}\left[r_i\frac{I_1}{I_0} - \frac{r_j}{2}\left(1 + \frac{I_2}{I_0}\right)\right] W_c \tag{156}$$

$$\frac{\partial}{\partial z_i}\left(\frac{I_1}{I_0}W_c\right) = -\frac{2}{h^2}(z_i - z_j)\frac{I_1}{I_0} W_c \tag{157}$$

$$\frac{\partial}{\partial r_i}\left(\frac{I_2}{I_0}W_c\right) = -\frac{2}{h^2}\left(r_i\frac{I_2}{I_0} - r_j\frac{I_1 + I_3}{2I_0}\right) W_c \tag{158}$$

We implement (154) and (158) in (149), it comes:

$$\begin{aligned} & -\frac{\partial}{\partial r_i}\sum_j \frac{m_j}{\rho_j^2}\left(\sigma_j^{rr}\frac{I_0 + I_2}{2I_0} + \sigma_j^{\theta\theta}\frac{I_0 - I_2}{2I_0}\right) W_c = \\ & \sum_j m_j W_c\left[\frac{\sigma_j^{rr}}{h^2}\left(r_i - r_j\frac{I_1}{I_0}\right) + \left(r_i\frac{I_2}{I_0} - r_j\frac{I_1 + I_3}{2I_0}\right) + \right. \\ & \left. \frac{\sigma_j^{\theta\theta}}{h^2}\left(r_i - r_j\frac{I_1}{I_0}\right) - \left(r_i\frac{I_2}{I_0} - r_j\frac{I_1 + I_3}{2I_0}\right)\right] \end{aligned} \tag{159}$$

We implement (157) in (150), it comes:

$$-\frac{\partial}{\partial z_i}\sum_j \frac{m_j}{\rho_j^2}\sigma_j^{rz}\frac{I_1}{I_0}W_c = \frac{2}{h^2}\sum_j m_j W_c\left(\frac{\sigma_j^{rz}}{\rho_j^2}(z_i - z_j)\frac{I_1}{I_2}\right) \tag{160}$$

We implement (154) in (151), it comes:

$$\frac{\sigma_i^{rr}}{\rho_i^2}\frac{\partial}{\partial r_i}\sum_j m_j W_c = \frac{\sigma_i^{rr}}{\rho_i^2}\sum_j m_j W_c\frac{2}{h^2}(r_i - r_j\frac{I_1}{I_0}) \tag{161}$$

We implement (155) in (152), it comes:

$$-\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial_i}\sum_j m_j W_c = \frac{\sigma_i^{rz}}{\rho_i^2}\sum_j m_j W_c\frac{2}{h^2}(z_i - z_j) \tag{162}$$

We can do the sum of (159), (160),(161), (162) and (153). Then:

$$\frac{\partial U^r}{\partial t} = \frac{1}{h^2}\sum_j m_j W_c\Bigg[\frac{\sigma_j^{rr}}{\rho_j^2}\bigg(r_i - 2r_j\frac{I_1}{I_0} + r_i\frac{I_2}{I_0} - r_j\frac{I_1}{2I_0} - r_j\frac{I_1}{2I_0}\bigg)$$
$$+\frac{\sigma_j^{\theta\theta}}{\rho_j^2}\bigg(r_i - r_j\frac{I_1}{I_0} + r_i\frac{I_2}{I_0} + r_j\frac{I_1}{2I_0} - r_j\frac{I_1}{2I_0}\bigg)$$
$$+2\frac{\sigma_j^{rz}}{\rho_j^2}(z_i - z_j)\frac{I_1}{I_0}$$
$$+2\frac{\sigma_i^{rr}}{\rho_i^2}(r_i - r_j\frac{I_1}{I_0})$$
$$+2\frac{\sigma_i^{rz}}{\rho_i^2}(z_i - z_j)\Bigg]$$

We can simplify the equation. Finally:

$$\frac{\partial U^r}{\partial t} = \frac{2}{h^2}\sum_j m_j W_c\Big[\frac{\sigma_j^{rr}}{\rho_j^2}\Big(r_i\frac{I_0+I_2}{2I_0} - r_j\frac{I_1}{I_0}\Big)$$
$$+\frac{\sigma_j^{\theta\theta}}{\rho_j^2}\Big(r_i\frac{I_0-I_2}{2I_0}\Big)$$
$$+\frac{\sigma_j^{rz}}{\rho_j^2}(z_i - z_j)\frac{I_1}{I_0}$$
$$+\frac{\sigma_i^{rr}}{\rho_i^2}\Big(r_i - r_j\frac{I_1}{I_0}\Big)$$
$$+\frac{\sigma_i^{rz}}{\rho_i^2}(z_i - z_j)\Big]$$

