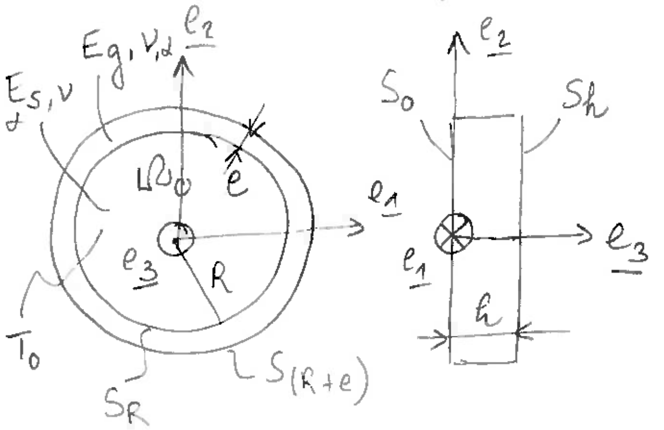
Examen Jeudi 30 novembre 2017



Champ de température imposé :

$$T(\underline{x}) = T_0 - A_2 (x_1^2 + x_2^2) \qquad A_2 > 0$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(\underline{e_1}, \underline{e_2})} \quad \text{cause CP}$$

1) loi de Hooke en thermoélasticité :

$$\underline{\underline{\sigma}} = \lambda_\beta \, tr(\underline{\underline{\varepsilon}}) \underline{\underline{Id}} + 2\mu_\beta \underline{\underline{\varepsilon}} - (3\lambda_\beta + 2\mu_\beta) \alpha \Delta T \, \underline{\underline{Id}}$$

⚠ $tr(\underline{\underline{\varepsilon}}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ alors que $tr(\underline{\underline{\varepsilon}}^{2D}) = \varepsilon_{11} + \varepsilon_{22}$.

On utilise la propriété $\sigma_{33} = 0 = \lambda_\beta (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu_\beta \varepsilon_{33} - (3\lambda_\beta + 2\mu_\beta) \alpha \Delta T$.

$$\Rightarrow \lambda_\beta (tr \, \underline{\underline{\varepsilon}}^{2D}) + (\lambda_\beta + 2\mu_\beta) \varepsilon_{33} - (3\lambda_\beta + 2\mu_\beta) \alpha \Delta T = 0$$

$$\Rightarrow \varepsilon_{33} = \frac{-\lambda_\beta}{\lambda_\beta + 2\mu_\beta} (tr \, \underline{\underline{\varepsilon}}^{2D}) + \frac{(3\lambda_\beta + 2\mu_\beta)}{(\lambda_\beta + 2\mu_\beta)} \alpha \Delta T$$

Pour alléger les notations on s'affranchit de l'indice β $\beta \in \{s, g\}$

$$\underline{\underline{\sigma}}^{2D} = \lambda \left[(tr \, \underline{\underline{\varepsilon}}^{2D}) - \frac{\lambda}{\lambda + 2\mu} (tr \, \underline{\underline{\varepsilon}}^{2D}) + \frac{(3\lambda + 2\mu)}{(\lambda + 2\mu)} \alpha \Delta T \right] \underline{\underline{Id}}^{2D} + 2\mu \underline{\underline{\varepsilon}}^{2D} - (3\lambda + 2\mu) \alpha \Delta T \, \underline{\underline{Id}}^{2D}$$

$$= 2\mu \left[\underline{\underline{\varepsilon}}^{2D} + \frac{\lambda}{(\lambda + 2\mu)} (tr \, \underline{\underline{\varepsilon}}^{2D}) \underline{\underline{Id}}^{2D} \right] + \underbrace{\left[\frac{\lambda (3\lambda + 2\mu)}{(\lambda + 2\mu)} - (3\lambda + 2\mu) \right]}_{(*)} \alpha \Delta T \, \underline{\underline{Id}}^{2D}$$

$$(*) = \frac{\lambda(3\lambda + 2\mu) - (\lambda + 2\mu)(3\lambda + 2\mu)}{\lambda + 2\mu} = \frac{-2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)}$$

$$\underline{\underline{\sigma}}^{2D} = 2\mu \left[\underline{\underline{\varepsilon}}^{2D} + \frac{\lambda}{\lambda + 2\mu} (tr \, \underline{\underline{\varepsilon}}^{2D}) \underline{\underline{Id}}^{2D} - \frac{(3\lambda + 2\mu)}{(\lambda + 2\mu)} \alpha \Delta T \, \underline{\underline{Id}}^{2D} \right]$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$\lambda + 2\mu = \frac{\cancel{2}E\nu + \cancel{2}E(1-2\nu)}{\cancel{2}(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\frac{\lambda}{\lambda+2\mu} = \frac{E\nu}{(1+\nu)(1-2\nu)} \times \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} = \frac{\nu}{(1-\nu)}$$

