Numerical solutions of differential equations

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Lecture 6

Hyperbolic Equations of first order - Part 3

Linearization

Linearization - Idea

Let

- $f : \mathbb{R} \to \mathbb{R}$ be a smooth flux
- and $v : \mathbb{R} \to \mathbb{R}$ and initial value of the form

$$v(x) = v_c + \varepsilon \tilde{v}(x).$$

where v_c is constant and $0 < \varepsilon \ll 1$ is a perturbation parameter.

We seek $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ with

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

Idea: If initial value is almost constant, we expect the solution $u(\cdot, t)$ to remain almost constant for every t.

Can we hence approximate the nonlinear equation by a linear equation?

Linearization - Idea

Idea: If initial value is almost constant, we expect the solution $u(\cdot, t)$ to remain almost constant for every t.

Can we hence approximate the nonlinear equation by a linear equation?

Why?

- Linear problem is mathematically simpler.
- We know how to handle linear hyperbolic equations.
- Useful applications, e.g.:
  Study how small perturbations around a constant state evolve in time (cf. HW2).
- Linearized problem tells us something about the stability of the non-linear problem.

Linearization - How?

We seek $u = u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ with

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{and} \qquad u(x,0) = v_c + \varepsilon \tilde{v}(x) \qquad (*)$$

How do we linearize?

Suppose that the solution to $(*)$ can be written as

$$u(x,t) = v_c + \varepsilon \tilde{u}(x,t).$$

Hence with $(*)$:

$$\begin{aligned} 0 &= \partial_t \left(v_c + \varepsilon \tilde{u}(x,t)\right) + \partial_x f(v_c + \varepsilon \tilde{u}(x,t)) = \varepsilon \partial_t \tilde{u}(x,t) + f'(v_c + \varepsilon \tilde{u}(x,t)) \varepsilon \partial_x \tilde{u}(x,t) \\ &= \varepsilon \partial_t \tilde{u}(x,t) + \left(f'(v_c) + \varepsilon \tilde{u}(x,t) f''(v_c) + \mathcal{O}(\varepsilon^2)\right) \varepsilon \partial_x \tilde{u}(x,t) \\ &= \varepsilon \left(\partial_t \tilde{u}(x,t) + f'(v_c) \partial_x \tilde{u}(x,t)\right) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Dropping the $\mathcal{O}(\varepsilon^2)$ term, we obtain the linearized equation

$$0 = \partial_t \tilde{u}(x,t) + f'(v_c) \partial_x \tilde{u}(x,t) \qquad \text{with} \qquad \tilde{u}(x,0) = \tilde{v}(x).$$

We obtain $u(x,t) \approx v_c + \varepsilon\, \tilde{u}(x,t)$.

Linearization - Summary

We seek $u = u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ with

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{and} \qquad u(x,0) = v_c + \varepsilon \tilde{v}(x)$$

Linearized problem:

Find $\tilde{u} = \tilde{u}(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ with

$$\partial_t \tilde{u}(x,t) + f'(v_c)\partial_x \tilde{u}(x,t) = 0 \qquad \text{with} \qquad \tilde{u}(x,0) = \tilde{v}(x).$$

The linearized approximation is given by

$$u(x,t) \approx v_c + \varepsilon\, \tilde{u}(x,t).$$

Linearization - Generalization to systems

System: Let

- $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ be a smooth flux
- and $v : \mathbb{R} \to \mathbb{R}^m$ and initial value of the form

$$\mathbf{v}(x) = \mathbf{v}_c + \varepsilon \tilde{\mathbf{v}}(x).$$

where $\mathbf{v}_c$ is constant and $0 < \varepsilon \ll 1$ is a perturbation parameter.

We seek $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^m$ with

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

Linearization - Generalization to systems

We seek $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^m$ with

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}_c + \varepsilon \tilde{\mathbf{v}}(x).$$

With

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \mathbf{f}_1(\mathbf{u}_1, \cdots, \mathbf{u}_m) \\ \vdots \\ \mathbf{f}_m(\mathbf{u}_1, \cdots, \mathbf{u}_m) \end{pmatrix}, \quad \mathbf{f}'(\mathbf{u}) = \begin{pmatrix} \frac{\partial \mathbf{f}_1(\mathbf{u})}{\partial \mathbf{u}_1} & \cdots & \frac{\partial \mathbf{f}_1(\mathbf{u})}{\partial \mathbf{u}_m} \\ \vdots & & \vdots \\ \frac{\partial \mathbf{f}_m(\mathbf{u})}{\partial \mathbf{u}_1} & \cdots & \frac{\partial \mathbf{f}_m(\mathbf{u})}{\partial \mathbf{u}_m} \end{pmatrix}$$

we can proceed as before: We seek $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^m$

$$\partial_t \tilde{\mathbf{u}}(x,t) + \mathbf{f}'(\mathbf{v}_c)\partial_x \tilde{\mathbf{u}}(x,t) = 0 \qquad \text{with} \qquad \tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{v}}(x).$$

The linearized approximation is given by

$$\mathbf{u}(x,t) \approx \mathbf{v}_c + \varepsilon\, \tilde{\mathbf{u}}(x,t).$$

Linearization

Note:

Sometimes there are features in the solution to a nonlinear problem that cannot be reproduced by linearized models.