Assignement 2 (SF2521) - Hyperbolic Equation

SF2521

Wave equation

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1 Introduction

We are in this report gonna study the hyperbolic equation, by looking to the stability of the schemes linked to it and the way of solving it numerically.

At the end of this report, we should have got a better understanding of the hyperbolic equations' behavior and all in all more hindsight in the way of studying and solving a hyperbolic differential equation.

2 Stability of numerical schemes

We consider this equation during this part

$$U^{n+1} = Q(t_n)U^n + \Delta F^n \tag{1}$$

$$U^0 = g$$

2.1 Work on the discrete Duhamel's Principle

We are in this part willing to work on the previous equation by expressing U in function of a theoretical mesh depending on variables such as $S_{\Delta t}$ and t_n. Moreover, we will discuss what this new equation is implying and which information on stability it can provide us.

First, we have to show that the following discrete principle holds :

$$U^n = S_{\Delta t}(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})F^\nu \tag{2}$$

with $t_n = n * \Delta_t$ and :

To show that this equation stands, we will combine the equation (1) and (2) and combine it with the supplementary equations implying S_{Δ_t}.

So, by implementing the equation (2) on the equality (1) we have a new equality :

$$U^{n+1} = Q(t_n)U^n + \Delta F^n \quad \text{leads to :}$$

$\rightarrow \quad S_{\Delta_{t(t_{n+1},0)*g}} + \Delta_t * \sum_{\nu=0}^{n} S_{\Delta t}(t_{n+1}, t_{\nu+1})F^\nu$

$$= Q(t_n) * (S_{\Delta_{t(t_n,0)*g}} + \Delta_t * \textstyle\sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})F^\nu) + \Delta_t * F^n$$

$\rightarrow$ Yet, as $\quad S_{\Delta t}(t_{n+1}, t_\mu) = Q(t_n)S_{\Delta t}(t_n, t\mu) \quad$ we have :

$$\rightarrow \quad S_{\Delta_t}(t_{n+1}, 0) * g = Q(t_n) * S_{\Delta_t}(t_n, 0) * g$$

We can now, removing from each part of the equality the concerned terms, rewrite the equality as :

$$\Delta_t * \sum_{\nu=0}^{n} S_{\Delta t}(t_{n+1}, t_{\nu+1})F^{\nu}$$

$$= \Delta_t * Q(t_n) * \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})F^{\nu} + \Delta_t * F^n$$

Because we want to make the link between the two sum terms, we are willing to consider the same intervals for the sums. To do so, we rewrite the left term :

$$\Delta_t * \sum_{\nu=0}^{n} S_{\Delta t}(t_{n+1}, t_{\nu+1})F^{\nu} = \Delta_t * \sum_{\nu=0}^{n-1} S_{\Delta t}(t_{n+1}, t_{\nu+1})F^{\nu} + \Delta_t * S_{\Delta_t}(t_{n+1}, t_{n+1})F^n$$

To sum up the overall equation, we are up to :

$$\Delta_t * \sum_{\nu=0}^{n-1} S_{\Delta t}(t_{n+1}, t_{\nu+1})F^{\nu} + \Delta_t * S_{\Delta_t}(t_{n+1}, t_{n+1})F^n =$$

$$\Delta_t * Q(t_n) * \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})F^{\nu} + \Delta_t * F^n$$

Yet, we once again have :

$$S_{\Delta t}(t_{n+1}, t_{\mu}) = Q(t_n) S_{\Delta t}(t_n, t\mu)$$

and so : $\Delta_t * \sum_{\nu=0}^{n-1} S_{\Delta t}(t_{n+1}, t_{\nu+1})F^{\nu} = \Delta_t * Q(t_n) * \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})F^{\nu}$

We finally get the following two last terms once we have removed the sum terms :

$$\Delta_t * S_{\Delta_t}(t_{n+1}, t_{n+1})F^n = \Delta_t * F^n$$

We recall that $S(t,t) = I$ and thus $S_{\Delta_t}(t_{n+1}, t_{n+1}) = I$, which eventually leads to :

$\Delta_t * I * F^n = \Delta_t * F^n$. Because this last equality is fulfilled, we can conclude on the fact that the equation (2) holds. We recall the discrete Duhamel's Principle applied to our system :

$$U^n = S_{\Delta t}(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})F^{\nu}$$

2.1.1 Inhomogeneous term F

To consider this inhomogeneous term F is like adding a term of wave considering different t0 for each F and thus "additional initial conditions".
In fact, if we have our function U at t, to get U(t+T) we consider our scheme considering the influence of F. If F is the force applied at a given time, it has impact on the speed U(t). Thus, we may consider this force as the one defining the initial speed for a greater iteration.
Eventually, to get U(t+T), we finally apply our scheme with no force term in the equation but rather in the initial conditions. We then do so as long as we want to iterate, be continuously redefining f after each iteration.

2.1.2 Solution

In the same way, and from the above point as well as the course, we know that the solution of an hyperbolic equation is the sum of all the waves composing it. Here, by having defined the term F as a variable acting as an initial condition, if we want to solve our system we have to do so for each F, and thus each "initial conditions"

With respect to the equation (2), we clearly see that we are considering U as a vector containing the sum of the initial value $S_{\Delta t}(t_n, 0)$ and the term $S_{\Delta t}(t_n, t_{\nu+1})$ linked to the function F. To sum-up, we see that we must consider the $S_{\Delta t}$ for which the initial conditions F are satisfied otherwise the term in the sum is null. Then, if the function satisfy the condition, we add it to the general sum, otherwise the function remains constant.

2.2 Norm

We are now on willing to show the influence of the dependence in time for our discrete equation. To do so, we begin by showing that :

if $\qquad ||S_{\Delta t}(t_\kappa, t_\nu)||_h \leq K * \exp^{\alpha(t_\kappa - t_\nu)}$

then $\qquad ||U^n||_h \leq K(\exp^{\alpha t_n} ||g||_h + \int_0^{t_n} \exp^{\alpha(t_n - s)} ds * max_{0 \leq \nu \leq n-1} ||F^\nu||_h)$

To show this will prove mathematically that studying our system by considering an inhomogeneous term won't imply a destructive behavior. Indeed, our function is maximized and thus the inhomogeneous term hasn't a negative influence on the solution's stability.

