Numerical solutions of differential equations

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Lecture 11

Entropy solutions

Weak Entropy Condition and Entropy Pairs

Reminder: Lax Entropy Condition

Let: u is weak solution to $\partial_t u + \partial_x f(u) = 0$ with some initial value; S is smooth curve in $\mathbb{R} \times \mathbb{R}^+$ along which u is discontinuous.

Let $(x_0, t_0) \in S$, $\quad u_l := \lim_{\delta \to 0} u(x_0 - \delta, t_0), \quad u_r := \lim_{\delta \to 0} u(x_0 + \delta, t_0)$

and $s := \frac{f(u_l) - f(u_r)}{u_l - u_r}$.

Then u fulfills the Lax Entropy Condition in (x_0, t_0) if and only if

$$f'(u_r) < s < f'(u_l).$$

- We only know: reasonable for convex fluxes. What in general?
- For numerical approximations the discontinuities are typically not given by smooth curves.
- How to guarantee that numerical method does right thing?

Weak entropy solutions

Goal:

- Derive a weak entropy condition that is easy to mimic on the discrete level.
- The weak entropy condition will be again in integral formulation.
- We wish to avoid the assumption of convexity, i.e. $f'' > 0$,
  Note: the linear flux $f(u) = au$ is not convex!
- However, if $f'' > 0$, the new entropy condition should be equivalent to the Lax entropy condition.

Motivation: Weak entropy condition

For $\varepsilon > 0$ we regard the viscose approximation

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon.$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be convex and smooth. Multiplying with $\eta'(u_\varepsilon)$:

$$\partial_t u_\varepsilon \, \eta'(u_\varepsilon) + \partial_x f(u_\varepsilon) \, \eta'(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon \, \eta'(u_\varepsilon).$$

Hence

$$\partial_t \eta(u_\varepsilon) + f'(u_\varepsilon) \, \eta'(u_\varepsilon) \, \partial_x u_\varepsilon = \varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2.$$

Define F such that $F' = f'\eta'$, then it follows with $\eta'' > 0$:

$$\partial_t \eta(u_\varepsilon) + \partial_x F(u_\varepsilon) \leq \varepsilon \partial_{xx} \eta(u_\varepsilon).$$

Passing formally to the viscosity limit $\varepsilon \to 0$ yields

$$\partial_t \eta(u) + \partial_x F(u) \leq 0.$$

Weak Entropy Condition

Definition (Entropy - Entropy Flux Pair and Entropy Solution)

1. Let $\eta : \mathbb{R} \to \mathbb{R}$ be convex and smooth and $F : \mathbb{R} \to \mathbb{R}$ a smooth function with

$$F' = f' \eta'$$

   Then (η, F) is called entropy - entropy flux pair for the conservation law.

2. A weak solution u fulfills the (weak) entropy condition if for all entropy pairs (η, F) and all $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ with $\phi \geq 0$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} (\eta(u)\, \partial_t \phi + F(u)\, \partial_x \phi) + \int_{\mathbb{R}} \eta(v_0)\, \phi(\cdot, 0) \geq 0.$$

   We call u the entropy solution.

Entropy Pair - Example

The pair (η, F) is an entropy pair if $\eta : \mathbb{R} \to \mathbb{R}$ is convex and smooth and if $F : \mathbb{R} \to \mathbb{R}$ is a smooth function with

$$F' = f'\eta'.$$

Example:

1. For given η, the entropy flux F is (up to a constant) given by

$$F(u) = \int_a^u \eta'(s) f'(s)\, ds, \qquad a \in \mathbb{R}.$$

2. If f is strictly convex, then $\eta(u) = f(u)$ is an entropy with entropy flux

$$F(u) = \int_0^u (f'(s))^2\, ds.$$

Uniqueness of entropy solutions

Theorem (Uniqueness)

1. Let u_1 and u_2 denote two entropy solutions for the initial values v_1 und v_2. Then we have for all $t > 0$ (a.e.)

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|v_1 - v_2\|_{L^1(\mathbb{R})} .$$

   In particular, this yields uniqueness of entropy solutions for the same initial value $v_1 = v_2$.

2. If u is an entropy solution and if $f'' > 0$, then u fulfills the Lax entropy condition at all discontinuities.

   Hence, both entropy conditions are equivalent in this case.

Kruzkov entropy pairs and Kruzkov entropy solution

Theorem (Kruzkov entropy condition)

- Let $\eta_\kappa(u) := |u - \kappa|$ and $F_\kappa(u) := \operatorname{sign}(u-\kappa)(f(u) - f(\kappa))$ with $\kappa \in \mathbb{R}$.

  The family $(\eta_\kappa, F_\kappa)_{\kappa\in\mathbb{R}}$ is called Kruzkov entropy pair.
- Let u be a weak solution. Then the following is equivalent:
  - (i) u is entropy solution fulfilling the weak entropy condition.
  - (ii) for all $\kappa \in \mathbb{R}$ and all $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_0^+)$ with $\phi \geq 0$ it holds:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} (\eta_\kappa(u)\, \partial_t \phi + F_\kappa(u)\, \partial_x \phi) + \int_{\mathbb{R}} (\eta_\kappa(v_0)\, \phi(\cdot, 0)) \geq 0$$

This reduces the “number” of entropy pairs significantly.