TP2: Nonlinear design of concrete bridge towers

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Introduction

In this report, we present the design of a concrete tower from non-linear theory. In this sense we will provide stability graphs and criteria for one to design such a system, together with an insight in the different methods used to get these criteria.

First and foremost, here is the model used to work on the system:

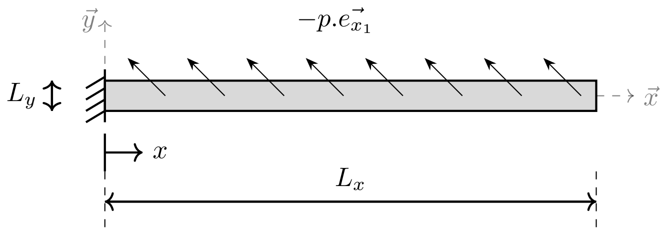


Figure 1: Slender beam model - inclined beam under its own weight
with $\underline{e}_{x_1} = \cos(\alpha)\,\underline{e}_x - \sin(\alpha)\,\underline{e}_y$.

The system is a slender beam whose geometry and material are defined by (some of which will slightly vary):

- $\rho = 2500\ kg.m^{-3}$
- $g = 9.81\ m.s^{-2}$
- $f_c = 45\ MPa$
- $E = 1100\ Pa$
- $\nu = 0.35$
- $\alpha = 0.0045\ rad$

Together with these informations we provide the variational problem we want to solve :

Find $\underline{u} \in \mathcal{C} \equiv \{\underline{u} : H^1(\Omega),\ \underline{u}(x_1, x_2)|_{x_1=0} = (0,0)$ and $\underline{u}'(x_1, x_2)|_{x_1=0} = (0,0)\}$ such that :

$$\underbrace{\int_\Omega \underline{\underline{\sigma}}(\underline{\underline{\epsilon}}(\underline{u})) : \underline{\underline{\epsilon}}(\underline{v})\,\mathrm{d}x\,\mathrm{d}y}_{\mathrm{a}(\underline{\mathrm{u}},\ \underline{\mathrm{v}})} = \underbrace{\int_\Omega \underline{b}\cdot\underline{v}\,\mathrm{d}x\,\mathrm{d}y}_{\mathrm{l}(\underline{\mathrm{v}})} \quad \forall \underline{v} \in \mathcal{C}_0 \equiv \mathcal{C} \tag{1}$$

with $\underline{b}$ the body forces.

1 Analytical approach

One way to design a beam is to first state its geometrical limits provided the material characteristics.

Q-1.1 In a first hand we deduce the maximum height of the tower given its compressive strength f_c and density, in the straight case $\boldsymbol{\alpha = 0°}$.

It comes that the stress in any point of the beam shouldn't be over the limit value f_c such that: $f_c = \dfrac{F}{S} = \dfrac{m\,g}{S} = \dfrac{\rho.L_x L_y L_z.g}{L_y L_z} \Longrightarrow L_x \approx 1835$ m.

This length L_x does not seem realistic if one considers the height of the Millau cable-stayed bridge (270 m) and the height of the Burj Khalifa — highest tower in the world (828 m). This huge value is partly due to the fact one neglected the beam's deformation in the model.

Q-1.2-a On another hand, it's possible to consider an approach from continuous mediums' theory while considering the lack of verticality. We thus introduce a deflection $\boldsymbol{\alpha \neq 0°}$:

$$\underline{R} = p\cos(\alpha)\,(x - L_x)\,\underline{e}_x - p\sin(\alpha)\,(x - L_x)\,\underline{e}_y \tag{2}$$

$$\underline{M} = \frac{p}{2}\sin(\alpha)\,(x - L_x)^2\,\underline{e}_z \tag{3}$$

From which one deduces:

$$\sigma_{xx} = \frac{N}{S} - y\frac{M_z}{I_{Gz}}$$

equation which now takes into account flexion in the beam.

