Numerical computation of the energy release rate

G-theta method

- Definition of ERR

$$\Gamma_h = \Gamma \cup [\boldsymbol{P}, \boldsymbol{P} + h\,\boldsymbol{t}]$$ crack set including the additional crack of length h in the tangent direction at the tip

$$\mathsf{P}(\Gamma_h) = \int_{\Omega\backslash\Gamma_h} \left(W(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)) - \boldsymbol{f}\cdot\boldsymbol{u}_h\right)\,\mathrm{d}x - \int_{\partial_N\Omega} \boldsymbol{T}\cdot\boldsymbol{u}_h \mathrm{d}s$$

Potential energy for an additional crack of length h

$$\mathsf{G} = -\left.\frac{d}{dh}\mathsf{P}(\Gamma_h)\right|_{h=0}$$

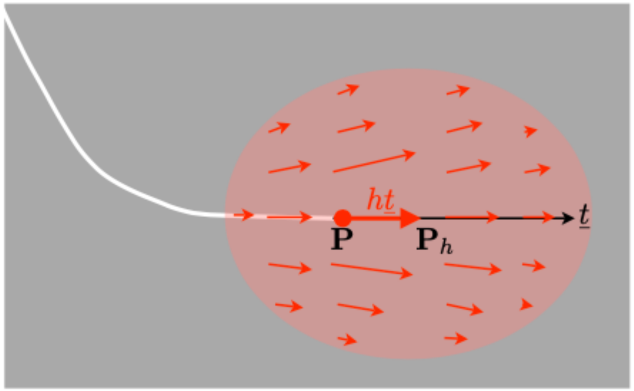
Energy release rate

- Given the solution for (u, σ) of the equilibrium problem for the given length h=0, and a θ-field such that

$$\boldsymbol{\theta} = \boldsymbol{t}, \quad \boldsymbol{\theta}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \quad \boldsymbol{\theta} = \boldsymbol{0} \text{ outside a neighbourhood of the tip where the crack is straight}$$

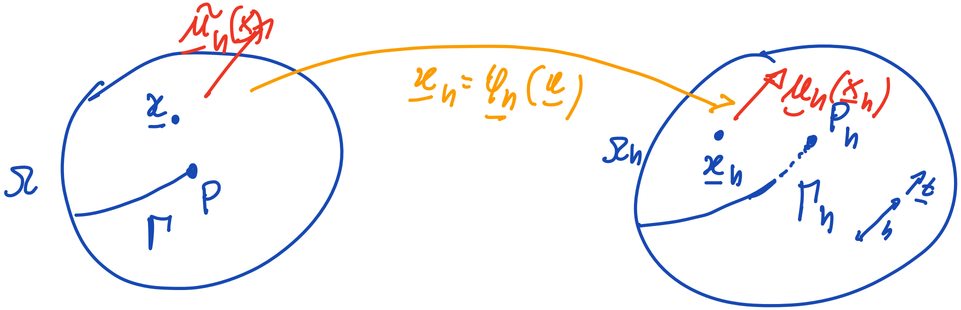
- The energy release rate is given by (the result is independent of θ

$$\boxed{\mathsf{G} = \int_{\Omega\backslash\Gamma} \left(\boldsymbol{\sigma} : (\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\theta}) - W(\boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{div}\boldsymbol{\theta} + \mathrm{div}(\boldsymbol{f}\otimes\boldsymbol{\theta})\cdot\boldsymbol{u}\right)\mathrm{d}x}$$



The formula can be generalised for curved cracks in 3d non-linear elastic materials
and pre-stresses. But the material constants and the loading should be regular close to the tip

PROOF FOR $\underline{f}, \underline{T} = 0$



$\tilde{\underline{u}}_h(\underline{x}) = \underline{u}_h(\underline{\varphi}_h(\underline{x}))$

