(Brittle) Fracture Mechanics

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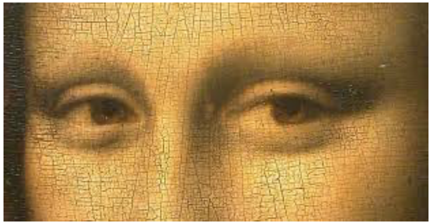
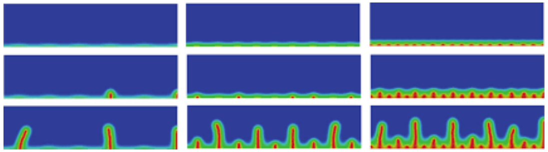
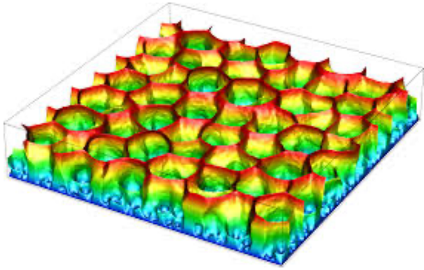






Acknowledgement: a large part of these slides is taken from the course of J.J. Marigo and K.Danas (Ecole Polytechnique), see the references

Tentative program (to update)

Lecture 1	Intro to Fracture, Stress concentrations, singularities (anti-plane)	21/9
Lecture 2	Stress singularities in plane elasticity, fracture modes, fracture toughness, Irwin criterion	28/9
Lecture 3	Energetic (variational) approach to fracture — Griffith's Theory: static problem	5/10
Lecture 4	Energetic (variational) approach to fracture — Griffith's Theory, Quasi-static evolution	12/10
Lecture 5	Numerical computation of the stress intensity factors I	19/10
Lecture 6	Numerical computation of the stress intensity factors II	26/10
Lecture 7	Examples	09/11
Lecture 8	Examples/Seminar	23/11
Final Exam (written)		30/11

I will probably give one Homework project at the end of october to do in groups of two students and final note will be calculate as

max (100% final exam 80% final exam + 20 % homework)

Energetic (variational) approach to fracture: the static problem

Content of Lecture 3

- The ingredient of the energetic (variational) approach to fracture
  - The crack set: the general case and the case of a single crack on a given crack path
  - The potential energy and the energy release rate
  - The surface energy, irreversibility condition

- Fracture as an energy minimisation problem
  - Formulation: the general case and the case of a single crack on a given crack path
  - First-order optimisation conditions for a single crack on a given crack path: the Griffith criterion

- Example: Peeling test

At the end of Lecture 3 you should be able to

- Write potential and surface energy for a brittle solid within the Griffith's theory

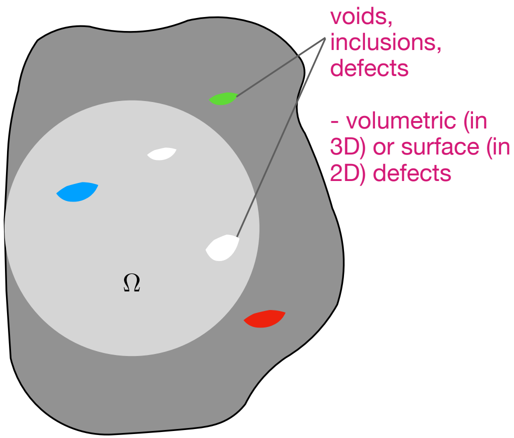
- Formulate the static variational problem for a single crack on a given crack path

- Derive the "Griffith" crack propagation criterion for a crack on a given path as a first-order optimality conditions

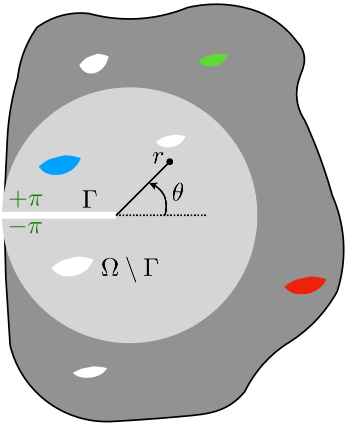
- Apply the variational approach to simple cases as DBC

What is a crack: a mathematical idealization

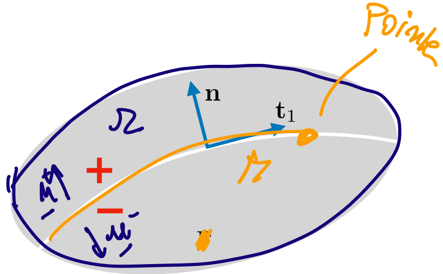
A crack is a **n-1 dimensional defect** in a n-dimensional domain where displacement can jump.
It is a surface in 3d, line in 2d, a point in 1d. The creation of crack dissipates some energy.