In order to show the above relation, we will exploit the equation (1). We thereby have the next inequality that stands :

$||U^n||_h = ||S_{\Delta t}(t_n, 0) * g + \Delta t \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1}) F^\nu||_h$

$\rightarrow \quad ||U^n||_h \leq ||S_{\Delta t}(t_n, 0)||_h * ||g||_h + \Delta t|| \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1}) F^\nu||_h$

From (2), we have :

$||S_{\Delta t}(t_a, t_b)||_h \leq K \exp^{\alpha(t_a - t_b)}$, and thus :

$||S_{\Delta t}(t_n, 0)||_h \leq K \exp^{\alpha(t_n - 0)}$ and so

$||U^n||_h \leq K \exp^{\alpha * t_n} ||g||_h + \Delta t|| \sum_{\nu=0}^{n-1} S_{\Delta_t}(t_n, t_{\nu+1}) F^\nu||_h$

We know take a look at the sum term :

$\Delta t|| \sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1}) F^\nu||_h = \Delta t \sum_{\nu=0}^{n-1} ||S_{\Delta t}(t_n, t_{\nu+1})||_h * \sum_{\nu=0}^{n-1} ||F^\nu||_h$

We naturally show that for a vector F depending on the variable ν :

$$\sum ||F^{\nu}||_h \leq max\ _{0\leq\nu\leq n+1}||F^{\nu}||_h$$

Furthermore, we have, considering the method of rectangles :

$||\sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})||_h \leq \int_{t(\nu(=0)+1)=t_1}^{t(\nu(=n-1)+1=t_n} |S_{\Delta t}(t_n, t_{\nu+1})|dt_{\nu+1}$

Yet, as we consider the inequality (2) we know that the absolute value of S_{Δ_t} is maximized by the value of the exponential term. Then, we have the following relation :

$||\sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})||_h \leq K * \int_{t_1}^{t_n} |exp^{\alpha(t_n - t_{\nu+1})}|dt_{\nu+1}$

Because we have exp ≥ 0, we can consider the following inequality, where the term t1 disappears :

$||\sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})||_h \leq K * \int_{t_0}^{t_n} exp^{\alpha(t_n - t_{\nu+1})}dt_{\nu+1}$

By setting $t_{\nu+1} = s$, we finally get the expected relation :

$||\sum_{\nu=0}^{n-1} S_{\Delta t}(t_n, t_{\nu+1})||_h \leq K * \int_{t_0=0}^{t_n} exp^{\alpha(t_n - s)}ds$

We eventually recall the equality find :

$$||U^n||_h \leq K(\exp^{\alpha t_n} ||g||_h + \int_0^{t_n} \exp^{\alpha(t_n - s)} ds * max_{0\leq\nu\leq n-1}||F^{\nu}||_h \qquad (3)$$

We recall that through this inequality we can now consider that if the stability of the homogeneous problem is fulfilled, then so it is with the inhomogeneous problem.

2.3 time

In this last question of the first part we are willing to understand even more how our discrete equation works by looking to its dependence in time. More precisely, we would like to know what would imply an α depending on the variable Δt.

Thus, we consider here $\alpha = \Delta t^{-1/2}$ in the equation (3). Yet, for α to depend on the variable Δt implies that α is also depending on the number of iteration n and t_n itself. More specifically, this new equality implies that, no matter how we deal with the number of iteration and t_n, in the equation (3) the variables will evolve the same way.
In fact, when we have α independent of the time variable Δt we can have impact on the function over the associated grid, but here we will have impact on the two exponent of the exponential and thus the impact on the grid will kind of cancel each other.

This implies that dealing with t_n no more implies acting on the grid as the exponent remain constant.

3 The Shallow Water Model

We are now working on the numerical solution associated to 2 differential equations describing our system :

$$\begin{cases} h_t + (hv)_x = 0 \\ (hv)_t + (hv^2 + \frac{1}{2}gh^2)_x = 0 \\ \forall (x,t) \in [0,L] * [0,\infty) \end{cases} \tag{4}$$

The idea will be to study the height and the speed of our wave using the above two equation. Even though we don't have a direct access to the variable v, we will be able to still get it by combining our two equations. Indeed, once our numerical scheme applied, we can access to v by setting hv/h –> v.
Before going deeper on the numerical resolution, we are to re-express the system, making appearing the initial and boundary conditions :

$$\begin{cases} h_t + (hv)_x = 0 & \forall (x,t) \in [0,L] * [0,\infty) \\ (hv)_t + (hv^2 + \frac{1}{2}gh^2)_x = 0 & \forall (x,t) \in [0,L] * [0,\infty) \\ & \\ h(x,0) = H + \epsilon \exp\left(-((x - L/2)^2)/\omega^2\right) & \forall x \in [0,L] \quad t = 0 \\ v(x,0) = 0 - - > (hv(x,0) = 0) & \forall x \in [0,L] \quad t = 0 \\ & \\ v(0,t) = v(L,t) = 0 - - > (hv(\partial x, t) = 0) & \forall\, \partial x \in 0, L \quad t \in [0,\infty) \end{cases} \tag{5}$$

We finally add the values of the different variables that are to be used to solve numerically our non-linear problem :

— L = 10m

— H = 1m

— g = 9.61m.s-2

— ω = 0.4m

3.1 Numerical Solution

We consider the Lax-Friedrichs method, applied on u = $(h, h{*}v)^T$ and defined as :

$$U_j^{n+1} = U_j^n - \frac{\Delta_t}{2 * \Delta_x} * (f(U_{j+1}) - f(U_{j-1}) - \alpha(U_{j+1} - 2 * U_j + U_{j-1})^n) \tag{6}$$

With $U_k^n = \begin{pmatrix} h \\ hv \end{pmatrix}$ and $f(U_k) = \begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}$

which can then be rewrite as :

$$U_k^n = \begin{pmatrix} u1 \\ u2 \end{pmatrix} \text{ and } f(U_k) = \begin{pmatrix} u2 \\ (u2^2)/u1 + \frac{1}{2}gu1^2 \end{pmatrix}$$

We here note that we used the fact that hv/h −> v to redefine our problem's variables.

We will see in the different question the considerations we do on the other method's variables : α, Δ_t and Δ_x.

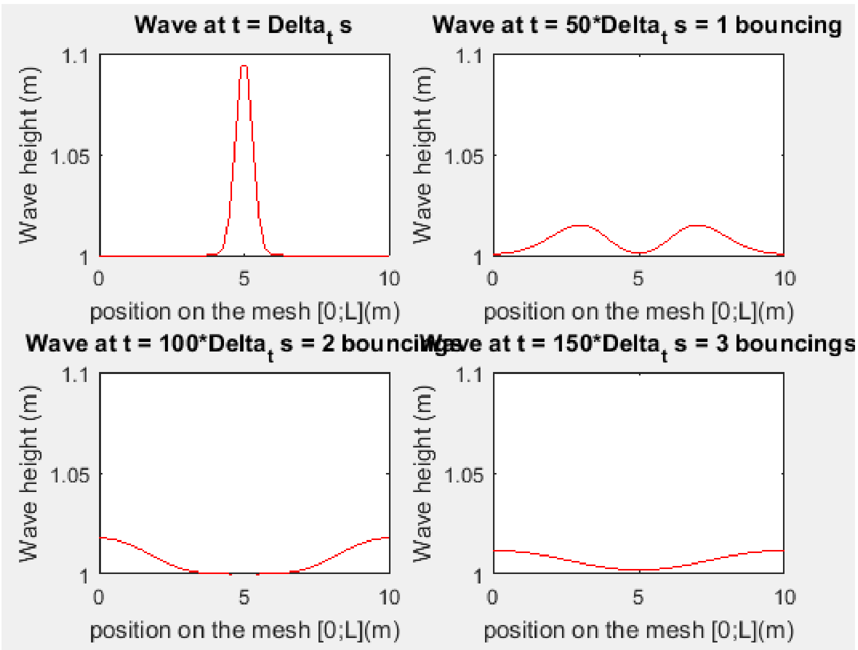
3.1.1

Considerations :