Q-1.2-b We deduce max normal stress:

$$|\sigma_{xx}|_{max} = \frac{p\cos(\alpha)}{S}\,|x - L_x|_{max} + |y|\,\frac{p\sin(\alpha)}{2\,I_{G_z}}\,|x - L_x|^2_{max}$$

which takes its max in $x = 0$ and $y = \pm\frac{L_y}{2}$.

Q-1.2-c Eventually in $x = 0$:

$$* \; y = \frac{-L_y}{2} \qquad \sigma_{xx}(0) = -\frac{p\cos(\alpha)}{S}L_x + \frac{L_y}{2}\frac{p\sin(\alpha)}{2I_{G_z}}L_x^2$$

$$* \; y = \frac{+L_y}{2} \qquad \sigma_{xx}(0) = -\frac{p\cos(\alpha)}{S}L_x - \frac{L_y}{2}\frac{p\sin(\alpha)}{2I_{G_z}}L_x^2$$

We then deduce that point $(0, \frac{+L_y}{2})$ will be associated to compression while traction may appear in $(0, \frac{-L_y}{2})$:

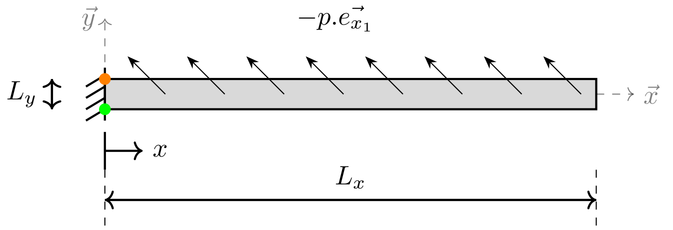


Figure 2: ● max. compressive stress ● max. tensile stress

As concrete doesn't respond well to traction, in addition to compressive strength criterion, we have a criterion on traction that one will be able to exploit for bridge design.

Q-1.2-d The two conditions on tensile and compression maxima give the following inequalities:

$$\begin{cases} \text{Traction:} & -\dfrac{p\cos(\alpha)}{S}L_x + \dfrac{L_y}{2}\dfrac{p\sin(\alpha)}{2I_{G_z}}L_x^2 < 0 \\[2ex] \text{Compression:} & -\dfrac{p\cos(\alpha)}{S}L_x - \dfrac{L_y}{2}\dfrac{p\sin(\alpha)}{2I_{G_z}}L_x^2 > -f_c \end{cases}$$

Q-1.2-e These inequalities are represented in figure 3, each representing a criterion one must ensure for bridge design. These two criteria enable one to determine a secure zone which is represented in blue in the graphic. In this area, both conditions are respected:

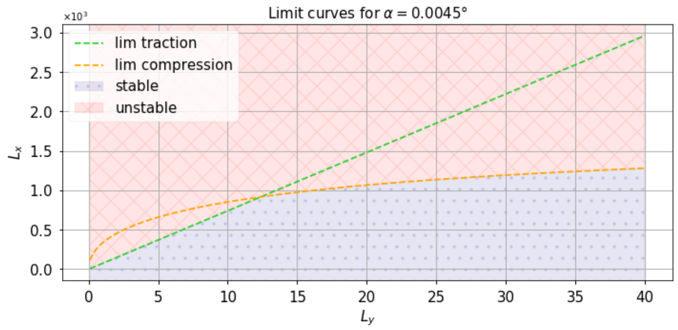


Figure 3: Limiting curves for bridge design

Q-1.2-f,g From figure 3 one can deduce dimension for one length provided the other one :

- $L_y = 10$ m $\rightarrow L_x \approx 735$ m. In that case, the limiting cause is the condition imposed on the maximum tensile stress (*i.e.* $\sigma_{xx}|_{x=0, y=\frac{-L_y}{2}} < 0$)
- $L_y = 20$ m $\rightarrow L_x \approx 1066$ m. In that case, the limiting cause is the condition imposed on the maximum compressive stress (*i.e.* $\sigma_{xx}|_{x=0, y=\frac{+L_y}{2}} < f_c$)

Q-1.2-h The value of L_z is here imposed at 1 such that we work by unit of thickness, and so we may change this value in 3D cases.

