High-Fidelity Simulations for Turbulent Flows

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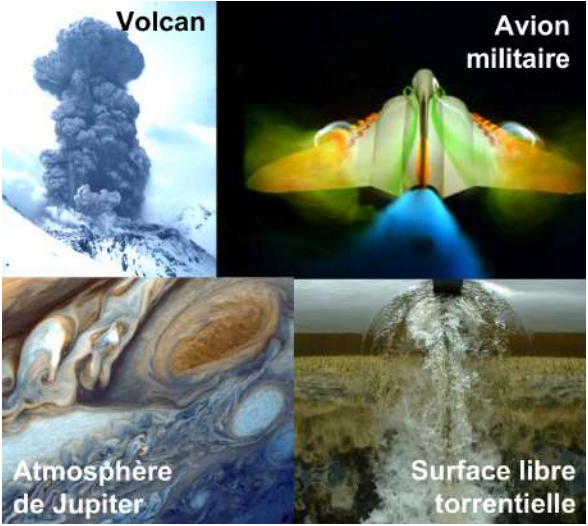
Part II

Modeling of the Navier–Stokes Equations

What is turbulence?



No rigorous definition..

- Irregular, seemingly random motions
- Non repeatability (sensitivity to IC)
- Large range of L and t scales
- 3D, unsteady, rotational
- Enhanced diffusion and dissipation
- Scale similarity

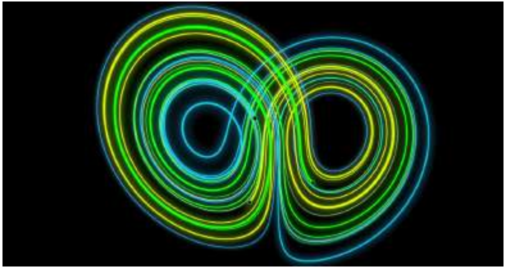
Features of Turbulence I

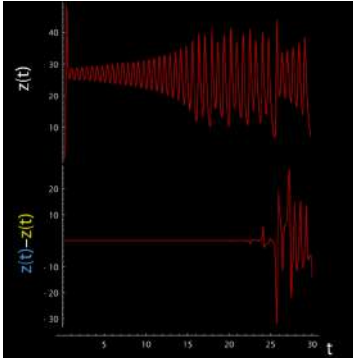
Sensitivity to initial conditions

Consider this simplified model for atmospheric convection (Lorenz system)

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - xz - y \\ \dot{z} = xy - \beta z \end{cases} \quad \text{with} \quad \sigma = 10, \beta = \frac{8}{3}, \rho = 35$$

- Infinitesimal perturbations (here, 10^{-5}) lead to completely different evolutions
- Chaotic solutions, deterministic but unpredictable
  - Butterfly effect, discovered by Lorenz in 1961 restarting from the middle a weather forecast: different rounding error of computer and printout (6 vs 3 digits)
- **Where do perturbations come from?**
  - "Physical" perturbations: impurities, small fluctuations of p, u or T, small geometry variations..
  - "Numerical" perturbations: rounding and truncation errors, initial and boundary conditions errors..

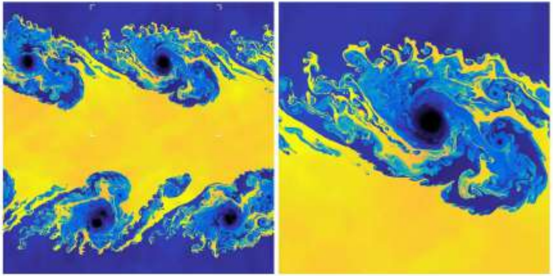




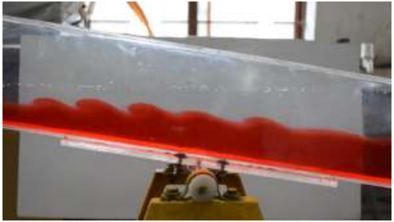
Features of Turbulence II

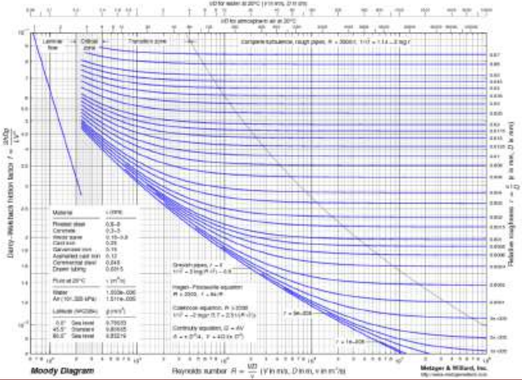
Scale Similarity

$Re \sim 10^3$, $Re \sim 10^4$, $Re \sim 10^8$



Enhanced Diffusivity





Features of Turbulence III

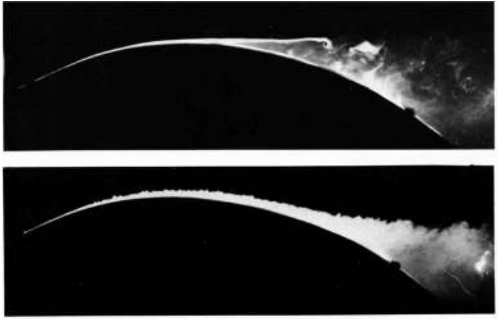
Large range of scales

Consider the 2D model equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = 0 \qquad \text{with} \qquad \begin{cases} u(x,y,t_0) = A\cos(k_x x)\sin(k_y y) \\ v(x,y,t_0) = B\sin(k_x x)\cos(k_y y) \end{cases} \quad (Ak_x + Bk_y = 0)$$

Taylor expansion: $u_i(x_i,t) = u_i(x_i,t_0) + (t-t_0)\left.\frac{\partial u_i}{\partial t}\right|_{t_0} + \ldots = u_i(x_i,t_0) - (t-t_0)\, u_j \left.\frac{\partial u_i}{\partial x_j}\right|_{t_0} \implies$

$$u(x,y,t) = A\cos(k_x x)\sin(k_y y) + (t-t_0)\frac{k_x A^2}{2}\left[\cos(2k_x x)\sin^2(k_y y) + \sin(2k_x x)\cos^2(k_y y)\right] + \ldots$$



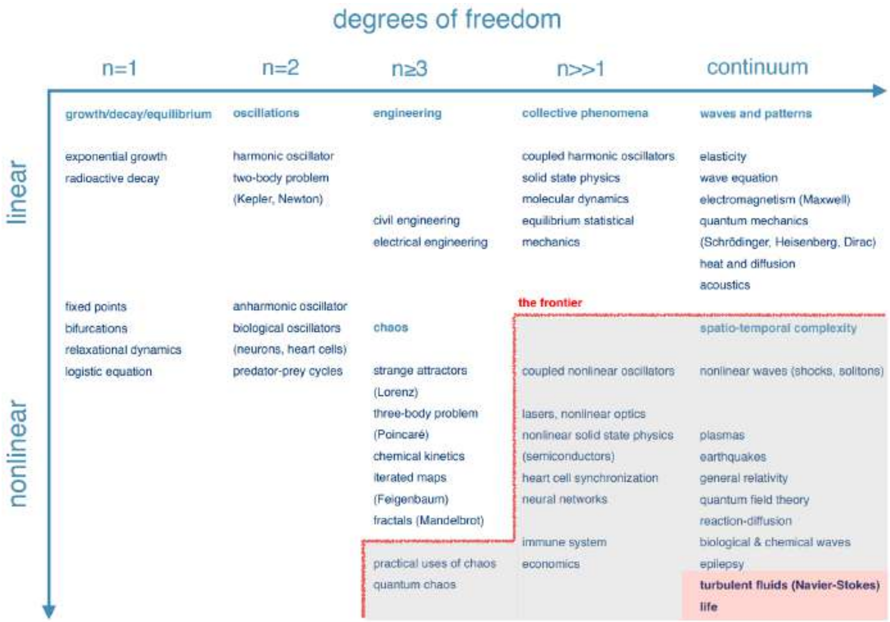
156. **Comparison of laminar and turbulent boundary layers.** The laminar boundary layer in the upper photograph separates from the crest of a convex surface (cf. figure 38), whereas the turbulent layer in the second photograph remains attached; similar behavior is shown below for a sharp corner. (Cf. figures 55-58 for a sphere.) Titanium tetrachloride is painted on the forepart of the model in a wind tunnel. Head 1982

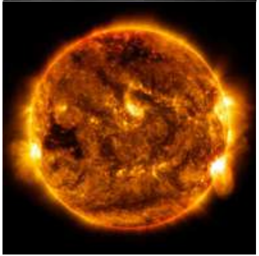
- High-order harmonics generated by non-linear term $\implies$ increase of active L and t scales
  - What is the smallest structure that could be activated?

Is turbulence good or bad?

- Most of practical flows are turbulent..
- ✔ Good for many applications
  - Delayed separation, enhanced mixing..
- ✘ Not desired in others
  - Increases drag, reduced efficiency..

Phenomena classifications





Milestones in turbulence

Osborne Reynolds (1842-1912)

- 1883: Laminar-to-turbulent transition (Re number)
- 1895: Reynolds decomposition

Ludwig Prandtl (1875-1953)

- 1904: First studies on boundary layers
- 1925: Mixing-length model for turbulent transport

Lewis Fry Richardson (1881-1953)

- 1922: Notions of vortex and energy cascade

Geoffrey Ingram Taylor (1886-1975)

- 1935: Statistical theory of Turbulence

Andrey Kolmogorov (1903-1987)

- 1941: K41 Theory: dimensional analysis, $-5/3$ law (energy spectrum)
- 1962: K62 Theory: scale invariance rupture, intermittency problem

Robert Kraichnan (1928-2008)

- 1967: Inverse energy cascade and Field Theory approach

Steven Orszag (1943-2011)

- 1966: Eddy-Damped Quasi-Normal Markovian (EDQNM) Approximation
- 1972: First DNS 3D on a 32^3 grid (Orszag & Patterson)

- 1948: Numerical weather forecast (von Neumann & Charney)
- 1963: Large-Eddy Simulation (LES) Technique (Smagorinsky)
- 1965: Fast-Fourier Transform (FFT) Algorithm (Cooley & Tukey)
- 1977: Cray-1 at the National Center for Atmoshperic Research
- 2002: DNS on a 4096^3 grid (Kaneda)
- 2020: DNS on a 8192^3 grid (`http://turbulence.pha.jhu.edu/`)

Navier–Stokes Equations

Equations:

- 1 mass conservation eq.
- 3 momentum balance eqs.
- 1 energy conservation eq.

$\Longrightarrow$ **5 equations**

We need $16 - 5 =$ **11 relations**

Unknowns:

- 1 density ρ
- 1 pressure p
- 1 temperature T
- 3 velocity components u, v, w
- 6 viscous stress tensor components σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}
- 3 heat flux components: q_x, q_y, q_z
- 1 energy (total, internal, enthalpy, ..) E

$\Longrightarrow$ **16 unknowns**

Thermal equation of state	$p = p(\rho, T)$		(+1 eq.)
Caloric equation of state	$e = e(\rho, T)$		(+1 eq.)
Fourier's Law	$q_i = -\lambda \dfrac{\partial T}{\partial x_i}$	with given $\lambda = \lambda(T)$ law	(+3 eq.)
Newtonian fluid + Stokes hypothesis	$\tau_{ij} = 2\mu \left(S_{ij} - \dfrac{1}{3} S_{kk} \delta_{ij} \right)$	with given $\mu = \mu(T)$ law	(+6 eq.)
		System closed!	16 eq.

Navier–Stokes Equations

Compressible Formulation (CNS)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u_i}{\partial x_i} = 0 \\ \dfrac{\partial \rho u_i}{\partial t} + \dfrac{\partial \rho u_i u_j}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \rho f_i + \dfrac{\partial \tau_{ij}}{\partial x_j} \\ \dfrac{\partial \rho E}{\partial t} + \dfrac{\partial [(\rho E + p) u_i]}{\partial x_i} = \rho f_i u_i + \dfrac{\partial (\tau_{ij} u_j)}{\partial x_i} - \dfrac{\partial q_i}{\partial x_i} \end{cases}$$

Incompressible Formulation (INS)

$$\begin{cases} \dfrac{\partial u_i}{\partial x_i} = 0 \\ \dfrac{\partial u_i}{\partial t} + \dfrac{\partial u_i u_j}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + f_i + \nu \dfrac{\partial^2 u_i}{\partial x_j^2} \end{cases}$$

$$\text{with} \quad q_i = -k\frac{\partial T}{\partial x_i} \qquad T_{ij} = -p\delta_{ij} + 2\mu\left[S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right] \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- **Thermodynamic modelling:**
  - Thermally perfect gas: $p = \rho R T$
  - Calorically perfect gas: $c_p, c_v = \text{const.}$

$$R = 287.1\,\frac{\text{J}}{\text{kg K}} \quad \text{and} \quad \gamma = 1.4$$

$$c_p = \frac{\gamma R}{\gamma - 1} = 1004.85\,\frac{\text{J}}{\text{kg K}}$$

$$c_v = \frac{R}{\gamma - 1} = 717.75\,\frac{\text{J}}{\text{kg K}}$$

- **Transport properties modelling:**
  - μ model: Sutherland's law $(100\,\text{K} < T < 1900\,\text{K})$

$$\mu(T) = \mu_{\text{ref}}\left(\frac{T}{T_{\text{ref}}}\right)^{\frac{3}{2}}\frac{(T_{\text{ref}} + S)}{(T + S)}$$

$$T_{\text{ref}} = 273.16\,\text{K},\, S = 110.4\,\text{K},\, \mu_0 = 1.711\cdot 10^{-5}\,\frac{\text{kg}}{\text{m s}}$$

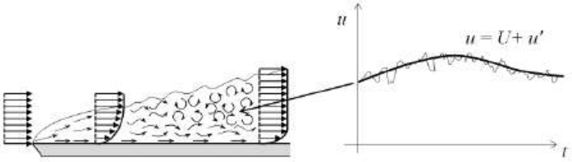
  - λ model: constant Prandtl assumption

$$\lambda = \frac{\mu c_p}{\text{Pr}} \qquad \text{with} \qquad Pr = 0.72$$

Statistical description: Reynolds averaging

A turbulent velocity field exhibits:

- Large spatial and temporal fluctuations
- "Smooth" and slowly-varying average
  - Statistical average applicable!
  - Statistical avg. $\equiv$ Ensemble avg. (ergodicity property)