— $\alpha = \frac{\Delta_x}{\Delta_t}$

— only a few iterations for the wave (we considered three bouncing and reflection at the boundaries)

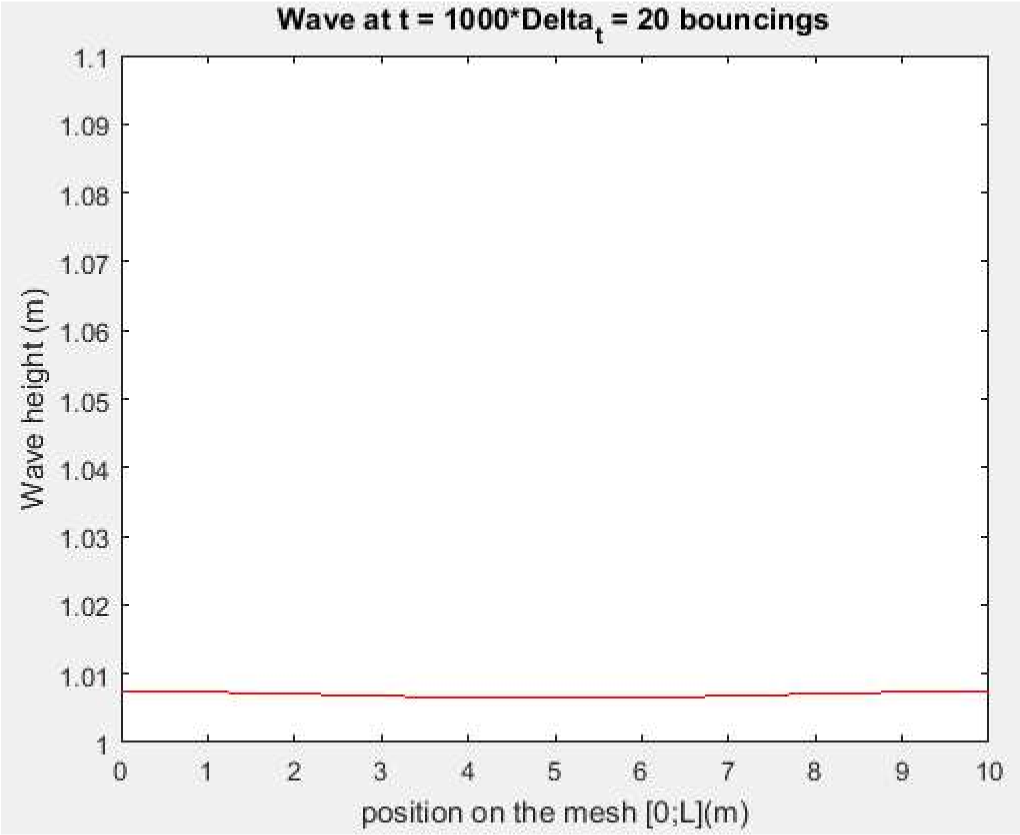
We are willing to run a simple code that would display the shape of the wave moving with the time and reflect at the boundaries.
Below, you have four plots showing the waves moving and bouncing at the boundaries before crossing each other :



(NOTE : you can have a look at it in action by looking in the Matlab code).

For what informations we can provide thanks to this plot, we already confirm visually the behavior of a wave that bounces on the walls and that get flatter with the time. We also see that the function is oscillating with damping. This may make the function tend toward a stable position after a certain time.

The final state (close to it) of the wave is as expected a wave with zero speed and that is almost flat :



To summarize the curve we obtained, we can physically see her as a wave that at the beginning is on the middle of a 2D plan, and that will thus split into two waves that will later on have interactions with each other.

We will go deeper now on the behavior of the wave by looking to its dependence with respect to variables as ϵ and α.

3.1.2

As we have now the basis for the wave equation, we want to have more hindsight on how behave our model depending on its variables.
First, we change epsilon by taking larger values and we take a look at the following characteristics :