Calculous of $\frac{\partial U^z}{\partial t}$

As for the radial axis, we can determine the axial acceleration thanks to the stress tensor. Basically,

$$\nabla\sigma^z = \frac{\partial\sigma^{zr*}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^{z\theta*}}{\partial\theta} + \frac{\partial\sigma^{zz*}}{\partial z} + \frac{\sigma^{zr*}}{r}$$

We remind that $\frac{\partial U^z}{\partial t} = \frac{\nabla\sigma^z}{\rho}$. It comes:

$$\frac{\partial U_z}{\partial t} = \frac{1}{\rho}\frac{\partial\sigma^{zr*}}{\partial r} + \frac{1}{\rho r}\frac{\partial\sigma^{z\theta*}}{\partial\theta} + \frac{1}{\rho}\frac{\partial\sigma^{zz*}}{\partial z} + \frac{\sigma^{zr*}}{\rho r} \tag{163}$$

In order to simplify the calculous, we will set:

$$\begin{cases} a &= \frac{1}{\rho}\frac{\partial\sigma^{zr*}}{\partial r} \\ b &= \frac{1}{\rho r}\frac{\partial\sigma^{z\theta*}}{\partial\theta} + \frac{\sigma^{zr*}}{\rho r} \\ c &= \frac{1}{\rho}\frac{\partial\sigma^{zz*}}{\partial z} \end{cases} \tag{164}$$

We will transform the terms of (164) into a SPH calcoulous thanks to (136):

$$\begin{cases} a &= -\frac{\partial}{\partial r_i}\sum(\frac{\sigma_i^{zr*}}{\rho_i^2} + \frac{\sigma_j^{zr*}}{\rho_j^2})m_jW_{ij} \\ b &= \frac{1}{r_i}\sum\Big[-\frac{\partial}{\partial\theta_i}\Big(m_jW_{ij}\Big(\frac{\sigma_i^{z\theta*}}{\rho_i^2} + \frac{\sigma_j^{z\theta*}}{\rho_j^2}\Big)\Big) \\ &\quad +\sigma_j^{zr*}\frac{m_jW_{ij}}{\rho_j^2}\Big] \\ c &= -\frac{\partial}{\partial z_i}\sum(\frac{\sigma_i^{zz*}}{\rho_i^2} + \frac{\sigma_j^{zz*}}{\rho_j^2})m_jW_{ij} \end{cases}$$

We will apply the contravariant tensor transformation:

$$\begin{cases} a &= -\frac{\partial}{\partial r_i}\sum(\frac{\sigma_i^{zr}c_i}{\rho_i^2} + \frac{\sigma_j^{zr}c_j}{\rho_j^2})m_jW_{ij} \\ b &= \frac{1}{r_i}\sum\Big[-\frac{\partial}{\partial\theta_i}\Big(\Big(\frac{s_i\sigma_i^{rz}}{\rho_i^2} + \frac{s_j\sigma_j^{rz}}{\rho_j^2}\Big)m_jW_{ij}\Big) + \frac{c_j\sigma_j^{rz}}{\rho_j^2}m_jW_{ij}\Big] \\ &= \frac{1}{r_i}\sum\Big[-\frac{\partial}{\partial\theta_i}\Big(\frac{s_i\sigma_i^{rz}W_{ij}}{\rho_i^2}\Big) - \frac{s_j\sigma_j^{rz}}{\rho_j^2}\frac{\partial W_{ij}}{\partial\theta_i}\Big) \\ &\quad +\frac{(c_j-c_j)\sigma_j^{rz}}{\rho_j^2}W_{ij}\Big]m_j \\ c &= -\frac{\partial}{\partial z_i}\sum(\frac{\sigma_i^{zz}}{\rho_i^2} + \frac{\sigma_j^{zz}}{\rho_j^2})m_jW_{ij} \end{cases}$$