uncracked body

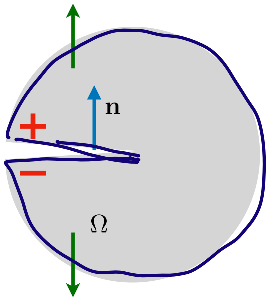


cracked body



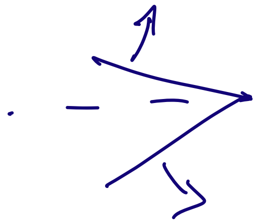
- Crack as internal boundary
  - Choice of normal **n**
  - Choice of “+” and “-“ side
  - Displacement jump

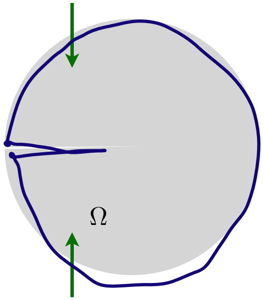
$$[\![\mathbf{u}]\!] = \mathbf{u}^+ - \mathbf{u}^-$$



open crack

- Crack faces are not in contact (open crack)
  - Displacement jump: $[\![\mathbf{u}]\!] \cdot \mathbf{n} \geq 0$
  - Traction free surface: $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$





closed crack

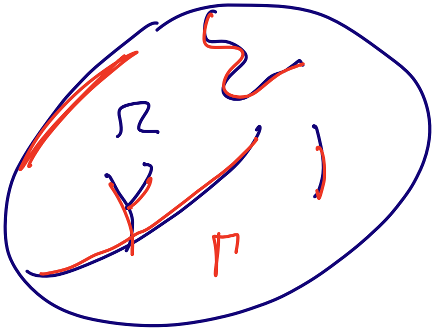
- Crack faces are in frictionless contact (closed crack)
  - Displacement jump: $[\![\mathbf{u}]\!] \cdot \mathbf{n} = 0 \qquad ([\![\mathbf{u}]\!] \cdot \mathbf{t} \neq 0)$
  - Traction at surface: $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} = \sigma_{nn}\, \mathbf{n}, \qquad \sigma_{nn} \leq 0 \qquad (\mathbf{T} \cdot \mathbf{t} = 0)$ frictionless

In the following we will neglect the non-interpenetration condition! $\underline{n} \cdot [\![u]\!] \gtreqless 0$

The ingredients of the variational approach

Griffith 1921, Francfort-Marigo 1998

Ω: The uncracked n-dimensional domain , Γ: The crack set of dimension n-1



IRREVERSIBILITY

Γ_0 : PRE-EXISTING CRACK SET

Γ : UNKNOWN CURRENT CRACK SET

$\Gamma \supseteq \Gamma_0 \rightsquigarrow$ ADMISSIBLE CRACK SETS

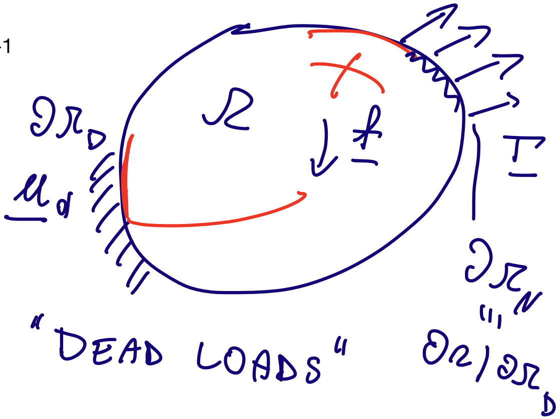
NO SELF-HEALING

Ω: The uncracked n-dimensional domain , Γ: The crack set of dimension n-1

$$P(\underline{u},\Gamma) = \overbrace{\int_{\Omega/\Gamma} \frac{1}{2}\, \underline{\underline{\varepsilon}}(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u})\, dx}^{\text{ELASTIC ENERGY}} \;-\; \underbrace{\overbrace{\left(\int_{\Omega/\Gamma} \underline{f} \cdot \underline{u}\, dx + \int_{\partial\Omega_N/\Gamma} \underline{T} \cdot \underline{u}\, ds \right)}^{\text{EXTERNAL WORK}}}_{W_{ext}(\underline{u})}$$

$W(\underline{u})$ (QUADRATIC wrt $\underline{u}$

$W_{ext}(\underline{u}) \rightarrow$ Linear wrt $\underline{u}$



Ω: The uncracked n-dimensional domain , Γ: The crack set of dimension n-1

We adopt the Griffith model: the energy required to create a crack is proportional to its surface

$$\underbrace{D(\Gamma)}_{\text{FRACTURE ENERGY}} = G_c \, \text{surface}(\Gamma)$$