$$3\lambda + 2\mu = \frac{3E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{(1+\nu)}$$

$$= \frac{3E\nu + E(1-2\nu)}{(1+\nu)(1-2\nu)} = \frac{E(1+\nu)}{(1+\nu)(1-2\nu)} = \frac{E}{(1-2\nu)}$$

$$\frac{3\lambda+2\mu}{\lambda+2\mu} = \frac{\cancel{E}}{\cancel{(1-2\nu)}} \times \frac{(1+\nu)\cancel{(1-2\nu)}}{\cancel{E}(1-\nu)} = \frac{(1+\nu)}{(1-\nu)}$$

$$\frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)} = \frac{E}{(1+\nu)} \times \frac{(1+\nu)}{(1-\nu)} = \frac{E}{(1-\nu)}$$

$$\boxed{\underline{\underline{\sigma}}^{2D} = C_1\left[\underline{\underline{\varepsilon}}^{2D} + C_2\left(\mathrm{tr}\,\underline{\underline{\varepsilon}}^{2D}\right)\underline{\underline{Id}}^{2D}\right] - C_3\,\alpha\,\Delta T\,\underline{\underline{Id}}^{2D}}$$

$C_1 = 2\mu = \dfrac{E}{1+\nu}$ (différent pour s et g)

$C_2 = \dfrac{\lambda}{\lambda+2\mu} = \dfrac{\nu}{1-\nu}$ (identique pour s et g)

$C_3 = \dfrac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)} = \dfrac{E}{(1-\nu)}$ (différent pour s et g)

2)
$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} = \underbrace{\begin{bmatrix} C_1(1+C_2) & C_1C_2 & 0\\ C_1C_2 & C_1(1+C_2) & 0\\ 0 & 0 & \frac{C_1}{2}\end{bmatrix}}_{} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{Bmatrix} - C_3\,\alpha\,\Delta T\begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

$$[A_{CP}] = C_1\begin{bmatrix}(1+C_2) & C_2 & 0\\ C_2 & C_1(1+C_2)) & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

3) <u>équilibre</u> (efforts volumiques négligés)

$$\underline{\mathrm{div}}\left(\underline{\underline{\sigma}}\right) = \underline{0}$$

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{21,1} + \sigma_{22,2} \end{cases} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \quad \forall \underline{x} \in (\overbrace{\Omega_0 \cup \Omega_g}^{=\Omega})$$

<u>conditions aux limites.</u>

- CL de symétrie en $x_2 = 0$.

$$\begin{cases} \underline{u} \cdot (-\underline{e_2}) = 0 \Rightarrow \boxed{u_2(x_1, 0) = 0 \ \forall x_1 \in [0, R+e]} \\ \underline{\sigma_T} = \underline{0} = \underline{\underline{\sigma}} \cdot \underline{n} - (\underline{n}\, \underline{\underline{\sigma}} \cdot \underline{n})\, \underline{n} \end{cases}$$

$$= \underline{\underline{\sigma}}(-\underline{e_2}) - (-\underline{e_2})\, \underline{\underline{\sigma}} \cdot (-\underline{e_2})(-\underline{e_2})$$

$$= \begin{vmatrix} -\sigma_{12} \\ -\sigma_{22} \end{vmatrix} + \begin{vmatrix} 0 \\ \sigma_{22} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \boxed{\sigma_{12}(x_1, 0) = 0 \ \forall x_1 \in [0, R+e]}$$

- De manière similaire en $x_1 = 0$ (sym $\rightarrow$ appui glissant)

$$\begin{cases} \underline{u} \cdot (-\underline{e_1}) = 0 \Rightarrow \\ \underline{\sigma_T}(0, x_2) = \underline{0} \Rightarrow \end{cases} \boxed{\begin{matrix} u_1(0, x_2) = 0 \ \forall x_2 \in [0, R+e] \\ \sigma_{21}(0, x_2) = 0 \ \forall x_2 \in [0, R+e] \end{matrix}}$$