Statistical average $\overline{f}(\vec{x},t)$ of a variable $f(\vec{x},t)$:

$$\overline{f}(\vec{x},t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f^{(i)}(\vec{x},t)$$

being $f^{(i)}$ the i-th sample.
From a practical point of view:

- Steady turbulent field, temporal average:

$$\overline{f}(\vec{x}) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f^{(i)}(\vec{x},t')\,\mathrm{d}t'$$

- Homogeneous turbulent field, spatial average:

$$\overline{f}(t) = \lim_{\mathcal{V}\to\infty} \frac{1}{\mathcal{V}} \iiint_{\mathcal{V}} f^{(i)}(\vec{x}',t)\,\mathrm{d}\vec{x}'$$

Reynolds averaging: $\boxed{f \equiv \overline{f} + f'}$

Properties:

1. Linearity: $\overline{f+g} = \overline{f} + \overline{g}$
2. Preservation of constants: $\overline{C^{te}} = C^{te}$ $(\implies \overline{\overline{f}} \equiv \overline{f})$
3. Commutation with derivatives: $\overline{\dfrac{\partial f}{\partial x}} = \dfrac{\partial \overline{f}}{\partial x}$

By definition:

- $\overline{f'} = 0 \qquad (\overline{f'} = \overline{f - \overline{f}} = \overline{f} - \overline{\overline{f}} = 0)$
- Product of 2 variables f and g:

$$fg \equiv (\overline{f} + f')(\overline{g} + g') = \overline{f}\,\overline{g} + \overline{f}g' + f'\overline{f} + f'g'$$

$$\implies \qquad \overline{fg} = \overline{\overline{f}\,\overline{g}} + \cancel{\overline{\overline{f}g'}} + \cancel{\overline{f'\overline{g}}} + \overline{f'g'} = \overline{f}\,\overline{g} + \overline{f'g'}$$

Statistical description: Favre averaging

Problem: $\overline{\rho f} = \overline{(\overline{\rho} + \rho')(\overline{u}_i + f')} = \overline{\rho}\,\overline{f} + \overline{\rho' f'}$

- Applying Reynolds avg. leads to a system of eqs. for the mean field different from the initial one!
- We apply **Favre** (or **density-weighted**) average:

$$\widetilde{f} = \frac{\lim\limits_{T\to\infty} \int_{t_0}^{t_0+T} \rho f \,\mathrm{d}t}{\lim\limits_{T\to\infty} \int_{t_0}^{t_0+T} \rho \,\mathrm{d}t} = \frac{\overline{\rho f}}{\overline{\rho}}$$

Obtaining

$$f = \widetilde{f} + f'' \qquad \text{with} \qquad f'' = f' - \frac{\overline{\rho' f'}}{\overline{\rho}}$$

with

$$\widetilde{f''} = \frac{1}{\overline{\rho}} \lim_{T\to\infty} \int_{t_0}^{t_0+T} \rho(f - \widetilde{f}) \,\mathrm{d}t$$

$$= \left(\frac{1}{\overline{\rho}} \lim_{T\to\infty} \int_{t_0}^{t_0+T} \rho f \,\mathrm{d}t \right) - \widetilde{f} = \widetilde{f} - \widetilde{f} = 0$$

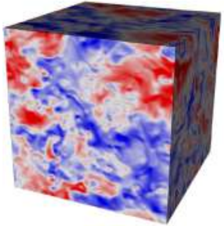
- Auxiliary relations:

$$\overline{f''} \neq 0$$
$$\overline{\rho f''} = 0$$
$$\overline{\rho \widetilde{f}} = \overline{\rho}\widetilde{f} = \overline{\rho f}$$

- Same properties of Reynolds averaging hold
- Simpler expressions for non-constant density flows
  - Examples:

$$\overline{\rho u T} = \overline{\rho}\,\overline{u}\,\overline{T} + \overline{\rho}\,\overline{u' T'} + \overline{\rho' u'}\,\overline{T} + \overline{\rho' T'}\,\overline{u} + \overline{\rho' u' T'}$$
$$\overline{\rho u T} = \overline{\rho}\widetilde{u}\widetilde{T} + \overline{\rho}\widetilde{u'' T''}$$

"Types" of turbulence



Symmetries and invariances of NS equations

Time shift	$(t, \vec{x}, \vec{u}) \to (t + a, \vec{x}, \vec{u})$
Space shift	$(t, \vec{x}, \vec{u}) \to (t, \vec{x} + \vec{a}, \vec{u})$
Galilean transform	$(t, \vec{x}, \vec{u}) \to (t, \vec{x} + \vec{a}t, \vec{u} + \vec{a})$
3D rotation	$(t, \vec{x}, \vec{u}) \to (t, \mathbf{R}\vec{x}, \mathbf{R}\vec{u})$
Scaling 1	$(t, \vec{x}, \vec{u}) \to (t, e^a\vec{x}, e^a\vec{u})$
Scaling 2	$(t, \vec{x}, \vec{u}) \to (e^a t, e^a\vec{x}, \vec{u})$
....	....

1. **Statistically Homogeneous Turbulence**: all statistics are invariant under translation of the coordinate system, i.e.

$$\overline{u_i'(x_k)u_j'(x_k)} = \overline{u_i'(x_k + \Delta x_k)u_j'(x_k + \Delta x_k)}, \quad i, j, k = 1, 2, 3$$

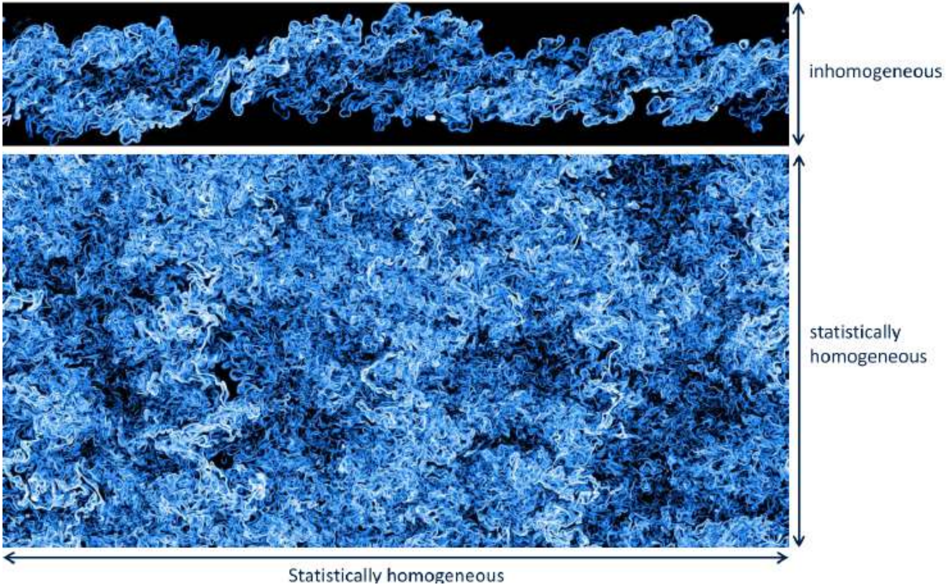
2. **Statistically Isotropic Turbulence**: all statistics are invariant under rotation and reflection of the coordinate system, i.e.

$$\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2} = u'^2, \qquad \overline{u_i'u_j'} = -\overline{u_i'u_j'} \implies \overline{u_i'u_j'} = 0 \quad \text{for } i \neq j$$

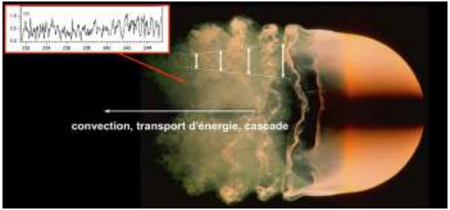
- Mean velocities $= 0$, no privileged directions
- Direct relation with k_t: $\overline{u_i'u_j'} = u'^2\delta_{ij} = \frac{2}{3}k_t\delta_{ij}$
- Homogeneous Isotropic turbulence (HIT): allows theoretical conclusions about turbulence, widely used for development of turbulence models

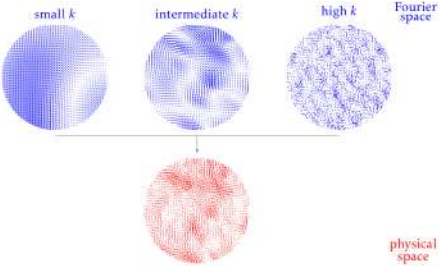
Shear flows

- Round jets, Flow around airfoil, flows in combustion chamber..



Two-point correlations





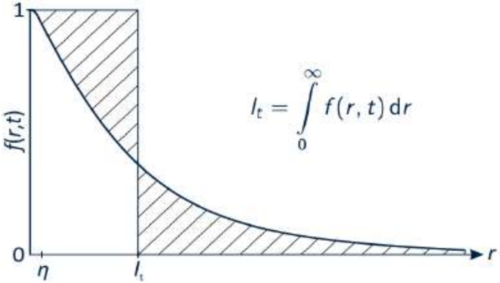
Largest physical (integral) scales:

$$\ell_t = \int_0^\infty f(r,t)\,\mathrm{d}t, \quad u' \sim \sqrt{\frac{2}{3} k_t}, \quad \tau_t = \frac{\ell_t}{u'}$$

Eddies exist at different length scales: How to determine the distribution of eddy size at a single point?

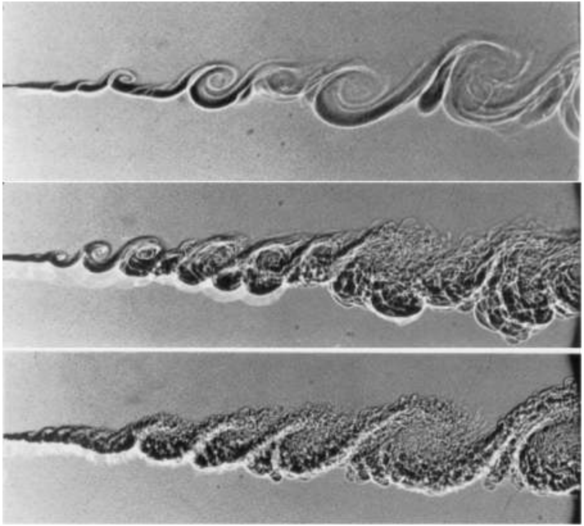
$\implies$ Two-point correlations!

- Measurement of velocity fluctuations, $u_i'(\vec{x}, t)$ and $u_j'(\vec{x} + \vec{r}, t)$
- **Two-point correlation:** $\boxed{R_{ij}(\vec{x}, \vec{r}, t) = \overline{u_i'(\vec{x}, t) u_j'(\vec{x} + \vec{r}, t)}}$
  - for HIT, $R_{ij}(\vec{x}, \vec{r}, t) = R_{ij}(r, t)$
- By normalization, **correlation function**: $f(r, t) = \dfrac{R(r, t)}{u_{\mathrm{rms}}^2(t)}$
  - Degree of correlation of stochastic signals

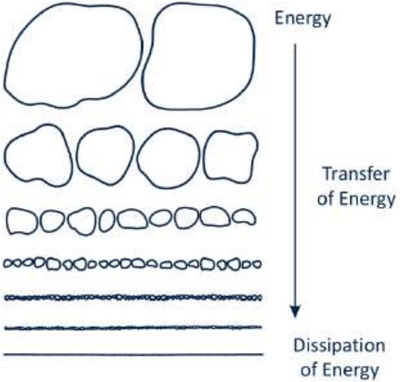


The energy cascade

- Dynamics dominated by big-size **coherent structures**
- Increase of *Re* ($\times 2$ for each picture):



Creation of smaller structures, large ones unchanged



$$\boxed{Re = \frac{u' \ell}{\nu}}$$

- Large eddies: $\ell_t \implies$ high-*Re*
  - Unstable due to inertial forces
  - Rupture and creation of smaller structures
  - Re_ℓ progressively lower
- The "cascade" stops at ℓ_η for which $Re_{\ell_\eta} = \dfrac{u_\eta \ell_\eta}{\nu} \approx 1$
  - This is called the Kolmogorov scale

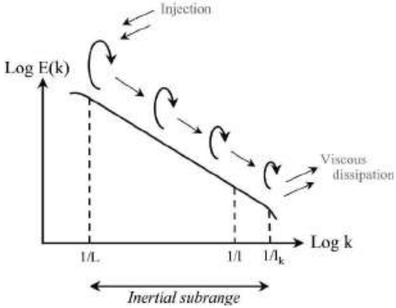
Kolmogorov's K41 Theory

First similarity hypothesis:
At sufficiently high-Re, small-scales eddies have a universal form, uniquely determined by ν and ε

- Leads to length, velocity and time scales by dimensional analysis of ν [m^{2}s^{-1}] and ε [m^{2}s^{-3}]:

	Large scales (energetic)	**Small scales** (dissipative)
Velocity	$u' \sim k_t^{1/2}$	$u_\eta = (\nu\varepsilon)^{1/4}$
Length	ℓ_t	$\ell_\eta = (\nu^3/\varepsilon)^{1/4}$
Time	$\tau_t \sim \ell_t/u'$	$\tau_\eta = (\nu/\varepsilon)^{1/2}$

Second similarity hypothesis:
At sufficiently high-Re, statistics of scale r in $\ell_\eta \ll r \ll \ell_t$ have a universal form, uniquely determined by ε, independent of ν

