Exo 2

La sol : $u(x,t) = (A\cos(\beta x) + B\sin(\beta x))(C\cos(\beta c t) + D\sin(\beta c t))$.

$u(0,t) = 0 \Rightarrow 0 = A(C\cos(c\beta t) + D\sin(c\beta t))$

$\Rightarrow \boxed{A = 0}.$

$u(\ell,t) = 0 \Rightarrow B\sin\beta\ell = 0 \quad (\boxed{B \neq 0})$

$\beta = \frac{n\pi}{\ell} \Rightarrow \lambda_n = \frac{n^2\pi^2}{\ell^2}.$

$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right),$

$\forall n = 1, 2, \ldots$

$\Rightarrow u_n(x,t) = \sin\left(\frac{n\pi x}{\ell}\right)\left(C\cos\left(\frac{n\pi c t}{\ell}\right) + D\sin\left(\frac{n\pi c t}{\ell}\right)\right)$

$\frac{\partial u_n}{\partial t}(x,0) = 0.$

$\Leftrightarrow 0 = \sin\left(\frac{n\pi x}{\ell}\right) D\,\frac{n\pi c}{\ell} \Rightarrow \boxed{D = 0}$

Par superposition :

$$u(x,t) = \sum_{n=1}^{+\infty} C_n \sin\frac{n\pi x}{\ell}\cos\left(\frac{n\pi c t}{\ell}\right)$$

$u(x,0) = a\sin\frac{\pi x}{\ell}\cos\frac{5\pi x}{\ell}.$

$\Leftrightarrow a\sin\frac{\pi x}{\ell}\cos\frac{5\pi x}{\ell} = \sum_{n=1}^{+\infty} C_n \sin\left(\frac{n\pi x}{\ell}\right)$

$\Leftrightarrow$

$$\Leftrightarrow \frac{a}{2}\left[\sin\frac{6\pi x}{\ell} - \sin\frac{4\pi x}{\ell}\right] = C_1 \sin\frac{\pi x}{\ell} +$$

$$+ C_2 \sin\frac{2\pi x}{\ell} + \dots\dots$$

Par identification :

$\Rightarrow C_1 = C_2 = C_3 = 0$; $C_4 = -\frac{a}{2}$, $C_5 = 0$,

$C_6 = \frac{a}{2}$, $C_7 = C_8 = \dots = 0$

$$\Rightarrow \boxed{u(x,t) = \frac{a}{2}\left[\sin\frac{6\pi x}{\ell}\cos\frac{6\pi c t}{\ell} - \sin\frac{4\pi x}{\ell}\cos\frac{4\pi c t}{\ell}\right].}$$

Exo 3 bis

$u_t = k^2 u_{xx}$; $u(x,0) = 25$.
$u(0,t) = 0$
$u(20,t) = 60$

(a)

$\frac{\partial}{\partial t}(3x) = 0 = k^2 \frac{\partial^2}{\partial x^2}(3x) \Rightarrow u_s$ sol. de (4).

$$\left.\begin{array}{l} u_s(0) = 3 \times 0 = 0 \\ u_s(20) = 3 \times 20 = 60 \end{array}\right\} \text{ C.L. OK.}$$

(b) $u(x,t) = v(x,t) + u_s(x)$

$\Rightarrow \frac{\partial}{\partial t}(v + u_s) = k^2 \frac{\partial^2}{\partial x^2}(v + u_s)$

$\Leftrightarrow \frac{\partial v}{\partial t} = k^2 \frac{\partial^2 v}{\partial x^2}$ (d'après Q a))

$u(0,t) = v(0,t) + u_s(0) = 0 \Rightarrow v(0,t) = 0$

$u(20,t) = v(20,t) + u_s(20) = 60 \Rightarrow v(20,t) = 60 - 60 = 0$

$\Rightarrow$ v vérifie l'éq. de la chaleur av. des C.L. homog.

De plus :

$$u(x,0) = v(x,0) + \underbrace{u_s(x)}_{3x} = 25$$

$$\Leftrightarrow v(x,0) = 25 - 3x \quad \text{C.i. pour } v.$$

(c) On résoud : $\begin{cases} v_t = k^2 u_{xx} \\ v(0,t) = v(20,t) = 0 \\ v(x,0) = 25 - 3x \end{cases}$

par sép. de var. $v(x,t) = X(x)T(t)$

$$\frac{T'}{k^2 T} = \frac{X''}{X} = -\beta^2 \qquad (\lambda = \beta^2)$$

$$\Rightarrow T(t) = Ce^{-\beta^2 k^2 t}$$

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$

$v(0,t) = 0 \Rightarrow \boxed{A = 0}$

$v(20,t) = 0 \Rightarrow B\sin(20\beta) = 0 \qquad B \neq 0$

$\Rightarrow 20\beta = n\pi \Rightarrow \beta = \frac{n\pi}{20}, \ \forall n = 1, 2, \ldots$

$(b_n = CB)$ $\Rightarrow v(x,t) = \sum_{n=1}^{+\infty} b_n e^{-\left(\frac{n\pi}{20}\right)^2 k^2 t} \sin\left(\frac{n\pi x}{20}\right)$

$$v(x,0) = 25 - 3x = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi x}{20}\right)$$

$$\Rightarrow b_n = \frac{2}{20} \int_0^{20} (25 - 3x)\sin\left(\frac{n\pi x}{20}\right) dx =$$