— Wave shape, wave amplitude

— Wave speed

— Wave collisions

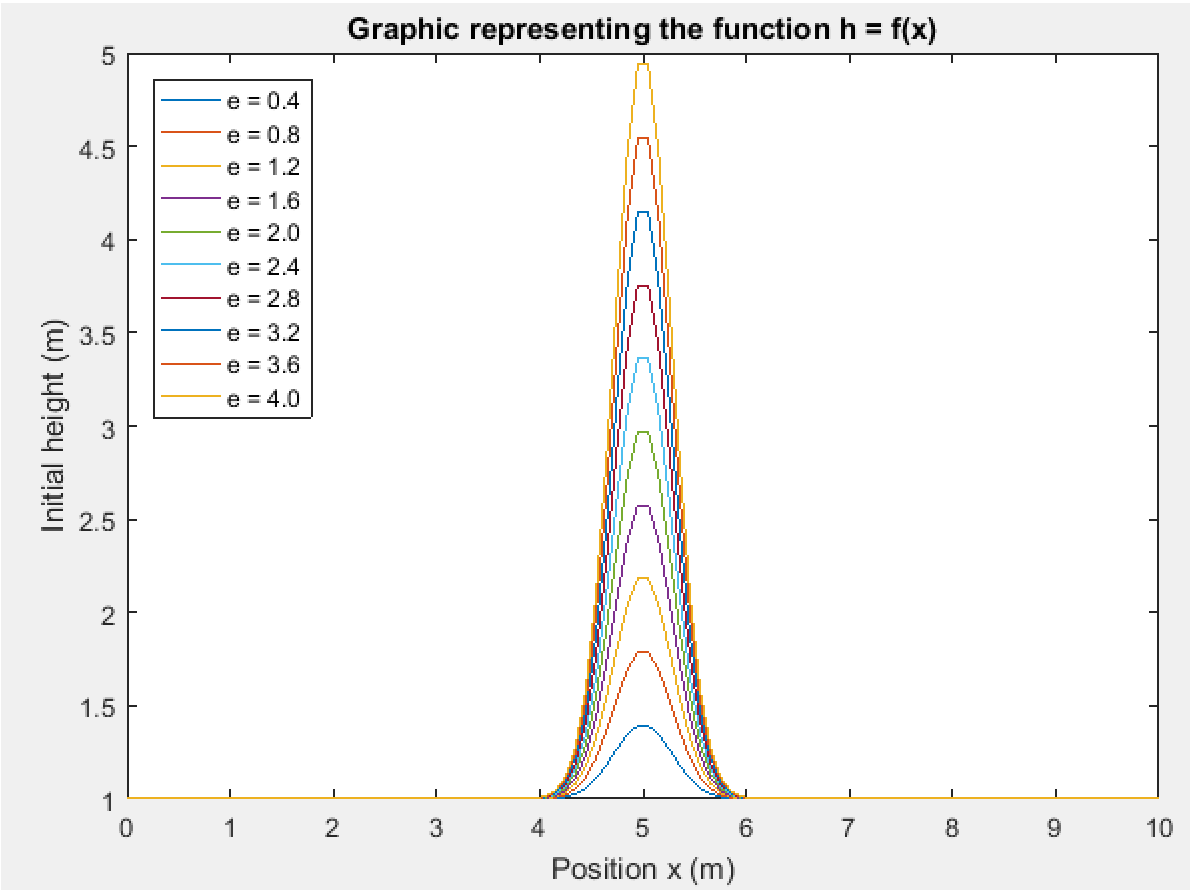
— Time-step

We decided to work on 10 different epsilon, defined as follows : $\epsilon = [0.4, 0.8, 1.2, 1.6, 2.0, 2.4, 2.8, 3.2, 3.6, 4.0]$;

We will now carry the study of each point by explaining the method used and then make a presentation of the results.

In a first time, we have been looking to the amplitude, and more generally to the wave's shape. To do so, we have been considering the same other characteristics as before except that we reduced the time step to ensure the stability of the solution (this point will be discussed in the following points).

In order to show the impact of epsilon over the wave's amplitude we decided to show it through the static and the dynamic case. For what deals with the static case, we display below the initial position and height of the wave with respect to epsilon :

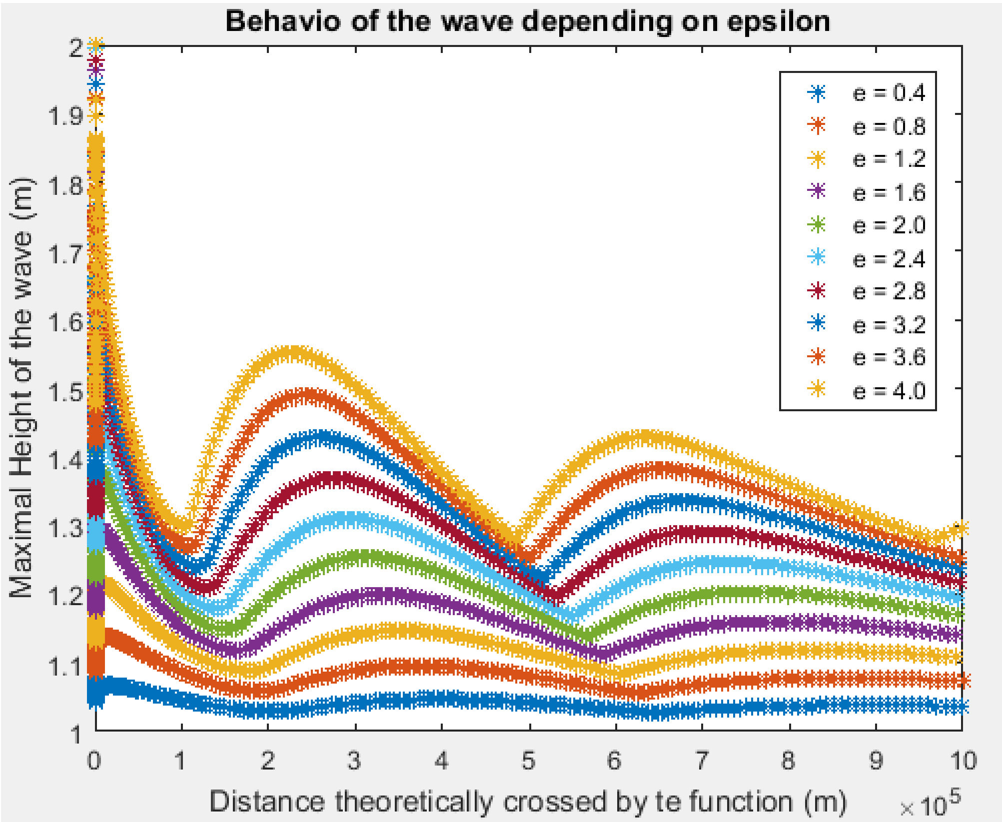


We can see that with the epsilon increasing, the initial height of the wave is increasing too. By looking to the expression of $h(x, 0) = H + \epsilon \exp -((x - L/2)^2)/\omega^2$, we see that the term ϵ appears in front of the exponential and thus it appears clear that by increasing this variable we have a direct impact on the height.

Moreover, we are now questioning the impact of it on the dynamic behavior of

the function. We thereby decided to draw the curve representing the shape of the wave across time as well as some graphics giving an equivalent information through a visual explanation.
In order to compare the different waves in an efficient way, we furthermore decided to only consider the maximum of the curve in one of the two direction. Indeed, as we consider an initial condition such that we ensure a central symmetry, we can consider any of both side to conclude on the overall behavior. Plus, to look at the maximum ensure to look at a point for which we have variations and that comes with a more comprehensive visual information. Finally, to work on these maximum, we have been creating a part of the program that aims in giving the crossed length rather than the position. This ensure that we don't have point superposing and that we can look at our function on a bigger scale.

In the graphic below we are then displaying the height of the waves depending on ϵ :



NOTE : the theoretical distance implies that we consider the height after a certain number of iteration. Thus, we consider the same number of iteration, and at each one, we go forward of Δ_x. It is then possible, depending on the speed that the length crossed isn't the same, it is rather the number of time-step that's equivalent vertically for each curve.

These curves represent as said the maximum took by the height depending on ϵ. Therefore, the pick we see for the maximum represent in fact the position at which

the height is the biggest, so the edges and the center in our case.
Thus, if this pick in the maximum doesn't happen at the same position, it implies that the epsilon got impact on the wave's speed. Besides, by doing so, it might have had impact on the shape. We indeed see that the wave doesn't oscillate as she did before for high epsilon but she rather shows period of sharp increase of the height followed by a slower diminishing.