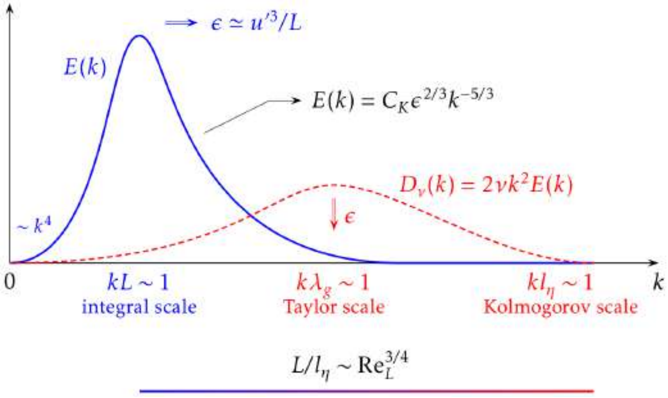


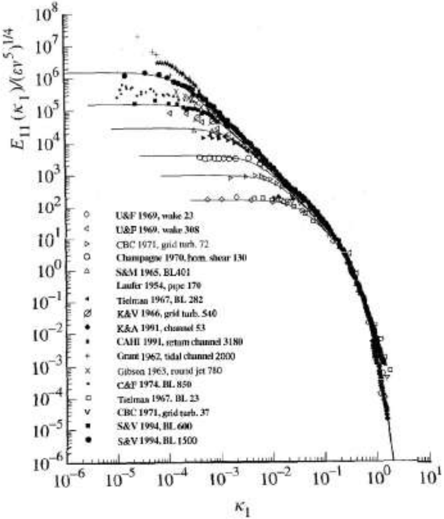
- Inertial equilibrium subrange, in which Prod. $\approx$ Diss. $\Longrightarrow \varepsilon \approx \dfrac{k_t}{\tau_t} \approx \dfrac{u'^2}{\ell_t/u'} = \dfrac{u'^3}{\ell_t}$ (independent of ν)

$$\begin{cases} \ell_t \approx \dfrac{u'^3}{\varepsilon} \\ \ell_\eta \approx \left(\dfrac{\nu^3}{\varepsilon}\right)^{1/4} \end{cases} \Longrightarrow \frac{\ell_t}{\ell_\eta} = \left(\frac{u'\ell_t}{\nu}\right)^{3/4} = Re_{\ell_t}^{3/4} \quad \text{and} \quad \begin{cases} \tau_t \approx \dfrac{\ell_t}{u'} \\ \tau_\eta \approx \left(\dfrac{\nu}{\varepsilon}\right)^{1/2} \end{cases} \Longrightarrow \frac{\tau_t}{\tau_\eta} = \left(\frac{u'\ell_t}{\nu}\right)^{1/2} = Re_{\ell_t}^{1/2}$$

Turbulent spectra

Kolmogorov's Turbulent kinetic energy spectrum: $\boxed{E(k) = c_K \varepsilon^{2/3} k^{-5/3}}$





A first cost estimation

Considering that:

- We need $\approx n = 3 \div 5$ points to discretize ℓ_η
- The number of grid points is $N \sim Re^{9/4}$ (shown later)
- Typical atmospheric scales are $u_0 \sim 10\,\dfrac{\mathrm{m}}{\mathrm{s}}$ $\quad \ell_t \sim 10^3\,\mathrm{m}$ $\quad \nu \sim 10^{-5}\,\dfrac{\mathrm{m}^2}{\mathrm{s}}$

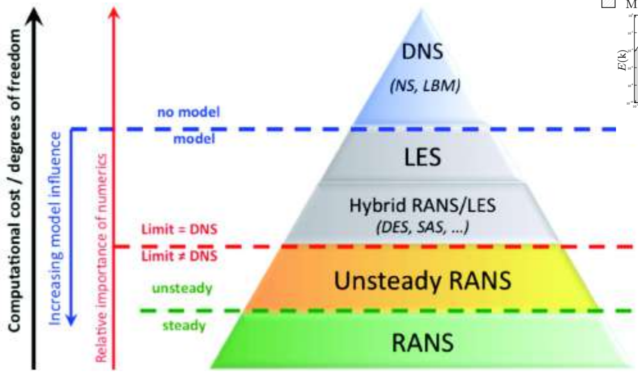
Estimate the **number of grid points needed to solve all spatial scales**. We obtain

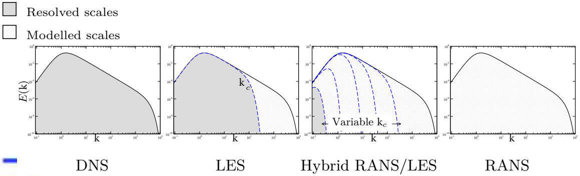
$$Re = \frac{u_0 \ell_t}{\nu} = \frac{10 \times 10^3}{10^{-5}} \approx 10^9 \qquad \begin{cases} \ell_t \approx 10^3\,\mathrm{m} \\ \ell_\eta = \dfrac{\ell_t}{Re^{3/4}} = \dfrac{10^3}{(10^9)^{3/4}} \approx 0.2\,\mathrm{mm} \end{cases} \qquad \begin{cases} \tau_t = \dfrac{L_t}{U_0} = \dfrac{10^3}{10} = 100\,\mathrm{s} \\ \tau_\eta = \dfrac{\tau_t}{Re^{1/2}} = \dfrac{100}{(10^9)^{1/2}} \approx 0.003\,\mathrm{s} \end{cases}$$

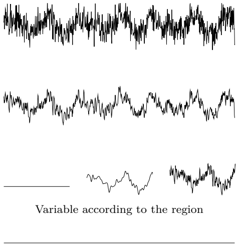
$$N \sim (nRe)^{9/4} = (3 \times 10^9)^{9/4} = 1.6 \cdot 10^{21}\ \text{points}$$

- This is just to simulate a $1\,\mathrm{km}^3$ cube of the atmosphere
- Think how big the atmosphere is
- Get depressed
- Not all is lost!

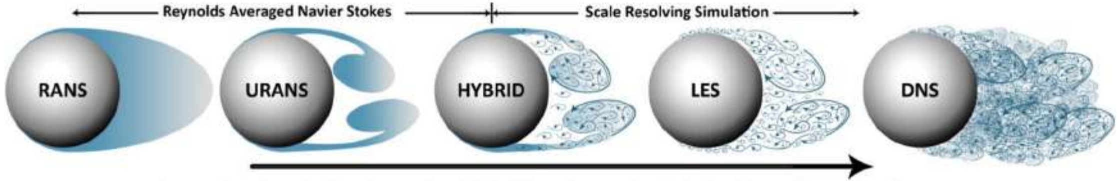
Hierarchy of CFD methods







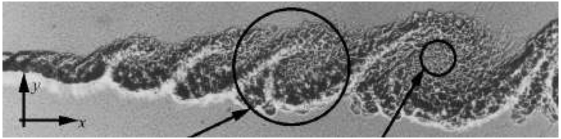
DNS	Avail.: 2080
LES	Avail.: 2070 Cost wall region ≃ DNS Avail. : 2045 with wall functions
Hybrid	Avail.: 2000 Rapid development
RANS	Avail.: 1985 Industrial standard



Direct Numerical Simulation (DNS)

- No modelling of turbulence
- Solve all spatial and temporal scales. Remember:

$$\frac{\ell_t}{\ell_\eta} \sim Re_{\ell_t}^{3/4} \qquad \frac{\tau_t}{\tau_\eta} \sim Re_{\ell_t}^{1/2}$$



largest eddies ℓ_t smallest eddies ℓ_η

Spatial Scales

- Grid-width comparable to ℓ_η
- Computational domain proportional to ℓ_t
- 1D example: Integral length ℓ_t discretized with N points, equispaced by h ($L = Nh$). We need:
  - The domain to be bigger than ℓ_t: $Nh \geq \ell_t$
  - The grid size to be smaller than ℓ_η: $h \leq \ell_\eta$

$$\ell_\eta N \geq Nh \geq \ell_t \implies N \geq \frac{\ell_t}{\ell_\eta} = Re_{\ell_t}^{3/4}$$

In 3D: $N_{3D} = N_{1D}^3 = Re_{\ell_t}^{9/4}$

- For $\mathrm{Re}_{\ell_t} = 10^6$, $\boxed{N_{3D} \approx 3 \cdot 10^{13} \text{ points}}$

✔ Extremely useful for fundamental research

✔ Test of turbulence models

Temporal Scales

- Time-step comparable to τ_η
- Total time of integration proportional to τ_t
- Example: Time-step based on CFL restriction

$$CFL = \frac{u' \Delta t}{\Delta x} \approx 1, \qquad \tau_t \sim \frac{\ell_t}{u'}$$

$$N_{ite} = \frac{\tau_t}{\Delta t} \sim \frac{\ell_t / u'}{\Delta x / u'} \sim \frac{\ell_t}{\ell_\eta} \sim Re_{\ell_t}^{3/4}$$

Total Cost: $N_{3D} \times N_{ite} \sim Re_{\ell_t}^{9/4} \times Re_{\ell_t}^{3/4} = \boxed{Re_{\ell_t}^3}$

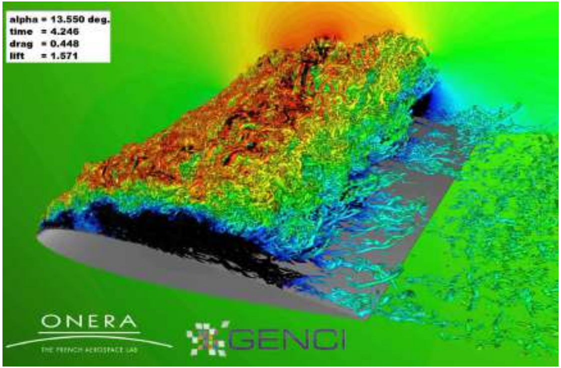
- Increasing $Re \times 10$ implies Cost $\times 1000$!

✘ Infeasible at large Re

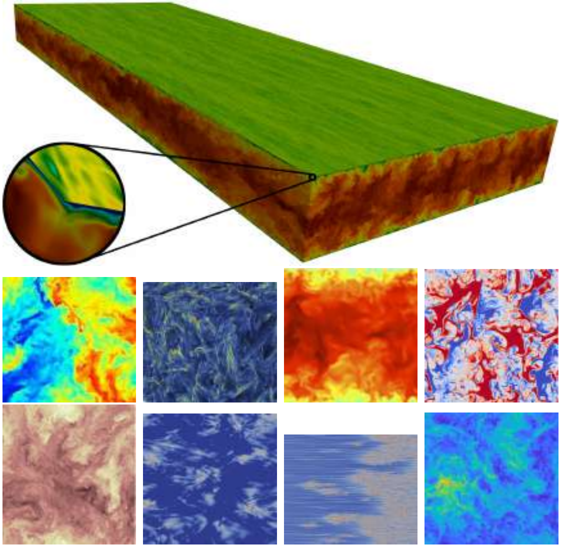
✘ Hard to implement in complex geometries

Direct Numerical Simulation (DNS)

- Examples shown in HPC session
- NACA0012 stall
  $N = 2 \cdot 10^8$ points, $M_\infty = 0.15$, Q-criterion



http://turbulence.pha.jhu.edu/

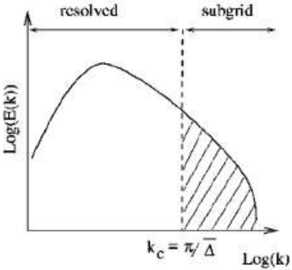


- Examples of large grid sizes:
  - Isotropic turbulence: 8192^3
  - Channel flow: $10240 \times 1536 \times 7680$
  - Boundary layer: $3320 \times 224 \times 2048$
- More than 1 PB of data available

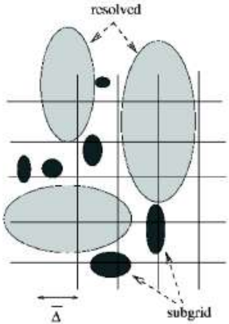
Large-Eddy Simulations (LES)

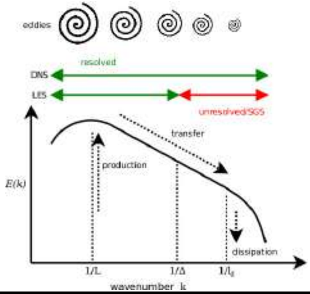
Large scales resolved and small scales modeled, κ_c defining the scales separation

Fourier space



Physical space





Comparison LES - DNS (Moin and Mahesh, 1998)

$$N_{\text{LES}} \sim \frac{0.4 N_{\text{DNS}}}{Re_\tau^{1/4}}$$

Re_H	Re_τ	N_{DNS}	N_{LES}
12 300	360	6.7×10^6	6.1×10^5
30 800	800	4.0×10^7	3.0×10^6
61 600	1450	1.5×10^8	1.0×10^7
230 000	4650	2.1×10^9	1.0×10^8

- LES, no walls: $N = Re^{0.5}$
- LES, walls: $N = Re^{2.4}$
- $N_{\text{RANS}} = 10^4$..

✔ Reduced computational effort (5-10 % w.r.t. DNS)

✔ Modeling part restricted to small (universal) scales

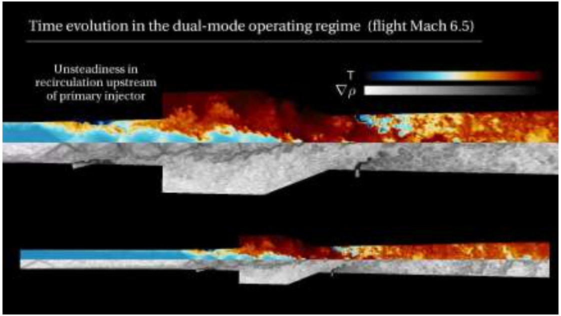
✔ Allows unsteady 3D computations of coherent structures

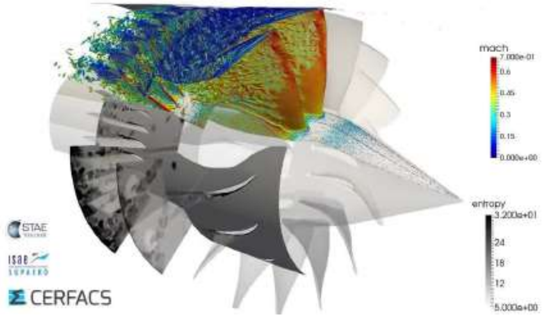
✘ Still expensive, notably for wall-bounded flows

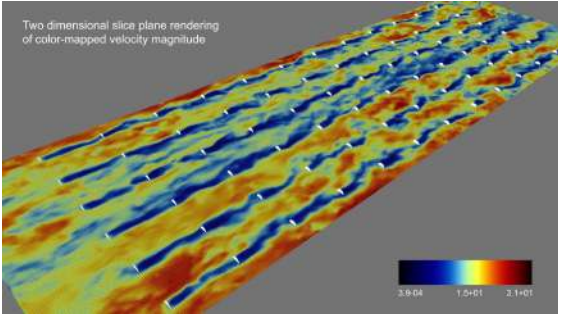
✘ Extension to complex geometries not trivial

✘ Filter interacts with discretization: **need for accurate schemes!**

Large-Eddy Simulations (LES)



