Numerical solutions of differential equations

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Course SF2521, 7.5 ECTS, VT18

General Finite Volumes Schemes of First Order

Monotone schemes

Monotone schemes

Properties and examples

Monotone schemes

- From the motivation we expect monotone schemes to convergence to the unique entropy solution.
- To make this precise, we require the notion of Total Variation (TV).

Total Variation

Definition (Total Variation)

For $u \in L^1([a, b])$ the total variation (TV) is defined as:

$$TV_{[a,b]}(u) := \sup_{\substack{a=x_0<\cdots<x_n=b \\ n\in\mathbb{N}}} \sum_{j=0}^{n-1} |u(x_{j+1}) - u(x_j)|$$

Analogously for “discrete functions” $\mathbf{Q} = (Q_j)_{j\in\mathbb{Z}}$:

$$TV_{[a,b]}(\mathbf{Q}) = \sum_{j, x_j\in[a,b]} |Q_{j+1} - Q_j|.$$

Total Variation - Examples

For $u \in L^1([a,b])$ the total variation (TV) is defined as:

$$TV_{[a,b]}(u) := \sup_{\substack{a=x_0<\cdots<x_n=b \\ n\in\mathbb{N}}} \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)|.$$

- $u(x) = x$ for $[a,b] = [0,1]$. $TV_{[0,1]}(u) = 1$.
- $u(x) = x$ for $[a,b] = [-1,1]$. $TV_{[-1,1]}(u) = 2$.
- $u(x) = \sin(\pi x)$ for $[a,b] = [0,1]$. $TV_{[0,1]}(u) = 2$.
- $u(x) = \sin(2\pi x)$ for $[a,b] = [0,1]$. $TV_{[0,1]}(u) = 4$.
- $u(x) = e^x$ for $[a,b] = [0,1]$. $TV_{[0,1]}(u) = e$.
- $u(x) = \sin(1/x)$ for $[a,b] = [0,1]$. $TV_{[0,1]}(u) = \infty$.
- u monotonically increasing, then $|u(x_{i+1}) - u(x_i)| = u(x_{i+1}) - u(x_i)$ and hence $TV_{[a,b]}(u) = u(b) - u(a)$.

Convergence of monotone schemes

Theorem

Let $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be an initial value which is only not zero on an interval $[a, b] \subset \mathbb{R}$ with $TV_{[a,b]}(v_0) < \infty$. The numerical initial condition is picked as

$$Q_j^0 := \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} v_0 \qquad \text{for } j \in \mathbb{Z}.$$

We consider a scheme Φ in conservation form, which is consistent and monotone. Time step size and mesh size are chosen such that

$$L \frac{\Delta t}{\Delta x} \le \frac{1}{2}, \qquad \text{where} \quad L := \sup_{\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2} \frac{|g(\mathbf{v}_1) - g(\mathbf{v}_2)|}{|\mathbf{v}_1 - \mathbf{v}_2|}.$$

Then, the numerical approximations given by

$$Q_j^{n+1} = \Phi(Q_{j-1}^n, Q_j^n, Q_{j+1}^n),$$

converge in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ to the unique entropy solution u.

Convergence of monotone schemes

Remark: Q_j^n converges in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ to u means

$$\int_V |\mathbf{Q}_{\Delta t, \Delta x}(x,t) - u(x,t)| \, dt \, dx \to 0 \qquad \text{for } \Delta t, \Delta x \to 0$$

for all bounded subsets $V \subset \mathbb{R} \times \mathbb{R}^+$ and where $\mathbf{Q}_{\Delta t, \Delta x}$ is the piecewise constant function with

$$\mathbf{Q}_{\Delta t, \Delta x}(x,t) = Q_j^n \qquad \text{if } x_j \leq x < x_{j+1} \text{ and } t_n \leq t < t_{n+1}.$$

Convergence of monotone schemes

Summary:

Scheme that is

- in conservation form,
- consistent + monotone
- and that fulfills the CFL condition

$$L \frac{\Delta t}{\Delta x} \leq \frac{1}{2},$$

is convergent to the entropy solution.

Monotone schemes - Example 1

Example 1: Consider the linear transport equation

$$\partial_t u + a\partial_x u = 0 \quad \text{with } a > 0$$

together with the scheme

$$Q_j^{n+1} = Q_j^n - a\frac{\Delta t}{\Delta x}(Q_j^n - Q_{j-1}^n) \quad \text{“backward differences”}$$

It holds:

$$Q_j^{n+1} = Q_j^n \underbrace{\left(1 - a\frac{\Delta t}{\Delta x}\right)}_{>0,\text{ if } \Delta t < \frac{\Delta x}{a}} + Q_{j-1}^n \underbrace{a\frac{\Delta t}{\Delta x}}_{>0}.$$

Hence, the scheme is monotone if the CFL condition $a\frac{\Delta t}{\Delta x} < 1$ is fulfilled.

Monotone schemes - Example 2

Example 2: Consider the linear transport equation

$$\partial_t u + a\partial_x u = 0 \quad \text{with } a > 0$$

together with the scheme

$$Q_j^{n+1} = Q_j^n - a\frac{\Delta t}{\Delta x}(Q_{j+1}^n - Q_j^n) \quad \text{“forward differences”}$$

It holds:

$$Q_j^{n+1} = Q_j^n \underbrace{\left(1 + a\frac{\Delta t}{\Delta x}\right)}_{>0} + Q_{j+1}^n \underbrace{a\frac{-\Delta t}{\Delta x}}_{<0}.$$

This contradicts the monotonicity.

Hence, the scheme is for $a > 0$ not monotone.

Monotone schemes - Example 3

Example 3:
The Lax-Friedrichs scheme and the Engquist-Osher scheme are both always monotone schemes.

This can be seen by a simple calculation.