Moreover, increasing the value of L_z would furthermore limit the value of L_x.

Q-1.3 We now consider the deformation of the tower under the applied bulk load `b0` which depends on `t` , a parameter that will regulate the intensity of the force.

2 Nonlinear quasi-static response analysis

In addition to consider the flexion of the beam, we here consider a numerical approach that works with quadratic 2D elements (i.e. T6 elements).

Q-2.1 First, we charge the beam along $\underline{e}_{x_1}$ (figure 2) and we observe deformation of the beam by looking to the normalized deflection $\frac{u_L}{L_x}$ at free end $x = L_x$:

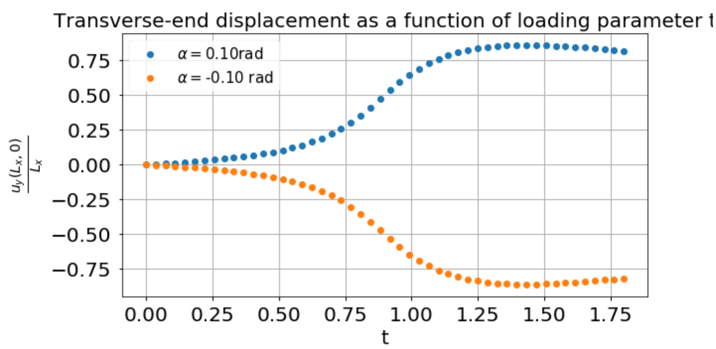


Figure 4: Bifurcation diagram, $\alpha = \pm 0.1$ rad

It appears the bifurcation diagram, as aimed by setting $\boldsymbol{\alpha \neq 0°}$.

Q-2.2 Let's see how α influences bifurcation:

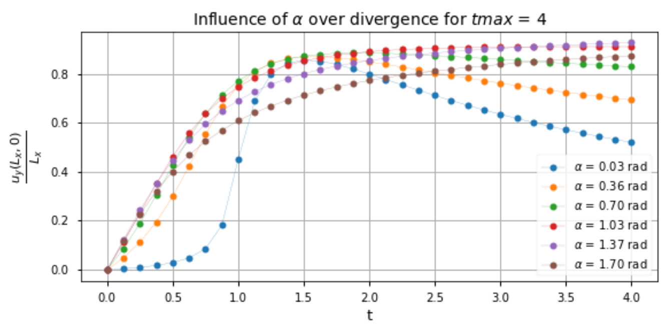


Figure 5: Bifurcation diagram as a function of α

One can see that increasing furthermore α makes bifurcation diagram closer to theoretical one. We then ensure buckling of the beam by setting a constant angle α.

Eventually, there is a limit when $\alpha \to \frac{\pi}{2}$ where the beam suffers a too high flexion (i.e. non-physical behavior).

Q-2.3 Let's take a moment to discuss numerical implementation and study of bifurcation .
As we work on a potentially unstable system, one must be careful in the way he/she modelizes and solves the problem 1.

We already mentioned the fact we use T6 elements for the mesh. This allows to account for non-linear relation and furthermore keep some reliability on the values, especially for strain and stress (i.e. continuity, precision, etc).

For the solving, we want to minimize energy

$$P = \int_\Omega \frac{\lambda}{2}.tr(\underline{\underline{E}})^2 + \mu \underline{\underline{E}} : \underline{\underline{E}} \, \mathrm{d}x - \int_\Omega \underline{b0}.\underline{u} \, \mathrm{d}x \quad \forall \underline{u} \in \mathcal{C}$$

where $\underline{\underline{E}}$ is the non-linear deformation tensor.