- Se libre d'efforts

$$x_1 = (R+e)\cos\theta$$
$$x_2 = (R+e)\sin\theta$$

$$\underline{n}_{ext} = \underline{e_r} = \cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2}$$

$$= \frac{x_1}{(R+e)}\underline{e_1} + \frac{x_2}{(R+e)}\underline{e_2}$$

$$\underline{\underline{\sigma}}\left(\sqrt{x_1^2 + x_2^2} = R\right) \cdot \left(\frac{x_1}{(R+e)}\underline{e_1} + \frac{x_2}{(R+e)}\underline{e_2}\right) = \underline{0}$$

- Continuité du déplacement et du vecteur contrainte.

$$\underline{u_g}(r=R) = \underline{u_s}(r=R) \qquad + \ \underline{\underline{\sigma}}_g(r=R) \cdot (\underline{e_r}) + \underline{\underline{\sigma}}_s(r=R) \cdot (-\underline{e_r}) = \underline{0}$$

4) Champs cinématiquement admissibles à 0.

$\mathcal{U}ad^0 = \{ \underline{v}(x_1, x_2)$ réguliers et continus tels que
$\begin{cases} v_1(0, x_2) = 0 & \forall x_2 \in [0, R+e] \\ v_2(x_1, 0) = 0 & \forall x_1 \in [0, R+e] \end{cases} \}$

5) formulation faible :

On cherche le champ solution $\underline{u} \in \mathcal{U}ad^0$, et on prend le champ test $\underline{v} \in \mathcal{U}ad^0$ également

$$\int_{\Omega} \underline{div}(\underline{\underline{\sigma}}).\underline{v}\, d\Omega = 0 = \underbrace{\int_{\Omega} div(\underline{\underline{\sigma}}.\underline{v})\, d\Omega}_{(*)} - \underbrace{\int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{grad}}(\underline{v})\, d\Omega}_{(**)}$$

$$= \int_{\partial\Omega} (\underline{\underline{\sigma}}.\underline{v}).\underline{n}_e\, dS - \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = 0$$

car $\underline{\underline{grad}}(\underline{v}) = \underbrace{\underline{\underline{\varepsilon}}(\underline{v})}_{sym} + \underbrace{\underline{\underline{\omega}}(\underline{v})}_{antisym}$ et $\underline{\underline{\sigma}}$ sym $\Rightarrow \underline{\underline{\sigma}} : \underline{\underline{\omega}} = 0$

travail par unité d'épaisseur

$$(*) \int_0^{R+e} (\underline{\underline{\sigma}}.(-\underline{e}_2)).\underline{v}\, dx_1 + \int_{S_e} \underbrace{(\underline{\underline{\sigma}}.\underline{n}_e)}_{=\underline{0}} \underline{v}\, dS + \int_0^{R+e} (\underline{\underline{\sigma}}(-\underline{e}_1)).\underline{v}\, dx_2$$

$=\underline{0}$ car surface libre

$$\begin{vmatrix} -\cancel{\sigma_{12}(x_1,0)}^{0} \\ -\sigma_{22}(x_1,0) \end{vmatrix} . \begin{vmatrix} v_1(x_1,0) \\ \cancel{v_2(x_1,0)}^{0} \end{vmatrix} \qquad \begin{vmatrix} -\sigma_{11}(0,x_2) \\ -\cancel{\sigma_{21}(0,x_2)}^{0} \end{vmatrix} . \begin{vmatrix} \cancel{v_1(0,x_2)}^{0} \\ v_2(0,x_2) \end{vmatrix}$$

$\Rightarrow$ le terme de gauche est nul

$$(**) \to \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = 0$$

$$\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = \sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{21} + \cancel{\sigma_{13}\varepsilon_{31}}^{0} + \sigma_{21}\varepsilon_{12} + \sigma_{22}\varepsilon_{22} + \cancel{\sigma_{23}\varepsilon_{32}}^{0} + \cancel{\sigma_{31}\varepsilon_{13}}^{0} + \cancel{\sigma_{32}\varepsilon_{23}}^{0} + \cancel{\sigma_{33}\varepsilon_{33}}^{0}$$

$$= \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + 2\sigma_{12}\varepsilon_{21} = \underline{\underline{\sigma}}^{2D} : \underline{\underline{\varepsilon}}^{2D}$$

$$\int_{\Omega_s} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega + \int_{\Omega_g} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\Omega = 0$$