→ length in 2d

G_c → Toughness (ténacité)

$$[G_c] = \frac{J}{m^2}$$

FRACTURE ENERGY ⇓

It is a dissipated energy because of the irreversibility condition

The ingredients of the variational approach: the total energy

Ω: The uncracked n-dimensional domain , Γ: The crack set of dimension n-1

$$\mathcal{E}(\underline{u}, \Gamma) := \underbrace{\mathcal{P}(\underline{u}, \Gamma)}_{\text{Potential energy}} + \underbrace{\mathcal{D}(\Gamma)}_{\text{FRACTURE ENERGY}}$$

Ω: The uncracked n-dimensional domain , Γ: The crack set of dimension n-1

$$\mathcal{C}(\Gamma) := \left\{ \underline{u} \in \underset{\text{“SMOOTH”}}{\underset{\downarrow}{H^1}}(\Omega \setminus \Gamma, \mathbb{R}^n),\ \underline{u} = \underline{u}_d \ \text{ on } \ \partial\Omega_D \setminus \Gamma \right\}$$

$$\mathcal{C}_0(\Gamma) := \left\{ \underline{u} \in H^1(\Omega \setminus \Gamma, \mathbb{R}^n),\ \underline{u} = \underline{0} \ \text{ on } \ \partial\Omega_D \setminus \Gamma \right)$$

$$\forall \underline{u} \in \mathcal{C}(\Gamma),\ \forall \underline{v} \in \mathcal{C}_0(\Gamma) \qquad \underline{u} + \underline{v} \in \mathcal{C}(\Gamma)$$

It is the variation of the potential energy when varying the crack set Γ, with FIXED LOADING

$$P(\Gamma) := \inf_{u \in \mathcal{C}(\Gamma)} P(u, \Gamma)$$

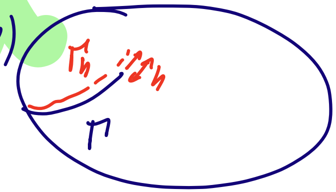
linear elastic problem with "unique" solution

(we assume that there are not "deconnected pieces" because of Γ)

"Taux de restitution d'énergie"

ENERGY RELEASE RATE

$$0 \leq G(\Gamma) := -\lim_{h \to 0} \frac{P(\Gamma_h) - P(\Gamma)}{\text{surface}(\Gamma_h) - \text{surface}(\Gamma)}$$



with $\Gamma_h \supseteq \Gamma$, $\Gamma_h \big|_{h=0} \equiv \Gamma$

Γ_h: One-parameter family of cracks

Si $\Gamma_2 \supseteq \Gamma_1$

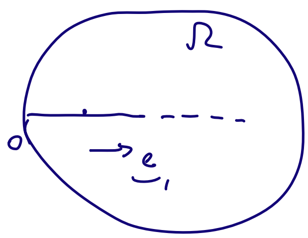
$$P(\Gamma_2) \leq P(\Gamma_1)$$

$$P(\Gamma_2) := \min_{u \in \mathcal{C}(\Gamma_2)} P(u, \Gamma) \leq \min_{u \in \mathcal{C}(\Gamma_1)} P(u,\Gamma) =: P(\Gamma_1)$$

$$P(\Gamma_1) := \min_{u \in \mathcal{C}(\Gamma_1)} P(u, \Gamma)$$

si $\Gamma_2 \supseteq \Gamma_1$ $\qquad \mathcal{C}(\Gamma_2) \supseteq \mathcal{C}(\Gamma_1)$

Simple crack with given crack path



$$\Gamma_\ell = \{ s\, \underline{e}_1,\ s \in (0, \ell) \}$$

$$G(\ell) = -\frac{dP(\ell)}{d\ell} = -P'(\ell)$$

The ingredients of the variational approach: summary

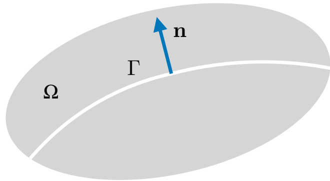
- **The crack set** Ω: The uncracked n-dimensional domain , Γ: The crack set of dimension n-1, Irreversibility condition $\Gamma \supseteq \Gamma_0$