Since the contravariant changes the basis in function of the particles, we choose $\theta_i = 0$. Since Wi_j is symmetric, we can write $\frac{\partial W_{ij}}{\partial\theta_i} = -\frac{\partial W_{ij}}{\partial\theta_j}$. It comes:

$$\begin{cases} a &= -\frac{\partial}{\partial r_i}\sum\Big[\frac{\sigma_i^{zr}}{\rho_i^2} + \frac{\sigma_j^{zr}c_j}{\rho_j^2}\Big]m_jW_{ij} \\ b &= \frac{1}{r_i}\sum m_j\frac{s_j\sigma_j^{rz}}{\rho_j^2}\frac{\partial W_{ij}}{\partial\theta_j} \\ c &= -\frac{\partial}{\partial z_i}\sum(\frac{\sigma_i^{zz}}{\rho_i^2} + \frac{\sigma_j^{zz}}{\rho_j^2})m_jW_{ij} \end{cases} \tag{165}$$

We implement (143), (144), (145) and (147) in (165) to take off the trigonometric functions. It comes:

$$\begin{cases} a &= -\sum\Big[\frac{\sigma_i^{zr}}{\rho_i^2} + \frac{\sigma_j^{zr}}{\rho_j^2}\frac{I_1}{I_0}\Big]m_j\frac{\partial W_{ij}}{\partial r_i} \\ b &= -\frac{2}{h^2}\sum r_j\frac{m_j}{\rho_j^2}\sigma_j^{rz}\frac{I_0-I_2}{2I_0}W_c \\ c &= -\sum(\frac{\sigma_i^{zz}}{\rho_i^2} + \frac{\sigma_j^{zz}}{\rho_j^2})m_j\frac{\partial W_c}{\partial z_i} \end{cases} \tag{166}$$

We import the results of the equations (166) in (163). It comes:

$$\frac{\partial U^z}{\partial t} = -\sum_j \frac{m_j}{\rho_j^2}\sigma_j^{rz}\frac{I_1}{I_0}\frac{\partial W_c}{\partial r_i} \tag{167}$$

$$-\sum \frac{\sigma_j^{zz}}{\rho_j^2}m_j\frac{\partial W_c}{\partial z_i} \tag{168}$$

$$-\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial r_i}\sum_j m_j W_c \tag{169}$$

$$-\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial z_i}\sum_j m_j W_c \tag{170}$$

$$-\frac{2}{h^2}\sum r_j\frac{m_j}{\rho_j^2}\sigma_j^{rz}\frac{I_0-I_2}{2I_0}W_c \tag{171}$$

We implement (154) in (167), it comes:

$$-\sum_j \frac{m_j}{\rho_j^2}\sigma_j^{rz}\frac{I_1}{I_0}\frac{\partial W_c}{\partial r_i} = \frac{2}{h^2}\sum_j m_j W_c\frac{\sigma_j^{rz}}{\rho_j^2}\left(r_i\frac{I_1}{I_0} - r_j\left(\frac{I_0+I_2}{2I_0}\right)\right) \tag{172}$$

We implement (155) in (168), it comes:

$$-\frac{\partial}{\partial z_i}\sum_j \frac{m_j}{\rho_j^2}\sigma_j^{zz}W_c = \frac{2}{h^2}\sum_j m_j W_c\left(\frac{\sigma_j^{zz}}{\rho_j^2}(z_i - z_j)\right) \tag{173}$$

We implement (154) in (169), it comes:

$$\frac{\sigma_i^{rr}}{\rho_i^2}\frac{\partial}{\partial r_i}\sum_j m_j W_c = \frac{2}{h^2}\frac{\sigma_i^{rr}}{\rho_i^2}\sum_j m_j W_c(r_i - r_j\frac{I_1}{I_0}) \tag{174}$$

We implement (155) in (170), it comes:

$$-\frac{\sigma_i^{rz}}{\rho_i^2}\frac{\partial}{\partial_i}\sum_j m_j W_c = \frac{2}{h^2}\frac{\sigma_i^{rz}}{\rho_i^2}\sum_j m_j W_c(z_i - z_j) \tag{175}$$

We can do the sum of (167), (168),(169), (170) and (171). Finally:

$$\frac{\partial U^z}{\partial t} = \frac{2}{h^2}\sum_j m_j W_c\left[\frac{\sigma_j^{rz}}{\rho_j^2}\left(r_i\frac{I_1}{I_0} - r_j\right) + \frac{\sigma_j^{zz}}{\rho_j^2}(z_i - z_j) + \frac{\sigma_i^{rz}}{\rho_i^2}\left(r_i - r_j\frac{I_1}{I_0}\right) + \frac{\sigma_i^{rz}}{\rho_i^2}(z_i - z_j)\right]$$

Those equations give the accelerations of every particle on every axis. The good part of having this kind of development is that it is easy to represent them in a computer program.