Pour chaque ω_β $\beta \in [s,g]$

$$\int_{\omega_\beta} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v})\, d\omega = \int_{\omega_\beta} \underline{\underline{\sigma}}^{2D} : \underline{\underline{\varepsilon}}^{2D}(\underline{v})\, d\omega$$

$$= \int_{\omega_\beta} \left(C_1^\beta \left[\underline{\underline{\varepsilon}}^{2D}(\underline{u}) + C_2 \left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{u}) \right) \underline{\underline{Id}}^{2D} \right] - C_3^\beta \alpha \Delta T \underline{\underline{Id}}^{2D} \right) : \underline{\underline{\varepsilon}}^{2D}(\underline{v})\, d\omega$$

$$= \int_{\omega_\beta} C_1^\beta \left[\underline{\underline{\varepsilon}}^{2D}(\underline{u}) + C_2 \left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{u}) \right) \underline{\underline{Id}}^{2D} \right] : \underline{\underline{\varepsilon}}^{2D}(\underline{v})\, d\omega.$$

$$- \int_{\omega_\beta} C_3^\beta \alpha \Delta T \underbrace{\underline{\underline{Id}}^{2D} : \underline{\underline{\varepsilon}}^{2D}(\underline{v})}_{= tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{v}) \;=\; \varepsilon_{11}(\underline{v}) + \varepsilon_{22}(\underline{v})}\, d\omega$$

$$= \int_{\omega_\beta} C_1^\beta \left[\underline{\underline{\varepsilon}}^{2D}(\underline{u}) : \underline{\underline{\varepsilon}}^{2D}(\underline{v}) + C_2 \left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{u}) \right) \left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{v}) \right) \right] d\omega.$$

$$- \alpha \int_{\omega_\beta} C_3^\beta \Delta T(\underline{x})\, tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{v})\, d\omega.$$

Formulation faible :

Trouver $\underline{u} \in \mathcal{U}ad^0$ tq. $\quad a(\underline{u},\underline{v}) = l(\underline{v}) \;\; \forall \underline{v} \in \mathcal{U}ad^0$

avec $a(\underline{u},\underline{v}) = C_1^s \int_{\omega_s} \left[\underline{\underline{\varepsilon}}^{2D}(\underline{u}) : \underline{\underline{\varepsilon}}^{2D}(\underline{v}) + C_2 \left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{u}) \right) \left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{v}) \right) \right] d\omega$

$$+ C_1^g \int_{\omega_g} [\text{ idem }]\, d\omega$$

$$l(\underline{v}) = + \alpha \left\{ \int_{\omega_s} C_3^s \Delta T(\underline{x})\, tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{v})\, d\omega + \int_{\omega_g} C_3^g \Delta T(\underline{x})\, tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{v})\, d\omega \right\}$$

Rq éventuellement développer

$\underline{\underline{\varepsilon}}^{2D}(\underline{u}) : \underline{\underline{\varepsilon}}^{2D}(\underline{v})$ et $\left(tr\, \underline{\underline{\varepsilon}}^{2D}(\underline{u}) \right)^2$

Formulation variationnelle

Soit $I(\underline{u}) = \frac{1}{2} a(\underline{u},\underline{u}) - \ell(\underline{u})$

$\Rightarrow$ trouver $\underline{u} \in \mathcal{U}_{ad}^0$ tq $I(\underline{u}) \leq I(\underline{v}) \ \forall \underline{v} \in \mathcal{U}_{ad}^0$

Unicité de la solution

$\underline{u}$ et $\underline{u}^*$ solutions $(\underline{u}-\underline{u}^*)$ l'est aussi

bilinéarité de a et linéarité de l donnent

$$a(\underline{u}-\underline{u}^*, \underline{u}-\underline{u}^*) = 0$$

$$\int_{\omega} C_1(\underline{x}) \left[\underline{\underline{\varepsilon}}^{2D}(\underline{u}-\underline{u}^*) : \underline{\underline{\varepsilon}}^{2D}(\underline{u}-\underline{u}^*) + C_2 \left(\mathrm{tr}\, \underline{\underline{\varepsilon}}^{2D}(\underline{u}-\underline{u}^*)\right)^2 \right] d\omega = 0$$

$$C_1(\underline{x}) = \begin{cases} C_1^s \text{ si } \underline{x} \in \omega_s \\ C_1^g \text{ si } \underline{x} \in \omega_g \end{cases}$$

$$\underline{\underline{\varepsilon}}^{2D} : \underline{\underline{\varepsilon}}^{2D} = \underbrace{\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{12}^2}_{\geq 0} \qquad \left(\mathrm{tr}\, \underline{\underline{\varepsilon}}^{2D}\right)^2 \geq 0$$