- **The potential energy:**

$$\mathcal{P}(\boldsymbol{u},\Gamma) = \underbrace{\overbrace{\int_{\Omega\setminus\Gamma} \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})\,\mathrm{d}x}^{\text{Elastic energy}}}_{W(\mathbf{u})} - \underbrace{\overbrace{\left(\int_{\Omega\setminus\Gamma} \boldsymbol{b}\cdot\boldsymbol{u}\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma} \boldsymbol{T}\cdot\boldsymbol{u}\,\mathrm{d}s\right)}^{\text{Work of external loads}}}_{W_{\text{ext}}(\mathbf{u})}$$

- **The surface energy:**

$$\mathcal{D}(\Gamma) = \int_\Gamma G_c(\boldsymbol{x},\boldsymbol{n})\,\mathrm{d}s \overset{\text{isotropic, homogenous toughness}}{\doteq} G_c\,\mathrm{surface}(\Gamma)$$



- **The total energy:**

$$\mathcal{E}(\boldsymbol{u},\Gamma) = \mathcal{P}(\boldsymbol{u},\Gamma) + \mathcal{D}(\Gamma)$$

- **The space of admissible displacements:**

We neglect here unilateral contact (non-interpenetration) condition $[\![\boldsymbol{u}]\!]\cdot\boldsymbol{n} \geq 0 \text{ on } \Gamma$

$$\mathcal{C}(\Gamma) \doteq \{\boldsymbol{u} \in H^1(\Omega\setminus\Gamma) : \quad \boldsymbol{u} = \boldsymbol{u}_d \text{ on } \partial\Omega_u \setminus \Gamma\}$$

- **The elastic energy release rate:**

$$\mathsf{P}(\Gamma) := \min_{u\in\mathcal{C}(\Gamma)} \mathcal{P}(\boldsymbol{u},\Gamma), \qquad \mathsf{G}(\Gamma) := -\lim_{h\to 0} \frac{\mathsf{P}(\Gamma_h) - \mathsf{P}(\Gamma)}{\mathrm{surface}(\Gamma_h) - \mathrm{surface}(\Gamma)} \geq 0$$

$\Gamma_h \supseteq \Gamma$

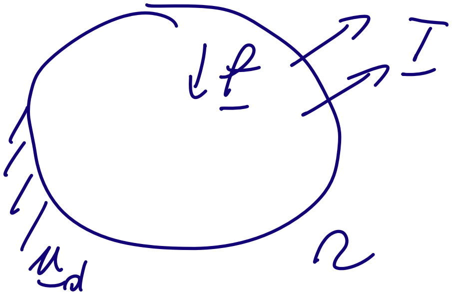
Fracture as an energy minimisation problem: the static case

Griffith 1921, Francfort-Marigo 1998

The static problem: the general case

For a fixed loading (static problem) and a preexisting crack set Γ_0, the "stable" cracked states $(\mathbf{u}^*, \Gamma^*)$ of the solid are defined as the solutions of the following minimisation problem:

$$(u^*, \Gamma^*) := \inf_{\substack{u \in \mathcal{C}(\Gamma) \\ \Gamma \supseteq \Gamma_0}} \underbrace{\mathcal{E}(u, \Gamma)}_{P(u,\Gamma) + D(\Gamma)} \quad , \left(\Gamma_0 \text{ "PRE-EXISTING" CRACK}\right)$$

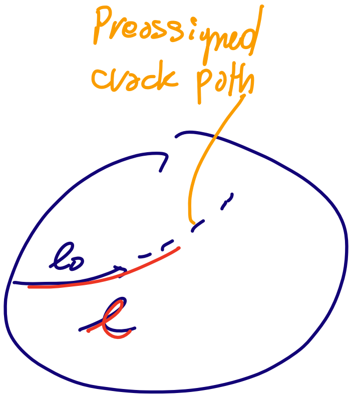


$$\boxed{\inf_{\substack{\Gamma \supseteq \Gamma_0 \\ u \in \mathcal{C}(\Gamma)}} \underbrace{\int_{\Omega/\Gamma} \frac{1}{2} \underline{\underline{\varepsilon}}(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(u)\, dx - W_{ext}(\underline{u})}_{P(u,\Gamma)} + \underbrace{G_c \,\mathrm{surface}(\Gamma)}_{D(\Gamma)}}$$

EXISTENCE OF SOLUTIONS (inf → min) can be proved for $W_{ext} = 0$

In general the solution is not unique

Consider a single crack propagating on a preassigned path.
We denote by ℓ_0 initial length and by ℓ the current unknown length.



$$\mathcal{E}(\underline{u}, \ell) = P(\underline{u}, \ell) + G_c\,\ell$$

$$\inf_{\substack{\ell \geq \ell_0 \\ u \in \mathcal{C}(\ell)}} \mathcal{E}(\underline{u}, \ell) = \inf_{\ell \geq \ell_0} \Big(\overbrace{\inf_{u \in \mathcal{C}(\ell)} P(\underline{u}, \ell)}^{P(\ell)} + G_c\,\ell \Big)$$