I.P.P

$$\frac{2}{20}\left[\left[(25-3x)\left(-\frac{20}{n\pi}\cos\left(\frac{n\pi x}{20}\right)\right)\right]_0^{20} - \right.$$

$$\left. - \int_0^{20}\left(-\frac{20}{n\pi}\cos\left(\frac{n\pi x}{20}\right)\right)\times(-3)\,dx\right]$$

$$= \frac{2}{20}\left[(25-60)\left(-\frac{20}{n\pi}\cos(n\pi)\right) - \right.$$

$$\left. - 25\left(-\frac{20}{n\pi}\right) - 0\right] =$$

$$= \frac{2}{20}\left[\frac{700}{n\pi}\times\cos(n\pi) + \frac{500}{n\pi}\right] =$$

$$= \frac{1}{n\pi}\left[70(-1)^n + 50\right]$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{+\infty}\left(\frac{50+70(-1)^n}{n\pi}\right) e^{-\left(\frac{n\pi}{20}\right)^2 k^2 t} \sin\frac{n\pi x}{20}$$

D'où :

$$u(x,t) = 3x + \sum_{n=1}^{+\infty} - 〃 - 〃 - 〃 -$$

Exo 4) Comme la temp. aux bords est indép. de θ,
on cherche $u(r,\theta) = R(r)$

On remplace ds $\Delta u = 0$

$$\Rightarrow rR''(r) + R'(r) = 0. \text{ éq. d'Euler}$$

de sol : $u(r,\theta) = R(r) = A \ln r + B$

$$u(a,\theta) = u_0 \Rightarrow u_0 = A \ln a + B$$

$$u(b,\theta) = u_1 \Rightarrow u_1 = A \ln b + B.$$

$$\Leftrightarrow u_1 - u_0 = A(\ln b - \ln a)$$

$$\Rightarrow A = \frac{u_1 - u_0}{\ln\left(\frac{b}{a}\right)}$$

$$B = u_1 - A \ln b = u_1 - \left(\frac{u_1 - u_0}{\ln\left(\frac{b}{a}\right)}\right) \cdot \ln b$$

$$= \frac{u_1 \ln\left(\frac{b}{a}\right) - (u_1 - u_0)\ln b}{\ln\left(\frac{b}{a}\right)} = \frac{u_0 \ln b - u_1 \ln a}{\ln\left(\frac{b}{a}\right)}$$

$$\Rightarrow u(r,\theta) = \frac{u_1 \ln\left(\frac{r}{a}\right) - u_0 \ln\left(\frac{r}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

Exo 3)
$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

i) $u(0,t) = u(l,t) = 0$

$$u(x,0) = \begin{cases} T_0 & , 0 < x \leq l/2 \\ 0 & , l/2 < x < 1. \end{cases}$$

ii) $u(x,t) = X(x)T(t)$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

Soit $\lambda = \beta^2$

$$T' + k\beta^2 T = 0 \Rightarrow T(t) = Ae^{-k\beta^2 t}$$

$$X'' + \lambda X = 0 \Rightarrow X(x) = B\cos(\beta x) + C\sin(\beta x).$$

avec $X(0) = X(l) = 0$

$$\Rightarrow X(0) = 0 \Rightarrow \boxed{B = 0}$$

$$X(l) = 0 \Rightarrow C\sin(\beta l) = 0 \quad (C \neq 0)$$

$$\Rightarrow \beta l = m\pi, \ \forall m = 1, 2, \ldots$$

$$\Rightarrow \beta = \frac{m\pi}{l} \Rightarrow \boxed{\lambda_m = \frac{m^2\pi^2}{l^2}} \quad \boxed{X_m(x) = \sin\left(\frac{m\pi x}{l}\right)}$$

$$u_m(x,t) = \sum_{m=1}^{+\infty} A_m e^{-k\frac{m^2\pi^2}{l^2}t} \cdot \sin\left(\frac{m\pi x}{l}\right)$$

$$u(x,0) = \sum_{m=1}^{+\infty} A_m \sin\left(\frac{m\pi x}{l}\right) = \begin{cases} T_0 , & 0 < x \leq l/2 \\ 0 , & l/2 < x < 1 \end{cases}$$

$$\Rightarrow A_m = \frac{2}{\ell}\int_0^{\ell} u(x,0)\sin\frac{m\pi x}{\ell}\,dx$$

$$= \frac{2}{\ell}\int_0^{\ell/2} T_0 \sin\frac{m\pi x}{\ell}\,dx$$

$$= \frac{2T_0}{\ell}\left[-\frac{\ell}{m\pi}\cos\frac{m\pi x}{\ell}\right]_0^{\ell/2}$$

$$= \frac{2T_0}{m\pi}\left[1-\cos\frac{m\pi}{2}\right]$$

$$= \frac{4T_0}{m\pi}\sin^2\left(\frac{m\pi}{4}\right).$$

$$\Rightarrow \boxed{u(x,t) = \frac{4T_0}{\pi}\sum_{m=1}^{+\infty}\frac{1}{m}\sin^2\left(\frac{m\pi}{4}\right)\sin\left(\frac{m\pi x}{\ell}\right)e^{-\frac{m^2\pi^2}{\ell^2}kt}}$$