Comme C_1^β sont non nuls, nécessairement $\underline{\underline{\varepsilon}}^{2D} = 0 \Rightarrow$ mvt de solide rigide

$$\Rightarrow (\underline{u}-\underline{u}^*) = \underline{a} \wedge \underline{x} + \underline{b} = \underline{\rho} \qquad \underline{\rho} = \begin{vmatrix} a_2 x_3 - a_3 x_2 + b_1 \\ a_3 x_1 - a_1 x_3 + b_2 \\ a_1 x_2 - a_2 x_1 + b_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} -a_3 x_2 + b_1 \\ a_3 x_1 + b_2 \\ \underbrace{a_1 x_2 - a_2 x_1 + b_3}_{=0 \text{ CP}} \end{vmatrix}$$

en $\begin{cases} x_2 = 0 & \rho_2(x_1, 0) = 0 \\ x_1 = 0 & \rho_1(0, x_2) = 0 \end{cases}$ $\qquad \rho(0,0) = 0 \Rightarrow b_1 = b_2 = 0$

$$\hookrightarrow \begin{cases} -a_3 x_2 = 0 & \forall x_2 \in [0, R+e] \\ +a_3 x_1 = 0 & \forall x_1 \in [0, R+e] \end{cases} \Rightarrow a_3 = 0$$

$\Rightarrow \underline{\rho} = \underline{0} \Rightarrow$ contradiction avec $\underline{u} \neq \underline{u}^*$

$\Rightarrow$ unicité de la solution

fonctions de forme du B3 1

- $N_k(x^{(i)}) = \delta_{ik}$
- image d'un nœud est un nœud
- Partition de l'unité

$*$ $N_1 = \xi_1 a^2 + \eta_1 a + \gamma_1 \Rightarrow \begin{cases} \xi_1 - \eta_1 + \gamma_1 = 1 \\ \gamma_1 = 0 \\ \xi_1 + \eta_1 + \gamma_1 = 0 \end{cases} \Rightarrow \begin{cases} \xi_1 - \eta_1 = 1 \\ \xi_1 + \eta_1 = 0 \end{cases}$

$\Rightarrow \xi_1 = \frac{1}{2} \quad \eta_1 = -\frac{1}{2} \quad \gamma_1 = 0$

$\Rightarrow N_1 = \frac{1}{2}[a^2 - a] \quad \boxed{N_1(a) = \frac{1}{2}a(a-1)}$

$*$ $\begin{cases} \gamma_2 = 1 \\ \xi_2 - \eta_2 + \gamma_2 = 0 \\ \xi_2 + \eta_2 + \gamma_2 = 0 \end{cases} \Rightarrow \begin{cases} \xi_2 - \eta_2 = -1 \\ \xi_2 + \eta_2 = -1 \end{cases} \Rightarrow 2\eta_2 = 0 \Rightarrow \eta_2 = 0 \quad \xi_2 = -1 \quad \boxed{N_2(a) = (1-a^2)}$

$*$ $\begin{cases} \xi_3 - \eta_3 + \gamma_3 = 0 \\ \gamma_3 = 0 \\ \xi_3 + \eta_3 + \gamma_3 = 1 \end{cases} \Rightarrow \begin{cases} \xi_3 - \eta_3 = 0 \\ \xi_3 + \eta_3 = 1 \end{cases} \Rightarrow \xi_3 = \frac{1}{2} \quad \eta_3 = \frac{1}{2} \quad \boxed{N_3(a) = \frac{1}{2}a(a+1)}$

représentation paramétrique

$s^{(1)} = \underset{(3\times1)}{\{N_1 \; N_2 \; N_3\}^t} \underset{(1\times3)}{\begin{Bmatrix} s^{(1)} \\ s^{(2)} \\ s^{(3)} \end{Bmatrix}}$ où les N_k sont les fonctions de forme de la B3.

$\{\underline{x}^{(2)}\} = \underset{(2\times12)}{\begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_6 \end{bmatrix}} \underset{(12\times1)}{\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_1^{(6)} \\ x_2^{(6)} \end{Bmatrix}}$

$J_e^{(1)} = \frac{ds}{da} = \sum_k \frac{dN_k(a)}{da} s^{(k)} = \frac{d}{da}\left[\frac{1}{2}a(a-1)\right]s^{(1)} + \frac{d}{da}\left[1-a^2\right]s^{(2)}$