ℓ_0 : initial lenght

ℓ : current lenght

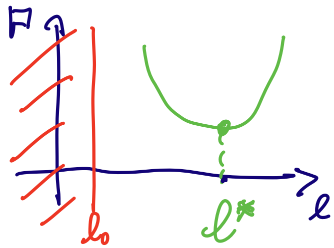
$$\inf_{\ell \geq \ell_0} \Big(\mathcal{E}(\ell) := P(\ell) + G_c\,\ell \Big)$$

Potential energy for a given crack length

$$\mathcal{E} : \ell \in \mathbb{R} \to \mathcal{E}(\ell) \in \mathbb{R}$$

Unilaterally constrained minimisation problem: Kuhn-Tucker optimality conditions

$F: \ell \in \mathbb{R} \to \overset{\to\text{reg.}}{F(\ell)} \in \mathbb{R}$, $\min\limits_{\ell \geq \ell_0} F(\ell)$

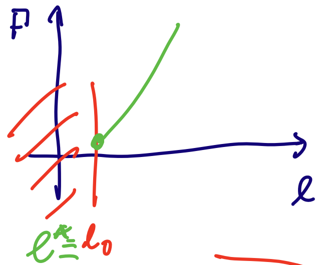


ℓ^* is such that

- $\boxed{\ell^* \geq \ell_0}$
- $F(\ell) - F(\ell^*) \geq 0 \quad \forall \ell \geq \ell_0$

$$\Downarrow \ \underline{\Delta\ell}$$

$$F'(\ell^*)(\ell - \ell^*) \geq 0$$



- cas 1 : $\boxed{\ell^* > \ell_0}$, $\Delta\ell \geq 0$ ou $\Delta\ell \leq 0 \Rightarrow F'(\ell^*) \geq 0$, $F'(\ell^*) \leq 0 \Rightarrow \boxed{F'(\ell^*) = 0}$

- cas 2 : $\boxed{\ell^* = \ell_0} \Rightarrow \Delta\ell \geq 0 \Rightarrow \boxed{F'(\ell^*) \geq 0}$

$F'(\ell^*) \geq 0$, $\ell^* - \ell_0 \geq 0$, KKT

$F'(\ell^*)(\ell^* - \ell_0) = 0$

$G(\ell) := -P'(\ell)$

$$\mathcal{E}(\ell) = \underline{P(\ell)} + \underline{G_c\ \ell}$$

$$\min_{\ell \geq \ell_0} \mathcal{E}(\ell) \implies \begin{cases} \mathcal{E}'(\ell^*) \geq 0 \\ \ell^* \geq \ell_0 \\ \mathcal{E}'(\ell^*)(\ell^* - \ell_0) = 0 \end{cases} \iff \begin{cases} P'(\ell^*) + G_c \geq 0 \\ \ell^* \geq \ell_0 \\ (P'(\ell^*) + G_c)(\ell^* - \ell_0) = 0 \end{cases}$$

GRIFFITH'S CRITERION

$$\iff \left\{ G^*(\ell) \leq G_c, \quad \ell^* \geq \ell_0, \quad (G(\ell^*) - G_c)(\ell^* - \ell_0) = 0 \right.$$

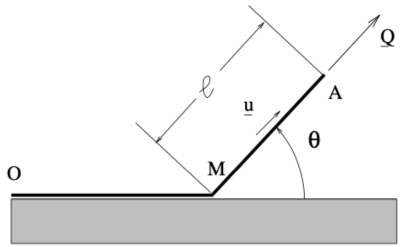
Comparison between Irwin's and Griffith's criteria

- If the material is isotropic and crack loaded in Mode I
  - The two criteria coincide
  - We can identify G_c by use of Irwin's criterion and experiments that measure the toughness (CT specimen, 3-point bending etc…)

$$\boxed{G_c = \frac{1-\nu^2}{E} K_{Ic}^2}$$

- If the material is anisotropic, mixed mode cracks, interface cracks…
  - Irwin's criterion is not applicable
  - Griffith's criterion is still applicable (but need for specific experiments to identify G_c in the case of anisotropic materials or interface cracks)

Example (TD): Peeling test



- Assume an adhesion surface energy Gc
- Assume that the adhesive is straight is the debonded part
  and that it is linear elastic with Young modulus E and thickness e
- Determine the critical peeling force for debonding
- Determine the “best peeling angle theta” to debond a scotch-tape
  with a minimal effort