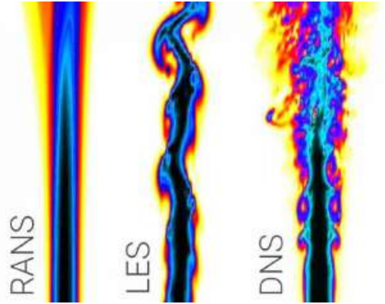


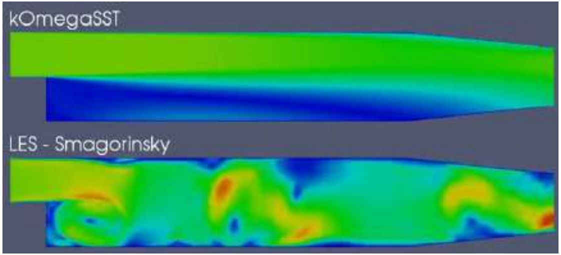


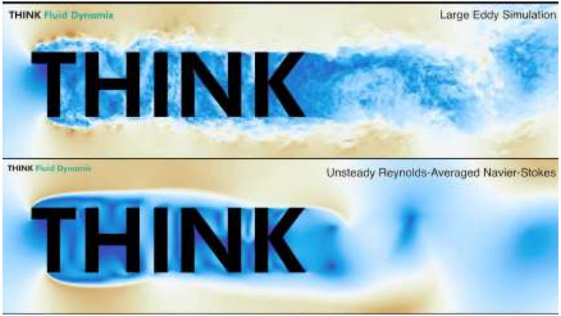
Reynolds-Averaged Navier–Stokes (RANS) simulations

All turbulent scales are modeled, only a statistical mean is solved

- ✔ Extremely cheap compared to DNS and LES
- ✔ Widely spread in traditional and industrial CFD codes
- ✔ Well adapted to complex geometries
- ✘ All the turbulent motions must be properly modeled
- ✘ Pragmatic “tuning” of model parameters is often required
- ✘ Results not always correspond to mean flow from experiments (requires the ergodic theorem to be valid)







Mean Flow equations

Incompressible NS, homogeneous fluid:

$$\begin{cases} \dfrac{\partial u_i}{\partial x_i} = 0 \\ \dfrac{\partial u_i}{\partial t} + \dfrac{\partial u_i u_j}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + \nu \dfrac{\partial^2 u_i}{\partial x_j^2} \end{cases} \Longrightarrow \begin{cases} \overline{\dfrac{\partial(\overline{u}_i + u_i')}{\partial x_i}} = 0 \\ \overline{\dfrac{\partial(\overline{u}_i + u_i')}{\partial t}} + \overline{\dfrac{\partial[(\overline{u_i} + u_i')(\overline{u}_j + u_j')]}{\partial x_j}} = -\dfrac{1}{\rho}\overline{\dfrac{\partial(\overline{p} + p')}{\partial x_i}} + \nu \overline{\dfrac{\partial^2(\overline{u}_i + u_i')}{\partial x_j^2}} \end{cases}$$

To find the transport equations for the mean and fluctuating quantities, we need to:

1. Replace the Reynolds decomposition
2. Average the equation

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial(\overline{u}_i + u_i')}{\partial x_i} = 0, \quad \frac{\partial(\overline{\overline{u}_i + u_i'})}{\partial x_i} = 0, \Longrightarrow \boxed{\frac{\partial \overline{u}_i}{\partial x_i} = 0}$$

Subtracting the equations $\dfrac{\partial u_i}{\partial x_i} = 0$ and $\dfrac{\partial \overline{u}_i}{\partial x_i} = 0$: $\boxed{\dfrac{\partial u_i'}{\partial x_i} = 0}$

- Linearity of continuity equation

$$\begin{cases} \dfrac{\partial \overline{u}_i}{\partial x_i} = 0 \\ \dfrac{\partial \overline{u}_i}{\partial t} + \dfrac{\partial(\overline{u}_i \overline{u}_j)}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial x_i} + \nu \dfrac{\partial^2 \overline{u}_i}{\partial x_j^2} - \dfrac{\partial \overline{u_i' u_j'}}{\partial x_j} \end{cases}$$

$\tau_{ij}^R = -\rho\overline{u_i' u_j'}$: **Reynolds stress tensor**

- Coupling between mean and fluct. field
- In general, $-\rho\overline{u_i' u_j'} \gg \tau_{ij}$ (apart from near-wall region)
- Unknown term, to be modeled

This system represents the "Reynolds–averaged Navier–Stokes" (RANS) equations

Can we derive an equation for the Reynolds stresses to close the system? Let's start from the kinetic energy

Kinetic energy of the mean field

By applying the Reynolds decomposition to the kinetic energy one has:

- $\hat{E}$: kinetic energy of the mean field
- k_t: turbulent kinetic energy (TKE)

$$\overline{E} = \frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\overline{(\overline{u}_i + u_i')(\overline{u}_i + u_i')} = \frac{1}{2}\overline{(\overline{u}_i\overline{u}_i + 2u_i'\overline{u}_i + u_i'u_i')}$$
$$= \frac{1}{2}(\overline{\overline{u}_i\overline{u}_i} + \cancel{2\overline{u_i'\overline{u}_i}} + \overline{u_i'u_i'}) = \frac{1}{2}(\overline{u}_i\overline{u}_i + \overline{u_i'u_i'}) = \hat{E} + k_t$$

Transport equation for the kinetic energy of the mean field $\hat{E}$

$$\overline{u}_i \times \left\{ \frac{\partial \rho \overline{u}_i}{\partial t} + \frac{\partial(\rho \overline{u}_i \overline{u}_j)}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau}_{ij}}{\partial x_j} - \frac{\partial \rho \overline{u_i'u_j'}}{\partial x_j} \right\}$$

$$\underbrace{\frac{\partial \frac{1}{2}\rho \overline{u}_i^2}{\partial t} + \frac{\partial \frac{1}{2}\rho \overline{u}_i^2 \overline{u}_j}{\partial x_j}}_{\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{2}\rho\overline{u}_i^2\right)} = \rho\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{\tau}_{ij}\frac{\partial \overline{u}_i}{\partial x_j} + \underbrace{\frac{\partial}{\partial x_j}\left(-\overline{u}_j\overline{p} + \overline{u}_i\overline{\tau}_{ij} - \overline{u}_i\rho\overline{u_i'u_j'}\right)}_{\text{transport terms}}$$

- For homogeneous isotropic turbulence:

$$\underbrace{\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{2}\rho\overline{u}_i^2\right)}_{\substack{\text{variation of } \hat{E} \\ \text{in the mean field}}} = \underbrace{\rho\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j}}_{\substack{\text{transfer to} \\ \text{turbulent field}}} - \underbrace{\overline{\tau}_{ij}\frac{\partial \overline{u}_i}{\partial x_j}}_{\substack{\text{dissipation of } \hat{E} \\ \text{for viscous effects}}}$$

Kinetic energy of the fluctuating field (1)

Transport equation for the turbulent kinetic energy k_t

$$\begin{cases} \dfrac{\partial \rho u_i}{\partial t} + \dfrac{\partial(\rho u_i u_j)}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \dfrac{\partial \tau_{ij}}{\partial x_j} \\ \qquad\qquad (-) \\ \dfrac{\partial \rho \overline{u}_i}{\partial t} + \dfrac{\partial(\rho \overline{u}_i \overline{u}_j)}{\partial x_j} = -\dfrac{\partial \overline{p}}{\partial x_i} + \dfrac{\partial \overline{\tau}_{ij}}{\partial x_j} - \dfrac{\partial \rho \overline{u_i' u_j'}}{\partial x_j} \end{cases} \Longrightarrow \frac{\partial \rho u_i'}{\partial t} + \frac{\partial}{\partial x_j}[\rho(u_i'\overline{u}_j + \overline{u}_i u_j' + u_i' u_j')] = -\frac{\partial p'}{\partial x_i} + \frac{\partial \tau_{ij}'}{\partial x_j} + \frac{\partial \rho \overline{u_i' u_j'}}{\partial x_j} \quad (\star)$$

$$\overline{u_i' \times (\star)} \quad \Longrightarrow \quad \frac{\overline{\mathrm{d}}}{\mathrm{d}t}(\rho k_t) = \underbrace{-\rho\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}}_{\text{Production}} - \underbrace{\overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}}}_{\text{Dissipation}} \underbrace{-\frac{1}{2}\frac{\partial}{\partial x_j}\overline{\rho u_i' u_i' u_j'} - \overline{u_i'\frac{\partial p'}{\partial x_i}} + \frac{\partial}{\partial x_j}\overline{u_i'\tau_{ij}'}}_{\text{Diffusion}}$$

- For homogeneous isotropic turbulence:

$$\underbrace{\frac{\overline{\mathrm{d}}}{\mathrm{d}t}(\rho k_t)}_{\substack{\text{Variation of } k_t \\ \text{in the mean field}}} = \underbrace{-\rho\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}}_{\substack{\text{energy transfer between} \\ \text{mean and fluctuating field}}} - \underbrace{\overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}}}_{\substack{\text{dissipation of } k_t \\ \text{for viscous effects}}}$$

Kinetic energy of the fluctuating field (2)

$$\frac{\overline{\mathrm{d}}}{\mathrm{d}t}(\rho k_t) = \underbrace{-\rho\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}}_{\substack{\text{Production}\\ \mathcal{P}}} \underbrace{-\overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}}}_{\substack{\text{Dissipation}\\ \rho\varepsilon}}$$

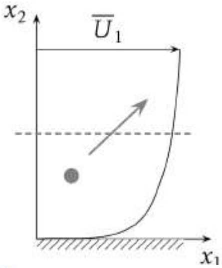
- Dissipation rate for k_t: $\rho\varepsilon \equiv \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}} = 2\mu\overline{S_{ij}'^2} = \frac{1}{2}\mu\overline{\left[\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right]^2} \geq 0$
- Meaning of production $\mathcal{P}$ term:

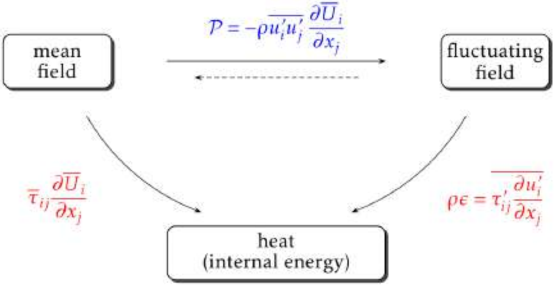
$$\text{if} \begin{cases} u_2' > 0 \\ u_1' < 0 \end{cases} \implies \overline{u_1' u_2'} < 0$$

$$\text{if} \begin{cases} u_2' < 0 \\ u_1' > 0 \end{cases} \implies \overline{u_1' u_2'} < 0$$

hence $\mathcal{P} > 0$!

- Closure problem not solved!
  - Triple correlations..
  - Derivation of equations for such correlations $\implies$ even higher correlations..





Derivation of FANS equations (1)

- Let's switch to the compressible case
- We use Favre filtering

Conservation of Mass

$$\overline{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i}} = 0$$

$$\overline{\frac{\partial \rho}{\partial t} + \frac{\partial [\rho(\widetilde{u}_i + u_i'')]}{\partial x_i}} = 0$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \left(\overline{\rho}\widetilde{u}_i\right)}{\partial x_i} + \cancel{\frac{\partial (\overline{\rho u_i''})}{\partial x_i}} = 0$$

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \left(\overline{\rho}\widetilde{u}_i\right)}{\partial x_i} = 0$$

Conservation of Momentum

$$\overline{\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_j\right)}{\partial x_j}} = \overline{-\frac{\partial p}{\partial x_i}} + \overline{\frac{\partial \tau_{ij}}{\partial x_j}}$$

$$\overline{\frac{\partial \rho(\widetilde{u}_i + u_i'')}{\partial t} + \frac{\partial}{\partial x_j}\left[\rho(\widetilde{u}_i + u_i'')(\widetilde{u}_j + u_j'')\right]} = \overline{-\frac{\partial p}{\partial x_i}} + \overline{\frac{\partial \tau_{ij}}{\partial x_j}}$$

$$\frac{\partial \overline{\rho \widetilde{u}_i}}{\partial t} + \cancel{\frac{\partial \overline{\rho u_i''}}{\partial t}} + \frac{\partial}{\partial x_j}\left[\overline{\rho \widetilde{u}_i \widetilde{u}_j} + \cancel{\overline{\rho \widetilde{u}_i u_j''}} + \cancel{\overline{\rho u_i'' \widetilde{u}_j}} + \overline{\rho u_i'' u_j''}\right] = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau}_{ij}}{\partial x_j}$$

$$\frac{\partial (\overline{\rho}\widetilde{u}_i)}{\partial t} + \frac{\partial \left(\overline{\rho}\widetilde{u}_i \widetilde{u}_j\right)}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\overline{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right]$$

Derivation of FANS equations (2)