To make solving more stable for now, we set the bulk force $b0 = \rho \, g$ using coefficient $\Gamma = 12 \, \dfrac{\rho \, g \, L_x}{E \, L_y^2}$ that derives from and provides a limit $\Gamma = 7.84$ below which the solution should be stable.

To finish, no matter what expression one uses, it's gonna be necessary to consider low load and time steps to ensure convergence of the solver. No much work has been made on this subject but one can guess a smooth loading together with a step-by-step solving in such a possibly-unstable case is necessary.

3 (Non-linear) Bridge design

We introduce a new criterion for one to get limiting L_x out of a given L_y : $\Gamma = 12.\frac{\rho.g.L_x^3}{E.L_y^2}$. The system then is said to be stable if $\Gamma < 7.8$ and unstable otherwise. [1]

In this part, the point will be to extend the theory of buckling to the non-linear analysis and compare Γ theory to numerical calculations.

We will consider the variables of except $E = 37E^9 \; Pa$. To consider another material with such a high Young modulus will force some changing in the code.

We then will work with mostly dimensionless variables in order to ensure convergence of the solving :

- $\overline{\rho.g} = \dfrac{(\rho.g)_{real}}{(\rho.g)^*} = \dfrac{(\rho.g)_{real}}{\sigma^*/x^*}$
- $\bar{f_c} = \dfrac{fc_{real}}{fc^*} = \dfrac{fc_{real}}{\sigma^*}$
- $\bar{E} = \dfrac{E_{real}}{E^*} = \dfrac{E_{real}}{\sigma^*}$

with $\sigma^* = E_{real}$ and $x^* = 1$.

To summarize, on one hand we will want to check Γ condition $\Gamma = 12.\frac{\rho.g.L_x^3}{E.L_y^2} < 7.8$ and on the other hand, we will check inequalities of part 1 in the non-linear case.

The study led to the following graph :

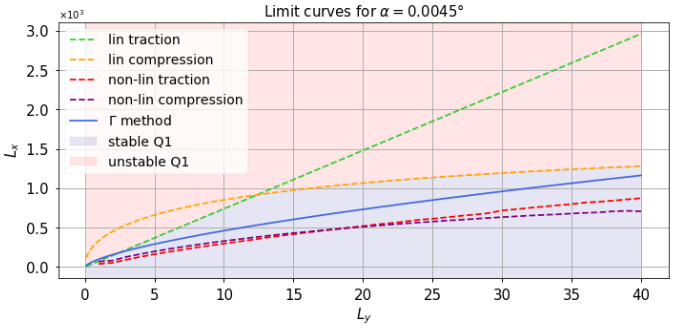


Figure 6: Limitation graphs with Γ method

We can see the fact that considering non-linear deformation brings stronger limitations on the values L_x/L_y can take. We moreover see that the criterion Γ is close to the numerical solution, at least closer than is the linear theory.

We eventually highlight the fact that non-linear terms are not neglectable in this case, and we deduced a new physical limit for the beam's design. Let's compare the different theory in some characteristic points.

Q-3.1 For $L_y = 10$m, the limit is in traction for both cases

- linear : $\boldsymbol{L_x \approx}$ **735m**
- Γ : $\boldsymbol{L_x \approx}$ **461m**
- non-linear : $\boldsymbol{L_x \approx}$ **298m**

Q-3.2 For $L_y = 20$m, the limit is in compression for both cases

- linear : $\boldsymbol{L_x \approx}$ **1066m**
- Γ : $\boldsymbol{L_x \approx}$ **732m**
- non-linear : $\boldsymbol{L_x \approx}$ **512m**

Q-3.3 For $L_y = 35$m, the limit once again is in compression for both cases

- linear : $L_x \approx 1241$m
- Γ : $L_x \approx 1063$m
- non-linear : $L_x \approx 680$m

To conclude on the criterion Γ, it is consistent for low values of L_y while not much for higher values. To have considered a non-linear theory highlights the fact the case is in fact very limit geometrically. Let's now see how the system behaves in the limit charging cases, no matter if one is at a geometrical limit.