$\forall \underline{v} \in \mathcal{E}_0(\Gamma_h)$

$\underline{u}_h : \underline{u}_h \in \mathcal{E}(\Gamma_h)$

$$\int_{\Omega_h/\Gamma_h} \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}_h(\underline{u}_h)) : \underline{\underline{\varepsilon}}_h(\underline{v})\, d\Omega_h = \int_{\Omega_h} \underline{f}\cdot\underline{v}\, d\Omega_h + \int \underline{T}\cdot\underline{v}\, ds$$

$$\underline{\underline{\varepsilon}}_h(\underline{u}_h) = \mathrm{sym}\left(\frac{\partial \underline{u}_h}{\partial \underline{x}_h}\right), \quad \underline{\varphi}_h(\underline{x}) = \underline{x} + h\,\underline{\theta}(\underline{x}), \quad \underline{\underline{F}}_h = \frac{\partial \underline{\varphi}_h}{\partial \underline{x}} = \underline{\underline{I}} + h\nabla\underline{\theta}$$

Useful relations

$$\frac{d}{dh}\boldsymbol{\varphi}_h\bigg|_{h=0} = \boldsymbol{\theta}, \qquad \frac{d}{dh}\boldsymbol{F}_h^{-1}\bigg|_{h=0} = -\nabla\boldsymbol{\theta}, \qquad \frac{d}{dh}\det\boldsymbol{F}_h\bigg|_{h=0} = \mathrm{div}\boldsymbol{\theta},$$

(we consider $\underline{f}, \underline{T} = \underline{0}$)

$$P(\Gamma_h) = \int_{\Omega_h/\Gamma_h} W(\underline{\underline{\varepsilon}}_h(\underline{u}_h))\, \underline{d\Omega_h} \quad , \quad G = -\left.\frac{dP(\Gamma_h)}{dh}\right|_{h=0}$$

$$P(\Gamma_h) = \int_{\Omega/\Gamma} W(\underline{\underline{\varepsilon}}_h(\underline{\tilde{u}}_h))(\det \underline{\underline{F}}_h)\, d\Omega \quad , \quad \underline{\underline{F}}_h = \frac{\partial \underline{x}_h}{\partial \underline{x}} \quad , \quad \underline{\underline{F}}_h^{-1} = \frac{\partial \underline{x}}{\partial \underline{x}_h}$$

$$\underline{\underline{\varepsilon}}_h(\underline{\tilde{u}}_h) = \left(\frac{\partial}{\partial \underline{x}_h} \underline{\tilde{u}}_h(\underline{x})\right)_{sym} = \left(\frac{\partial \underline{\tilde{u}}_h}{\partial \underline{x}} \cdot \overbrace{\frac{\partial \underline{x}}{\partial \underline{x}_h}}^{\underline{\underline{F}}_h^{-1}}\right)_{sym} = \left(\underline{\underline{\nabla}}\underline{\tilde{u}}_h \cdot \underline{\underline{F}}_h^{-1}\right)_{sym}$$

$$\left.\frac{dP(\Gamma_h)}{dh}\right|_{h=0} = \int_{\Omega/\Gamma} \frac{d}{dh}\left[W\left(sym(\underline{\underline{\nabla}}\underline{\tilde{u}}_h \cdot \underline{\underline{F}}_h^{-1})\right) \det \underline{\underline{F}}_h\right]_{h=0} d\Omega$$

$$\boxed{\frac{d}{dh}(\;) = (\dot{\;})}$$

$$= \int_{\Omega/\Gamma} W\left(sym(\underline{\underline{\nabla}}\underline{\tilde{u}}_h \cdot \underline{\underline{F}}_h^{-1})\right)\Big|_{h=0} \cdot (\det \underline{\underline{F}}_h)^{\cdot}\Big|_{h=0} + \frac{d}{dh}\left(W(sym(\underline{\underline{\nabla}}\underline{\tilde{u}}_h \cdot \underline{\underline{F}}_h^{-1}))\right)\Big|_{h=0} \det \underline{\underline{F}}_h\Big|_{h=0} d\Omega$$

$$\int_{\Omega/\Gamma} \underbrace{\frac{\partial W}{\partial \underline{\underline{\varepsilon}}}}_{\underline{\underline{\sigma}}} : \left((\underline{\underline{\nabla}}\dot{\underline{\tilde{u}}}_h) \cdot \underline{\underline{F}}_h^{-1} + (\underline{\underline{\nabla}}\underline{u}_h) \cdot \dot{\underline{\underline{F}}}_h^{-1}\right)\Big|_{h=0} \cancel{\det \underline{\underline{F}}_h}\Big|_{h=0}^{1} d\Omega$$