Conservation of Energy

$$\overline{\frac{\partial \rho E}{\partial t}} + \overline{\frac{\partial \rho H u_j}{\partial x_j}} = \overline{\frac{\partial \tau_{ji} u_i}{\partial x_j}} - \overline{\frac{\partial q_j}{\partial x_j}}$$

$$\overline{\frac{\partial \rho E}{\partial t}} = \overline{\frac{\partial}{\partial t}\left[\rho\left(\widetilde{e} + e'' + \frac{1}{2}(\widetilde{u}_i + u_i'')(\widetilde{u}_i + u_i'')\right)\right]}$$
$$= \overline{\frac{\partial}{\partial t}\left[\rho\left(\widetilde{e} + \frac{1}{2}\widetilde{u}_i\widetilde{u}_i\right) + \rho\frac{1}{2}u_i''u_i''\right]}$$
$$= \frac{\partial}{\partial t}\left[\overline{\rho}\left(\widetilde{e} + \frac{1}{2}\widetilde{u}_i\widetilde{u}_i\right) + \overline{\rho}\frac{1}{2}\widetilde{u_i''u_i''}\right]$$
$$= \frac{\partial}{\partial t}\left[\overline{\rho}\left(\widetilde{e} + \frac{1}{2}\widetilde{u}_i\widetilde{u}_i + \frac{1}{2}k_t\right)\right] = \frac{\partial \overline{\rho}\widetilde{E}}{\partial t}$$

$$\overline{\frac{\partial(\tau_{ji} u_i)}{\partial x_j}} = \overline{\frac{\partial}{\partial x_j}[\tau_{ji}(\widetilde{u}_i + u_i'')]}$$
$$= \frac{\partial}{\partial x_j}\left(\overline{\tau}_{ji}\widetilde{u}_i + \overline{\tau_{ji}u_i''}\right)$$

$$\overline{\frac{\partial q_j}{\partial x_j}} = \frac{\partial \overline{q}_j}{\partial x_j}$$

$$\overline{\frac{\partial \rho H u_j}{\partial x_j}} = \overline{\frac{\partial}{\partial x_j}\left[\rho(\widetilde{u}_j + u_j'')\left(\widetilde{h} + h'' + \frac{1}{2}(\widetilde{u}_i + u_i'')(\widetilde{u}_i + u_i'')\right)\right]}$$
$$= \overline{\frac{\partial}{\partial x_j}\left[\rho(\widetilde{u}_j + u_j'')\left(\widetilde{h} + h'' + \frac{1}{2}\widetilde{u}_i\widetilde{u}_i + u_i''\widetilde{u}_i + \frac{1}{2}u_i''u_i''\right)\right]}$$
$$= \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u}_j\widetilde{h} + \overline{\rho}\widetilde{u}_j\frac{1}{2}\widetilde{u}_i\widetilde{u}_i + \overline{\rho}\widetilde{u}_j k_t + \overline{\rho u_j'' h''} + \overline{\rho u_j'' u_i''}\widetilde{u}_i + \frac{1}{2}\overline{\rho u_j'' u_i'' u_i''}\right)$$
$$= \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u}_j\widetilde{H} + \overline{\rho u_j'' h''} + \overline{\rho u_j'' u_i''}\widetilde{u}_i + \frac{1}{2}\overline{\rho u_j'' u_i'' u_i''}\right)$$

$$\frac{\partial \overline{\rho}\widetilde{E}}{\partial t} + \frac{\partial \overline{\rho}\widetilde{u}_j\widetilde{H}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\overline{\tau}_{ji} - \overline{\rho u_j'' u_i''})\widetilde{u}_i\right] - \frac{\partial}{\partial x_j}\left(\overline{q}_j + \overline{\rho u_j'' h''}\right) + \frac{\partial}{\partial x_j}\left(-\frac{1}{2}\overline{\rho u_j'' u_i'' u_i''} + \overline{\tau_{ji}u_i''}\right)$$

Derivation of FANS equations (3)

$$\begin{cases} \dfrac{\partial \overline{\rho}}{\partial t} + \dfrac{\partial\left(\overline{\rho}\widetilde{u}_i\right)}{\partial x_i} = 0 \\ \dfrac{\partial (\overline{\rho}\widetilde{u}_i)}{\partial t} + \dfrac{\partial\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j\right)}{\partial x_j} = -\dfrac{\partial \overline{p}}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left[\overline{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right] \\ \dfrac{\partial \overline{\rho}\widetilde{E}}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j\widetilde{H}}{\partial x_j} = \dfrac{\partial}{\partial x_j}\left[(\overline{\tau}_{ji} - \overline{\rho u_j'' u_i''})\widetilde{u}_i\right] - \dfrac{\partial}{\partial x_j}\left(\overline{q}_j + \overline{\rho u_j'' h''}\right) + \dfrac{\partial}{\partial x_j}\left(-\dfrac{1}{2}\overline{\rho u_j'' u_i'' u_i''} + \overline{\tau_{ji} u_i''}\right) \end{cases}$$

$$\begin{cases} \dfrac{\partial \overline{\rho}}{\partial t} + \dfrac{\partial\left(\overline{\rho}\widetilde{u}_i\right)}{\partial x_i} = 0 \\ \dfrac{\partial (\overline{\rho}\widetilde{u}_i)}{\partial t} + \dfrac{\partial\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j\right)}{\partial x_j} = -\dfrac{\partial \overline{p}}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left[2\overline{\mu}\widetilde{S}_{ij}^D - \overline{\rho u_i'' u_j''}\right] \\ \dfrac{\partial \overline{\rho}\widetilde{E}}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j\widetilde{H}}{\partial x_j} = \dfrac{\partial}{\partial x_j}\left[(2\overline{\mu}\widetilde{S}_{ji}^D - \overline{\rho u_j'' u_i''})\widetilde{u}_i\right] - \dfrac{\partial}{\partial x_j}\left(\dfrac{\overline{\mu}}{Pr}\dfrac{\partial \widetilde{h}}{\partial x_j} + \overline{\rho u_j'' h''}\right) + \dfrac{\partial}{\partial x_j}\left(-\dfrac{1}{2}\overline{\rho u_j'' u_i'' u_i''} + \overline{\tau_{ji} u_i''}\right) \end{cases}$$

✔ Terms closed

✘ Terms needing closure..

1. Reynolds stress
2. Reynolds heat flux
3. Turbulent transport and work
4. Instances of k_t

Assumptions:

1. $\overline{\tau}_{ij} = \overline{2\rho\nu\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)} = \overline{2\rho\nu S_{ij}^D} = 2\overline{\rho}\widetilde{\nu}\widetilde{S}_{ij}^D + \cancel{\overline{2\rho\nu'' S_{ij}^{D''}}} = 2\overline{\mu}\widetilde{S}_{ij}^D$
2. $\overline{p} = \overline{\rho R T} = \overline{\rho} R \widetilde{T}, \qquad \widetilde{e} = c_v \widetilde{T}, \qquad \widetilde{h} = \widetilde{e} + \dfrac{\overline{p}}{\overline{\rho}} = c_p \widetilde{T}$
3. $\overline{q}_j = -\overline{\lambda \dfrac{\partial T}{\partial x_j}} = -\overline{\dfrac{\mu}{Pr}\dfrac{\partial h}{\partial x_j}} = -\dfrac{1}{Pr}\left(\overline{\rho}\widetilde{\nu}\dfrac{\partial \widetilde{h}}{\partial x_j} + \cancel{\overline{\rho\nu''\dfrac{\partial h''}{\partial x_j}}}\right) = \dfrac{\overline{\mu}}{Pr}\dfrac{\partial \widetilde{h}}{\partial x_j}$
4. $\overline{\mu} = \overline{\mu_{\mathrm{ref}}\left(\dfrac{T}{T_{\mathrm{ref}}}\right)^{3/2}\dfrac{T_{\mathrm{ref}} + S}{T_{\mathrm{ref}} + S}} \approx \mu_{\mathrm{ref}}\left(\dfrac{\widetilde{T}}{T_{\mathrm{ref}}}\right)^{3/2}\dfrac{T_{\mathrm{ref}} + S}{\widetilde{T}_{\mathrm{ref}} + S}$

The Boussinesq hypothesis

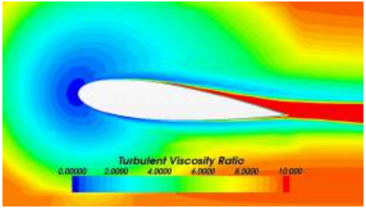
How to model the Reynolds stress tensor $\tau_{ij}^R = \overline{\rho u_i'' u_j''}$? Two main ideas:

1. Try to derive a transport equation for τ_{ij}^R
   - Already tried, closure problem remains
   - RSM (second-moment closures), see later
2. Use a "turbulent" viscosity: Boussinesq hyp. (1877)
   - Analogy to Newtonian fluid approach for molecular shear stress
   - Gradient transport model (first-order model, providing second moments as function of first)

$$\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}$$

$$\tau_{ij}^R = -\overline{\rho u_i'' u_j''} = 2\mu_t \widetilde{S}_{ij} - \frac{2}{3}\overline{\rho} k_t \delta_{ij}$$

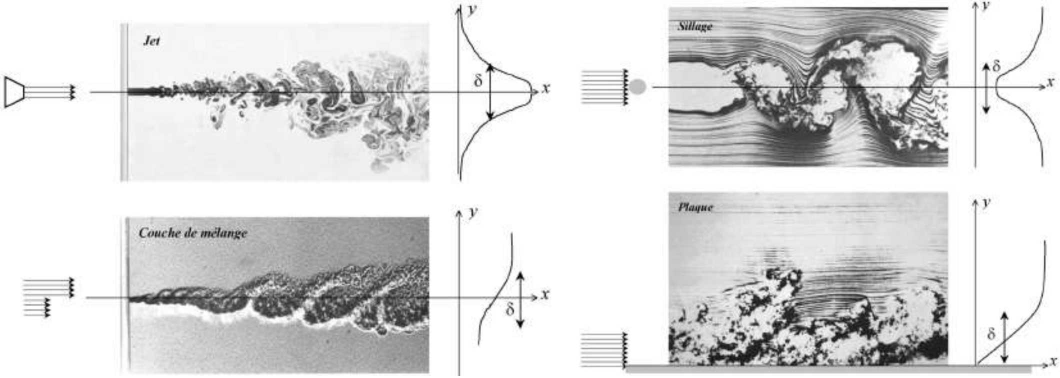
- $\mu_t = \mu_t(\vec{x}, t)$: **eddy viscosity**
  - Property of the flow, not of the fluid
  - In general, $\mu_t \gg \mu$, apart from near-wall regions
  - $2/3\overline{\rho}k_t\delta_{ij}$ ensures correct definition of k_t



Weaknesses of Boussinesq approximation:

- $\mathcal{P} = 2\mu_t \overline{S_{ij}^2} \geq 0$ always
- Reduction from 6 to 1 unknown, but closure model always needed
- Linear relation between μ_t and $\overline{S}_{ij}$!
  - When $\overline{S}_{ij} = 0$, $\tau_{ij}^R = 0$ instantaneously (violation of causality)
  - ✘ Axisymmetric, stratified, rotating, separated, shocked flows, 3D flows, strong curvatures, impingement, axial strain..
  - ✔ Fixes for rotation, curvature, ν_t limiters..

Weakly non-parallel shear flows



Hypothesis:

- Steady, incompressible flow
- $\overline{u} = \overline{u}(y), \quad \overline{v} = 0$
- Small streamwise fluctuations (i.e., $\dfrac{\partial(\bullet)}{\partial x} \ll \dfrac{\partial(\bullet)}{\partial y}$)
- τ_{tot} must nullify when velocity profile achieves a local extremum

RANS for steady incompressible flow:

$$\cancel{\rho\overline{u}\frac{\partial \overline{u}}{\partial x}} + \cancel{\rho\overline{v}\frac{\partial \overline{u}}{\partial y}} = -\cancel{\frac{\partial \overline{p}}{\partial x}} + \cancel{\frac{\partial(\rho\overline{u'u'})}{\partial x}} + \frac{\partial}{\partial y}\left(\overline{\tau}_{xy} - \rho\overline{u'v'}\right)$$

Total shear stress:

$$\tau_{\text{tot}} = \overline{\tau}_{xy} - \rho\overline{u'v'} = \mu\frac{\partial \overline{u}}{\partial y} + \mu_t\frac{\partial \overline{u}}{\partial y} = (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

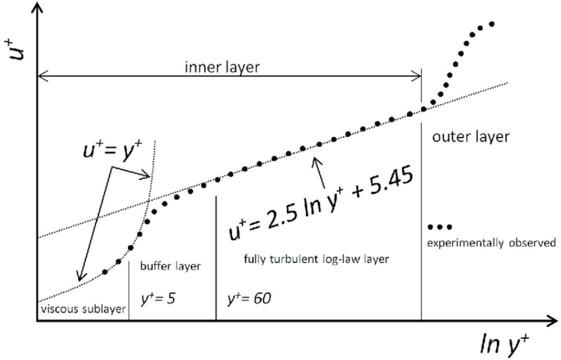
Law of the wall for a ZPG turbulent boundary layer

$$\overbrace{\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}}^{\text{inertia}}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}-\rho\overline{u'v'}\right)=\overbrace{\frac{\partial \tau_{\text{tot}}}{\partial y}}^{\text{friction}}$$

- Far from wall, inertia $\gg$ friction (**Outer Region**)
- Near the wall, inertia $\ll$ friction (**Inner Region**):

$$\frac{\partial \tau_{\text{tot}}}{\partial y}\approx 0 \implies \tau_{\text{tot}}=\text{C}^{\text{te}}$$

Two zones may be identified:



1) Viscous sublayer

Very close to the wall, laminar friction dominant since $u, v \to 0$. Hence

$$\mu\frac{\partial^2 u}{\partial y^2}\approx 0 \quad\implies\quad u=Cy \quad \text{with} \quad C=\frac{\tau_w}{\mu}$$

We introduce the friction velocity

$$u_\tau=\sqrt{\frac{\tau_w}{\rho}} \quad\implies\quad u^+=\frac{u}{u_\tau} \qquad y^+=\frac{\rho u_\tau y}{\mu}=Re^+$$

$$u=Cy \implies \boxed{u^+=y^+}$$

2) Logarithmic Region

Turbulent friction increases progressively and dominates laminar one. Since $\tau_{\text{tot}}=\text{C}^{\text{te}}$ and $u_\tau=\text{C}^{\text{te}}$, by dimensional analysis:

$$\frac{\partial u}{\partial y}\sim\frac{\mathcal{O}(u)}{\mathcal{O}(y)}=A\frac{u_\tau}{y^+} \quad \text{with} \quad A=\frac{1}{\kappa}$$

$$u=\frac{u_\tau}{\kappa}\int\frac{\text{d}y^+}{y^+} \implies \frac{u}{u_\tau}=\boxed{u^+=\frac{1}{\kappa}\log(y^+)+B}$$

with $\kappa\approx 0.41$ (von Karman constant) and $B\approx 5.4$

Closure for FANS

1. Reynolds stress tensor:

$$\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}$$

$$-\overline{\rho u_i'' u_j''} = 2\mu_t \widetilde{S}_{ij} - \frac{2}{3}\overline{\rho} k_t \delta_{ij}$$

2. Reynolds heat flux:

$$\overline{q}_j = \frac{\overline{\mu}}{Pr}\frac{\partial \widetilde{h}}{\partial x_j}$$

$$\overline{\rho u_j'' h''} = -\lambda_t \frac{\partial \widetilde{T}}{\partial x_j} = \frac{\mu_t}{Pr_t}\frac{\partial \widetilde{h}}{\partial x_j}$$

- Usually, $Pr_t = 0.9$

3. Turbulent kinetic energy: several approaches depending on the turbulence model:
   - Part of the solution (e.g., k_t-ε, k_t-ω, ..)
   - Not part of the solution (e.g., S-A), neglected
   - Relation with μ_t (Bradshaw's assumption)