3.5.2 Calculous of the constitutive equations for the strain rates

The constitutive equations of the strain rates can be calculated as well thanks to the SPH method and the velocity vectors of the particles.We remind that $\underline{U}^* = \begin{bmatrix} U^{r*} \\ 0 \\ U^{z*} \end{bmatrix}$.

The variation of the stress over the time can be determined thanks to the sum of the velocity gradient and its transposed. It comes:

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{2}(gradU + grad^T U)$$

Since $\underline{\underline{gradU}} = \begin{bmatrix} \frac{\partial U^{r*}}{\partial r} & 0 & \frac{\partial U^{r*}}{\partial z} \\ 0 & \frac{1}{r}\left(\frac{\partial U^{\theta*}}{\partial \theta} + U^{r*}\right) & 0 \\ \frac{\partial U^{z*}}{\partial r} & 0 & \frac{\partial U^{z*}}{\partial z} \end{bmatrix}$

It comes:

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{2}\begin{bmatrix} 2\frac{\partial U^{r*}}{\partial r} & 0 & \frac{\partial U^{z*}}{\partial r} + \frac{\partial U^{r*}}{\partial z} \\ 0 & \frac{2}{r}\left(\frac{\partial U^{\theta*}}{\partial \theta} + U^{r*}\right) & 0 \\ \frac{\partial U^{z*}}{\partial r} + \frac{\partial U^{r*}}{\partial z} & 0 & 2\frac{\partial U^{z*}}{\partial z} \end{bmatrix} \tag{176}$$

We can use the covariance applied to the vectors to simplify the equations. As for the contravarient tensor, we will apply the covariance because of the change of basis of the particles. The covariance equation comes: $U^* = PU$.

We remind that $P = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Finally it comes:

$$\underline{U^*} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} U^r \\ 0 \\ U^z \end{bmatrix} = \begin{bmatrix} cU^r \\ -sU^r \\ U^z \end{bmatrix} \tag{177}$$

Using (176), we can set:

$$\begin{cases} \frac{\partial \epsilon^{rr}}{\partial t} &= \frac{\partial U^{r*}}{\partial r} \\ \frac{\partial \epsilon^{zz}}{\partial t} &= \frac{\partial U^{z*}}{\partial z} \\ \frac{\partial \epsilon^{rz}}{\partial t} &= \frac{1}{2}\left(\frac{\partial U^{r*}}{\partial z} + \frac{\partial U^{z*}}{\partial r}\right) \\ \frac{\partial \epsilon^{\theta\theta}}{\partial t} &= \frac{\partial U\theta^*}{\partial \theta} + \frac{U^{r*}}{r} \end{cases} \tag{178}$$

We will use the method of the equation (129) to improve the accuracy of the final results. It comes:

$$
\begin{cases}
\dfrac{\partial \epsilon^{rr}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (U_i^{r*} - U_j^{r*})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{zz}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (U_i^{z*} - U_j^{z*})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{rz}}{\partial t} & = -\dfrac{1}{2}\dfrac{\partial}{\partial z_i}\sum (U_i^{r*} - U_j^{r*})\dfrac{m_j}{\rho_j}W_{ij} \\
 & -\dfrac{1}{2}\dfrac{\partial}{\partial r_i}\sum (U_i^{z*} - U_j^{z*})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{\theta\theta}}{\partial t} & = \dfrac{1}{r_i}\Big[\sum \dfrac{\partial}{\partial \theta_j}(U_j^{\theta*} - U_i^{\theta*})\dfrac{m_j}{\rho_j}W_{ij} + \\
 & \sum U_j^{r*}\dfrac{m_j}{\rho_j}W_{ij}\Big]
\end{cases}
\tag{179}
$$