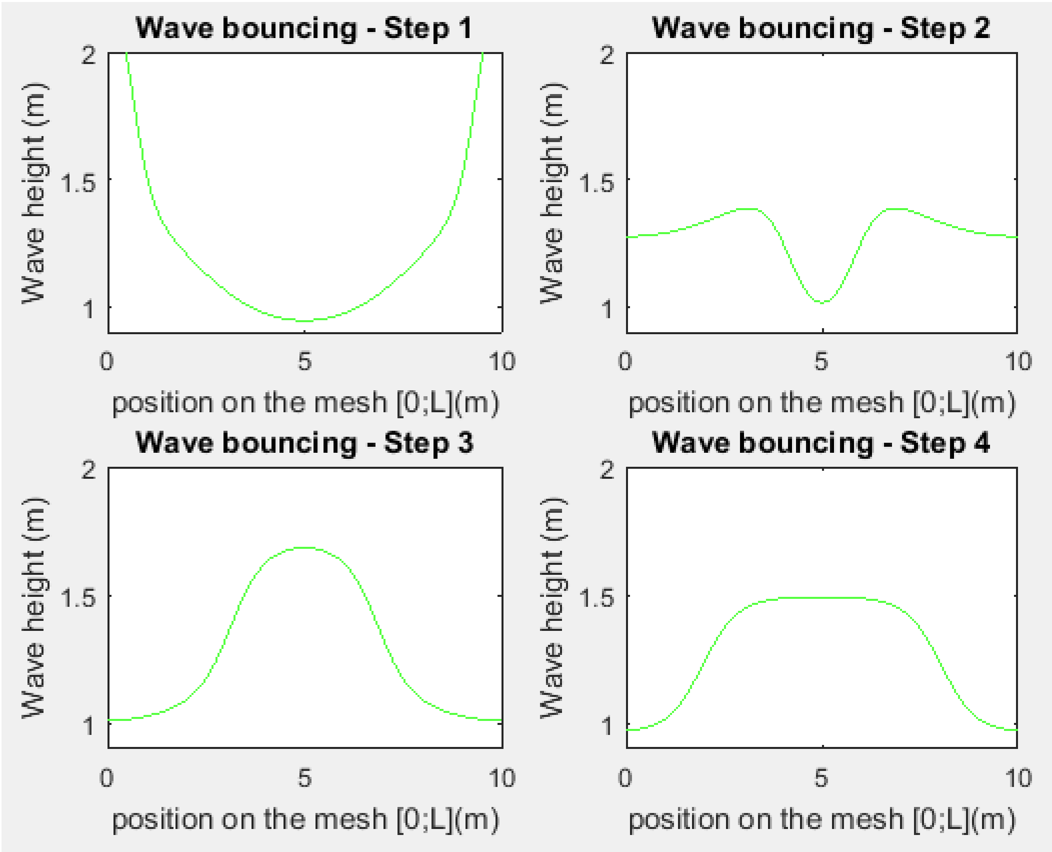
Thence, a first observation that far is that these curves give informations on the wave's shape as well as on the speed.
In a first time, we take a look at the variation of the shape regarding to ϵ. We know that bigger epsilon is, bigger the variations in height get. We also see that the curve is obviously still suffering damping for every epsilon. What's new that we are about to see is how this new shape impact on the behavior of the wave and more specifically the waves interactions and bouncings.
We had in the first question a curve behaving as an oscillating function with damping. This damping as 2 explanation :

- The modelisation of losses in the energy;
- The interaction between the curves that can lead into a positive or a negative damping (increasing the height a lot or just a little here).

This phenomenon of damping can be explained by recalling the framework of our study : we consider a constant grid, and with it a constant volume that the water can fill, and thus if we increase ϵ, we increase the amount of water in the tank and we have to deal with more complex bouncing. Indeed, as we have more water in a constant available volume, we increase the interaction between the waves on both sides. Moreover, the "stability" of the wave decreases with its height increasing.

To sum-up, we have more volume which implies a kind of instability. The effect of this is that the wave won't necessarily push themselves upward, but they might also cancel the speed of each other.

In the graph that follows we try to illustrate this phenomenon resulting on a possible negative damping of the wave :



On this graph we see that the wave no more behave as previously. In fact we have 4 main steps :

— The waves bounce in each side on the edges, calling the fact that they will come back toward the center and enter in contact soon ;

— The wave both have their own height creating a hole on the middle. Basically, bigger this hole is, sharper will be this increase of height ;

— We are now at a step which gather an enormous amount of "water". The interaction between the waves created this bump on the middle that take more place than before ;

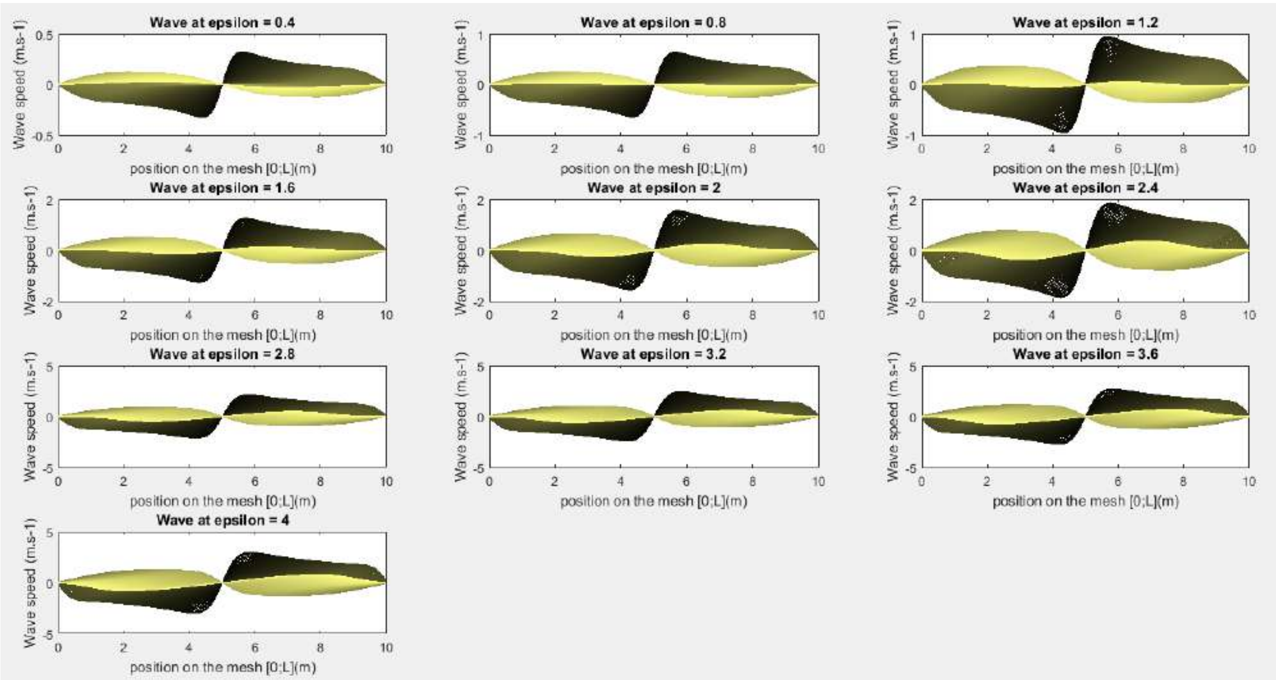
— In this last step, we see that the waves have kind of pushed each other by interacting, which results here in a big amount of water falling rather that spreading.

To conclude on this part, as we have more volume for the wave, the interaction don't happen anymore at two distinct points (middle and edges) but rather at ranges, modifying and multiplying their way of interacting. We here showed one of them that present the modification suffered by the wave due to the influence of epsilon.