4. Turbulent transport and work:
   - For models where k_t is not available, neglected
   - For models where k_t is available, generally something like:

$$-\overline{\rho u_j'' \frac{1}{2} u_i'' u_i''} + \overline{\tau_{ji} u_i''} = \left(\overline{\mu} + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k_t}{\partial x_j}$$

   - For low-Ma flows often neglected, as the term $\frac{2}{3}\overline{\rho} k_t \delta_{ij}$ in τ_{ij}^R ($\overline{\rho} k_t \ll \overline{p}$)

Closed FANS equations

$$\begin{cases}
\dfrac{\partial \overline{\rho}}{\partial t} + \dfrac{\partial\left(\overline{\rho}\widetilde{u}_i\right)}{\partial x_i} = 0 \\
\dfrac{\partial(\overline{\rho}\widetilde{u}_i)}{\partial t} + \dfrac{\partial\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j\right)}{\partial x_j} = -\dfrac{\partial \overline{p}}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left[2(\overline{\mu} + \mu_t)\widetilde{S}_{ij} - \dfrac{2}{3}\overline{\rho} k \delta_{ij}\right] \\
\dfrac{\partial \overline{\rho}\widetilde{E}}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j\widetilde{H}}{\partial x_j} = \dfrac{\partial}{\partial x_j}\left[\left(2(\overline{\mu} + \mu_t)\widetilde{S}_{ij} - \dfrac{2}{3}\overline{\rho} k \delta_{ij}\right)\widetilde{u}_i\right] + \dfrac{\partial}{\partial x_j}\left[\left(\dfrac{\overline{\mu}}{Pr} + \dfrac{\mu_t}{Pr_t}\right)\dfrac{\partial \widetilde{h}}{\partial x_j}\right] + \dfrac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \dfrac{\mu_t}{\sigma_k}\right)\dfrac{\partial k_t}{\partial x_j}\right]
\end{cases}$$

General principles for turbulence modeling

Objectives of the turbulence models

- As cheap as possible
  (overnight computations, parametric studies, ..)
- Predictive (no *a priori* knowledge of the solution)
- Representing at best the flow physics

Some **principles** exist for their development

✔ Provide a method/mathematical framework

✔ Give physical constraints enabling the modeler to make relevant choices

✘ Not sufficient for ensuring that the models will "work properly"

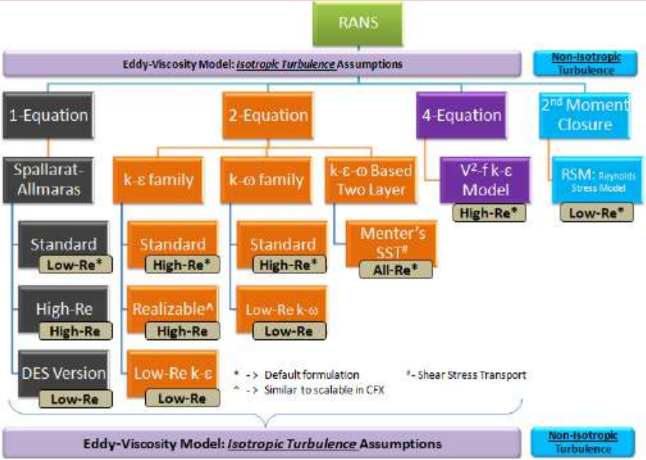
✘ Not necessarily compatible with each other

✘ Some of them can be considered more important, but impossible to rank them

List of more common principles

1. Closure: the model must consist in a closed system of equations (as many equations as unknowns)
2. Dimensional homogeneity: the model for a quantity must be of the same dimension as the quantity itself
3. Completeness: it does not require any *a priori* knowledge of the flow (e.g., no ℓ_{mix})
4. Objectivity: independent on the reference frame
5. Universality: it should theoretically be applicable to all flows (or at least to a wide class of flows)
6. Realizability: the moments of order 1 and 2 have precise mathematical properties, e.g., τ_{ij}^R is a positive-semidefinite tensor
   ($\implies$ eigenvalues and principal invariants positive)
7. Consistency with boundary layer theory: it is expected to reproduce at least a standard BL
8. Numerical robustness: it should be robust (contrast with ease of use / quality of physics representation)

Overview of RANS Models



URANS: Unsteady-RANS

- ▶ Study of long-term periodical oscillations / time-dependent effects
  - Bluff bodies, IC Engines, Helicopter rotors, Transonic airfoils,..
- ✔ Ok with various eddy-viscosity models
- ✔ Successfully combined with LES or DES
- ✘ Model type and time step can have a significant impact on results

Difference Low-*Re* / High-*Re* / All-*Re* related to wall treatment (explained later)

Eddy-Viscosity models (EVM) ν_t

1. 0-equations (Algebraic) models (Prandtl, Baldwin-Lomax, ..) $l_m^2|\overline{S}_{ij}|$
2. 1-equation models
   - TKE models (Prandtl-Kolmogorov) $C l_{pk}\sqrt{k_t}$
   - Spalart-Allmaras $\nu_{\text{sa}} f_{v1}$
3. 2-equation models
   - k_t-ε models (Jones-Launder, RNG, Realizable,..) $C_\mu \dfrac{k_t^2}{\varepsilon}$
   - k_t-ω models (Wilcox., Menter's SST) $\rho\dfrac{k_t}{\omega}$

Can be linear (EVM) or non-linear (NEVM)

Reynolds Stress Models

- ▶ Transport equations for the Reynolds stress tensor + length scale

Algebraic models

- μ_t computed as a function of a suitable "mixing" length
- Based on analogies: for μ with kinetic theory of gases, for τ between turbulent and molecular transport:

$$\tau_{\text{mol}} = \mu \frac{\partial u}{\partial y} \quad \text{with} \quad \mu = \frac{1}{3}\rho \ell_{mfp} v_{mol}$$

$$\tau_{\text{turb}} = \mu_t \frac{\partial \overline{u}}{\partial y} \quad \text{with} \quad \mu_t = \rho \ell_{mix} v_{turb}$$

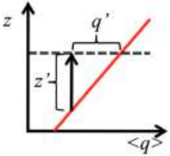
where $v_{\text{turb}} = c_1 \ell_{\text{mix}} \left| \frac{\partial \overline{u}}{\partial y} \right|$ and $\ell_{\text{mix}} = c_2 y$

  - c_1 and c_2 to be specified
  - μ_t is part of the solution, since it depends on $\frac{\partial \overline{u}}{\partial y}$

- ℓ_{mix} depends upon flow configuration
  - Fundamental difference w.r.t. mean free path
  - $\propto$ characteristic length (case by case)
  - Typically, function of distance from nearest wall

  ↪ Problems for complex geometries, detached flows, unstructured solvers..
  - Debatable performances...

Prandtl's mixing length



$$u_t = \ell_{\text{mix}} \left| \frac{\partial u}{\partial y} \right| \implies \mu_t = \rho \ell_{\text{mix}}^2 \left| \frac{\partial u}{\partial y} \right|$$

For a general quantity q of a particle, with an assumed linear profile $q = q(y)$:

- Turbulent eddy moves the particle by an amount y' towards a level y: $q' = \left(\frac{\partial \overline{q}}{\partial y} \right) y'$
- To move up, the particle has a velocity $v' \propto u'$, thus

$$\text{if } \frac{\partial \overline{u}}{\partial y} \begin{cases} > 0: & v' = Cu' \\ < 0: & v' = -Cu' \end{cases} \implies \text{combining, } v' = C \left| \frac{\partial \overline{u}}{\partial y} \right| y'$$

- A kinematic flux can be formed: $\overline{v'q'} = -Cy'^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \frac{\partial \overline{q}}{\partial y}$

  Prandtl assumed $C = 1$ and called y' the mixing length:

$$\overline{v'q'} = -\ell_{\text{mix}}^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \frac{\partial \overline{q}}{\partial y} \implies \boxed{\overline{u'v'} = -\ell_{\text{mix}}^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \frac{\partial \overline{u}}{\partial y} = \nu_t \frac{\partial \overline{u}}{\partial y}}$$

- To be valid whatever the orientation of the shear, generalized as $\nu_t = \ell_{\text{mix}}^2 \sqrt{2 S_{ij} S_{ij}}$

Mixing length and Van Driest damping functions

For **free-shear flows**, ℓ_{mix} proportional to the layer width:

$$\ell_{\text{mix}} = \begin{cases} 0.071\,\delta & \text{for mixing layer} \\ 0.098\,\delta & \text{for plane jet} \\ 0.080\,\delta & \text{for round jet} \\ 0.180\,\delta & \text{for plane wake} \\ \dots \end{cases}$$

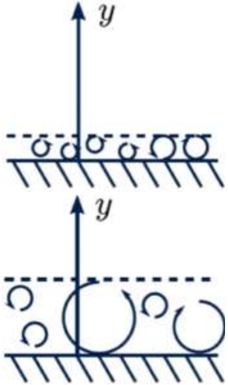
For **wall-bounded flows**, the walls block the maximum size of the eddies

- ℓ_t related to distance from the wall
  - Prandtl supposed that $\ell_t = \kappa y$ with $\kappa = 0.41$, thus

$$\ell_t = \min(\kappa y, 0.09\delta)$$

- Moreover, viscosity dampens the eddies in the viscous layer and reduces their size
  - The Van Driest damping function accounts for this effect:

$$\ell = \kappa y \left[1 - \exp\left(-\frac{y^+}{A^+}\right)\right] \quad \text{with} \quad A^+ = 26.0$$



Baldwin-Lomax Model

$$\mu_t = \begin{cases} \mu_{t,\text{inner}} = \rho \ell^2 |\Omega| & \text{if } y \leq y_{\text{cross}} \\ \mu_{t,\text{outer}} = \rho K C_{\text{cp}} F_{\text{wake}} F_{\text{kleb}}(y) & \text{if } y > y_{\text{cross}} \end{cases}$$

with $y_{\text{cross}} = \min(y)$ where $\mu_{t,\text{inner}} = \mu_{t,\text{outer}}$

$$|\Omega| = \sqrt{2\Omega_{ij}\Omega_{ij}}$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\ell = \kappa y \left[1 - \exp\left(\frac{-y^+}{A^+}\right)\right]$$

$$F_{\text{wake}} = \min\left(y_{\max} F_{\max},\ C_{\text{wk}} y_{\max} \frac{u_{\text{diff}}^2}{F_{\max}}\right)$$

$$u_{\text{diff}} = \max\left(\sqrt{u_i u_i}\right) - \min\left(\sqrt{u_i u_i}\right)$$

$$F_{\text{kleb}} = \left[1 + 5.5\left(\frac{y C_{\text{kleb}}}{y_{\max}}\right)^6\right]^{-1}$$

Evolutions of the base model:

- Van Driest (1956) wall damping
- Clauser (1956) defect layer modification
- Corrsin and Kister (1954) intermittency modification

$y_{\max}$ and $F_{\max}$ determined by the maximum of

$$F(y) = y|\Omega| \left[1 - \exp\left(\frac{-y^+}{A^+}\right)\right]$$

$$A^+ = 26, \quad C_{\text{cp}} = 1.6, \quad C_{\text{kleb}} = 0.3, \quad C_{\text{wk}} = 0.25, \quad \kappa = 0.4, \quad K = 0.0168$$

✔ Good results for simple (attached) flows

✔ Fast and robust

✘ Not general

✘ Only depends on local flow properties
$\Longrightarrow$ no abrupt flow variations
(pressure gradients, separations, complex flows..)

Transport equation models

Turbulence does not adapt instantaneously to abrupt variations! How to take into account flow history?

- **Idea**: write transport equations for given properties. Obviously, start from k_t ($\overline{u_i'' \times \text{Momentum}}$)

$$\frac{\partial \overline{\rho} k_t}{\partial t} + \frac{\partial \overline{\rho} \widetilde{u}_j k_t}{\partial x_j} = \overbrace{\overline{\rho u_i'' u_j''} \frac{\partial \widetilde{u}_i}{\partial x_j}}^{\text{Production}} - \overbrace{\overline{\tau_{ji} \frac{\partial u_i''}{\partial x_j}}}^{\text{Dissipation}} + \frac{\partial}{\partial x_j}\left[\overbrace{\overline{\tau_{ji} u_i''}}^{\text{Molecular work}} - \overbrace{\overline{\rho u_j'' \frac{1}{2} u_i'' u_i''}}^{\text{Turbulent transport}} \right] - \overbrace{\frac{\partial \overline{p' u_i''}}{\partial x_i}}^{\text{Pressure diffusion}} - \overbrace{\overline{u_i''} \frac{\partial \overline{p}}{\partial x_i}}^{\text{Pressure work}} + \overbrace{\overline{p' \frac{\partial u_i''}{\partial x_i}}}^{\text{Dilatation work}}$$

$$= \left(2\mu_t \widetilde{S}_{ij} - \frac{2}{3}\overline{\rho} k_t \delta_{ij}\right) \frac{\partial \widetilde{u}_i}{\partial x_j} - \overline{\rho}\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k_t}{\partial x_j}\right] = \mathcal{P} - \overline{\rho}\varepsilon + \mathcal{D} + \cancel{\Phi}$$

- Production: Boussinesq + Newtonian fluid

$$\tau_{ij}^R = 2\mu_t \widetilde{S}_{ij} - \frac{2}{3}\overline{\rho} k_t \delta_{ij}$$

- Dissipation: $\varepsilon = f(k_t, \ell)$. Either:
  - Algebraic relation: $\varepsilon = C_D \dfrac{k_t^{3/2}}{\ell}$ (1-eq. model)
  - Supply 2$^{\text{nd}}$ PDE (2-eq. model)

  Therefore, $\mu_t = \rho C v_k \ell_k = \rho C k^{1/2} \dfrac{k^{3/2}}{\varepsilon} = \rho C \dfrac{k_t^2}{\varepsilon}$

- Turbulent transport and work: same as previously shown in FANS closure (for consistency)
- Pressure terms: no standards, often neglected

✔ Include history effect w.r.t. 0-equation models

✘ Need to set a turbulent length..