We will apply the covarient vector (176) on the equations:

$$
\begin{cases}
\dfrac{\partial \epsilon^{rr}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum c(U_i^{r} - U_j^{r})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{zz}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{rz}}{\partial t} & = -\dfrac{1}{2}\dfrac{\partial}{\partial z_i}\sum c(U_i^{r} - U_j^{r})\dfrac{m_j}{\rho_j}W_{ij} - \\
 & \dfrac{1}{2}\dfrac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{\theta\theta}}{\partial t} & = \dfrac{1}{r_i}\Big[-\sum \dfrac{\partial}{\partial \theta_j}\Big(s(U_j^{r} - U_i^{r})\dfrac{m_j}{\rho_j}W_{ij}\Big) + \\
 & \sum cU_j^{r}\dfrac{m_j}{\rho_j}W_{ij}\Big]
\end{cases}
\tag{180}
$$

Since the covariant changes the basis in function of the particles, we choose $\theta_i = 0$. It comes:

$$
\begin{cases}
\dfrac{\partial \epsilon^{rr}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (cU_i^{r} - U_j^{r})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{zz}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{rz}}{\partial t} & = -\dfrac{1}{2}\dfrac{\partial}{\partial z_i}\sum (cU_i^{r} - U_j^{r})\dfrac{m_j}{\rho_j}W_{ij} - \\
 & \dfrac{1}{2}\dfrac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\dfrac{m_j}{\rho_j}W_{ij} \\
\dfrac{\partial \epsilon^{\theta\theta}}{\partial t} & = \dfrac{1}{r_i}\Big[-\sum \dfrac{\partial}{\partial \theta_j}\Big(sU_j^{r}\dfrac{m_j}{\rho_j}W_{ij}\Big) + \\
 & \sum cU_j^{r}\dfrac{m_j}{\rho_j}W_{ij}\Big] \\
 & = \dfrac{1}{r_i}\Big[-\sum \dfrac{\partial}{\partial \theta_j}\Big(sU_j^{r}\dfrac{m_j}{\rho_j}\dfrac{\partial W_{ij}}{\partial \theta_j}\Big) + \\
 & \sum (c-c)U_j^{r}\dfrac{m_j}{\rho_j}W_{ij}\Big]
\end{cases}
\tag{181}
$$

We implement (143) and (146) in (181). It comes:

$$
\begin{cases}
\dfrac{\partial \epsilon^{rr}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (U_i^{r} - U_j^{r}\dfrac{I_1}{I_0})\dfrac{m_j}{\rho_j}W_c \\
\dfrac{\partial \epsilon^{zz}}{\partial t} & = -\dfrac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\dfrac{m_j}{\rho_j}W_c \\
\dfrac{\partial \epsilon^{rz}}{\partial t} & = -\dfrac{1}{2}\dfrac{\partial}{\partial z_i}\sum (U_i^{r} - U_j^{r}\dfrac{I_1}{I_0})\dfrac{m_j}{\rho_j}W_c - \\
 & \dfrac{1}{2}\dfrac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\dfrac{m_j}{\rho_j}W_c \\
\dfrac{\partial \epsilon^{\theta\theta}}{\partial t} & = \dfrac{2}{h^2}\sum U_j^{r}\dfrac{m_j}{\rho_j}r_jW_c\dfrac{I_0 - I_2}{2I_0}
\end{cases}
\tag{182}
$$

We implement (154) in $\dfrac{\partial \epsilon^{rr}}{\partial t}$, it comes:

$$
\frac{\partial \epsilon^{rr}}{\partial t} = \sum \Big(U_i^r(r_i - r_j\frac{I_1}{I_0}) - U_j^r(r_i\frac{I_1}{I_0} - \frac{r_j}{2}(1 + \frac{I_2}{I_0}))\Big)\frac{m_j}{\rho_j}W_c
\tag{183}
$$

We implement (154) in $\dfrac{\partial \epsilon^{zz}}{\partial t}$, it comes:

$$
\frac{\partial \epsilon^{zz}}{\partial t} = \frac{2}{h^2}\sum (U_i^z - U_j^z)(z_i - z_j)\frac{m_j}{\rho_j}W_c
\tag{184}
$$

We implement (148) in $\dfrac{\partial \epsilon^{rz}}{\partial t}$, it comes:

$$
\frac{\partial \epsilon^{rz}}{\partial t} = \frac{1}{h^2}\sum W_c\frac{m_j}{\rho_j}\Big((U_i^r - U_j^r\frac{I_1}{I_0})(z_i - z_j) + (U_i^z - U_j^z)(r_i - r_j\frac{I_1}{I_0}\Big)
\tag{185}
$$

This demonstration gives the calculous of the strains velocity that are not equals to zero for a cylindric system.