To sum up, the observation of these curves told us more on the wave's behavior (the fact that it was influenced by epsilon, and a brief explanation of it) as well as the impact of epsilon over the interaction on the waves. We recall that it is at this point interesting to take a look at the code as it provides a visual information through time.

One the other hand, we have the impact on the speed. We already called the fact that having pick at different positions was necessarily implying a modification on the speed. We also understand with the last point that the sharp increase of height will be linked to an increasing of the speed while the falling implies a decreasing of the speed.

To conclude on this part, we decided to show the global shape of the speed on one loop. It allows to make sure we don't look at a particular case of the speed and to have more hindsight on the speed behavior. To do so, we gathered the different shape for the speed in each point, considering graphs for each epsilon. The result of this is summarized in the subplots below :



What we were willing to show here is :

— The fact that when considering the superposed shape of the speed, we see that it is constant, no matter epsilon ;

— While the shape is fairly identical, what changes is the amplitude taken by the speed.

We have finally a speed that is indeed increasing, and to explain briefly its shape, as you might think, the speed takes negative value when we are willing to move the wave toward the left and positive value for the right. We also have a speed that is indeed increasing where the shock between the waves happen, which confirm the observations made on the wave's shape.

To finish with this part and as mentioned before, we are gonna take a few lines to present the link between epsilon and the model's stability. As we decided to take a look at a variable having impact on the initial height, we saw that the repercussions was nevertheless not restricted to the amplitude. We then have seen that epsilon had influence on how behave our function, and by the same way, that led us into reconsidering the time-step to ensure stability. In fact, as we were dealing with sharp variations and non constant behaviors, to diminish the time step was providing us more freedom in the plots, and it was mainly ensuring the stability for sudden variations, avoiding for example possible bump leading to unsteadiness and then to a destructive behavior.

3.1.3

More than epsilon that impacts the stability of the solution, we have the variable alpha that has the same negative repercussion. This means that alpha too have an impact on the function's stability and may lead to unsteadiness or even a destructive behavior.

To work on this variable, we decided to take back $\epsilon = 0.1$ and to work on multiple α defined as follows : $\alpha = C_0 * \frac{\Delta_x}{\Delta_t}$ with $C0 \in [0.5, 1.5]$.

In our code, we decided to define the ratio $\frac{\Delta_x}{\Delta_t}$ such that it is equal to 10. This will allow us to have a better understanding of the range for the solution stability.

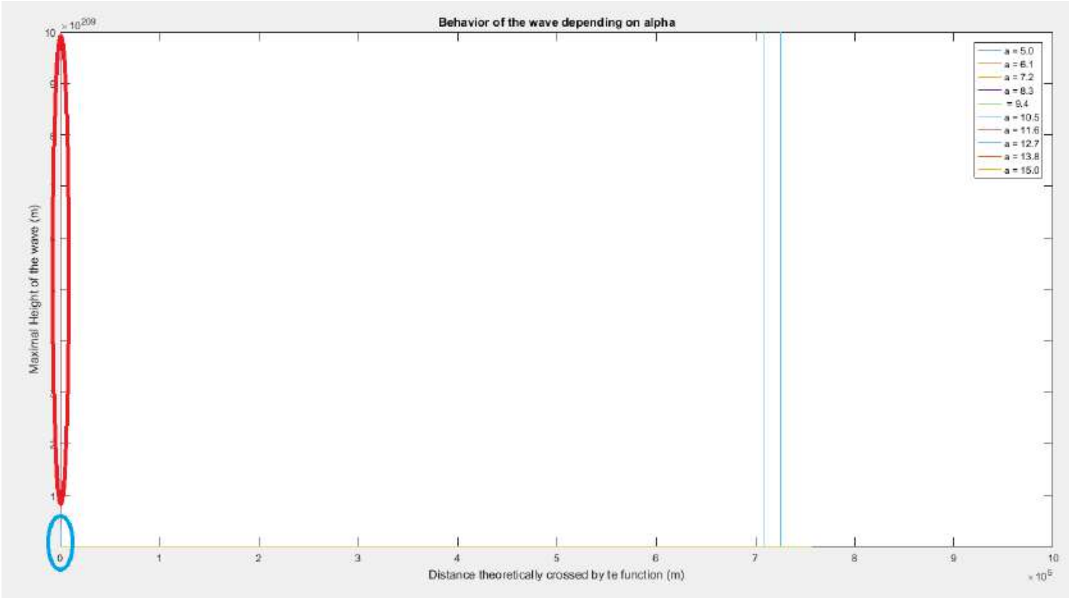
Indeed, a criteria necessary to get a stable solution is the CFL condition defined by :

$|a|\frac{\Delta_t}{\Delta_x} \leq 1$. We thus need to ensure $\frac{1}{10} \leq \frac{1}{a}$ and finally $a \leq 10$. Because we have a $= 1$, the inequality is automatically verified, assuming we keep Δ_x greater than Δ_t. A bigger assumption that is to be maid is on alpha. In fact, as it is also being multiplied by the ratio $\frac{\Delta_t}{\Delta_x}$ on our equation, it is also necessary to ensure that $\alpha \leq 10$ stands. Because alpha is this far defined by $\alpha = \frac{\Delta_x}{\Delta_t}$, this new inequality isn't automatically verified and we might encounter unstable cases.

Yet, with $\alpha = C0 * \frac{\Delta_x}{\Delta_t}$ we will obtain $\alpha \in [5, 15]$.

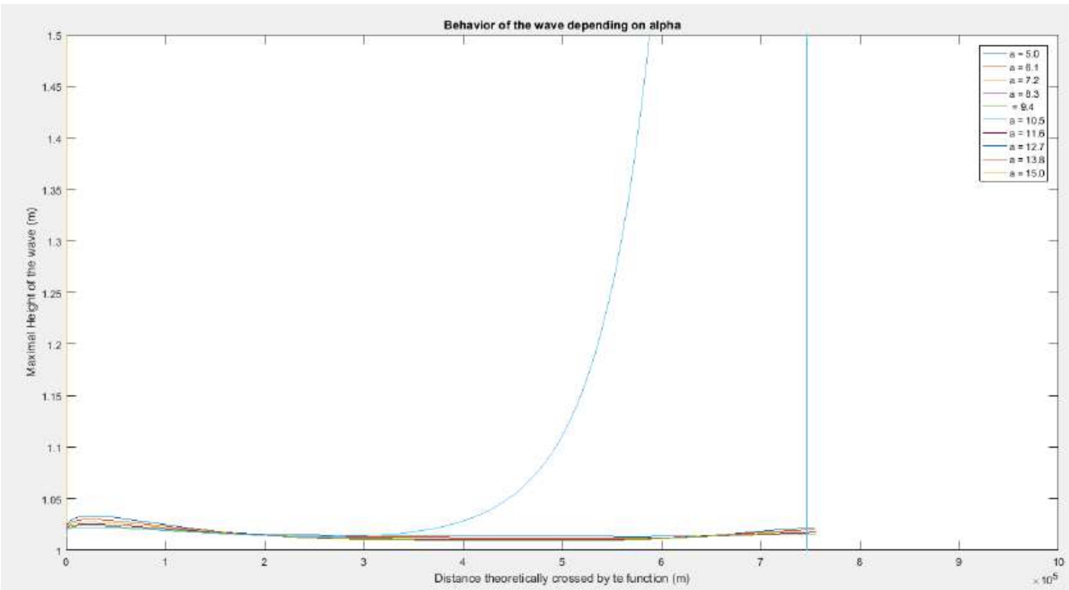
It is now clear that even if the CFL condition isn't sufficient to conclude on a stable system, we will be able to conclude on the fact that for $\alpha \geq 10$ we will get instability solution.