$$= \int_{\Omega/\Gamma} W(\underline{\underline{\varepsilon}}(\underline{u})) \underbrace{(\det \underline{\underline{F}}_h)^{\cdot}\Big|_{h=0}}_{\operatorname{div} \underline{\theta}} + \underline{\underline{\sigma}} : \left[\underbrace{(\underline{\underline{\nabla}}\dot{\underline{\tilde{u}}}_h)\Big|_{h=0}}_{} + \underline{\underline{\nabla}}\underline{u} \cdot \underbrace{(\dot{\underline{\underline{F}}}_h^{-1})\Big|_{h=0}}_{-\underline{\underline{\nabla}}\underline{\theta}}\right] d\Omega$$

$$\underline{\tilde{u}}_h = \underline{u}_h\Big|_{h=0} = \underline{u}$$

$$\int \underline{\underline{\sigma}} \cdot \underline{\underline{\nabla}}\dot{\underline{u}}_h\, d\Omega = 0 \quad \text{because of the equilibrium}$$

$$G = -\frac{dP}{d\eta}(\Gamma_\eta)\Big|_{\eta=0} = \int_\Omega \left[\underline{\underline{\sigma}} : \left(\underline{\underline{\nabla \underline{u}}} \cdot \underline{\underline{\nabla \underline{\theta}}} \right) - W(\underline{\underline{\varepsilon}}(\underline{u}))\, \operatorname{div} \underline{\theta} \right] d\Omega$$

(1) $\left.\dfrac{d}{dh}\det(\underline{\underline{F}}_h)\right|_{h=0} = \left.\dfrac{d}{dh}\det(\underline{\underline{I}} + h\underline{\underline{\nabla}}\underline{\theta})\right|_{h=0}$

$= \left.\dfrac{d}{dh}\left(1 + h\,\mathrm{tr}(\underline{\underline{\nabla}}\underline{\theta}) + o(h)\right)\right|_{h=0}$

$= \mathrm{div}\,\underline{\theta}$

(2) $\left.\dfrac{d\underline{\underline{\nabla}}\underline{\tilde{u}}_h}{dh}\right|_{h=0} = \underline{\underline{\nabla}}\,\underline{\mathring{u}}_h$

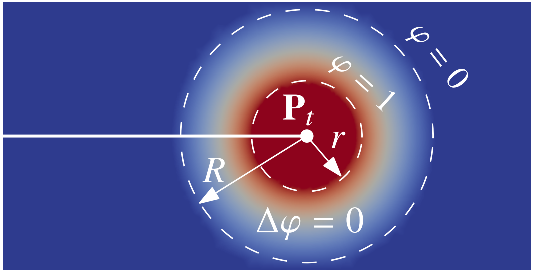
(3) $\left.\dfrac{d\underline{\underline{F}}_h^{-1}}{dh}\right|_{h=0} =$

$\dfrac{d}{dh}\left(\underline{\underline{F}}_h^{-1} \cdot \underline{\underline{F}}_h = \underline{\underline{I}}\right)$

$\Rightarrow \underline{\underline{\mathring{F}}}_h^{-1} \cdot \underline{\underline{F}}_h + \underline{\underline{F}}_h^{-1} \cdot \underline{\underline{\mathring{F}}}_h = 0$

$\left.\underline{\underline{\mathring{F}}}_h^{-1}\right|_{h=0} = \left. - \underline{\underline{F}}_h^{-1}\,\underline{\underline{\mathring{F}}}_h\,\underline{\underline{F}}_h^{-1}\right|_{h=0} = -\underline{\underline{\nabla}}\,\underline{\theta}$

We can solve the following auxiliary scalar problem (see also the code)