3.5.3 Calculous of the curls in an axisymmetric system

The SPH method allows to determine the curls as well. In our case, the only non-zero curl is on the θ-axis. It comes:

$$
\frac{1}{2}\underline{rot}(\underline{U}) = \begin{bmatrix}\Omega^r \\ \Omega^\theta \\ \Omega^z\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\dfrac{\partial}{\partial r} \\ 0 \\ \dfrac{\partial}{\partial z}\end{bmatrix} \wedge \begin{bmatrix}U^{r*} \\ 0 \\ U^{z*}\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ \dfrac{\partial U^{r*}}{\partial z} - \dfrac{\partial U^{z*}}{\partial r} \\ 0\end{bmatrix}
\tag{186}
$$

Finally, $\Omega^\theta = \dfrac{1}{2}\Big(\dfrac{\partial U^{r*}}{\partial z} - \dfrac{\partial U^{z*}}{\partial r}\Big)$

We can determine this equation using (129). Plus, $\nabla_j W_{ij} = -\nabla_i W_{ij}$. It comes:

$$
\Omega^\theta = \frac{1}{2}\Big[-\frac{\partial}{\partial z_i}\sum (U_i^{r*} - U_j^{r*})\frac{m_j}{\rho_j}W_{ij} + \frac{\partial}{\partial r_i}\sum (U_i^{z*} - U_j^{z*})\frac{m_j}{\rho_j}W_{ij}\Big]
\tag{187}
$$

We will apply the covarient vector (176) on the equations:

$$
\Omega^\theta = \frac{1}{2}\Big[-\frac{\partial}{\partial z_i}\sum (cU_i^{r} - cU_j^{r})\frac{m_j}{\rho_j}W_{ij} + \frac{\partial}{\partial r_i}\sum (U_i^{z} - U_j^{z})\frac{m_j}{\rho_j}W_{ij}\Big]
\tag{188}
$$

Since the covariant changes the basis in function of the particles, we choose $\theta_i = 0$. It comes:

$$\Omega^\theta = \frac{1}{2}\Big[-\frac{\partial}{\partial z_i}\sum(U_i^r - cU_j^r)\frac{m_j}{\rho_j}W_{ij} + \frac{\partial}{\partial r_i}\sum(U_i^z - U_j^z)\frac{m_j}{\rho_j}W_{ij}\Big] \tag{189}$$

We implement (143) in (189), it comes:

$$\Omega^\theta = \frac{1}{2}\Big[-\frac{\partial}{\partial z_i}\sum(U_i^r - U_j^r\frac{I_1}{I_0})\frac{m_j}{\rho_j}W_c + \frac{\partial}{\partial r_i}\sum(U_i^z - U_j^z)\frac{m_j}{\rho_j}W_c\Big] \tag{190}$$

We implement (154), (155) and (157) in (190). It comes:

$$\Omega^\theta = \frac{1}{h^2}\sum\Big[(U_i^r - U_j^r\frac{I_1}{I_0})(z_i - z_j) - (U_i^z - U_j^z)(r_i - r_j\frac{I_1}{I_0})\Big]\frac{m_j}{\rho_j}W_c \tag{191}$$

As we saw, the SPH method allows to calculate every terms we need for a given system. In an axisymmetric domain, the acceleration of the particles can be calculated thanks to the stress tensor, but the strain and the curl can be calculated as well thanks to the velocity. In a computer program, the stress and the velocity are the terms that the user can choose (generally, the density depends on the mass of a particle, which is given depending on the mass of a particle of the system). Some other terms can be calculated thanks to the SPH method like the energy equation using the velocity as variable.

Conclusion

As a recent method, a lot of researches are currently being worked on in order to solve the SPH method's drawbacks. We divided our researches into two parts. Firstly, we studied the SPH method and set some basic assumptions. Then, the researches have been focused on the boundary condition issues for a 1D and 2D cases. We gave the required parameters' expressions of the ghost particles in order to implement this method to solve the problem of consistence at the boundaries. Finally, we did the demonstrations to determine the particles' accelerations, the constitutive equations for the strain rate and the curls for the SPH method for any axi-symmetric system. Of course, our researches are not exhaustive since the SPH method can be applied for any function. It was then impossible to work on every mecanical systems. The SPH method is very useful despite the long computing time because it is capable of solving problems which are impossible or difficult to solve with regular mesh methods. Indeed, this method determines the particles' parameters for gases impact.

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