4 Stability Analysis

In this part we develop the idea of buckling and stability of the structure.

Bifurcation point One can reach the first bifurcation point P1 considering $\alpha = 0\,\mathrm{rad}$ (blue curve). Slightly increasing α (up to 0.0045 rad — orange curve) is a way to reach the second bifurcation point P2 (*cf* figure 7): one can notice a displacement jump between the two angles at $t = 1$.

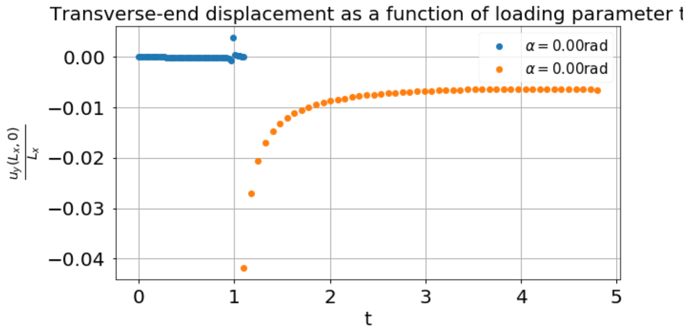
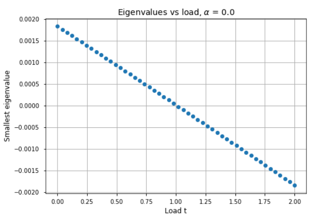


Figure 7: Second bifurcation point

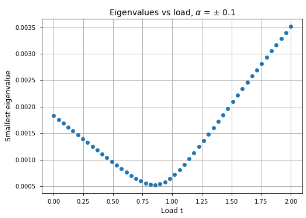
Stability of the branches One considers the bifurcation diagram (4) and computes the associated eigenvalues to determine the stability of its different branches. Indeed, "stability depends on the minimal eigenvalue of the Hessian matrix". This matrix — second derivative of the energy — is the jacobian in the code.

One studied the eigenvalues for three possible cases: $\alpha = 0\,\mathrm{rad}$ and $\alpha = \pm 0.1\,\mathrm{rad}$ (*cf* figure 8). Their sign determine the stability of the branch: positive eigenvalues are linked to stable bifurcation branches while negative ones are associated to unstable bifurcation branches.

Hence, one can deduce from the figures below branches linked to $\alpha = \pm 0.1\,\mathrm{rad}$ are stable whatever the load is. The case $\alpha = 0\,\mathrm{rad}$ delivers positive eigenvalues until $t = 1$ which lead to a stable branch. Beyond this point, eigenvalues are negative which is linked to an unstable branch.



(a) $\alpha = 0\,\mathrm{rad}$



(b) $\alpha = \pm 0.1\,\mathrm{rad}$

Figure 8: Study of the eigenvalues' sign

This can be interpreted as follows: when the beam is at $\alpha = 0$, a small inference can change the equilibrium of the structure when it is submitted to a certain load, explaining consequently the unstable equilibrium.

Conclusion & Perspectives

Throughout this paper we mainly studied the influence of α over the end-displacement in linear and non-linear cases, as well as the stability.

Increasing α leads to a general behaviour of the solution and a limit angle can then be determined (the normalized end-displacement doesn't grow as rapidly than for the previous values of α).

This project provides us an overview of several criteria used to dimension a structure as well as the stability analysis (sustainability and safety of the structure).

Finally, not much work has been done on the convergence of the solver but one can already determine it depends on the load and on time steps.

References

[1] "Opera Magistris (Elements of Applied Mathematics)" v3.7 online reference Chapter 15, section 2.2.4.1, ISBN 978-2-8399-0932-7. Available on: https://archive.org/details/OperaMagistris