$+ \frac{d}{da}\left[\frac{1}{2}a(a+1)\right]s^{(3)}$

$$J_e^{(1)} = \left(a - \frac{1}{2}\right) s^{(1)} - 2a\, s^{(2)} + \left(a + \frac{1}{2}\right) s^{(3)}$$

$$= \frac{1}{2} \underbrace{s^{(3)} - s^{(1)}}_{L_e = \frac{R\pi}{2}} + a \Big[\underbrace{s^{(3)} - 2 s^{(2)} + s^{(1)}}_{=0} \Big] \quad \text{cas part } 1/4 \text{ cercle} \quad \Rightarrow \boxed{J_e^{(1)} = \frac{L_e}{2}}$$

sur (2) :

$$x_1 = 4a(1 - a_1 - a_2)\frac{R\sqrt{2}}{2} + a_1(2a_1 - 1)R + 4a_1 a_2 \frac{R\sqrt{2}}{2}$$

$$= R\left[2a_1\sqrt{2}(1 - a_1 - a_2) + a_1(2a_1 - 1) + 2a_1 a_2 \sqrt{2}\right]$$

$$= R\left[2a_1\sqrt{2} - 2a_1^2\sqrt{2} - 2a_1 a_2\sqrt{2} + 2a_1^2 - a_1 + 2a_1 a_2 \sqrt{2}\right]$$

$$= R\left[(2\sqrt{2} - 1)a_1 + [2 - 2\sqrt{2}]a_1^2\right]$$

$$= a_1 R\left[(2\sqrt{2} - 1) + (2 - 2\sqrt{2})a_1\right]$$

$$\boxed{\begin{gathered} x_1 = a_1 R\left[p + q a_1\right] \\ p = 2\sqrt{2} - 1 \\ q = 2(1 - \sqrt{2}) \end{gathered}}$$

$$x_2 = 4a_1(1-a_1-a_2)\times 0 + 0 + 4a_1a_2\frac{R}{\sqrt{2}} + a_2(2a_2-1)R + 4a_2(1-a_1-a_2)\frac{R}{\sqrt{2}}.$$

$$= R\left[\frac{4a_1a_2}{\sqrt{2}} + a_2(2a_2-1) + \frac{4}{\sqrt{2}}a_2(1-a_1-a_2)\right]$$

$$= R\left[2\sqrt{2}a_1a_2 + a_2(2a_2-1) + 2\sqrt{2}a_2(1-a_1-a_2)\right]$$

$$= R\left[\cancel{2\sqrt{2}a_1a_2} + 2a_2^2 - a_2 + 2\sqrt{2}a_2 \cancel{-2\sqrt{2}a_2a_1} - 2\sqrt{2}a_2^2\right]$$

$$= R\left[(2-2\sqrt{2})a_2^2 + (2\sqrt{2}-1)a_2\right].$$

$$\boxed{x_2 = a_2R\left[p + q\,a_2\right]}$$

1|2 ont même rôle

Matrice Jacobienne: $J_{ij} = \frac{\partial x_i}{\partial a_j} = \sum_k \frac{\partial N_k}{\partial a_j} x_i^{(k)}$.

$$[J] = \begin{bmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} \end{bmatrix} =$$

$$\begin{cases} \alpha = p \\ \beta = q \end{cases}$$

$$\frac{\partial x_1}{\partial a_1} = R[\alpha + \beta a_1] + a_1R[\beta].$$
$$= R\alpha + \beta R a_1 + \beta R a_1.$$
$$= R[\alpha + 2\beta a_1]$$
$$= R[\alpha + 2\beta a_1]$$

$$\frac{\partial x_1}{\partial a_2} = 0$$

$$\frac{\partial x_2}{\partial a_1} = 0$$

$$\frac{\partial x_2}{\partial a_2} = R[\alpha + 2\beta a_2].$$

$$[J] = R\begin{bmatrix} \alpha + 2\beta a_1 & 0 \\ 0 & \alpha + 2\beta a_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} J &= R^2[\alpha + 2\beta a_1][\alpha + 2\beta a_2] \\ &= f(a_1, a_2) \end{aligned}}$$

$\omega_e = \int_{\omega_e} d\omega_e = \int_{\Delta e} J(a_1, a_2)\, da_1 da_2$ ~~6~~ 10

Méthode de Gauss.

T6 → 3 pt de Gauss (pas d'IR) / Hammer.