✘ k_t 1-equation model abandoned for SA approach

Remark: **gradient-diffusion** relation commonly used between turbulent flux of a quantity and its corresponding mean gradient:

$$-\rho\overline{v'\phi'} = \Gamma_t \frac{\partial \phi}{\partial y} \quad \text{with} \quad \Gamma_t = \frac{\mu_t}{\sigma_t}$$

- Γ turbulent diffusivity, σ_t turbulent Prandtl number
- $\sigma_t \approx 1$ (the same turbulent eddies are responsible for transporting momentum and other scalars)

Spalart-Allmaras Model

- Transport equation directly based on a "modified" turbulent viscosity, ν_{sa}
- More info at `https://turbmodels.larc.nasa.gov/spalart.html`

$$\frac{\partial \nu_{sa}}{\partial t} + \widetilde{u}_j \frac{\partial \nu_{sa}}{\partial x_j} = c_{b1}(1 - f_{t2}) S_{sa} \nu_{sa} - \left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2} \right] \left(\frac{\nu_{sa}}{d} \right)^2 + \frac{1}{\sigma} \left[\frac{\partial}{\partial x_j} \left((\widetilde{\nu} + \nu_{sa}) \frac{\partial \nu_{sa}}{\partial x_j} \right) + c_{b2} \frac{\partial \nu_{sa}}{\partial x_k} \frac{\partial \nu_{sa}}{\partial x_k} \right]$$

$$\nu_t = \nu_{sa} f_{v1} \qquad S_{sa} = \Omega + \frac{\nu_{sa}}{\kappa^2 d^2} f_{v2} \qquad \Omega = \sqrt{2 \widetilde{\Omega}_{ij} \widetilde{\Omega}_{ij}} \qquad \widetilde{\Omega}_{ij} = \frac{1}{2} \left(\frac{\partial \widetilde{u}_i}{\partial x_j} - \frac{\partial \widetilde{u}_j}{\partial x_i} \right)$$

$$\chi = \frac{\nu_{sa}}{\widetilde{\nu}} \qquad f_{t2} = c_{t3} \exp\left(-c_{t4} \chi^2 \right) \qquad f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \qquad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

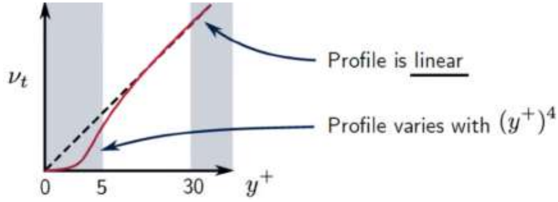
$$f_w = g \left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right)^{1/6} \qquad g = r + c_{w2}(r^6 - r) \qquad r = \min\left(\frac{\nu_{sa}}{S_{sa} \kappa^2 d^2}, 10 \right)$$

$$c_{b1} = 0.1355 \quad c_{b2} = 0.622 \quad \sigma = 2/3 \quad \kappa = 0.41 \quad c_{v1} = 7.1 \quad c_{t3} = 1.2 \quad c_{t4} = 0.5$$

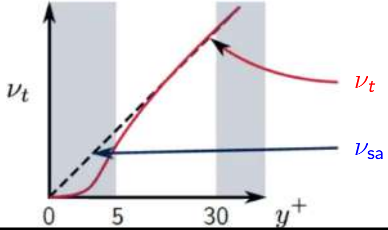
$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{1 + c_{b2}}{\sigma} \quad c_{w2} = 0.3 \quad c_{w3} = 2$$

- $c_{w1} f_w$ models destruction due to inviscid blocking (pressure fluctuation damping)
- Viscous destruction is accounted for in the definition of ν_t (by means of f_{v1})
- Additional nonlinear term $c_{b2} \frac{\partial \nu_{sa}}{\partial x_k} \frac{\partial \nu_{sa}}{\partial x_k}$ important at the edge of the turbulent region, where diffusion dominates.
- **Why not solving directly for ν_t?** $\implies$ because of ν_t behaviour near the wall!

Spalart-Allmaras Model - cont'd



- The ν_t profile is
  - **linear** in the log region: $\dfrac{\nu_t}{\nu} = \kappa y^+$
  - **quartic** in the viscous layer, $\dfrac{\nu_t}{\nu} \sim (y^+)^4$
    - Lot of cells needed to represent such quantity!
    - ν_{sa} reproduces viscous damping thanks to f_{v1}
    - Far from the wall, $f_{v1} \to 1$ and $\nu_{sa} = \nu_t$



What about BCs?

- At the wall, $\nu_{sa} = 0$
- In the freestream, $\nu_{sa} = \nu_t$. Thus, at the inlet:

$$\nu_{sa} = \nu_t = \frac{C_\mu k_t^2}{\varepsilon} \quad \text{or} \quad \nu_{sa} = \nu_t = \frac{k_t}{\omega}$$

with k_t and ω calculated from:

- A length scale ℓ (e.g., 10% of the wing chord)
- The turbulence intensity I (e.g., 5%)

$$k_t = \frac{3}{2} u_\infty^2 I^2 \qquad \varepsilon = C_\mu \frac{k_t^{3/2}}{\ell}$$

✔ Cheap and easy to implement

✔ Attached wall bounded flows, mild separation and recirculation

✘ Massively separated flows, free shear flows, decaying turbulence..

k_t-ε model

- Transport of k_t and of its dissipation rate ε. Jones and Launder (1972), Launder and Sharma (1974), ...

$$\begin{cases} \dfrac{\partial \overline{\rho} k_t}{\partial t} + \dfrac{\partial \overline{\rho} \widetilde{u}_j k_t}{\partial x_j} = \tau_{ij}^R \dfrac{\partial \widetilde{u}_i}{\partial x_j} - \overline{\rho}\varepsilon + \dfrac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \dfrac{\mu_t}{\sigma_k}\right)\dfrac{\partial k_t}{\partial x_j}\right] \\ \dfrac{\partial \overline{\rho}\varepsilon}{\partial t} + \dfrac{\partial \overline{\rho} \widetilde{u}_j \varepsilon}{\partial x_j} = \overline{\rho} C_{\varepsilon 1} \dfrac{\varepsilon}{k_t} \tau_{ij}^R \dfrac{\partial \widetilde{u}_i}{\partial x_j} - C_{\varepsilon 2} \overline{\rho} \dfrac{\varepsilon^2}{k_t} + \dfrac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \dfrac{\mu_t}{\sigma_\varepsilon}\right)\dfrac{\partial \varepsilon}{\partial x_j}\right] \end{cases}$$

$$\mu_t = \rho C_\mu \frac{k_t^2}{\varepsilon} \qquad C_\mu = 0.09$$
$$C_{\varepsilon 1} = 1.44 \qquad \sigma_k = 1.0$$
$$C_{\varepsilon 2} = 1.92 \qquad \sigma_\varepsilon = 1.3$$

✔ Simple to implement, robust, predictable behaviour
✔ Independence from free-stream values
✔ Correctly reproduces shear stress in free shear flows
✔ Applicable to wider range of flows than SA

✘ Linearity of the model (no complex 3D flows)
✘ Not sensitive to streamline curvatures and rotations
✘ Difficult to impose BCs: low-Re corrections needed
  ✘ Equations not valid in viscous layer
  ✘ ε^2/k_t singular at the wall

Improvements:

RNG k_t-ε

- Modified ε-eq. and Coeffs derived analytically based on Renormalization Group Theory
  ✔ Rapidly strained flows, moderate swirl
  ✔ Flows with streamline curvature

Realizable k_t-ε

- Modified ε-eq. taking into account physical constraints
  1. $\det(\overline{u_i' u_j'}) \geq 0$
  2. $\overline{u_i' u_i'} \geq 0$ (positive normal stresses)
  3. $|\overline{u_i' u_j'}|^2 \leq \overline{u_i^2}\,\overline{u_j^2}$ (Cauchy-Schwartz inequality)

✔ Recirculation, rotation, separation
✔ Often more accurate and easier to converge than RNG

k_t-ω Wilcox model

- 2^{nd} transport equation for the specific dissipation rate (a.k.a. turbulence frequency): $\omega = \dfrac{\varepsilon}{C_\mu k_t}$
- More info at `https://turbmodels.larc.nasa.gov/wilcox.html`

$$\begin{cases} \dfrac{\partial \overline{\rho} k_t}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j k_t}{\partial x_j} = \tau_{ij}^R \dfrac{\partial \widetilde{u}_i}{\partial x_j} - \beta^* \overline{\rho} k_t \omega + \dfrac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \sigma_k \dfrac{\overline{\rho} k_t}{\omega}\right)\dfrac{\partial k_t}{\partial x_j}\right] \\ \dfrac{\partial \overline{\rho}\omega}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j \omega}{\partial x_j} = \alpha \dfrac{\omega}{k_t}\tau_{ij}^R \dfrac{\partial \widetilde{u}_i}{\partial x_j} - \beta \overline{\rho}\omega^2 + \dfrac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \sigma_\omega \dfrac{\overline{\rho} k_t}{\omega}\right)\dfrac{\partial \omega}{\partial x_j}\right] + \sigma_d \dfrac{\overline{\rho}}{\omega}\dfrac{\partial k_t}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} \end{cases}$$

$$\beta = \beta_0 f_\beta, \qquad f_\beta = \frac{1+85\chi_\omega}{1+100\chi_\omega}, \qquad \chi_\omega = \left|\frac{\Omega_{ij}\Omega_{jk}\hat{S}_{ki}}{(\beta^*\omega)^3}\right|, \qquad \hat{S}_{ki} = \widetilde{S}_{ki} - \frac{1}{2}\frac{\partial \widetilde{u}_m}{\partial x_m}\delta_{ki}$$

$$\sigma_d = \begin{cases} 0 & \dfrac{\partial k_t}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} \le 0 \\ \sigma_{do} & \dfrac{\partial k_t}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} > 0 \end{cases}, \qquad \mu_t = \frac{\overline{\rho} k_t}{\hat{\omega}}, \qquad \hat{\omega} = \max\left(\omega, C_{\lim}\sqrt{\frac{2}{\beta^*}\widetilde{S}_{ij}\widetilde{S}_{ij}}\right)$$

$$\alpha = 0.52, \quad \beta^* = 0.09, \quad \sigma_\omega = 0.5, \quad \sigma_k = 0.6, \quad \sigma_{do} = 0.125, \quad C_{\lim} = 0.875$$

✔ Better then k_t-ε for external aerodynamics and turbomachinery

✔ Correct behavior in the viscous layer

✔ APG BL, free shear, low-Re flows, separation

✔ Production indep. on k_t since $\alpha \dfrac{\omega}{k_t} P = 2\alpha S_{ij}S_{ij}$ (numerical robustness)

✘ Transition/separation predicted excessive/early

✘ $\omega(y \to 0) \sim \dfrac{6\nu}{\beta y^2} \to \infty$ at the wall

✘ Sensitive to inlet and far-field values of ω! k_t-ε was better $\Longrightarrow$ **hybridization**

Replacing $\varepsilon = C_\mu k_t \omega$ in the ε equation leads to appearance of the **cross-diffusion term**

k_t-ω SST, Menter's Shear-Stress Transport model

- Blending of k_t-ε and k_t-ω. More info at `https://turbmodels.larc.nasa.gov/sst.html`

$$\begin{cases} \dfrac{\partial \overline{\rho} k_t}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j k_t}{\partial x_j} = \tau_{ij}^R \dfrac{\partial \widetilde{u}_i}{\partial x_j} - \beta^* \overline{\rho} k \omega + \dfrac{\partial}{\partial x_j}\left[(\overline{\mu} + \sigma_k \mu_t) \dfrac{\partial k}{\partial x_j} \right] \\ \dfrac{\partial \overline{\rho}\omega}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j \omega}{\partial x_j} = \dfrac{\alpha}{\nu_t} \tau_{ij}^R \dfrac{\partial \widetilde{u}_i}{\partial x_j} - \beta \overline{\rho} \omega^2 + \dfrac{\partial}{\partial x_j}\left[(\mu + \sigma_\omega \mu_t) \dfrac{\partial \omega}{\partial x_j} \right] + 2(1 - F_1) \dfrac{\overline{\rho}\sigma_{\omega 2}}{\omega} \dfrac{\partial k_t}{\partial x_j} \dfrac{\partial \omega}{\partial x_j} \end{cases}$$

$$\sigma_k = F_1 \sigma_{k1} + (1 - F_1)\sigma_{k2}, \qquad \sigma_\omega = F_1 \sigma_{\omega 1} + (1 - F_1)\sigma_{\omega 2}, \qquad \beta = F_1 \beta_1 + (1 - F_1)\beta_2$$

$\sigma_{k1}, \sigma_{\omega 1}, \beta_1$ from k–ω

$$\alpha = F_1 \alpha_1 + (1 - F_1)\alpha_2, \qquad \alpha_1 = \frac{\beta_1}{\beta^*} - \sigma_{\omega 1} \frac{\kappa^2}{\sqrt{\beta^*}}, \qquad \alpha_2 = \frac{\beta_2}{\beta^*} - \sigma_{\omega 2} \frac{\kappa^2}{\sqrt{\beta^*}}$$

$\sigma_{k2}, \sigma_{\omega 2}, \beta_2$ from k–ε

$$F_1 = \tanh(\mathrm{arg}_1^4), \qquad \mathrm{arg}_1 = \min\left[\max\left(\frac{\sqrt{k_t}}{0.09 \omega d}, \frac{500 \widetilde{\nu}}{d^2 \omega} \right), \frac{4 \overline{\rho} \sigma_{\omega 2} k_t}{\mathrm{CD}_{k_t \omega} d^2} \right]$$

$$\beta^* = 0.09 \qquad \kappa = 0.41$$

$$\mu_t = \frac{a_1 \overline{\rho} k_t}{\max(a_1 \omega, \Omega F_2)}, \qquad F_2 = \tanh(\mathrm{arg}_2^2), \qquad \mathrm{arg}_2 = \max\left(\frac{2\sqrt{k_t}}{0.09 \omega d}, \frac{500 \widetilde{\nu}}{d^2 \omega} \right)$$

- Terms in green are different from standard $k_t - \omega$ model; blending of $\phi = \phi_1 F_1 + \phi_2 (1 - F_1)$
- Model driven by F_1 $\begin{cases} \to 1 \text{(near-wall region): term disappears, } k_t\text{-}\omega \text{ recovered} \\ \to 0 \text{(far from walls): term active, } k_t\text{-}\varepsilon \text{ recovered with variable change } \omega = \dfrac{\varepsilon}{k_t} \end{cases}$
- Bound on μ_t avoid overestimation of shear stress in APG BLs

✔ Benefits of combined k_t-ε and k_t-ω

✘ Dependency on wall distance: less suitable for free shear flows

EVM and NEVM

Eddy-viscosity models (EVM)

- ν_t based on turbulence scalars determined by solving transport equations
- (Deviatoric) turbulent stress $\propto$ mean strain