We are displaying the following graph representing the different height taken by the wave depending on alpha and that we will comment later on :



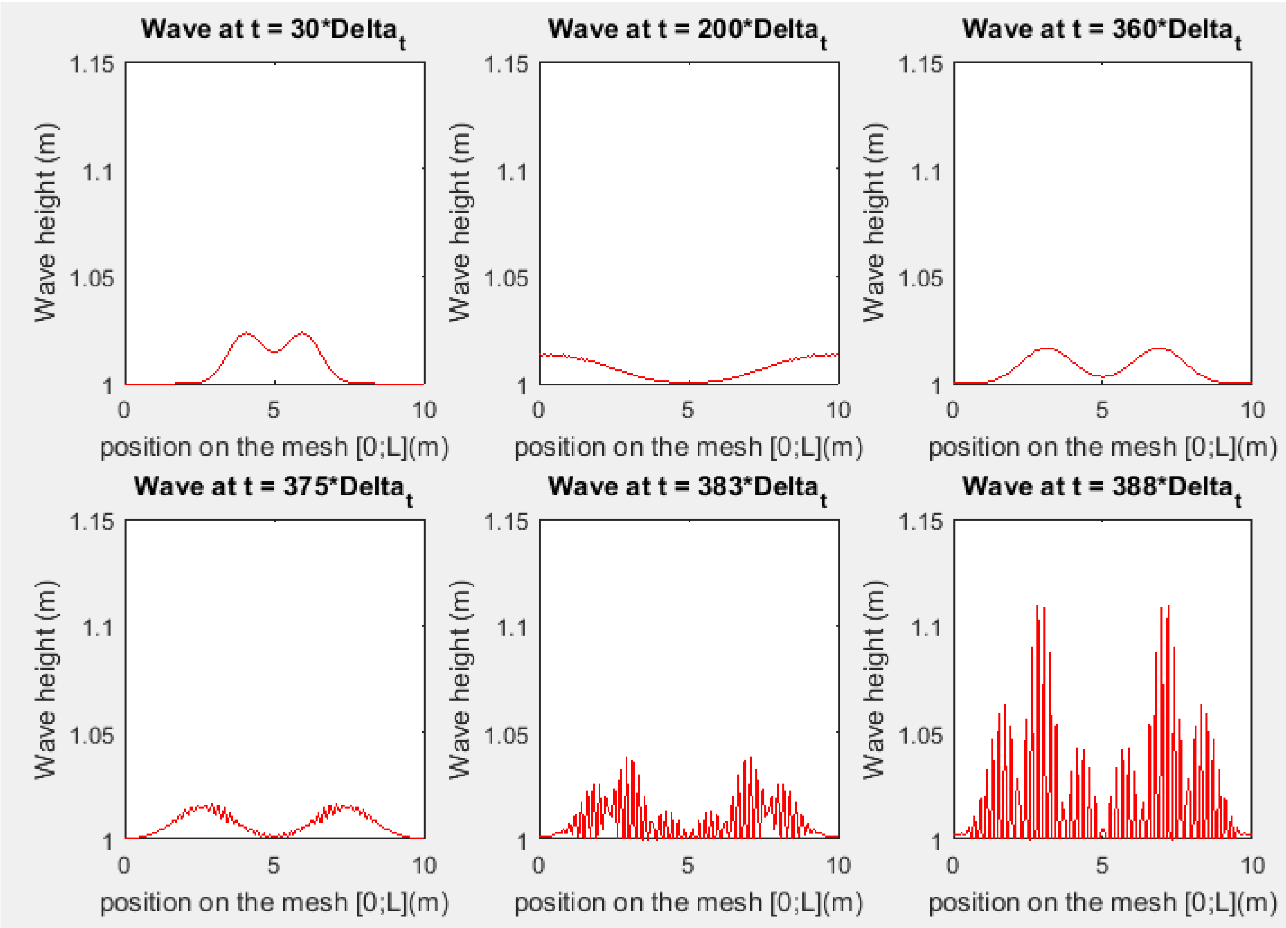
The graph is representing the height of the wave depending on alpha. We thus should have 10 curves, but we here see only a few lines. In fact, has you may have noticed, the y axis is taking tremendous values until 10^{200} m for some wave's height (3 visible). Besides, we can see that 2 of these 3 waves are at $x = 0$. This is because, in order to compute, we had to fix the matrices for the corresponding alpha.

We indeed had either close to infinite values or non defined values (NaN). We thus decided to put all the non defined values at zero to then see which alpha were presenting this irregularity. It comes out that the α superior to 10.5 have a destructive behavior. More precisely, on our graph we see that $\alpha = 11.6$ and $\alpha = 12.7$ are two lines at zero. They in fact look to overwrite the line beneath it.
To make sure of this, we will continue our analysis of this graph by zooming in it :



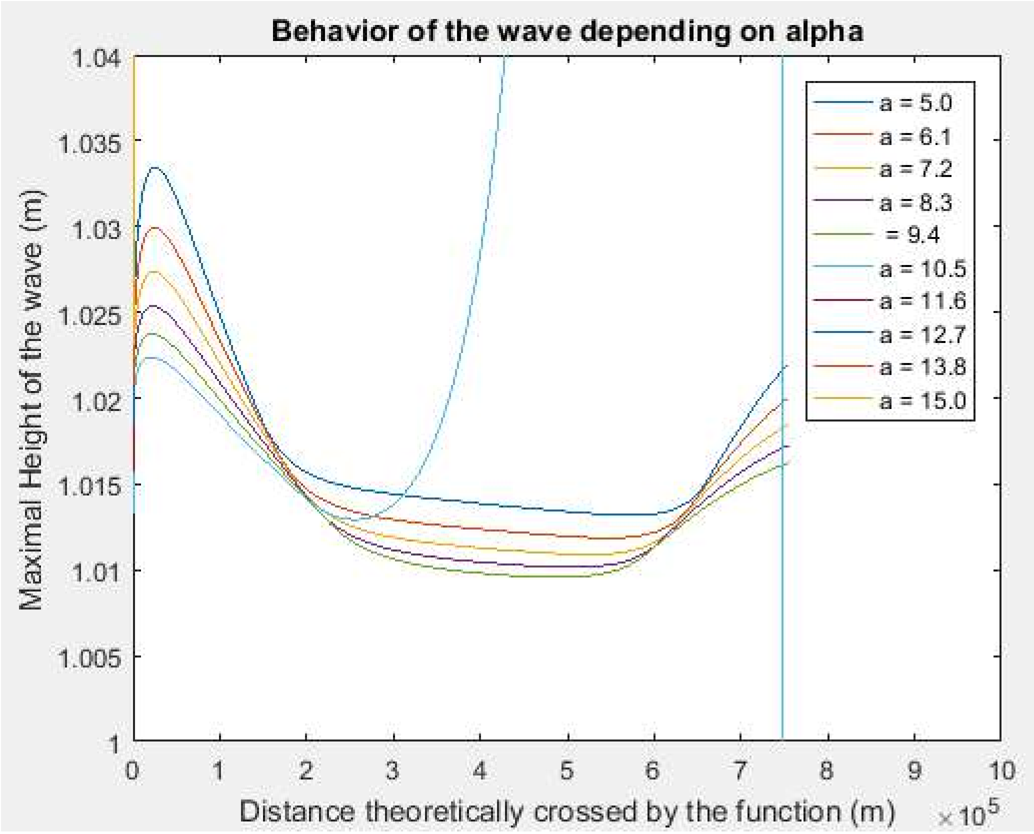
This zoomed graph show a few lines close to the bottom and an other one that