Fundamental link between the derivative of the elastic energy and the singularity

$$\boxed{G = \frac{1-\nu^2}{E}(K_I^2 + K_{II}^2)}$$

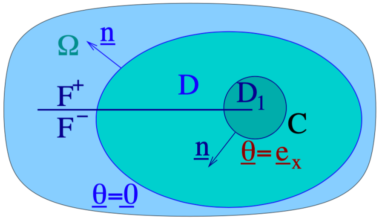
Plane-strain

$$\boxed{G = \frac{1}{2\mu}K_{III}^2}$$

Anti-plane

Proof: see pag 196 Marigo (use G-theta integral and singular solutions)

$$J = \int_C \left(\frac{1}{2} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}) n_x - \boldsymbol{u}_x . \boldsymbol{T}(\boldsymbol{n}) \right) ds$$



- More restrictives hypotheses: straight cracks, no volume forces, … to have the integral independent of the contour
- Less accurate in numerical computations wrt a volume integral

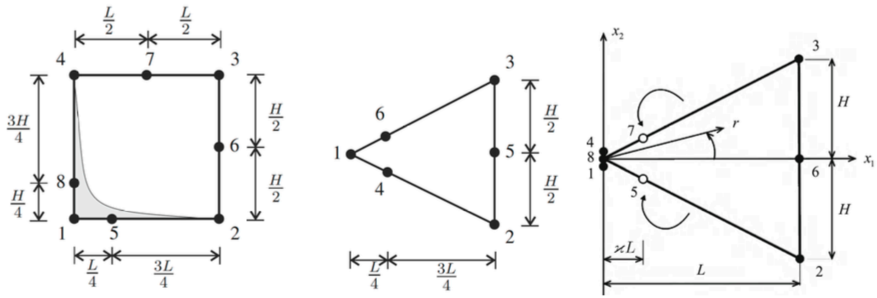
How to extract the stress intensity factors

- We can calculate the stress intensity factors for the single modes using the following integrals where $\mathbf{u}_{\mathrm{I}}$ and $\mathbf{u}_{\mathrm{II}}$ are the expressions for the displacement of the singular solutions in mode I and mode II

$$K_{\mathrm{I}} = -\frac{E}{4(1-\nu^2)} \int_{\Omega} \left(\frac{1}{2} (\boldsymbol{\sigma} : \nabla \boldsymbol{u}_{\mathrm{I}}) \, \mathrm{div} \boldsymbol{\theta} - \boldsymbol{\sigma} . (\nabla \boldsymbol{u}_{\mathrm{I}} . \nabla \boldsymbol{\theta}) \right) d\Omega$$

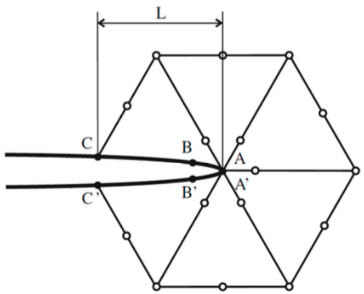
$$K_{\mathrm{II}} = -\frac{E}{4(1-\nu^2)} \int_{\Omega} \left(\frac{1}{2} (\boldsymbol{\sigma} : \nabla \boldsymbol{u}_{\mathrm{II}}) \, \mathrm{div} \boldsymbol{\theta} - \boldsymbol{\sigma} . (\nabla \boldsymbol{u}_{\mathrm{II}} . \nabla \boldsymbol{\theta}) \right) d\Omega$$

- The proof is easy with a nice trick (see Suquet page 79)



(a) Quarter-point quadrilaterals (b) Quarter-point triangles (c) Quarter-point degenerate quadri-laterals

Figure 3.3: Types of 2D quarter-point elements (adapted from Kuna 2012)



$$u\left(r\right) \cong u^{1} + \left(4u^{3} - 3u^{1} - u^{2}\right)\sqrt{\frac{r}{L}} + 2\left(u^{1} + u^{2} - 2u^{3}\right)\frac{r}{L}$$

$$\varepsilon\left(r\right) = \frac{du}{dr} \cong \left(2u^{3} - \frac{3}{2}u^{1} - \frac{1}{2}u^{2}\right)\frac{1}{\sqrt{Lr}} + \frac{2\left(u^{1} + u^{2} - 2u^{3}\right)}{L}$$