✔ Simply to code, robust thanks to extra viscosity

✔ Theoretically supported in some simple but common flow configurations

✔ Effective in many engineering flows

✘ Dependence on single scalar! Fails for strongly anistropic flows and when more than one stress component has an effect on the mean flow

Example: in the log region of a BL, the stress are

$$\overline{u'^2} : \overline{v'^2} : \overline{w'^2} = 1 : 0.4 : 0.6$$

EVM would predict all of these to be equal to $\frac{2}{3}k_t$

- Not particularly important for simple shear flows (but it is for complex flows).
  Possible solutions: **NEVM** and **RSM**

Non-linear eddy-viscosity models (NEVM)

- Halfway between EVM and RSM
- τ_{ij}^R non-linear function of mean strain and vorticity

Defining the anisotropy tensor $a_{ij} = \frac{\tau_{ij}^R}{k_t} = \frac{\overline{u_i' u_j'}}{k_t} - \frac{2}{3}\delta_{ij}$

The dimensionless $\overline{S_{ij}}$ and $\overline{\Omega_{ij}}$ are (e.g., for k_t-ε):

$$\overline{s_{ij}} = \frac{k_t}{\varepsilon}\overline{S_{ij}} \qquad \overline{\omega_{ij}} = \frac{k_t}{\varepsilon}\overline{\Omega_{ij}}$$

Then one has:

$$a_{ij} = \begin{cases} -2C_\mu \overline{s_{ij}} & \text{for EVM} \\ -2C_\mu \overline{s_{ij}} + \boxed{NL(\overline{s_{ij}}, \overline{\omega_{ij}})} & \text{for NEVM} \end{cases}$$

✔ Huge improvement for certain important flows

✔ Only slightly more expensive than EVM

✘ Don't accurately represent the real production and advection processes

✘ Little theoretical foundation in complex flows

Reynolds Stress Models (RSM)

Derive an exact equation for the Reynolds stresses:

$$\overline{u_j' \times \left[\mathcal{N}(u_i) - \overline{\mathcal{N}(u_i)}\right] + u_i' \times \left[\mathcal{N}(u_j) - \overline{\mathcal{N}(u_j)}\right]}$$

$$\overbrace{\frac{\partial \rho \overline{u_i' u_j'}}{\partial t}}^{\substack{\text{Local time}\\ \text{Derivative}}} + \overbrace{\frac{\partial \rho \overline{u}_k \overline{u_i' u_j'}}{\partial x_k}}^{\substack{\text{Convection}\\ C_{ij}}} = \overbrace{-\rho\left(\overline{u_i' u_k'}\frac{\partial \overline{u}_j}{\partial x_k} + \overline{u_j' u_k'}\frac{\partial \overline{u}_i}{\partial x_k}\right)}^{\substack{\text{Production}\\ P_{ij}}} \overbrace{-2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}^{\substack{\text{Dissipation}\\ \varepsilon_{ij}}} + \overbrace{\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}^{\substack{\text{Pressure-strain}\\ \phi_{ij}}} + \overbrace{\frac{\partial}{\partial x_k}\left(\mu\frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right)}^{\substack{\text{Molecular}\\ \text{diffusion}, D_{L,ij}}}$$

$$+ \underbrace{\frac{\partial}{\partial x_k}\left[\rho\overline{u_i' u_j' u_k'} - \overline{p'\left(\delta_{kj} u_i' + \delta_{ik} u_j'\right)}\right]}_{\substack{\text{Turbulent Diffusion}\\ D_{T,ij}}} = P_{ij} + \varepsilon_{ij} + \phi_{ij} + D_{L,ij} + D_{T,ij}$$

- ✔ 7 additional equations (6 stresses + length scale)
  - C_{ij} and P_{ij} are exact, no model required!
- ✘ 22 new unknowns (red terms require modeling)
- ▶ Speziale-Sarkar-Gatzki, Launder-Reece-Rodi, ...
- ▶ Algebraic RSM, Differential RSM, Explicit RSM..

- ✔ Anisotropy of μ_t and $\tau_{ij}^R \implies$ ok for Rotation, high strain rate, swirl, strong 3D effects, curvature..
- ✘ Computationally more expensive
- ✘ Less robust (no μ_t) and tougher to converge due to tight coupling with momentum

Near-wall region

Viscous effects

Local: contact needed

- **Strong mean velocity gradients**
  - Turbulence production peak
  - Damping of all components of u_i'
  - Narrowing of the turbulence spectrum (vanishing of the inertial zone)

Non-viscous effects

Non-local: the wall is felt at distance

- **Wall Echo**: Reflection of p fluctuations on the wall
  - Increased fluctuations ($\Longrightarrow$ larger redistribution)
- **Blocking effect**: Wall-normal fluctuations generate high-p zones that tend to
  - Slow down the flow in the wall-normal direction
  - Deviate it in wall-parallel directions

What about the "universal" behaviour?

- Obtained based on very **strict hypotheses** (high-Re, ZPG flat plates, ..)
- In real flows, it is often perturbed or completely modified (e.g., at separation)

Consequences for modeling?

- Models based on hypotheses not always applicable in the near-wall region. What can be done?
  1. Force the model to behave correctly introducing corrections depending on wall distance or Re_t
     $\Longrightarrow$ **Low-Re models**
  2. Use models that work properly near the walls (e.g., k_t-ω, kind of a low-Re model)
  3. Avoid the resolution of the near-wall region
     $\Longrightarrow$ **High-Re models + wall functions**
  4. Prescribing a turbulent scale near the walls
     $\Longrightarrow$ **Two-layers models**
  5. Reconsidering the hypotheses used in the derivation of the models (e.g., elliptic relaxation)

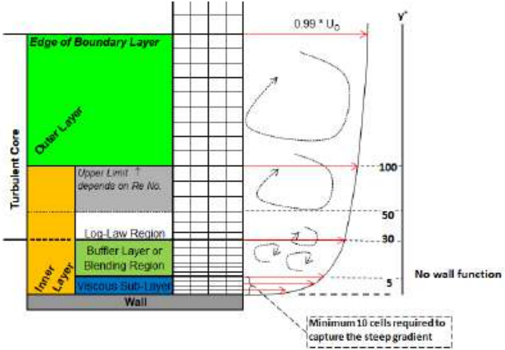
High-*Re* vs low-*Re* models

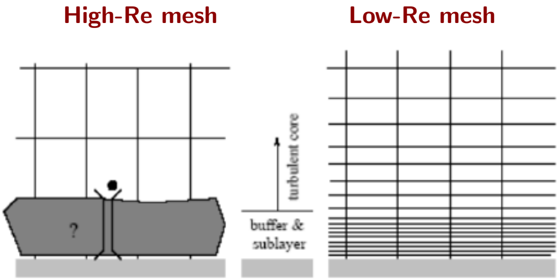
- Notion of high/low-*Re* refers to the **turbulent Re**:

$$Re_t = \frac{\text{Turbulent forces}}{\text{Viscous forces}} = \frac{\nu_t}{\nu}$$

  - Local: variable inside the flow, $\rightarrow 0$ at the wall
  - There is always a low-*Re* region close to the wall
    There can be others elsewhere!

- **High-Re** model: not integrable down to the wall (correction with wall functions needed)
- **Low-Re** model: can be applied down to the wall (with or without damping functions)
- **All-Re** model: model consistent for all y^+ (e.g., Adaptive wall functions)





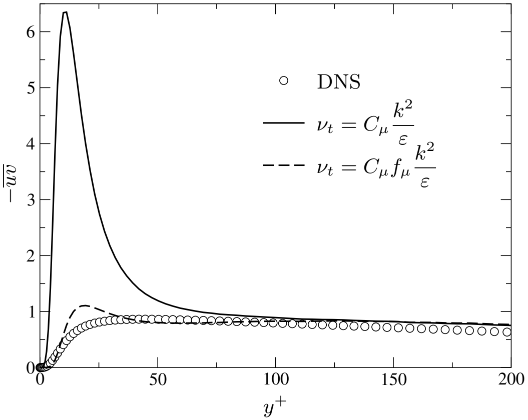
Low-Re models: damping functions (DF)

Example: Low-Re k_t-ε model

- Coefficients C_1, C_2, C_μ damped with f_1, f_2, f_μ
  - f_μ can be $f(Re_t)$, $f(d_w)$, or both

$$\begin{cases} f_1 = 1 \\ f_2 = 1 - 0.3\exp(-Re_t^2) \\ f_\mu = \exp\left[\dfrac{-3.4}{(1 + Re_t/50)^2}\right] \end{cases}$$

$$Re_t = \frac{\rho k_t^{1/2}\ell_t}{\mu} = \frac{\rho k_t^{1/2}}{\mu}\left(\frac{k_t^{3/2}}{\varepsilon}\right) = \frac{\rho k_t^2}{\mu\varepsilon}$$



✔ Some models are intrinsically low-Re (e.g., k_t-ω)

✔ DF applicable to most models (also RSM)

✘ Viscous sublayer must be well resolved ($y^+ < 1$)

✘ DF are very empirical $\Longrightarrow$ lack of universality

High-*Re* models: wall functions (WF)

- Viscous sublayer modeled (first point at $30 < y^+ < 100$ and $y < 0.1\delta$). Examples for k_t-ε:

One-scale approach

- Compute u_τ with iterative method from:

$$\frac{u}{u_\tau} = \frac{1}{\kappa} \ln \frac{y u_\tau}{\nu} + C \tag{1}$$

(e.g. with Newton–Raphson)

- Use u_τ to evaluate τ_w, k_t and ε:

$$\tau_w = \mu \frac{\partial u}{\partial y} = \rho u_\tau^2, \quad \varepsilon = \frac{u_\tau^3}{\kappa y}, \quad k_t = C_\mu^{-1/2} u_\tau^2$$

✔ Good results in situations close to ideal conditions

✔ Substantial reduction of number of grid points

✔ Simpler mesh generation

✔ Robustness: can be used all the time (this does not guarantee the quality of the results)

Two-scale approach

- u_τ evaluated from (1)
- Instead of imposing k_t at the wall, $\partial k / \partial y = 0$
- A second velocity scale, u_k, is evaluated from

$$u_k^2 = C_\mu^{1/2} k_t$$

where k_t is taken at the point closest to the wall

- Friction imposed as $\tau_w = \mu \frac{\partial u}{\partial y} = \rho u_\tau u_k$

✘ Location of the discretization point closest to the wall must be well controlled

✘ Hypothesis of equilibrium conditions ($\mathcal{P} = \varepsilon$)

✘ Relations does not hold at stagnation, separation, reattachment, recirculation, APG BL, curved wall.. for example, for separation/reattachment:

$$u_\tau = 0 \implies k_t = \varepsilon = 0$$

Wall treatment: High-Re

1. **Standard wall function**: use log-law to compute BCs
2. **Non-equilibrium wall function**: improved models for flows with separations, reattachments, high ∇p..
3. **Zonal model**: regions distinguished by a wall-distance-based Reynolds: $Re_y = \dfrac{\rho\sqrt{k_t}y}{\mu}$
   - $Re_y > 200$: turbulent core region, regular models
   - $Re_y \leq 200$: viscosity-affected region: only k_t eq. solved, ε from a Re_y-dependent correlation
   - Damping functions also used for the turbulent viscosity
4. **Two-layers models**: check first cell position

$$y^* = \frac{\rho C_\mu^{1/4} k_t^{1/2} y}{\mu} \implies \begin{cases} u^* = \dfrac{1}{\kappa}\ln(Ey^*) & \text{if } y^* \geq 11.225 \\ u^* = y^* & \text{if } y^* \leq 11.225 \end{cases} \implies \tau_w = \frac{\rho u C_\mu^{1/4} k_t^{1/2}}{u^*}$$

   E: empirical const. (9.8 for smooth walls)

5. **Enhanced Wall Treatments**: blending of two-layers models:

$$u^+ = \exp(\Gamma) u^+_{\text{lam}} + \exp\left(\frac{1}{\Gamma}\right) u^+_{\text{turb}} \qquad \text{with} \qquad \Gamma = -\frac{0.01(y^+)^4}{1+5y^+}$$

6. **Scalable wall functions**..

Conclusions

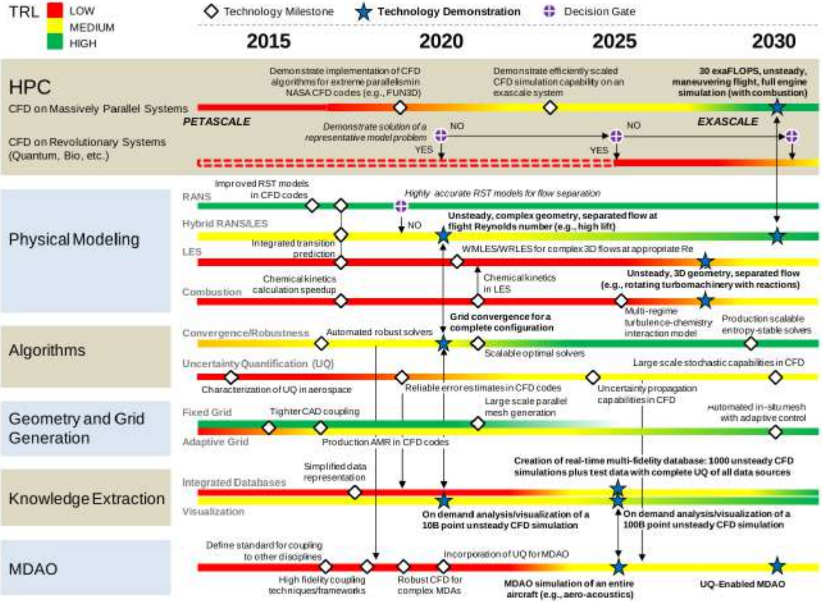
No universally accepted turbulence model that works for all applications

- Accuracy and effectiveness of each model varies depending on application
- Choice of the model will depend on
  1. Flow Physics
  2. Established practice for the class of problem
  3. Level of accuracy required
  4. Availability of models & functions in the CFD software
  5. Available computation resources
  6. Amount of time to carry out the simulation
- Research modeling choices for your class of problems
  - ✔ Choose the model tweaked specifically for this type
  - ✘ Don't use if it has been modified for another problem

Try to make the mesh either coarse or fine enough to avoid placing the wall-adjacent cells in the buffer layer ($5 < y^+ < 30$)!

Roadmap to 2030

"CFD Vision 2030 Study: A path to Revolutionary Computational Aerosciences", NASA/CR–2014-218178



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