PG

a_1	a_2
1/6	1/6
2/3	1/6.
1/6	2/3

$w_g = \frac{1}{6}$ pour tout le monde.

$$\int_{\Delta e} J(a_1,a_2)\, da_1 da_2 \simeq \frac{1}{6} R^2 \left[\left(\alpha+\frac{2\beta}{6}\right)\left(\alpha+\frac{2\beta}{6}\right)+2\left(\alpha+\frac{4\beta}{3}\right)\left(\alpha+\frac{2\beta}{6}\right)\right]$$

$$\simeq \frac{R^2}{6}\left[\left(\alpha+\frac{\beta}{3}\right)^2 + 2\left(\alpha+\frac{4\beta}{3}\right)\left(\alpha+\frac{\beta}{3}\right)\right].$$

$$\simeq \frac{R^2}{6}\left(\left[\alpha^2+\frac{2\beta}{3}\alpha+\frac{\beta^2}{9}+2\left[\alpha^2+\frac{\alpha\beta}{3}+\frac{4\alpha\beta}{3}+\frac{4\beta^2}{9}\right]\right)\right.$$

$$\simeq \frac{R^2}{6}\left(3\alpha^2+\frac{12\alpha\beta}{3}+\frac{9\beta^2}{9}\right).$$

$$\simeq \frac{R^2}{6}\left(3\alpha^2+4\alpha\beta+\beta^2\right).$$

$\alpha^2 = (2\sqrt{2}-1)^2 = 8-4\sqrt{2}+1$

$\beta^2 = (2-2\sqrt{2})^2 = 4-8\sqrt{2}+8$

$\alpha\beta = (2\sqrt{2}-1)(2-2\sqrt{2}) = 4\sqrt{2}-8-2+2\sqrt{2}.$

$$\simeq \frac{R^2}{6}\left(24-12\sqrt{2}+3+16\sqrt{2}-32-\cancel{8}+\cancel{8}\sqrt{2}+4-\cancel{8}\sqrt{2}+\cancel{8}\right.$$

$$\simeq \frac{R^2}{6}\left(-1+4\sqrt{2}\right).$$

$\frac{4\sqrt{2}-1}{6} \simeq 0{,}776 < \frac{\pi}{4} \simeq 0{,}79.$

⇒ surface légèrement inférieure au quart de cercle.

$$\varepsilon(s) = \frac{du(s)}{ds} = \frac{du}{da}\underbrace{\frac{da}{ds}}_{1/J_e(s)}$$

$$\varepsilon(s) = \frac{1}{J_e}\frac{du}{da} = \frac{1}{J_e}\sum_k \frac{dN_k(a)}{da}u^{(k)}$$

$$\Rightarrow \varepsilon(s) = \frac{1}{J_e}\underbrace{\left\{\frac{dN_1}{da} \quad \frac{dN_2}{da} \quad \frac{dN_3}{da}\right\}}_{\{D_N\}^t}\underbrace{\begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{Bmatrix}}_{\{U_e^{(1)}\}_{loc}}$$

$$\{B_e^{(1)}\}^t = \frac{1}{J_e}\{D_N\}^t = \boxed{\frac{1}{J_e}\left\{\left(a-\frac{1}{2}\right) \quad -2a \quad \left(a+\frac{1}{2}\right)\right\} = \{B_e^{(1)}\}^t}$$

$$\sigma(s) = \frac{E_g}{J_e}\left\{\left(a-\frac{1}{2}\right) \quad -2a \quad \left(a+\frac{1}{2}\right)\right\}\begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{Bmatrix} \quad \text{car } \sigma(s) = E_g\,\varepsilon(s)$$

Énergie de déformation

$$\mathcal{E} = \int_{(1)} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}\, dV = h_e\int_0^{L_e}\sigma(s)\,\varepsilon(s)\,ds = h_e\int_{-1}^{1}\sigma(1)\,\varepsilon(1)\,J_e\,da$$

$$= h_e\int_{-1}^{1}\left(\frac{E_g}{J_e}\{D_N\}^t\{U_e^{(1)}\}_{loc}\right)\left(\frac{1}{J_e}\{D_N\}^t\{U_e^{(1)}\}_{loc}\right)J_e\,da$$

A_g

$$= \{U_e^{(1)}\}_{loc}^t\underbrace{\frac{E_g h_e}{J_e}\int_{-1}^{1}\{D_N\}\{D_N\}^t\,da}_{[K_e^{(1)}]_{loc}}\{U_e^{(1)}\}_{loc}$$

$\text{Dim} = (3\times 3)$

$$M(a) = \{D_N\}\{D_N\}^t = \begin{Bmatrix} \left(a-\frac{1}{2}\right) \\ -2a \\ \left(a+\frac{1}{2}\right) \end{Bmatrix}\left\{\left(a-\frac{1}{2}\right) \quad -2a \quad \left(a+\frac{1}{2}\right)\right\}$$

Polynômes d'ordre 2 en a $\Rightarrow$ il faut que $2NPG - 1 \geq d$

$NPG \geq \frac{d+1}{2} = \frac{3}{2}$ $\Rightarrow$ il faut au moins 2 PG

Second membre

Pour la structure :

$$\ell_1(\underline{v}) = \alpha C_3^1 \int_{\Omega_1} \Delta T(\underline{x}) \left(\mathrm{tr}\,\underline{\underline{\varepsilon}}^{2D}(\underline{v})\right) d\Omega$$

$$\mathrm{tr}\,\underline{\underline{\varepsilon}}^{2D} = \underline{\underline{\varepsilon}}^{2D} : \underline{\underline{Id}}^{2D} = \{\varepsilon\}^t \begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

$$= \alpha C_3^1 \int_{\Omega_1} \Delta T(\underline{x}) \{\varepsilon\}^t \begin{Bmatrix}1\\1\\0\end{Bmatrix} d\Omega$$

$$= \alpha C_3^1 \{U_e^{(2)}\}^t \int_{(2)} [B_e^{(2)}]^t \Delta T(\underline{x}) J_e \, da_1 da_2 \begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

$$= \{U_e^{(2)}\}^t \{F_e^{(2)}\} \quad \text{avec} \quad \{F_e^{(2)}\} = \alpha C_3^1 \int_{(2)} [B_e^{(2)}]^t \Delta T(\underline{x}) J_e \, da_1 da_2 \begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

$$\Delta T = -A_2 (x_1^2 + x_2^2) = -A_2 \{x_1 \; x_2\} \begin{Bmatrix}x_1\\x_2\end{Bmatrix}$$

$$= -A_2 \{\underline{x}\}^t \{\underline{x}\}$$

$$= -A_2 \{X_e^{(2)}\}^t [N_e^{(2)}]^t [N_e^{(2)}] \{X_e^{(2)}\}$$

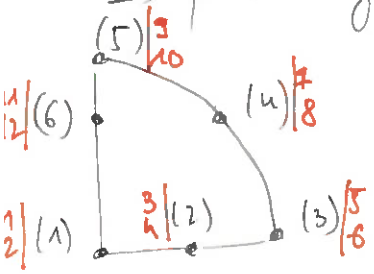
$$\boxed{\{F_e^{(2)}\} = -A_2 \alpha C_3^1 \int_{(2)} [B_e^{(2)}]^t \{X_e^{(2)}\}^t [N_e^{(2)}]^t [N_e^{(2)}] \{X_e^{(2)}\} J_e \, da_1 da_2 \begin{Bmatrix}1\\1\\0\end{Bmatrix}}$$

Assemblage

$$\mathrm{Dim}\, [K_e^{(1)}]_{(\underline{e}_1, \underline{e}_2)} = (6 \times 6)$$

$$\mathrm{Dim}\, [K] = (12 \times 12) \qquad \mathrm{Dim}\, \{F\} = (12 \times 1)$$

On n'a aucun $\Omega^{(i)} \cap \Omega^{(j)} = \emptyset$ $\forall i,j$ les noeuds de la structure
$\Rightarrow$ pas de zéro dans la matrice



$$\mathrm{Connec} = \begin{array}{cccccc} 1 & 2 & 3 & & & \\ \end{array}$$
$$\begin{bmatrix} (3) & (4) & (5) & & & \\ (1) & (2) & (3) & (4) & (5) & (6) \end{bmatrix}$$
$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$[K_e^{(1)}] = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & & & \\ 2 & & & & & & \\ 3 & & & k_{33} & & & \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array}$$

$[K]_{77} \rightarrow$ ddl $7 = u_1^{(4)}$

$$\boxed{[K]_{77} = [K_e^{(1)}]_{33} + [K_e^{(2)}]_{77}}$$

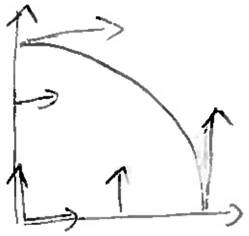
Les composantes connues de $\{U\}$ sont :

$U_1 = u_1^{(1)}$ $U_2 = u_2^{(1)}$ $U_4 = u_2^{(2)}$ $U_6 = u_2^{(3)}$ $U_9 = u_1^{(5)}$ et $U_{11} = u_1^{(6)}$

Dim $[C] = (6 \times 12)$

Résolution cf cours

Réactions :



aux extrémités contribut° de 2 éléments.