does a bump. This blue line corresponds as said at $\alpha = 10.5$ and we observe that it represents the alpha limit considering our vector α. Thus, above this value the behavior of the wave isn't numerically stable considering our ratio $\frac{\Delta_x}{\Delta_t}$.
To have a more comprehensive approach on this aspect, we show below the behavior of the limit wave in a 2D graph :



While in the first subplot the wave begin to extend toward the side and that it comes back to the middle on the third one, we see that it is where the instability appears. As the graph told us, the first part of the propagation is going as usual but we then have the apparition of an instability that in a few time step transforms into a destructive behavior, ending up with values as in the above graphs.

Finally, this first part of the analysis is just confirming what we knew about numerical analysis and more specifically α : the stability of a method is bounded to the variable defining it and with even the fewest variations come big response from the function. Anyway, we will now take a look to the apparently stable solution to conclude on the influence of alpha on the method used. We then present the following graph that is once again a zoom of the previous one :



On this last graph, we can determine that alpha has mainly an impact on the stability of the solution as it has the same shape for every alpha. Thus, event if the height increases with alpha getting lower, we observe that the shape is very similar between the different curves and that contrary to ϵ, α has no influence on the wave's behavior but rather on its height's value.

To work on an explicit scheme shows its possible negative aspect as the fact of dealing with unstable solution.

4 Linearization

To facilitate the work done on the system we are solving, we are in this part trying to approximate the non-linear equation by a linear equation ; which would aim in simplifying the resolution.

We will consider the following equality that provides more hindsight on the method that will be applied here :

$$[u(x,t)[1]; u(x,t)[2]]^T = [v(x)[1]; v(x)[2]]^T + \epsilon * [\tilde{u}(x,t)[1]; \tilde{u}(x,t)[2]]^T$$

with $[v(x)[1]; v(x)[2]]^T$ a constant state that we will denote the initial state of u(x,t) and taht we will detail on the following parts.

We will first show the pertinence of this method by looking to its consistence and what solution it gives and then we will make a numerical application.

4.1 Consistency

To take a consistent couple (h_0, v_0) here amounts to take a point close enough from initial point such that the truncation error τ is close to zero for $\Delta_t - > 0, \Delta_x - > 0$. Moreover, it would be necessary to carry this for the entire function to show consistency, but as we only want consistency for a defined couple, we can show that the truncation error is low for a point close to the initial condition.

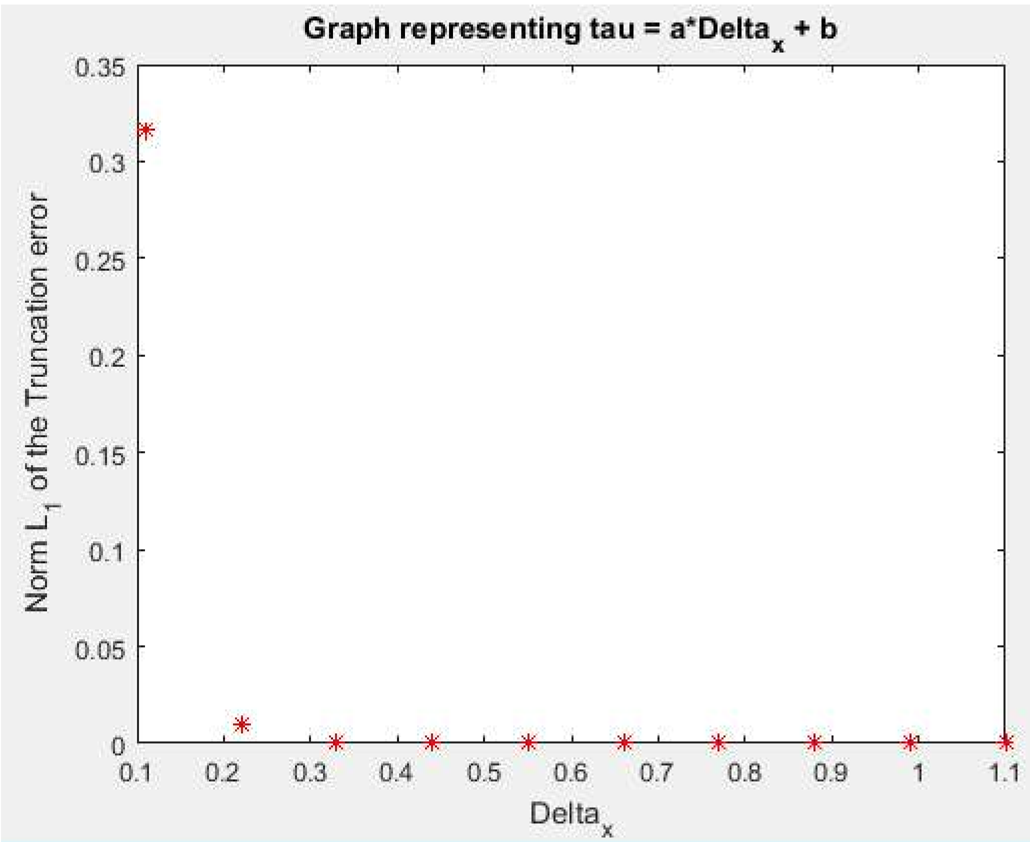
Thus, we apply the truncation error function to (h_0, v_0) :

$$\tau^n = \frac{u_j^{n+1} - u_j^n}{\Delta_t} + \frac{a}{2}\frac{f(u_{j+1}) - f(u_{j-1}) - \alpha(u_{j+1} - 2 * u_j + u_{j-1})}{\Delta_x}$$

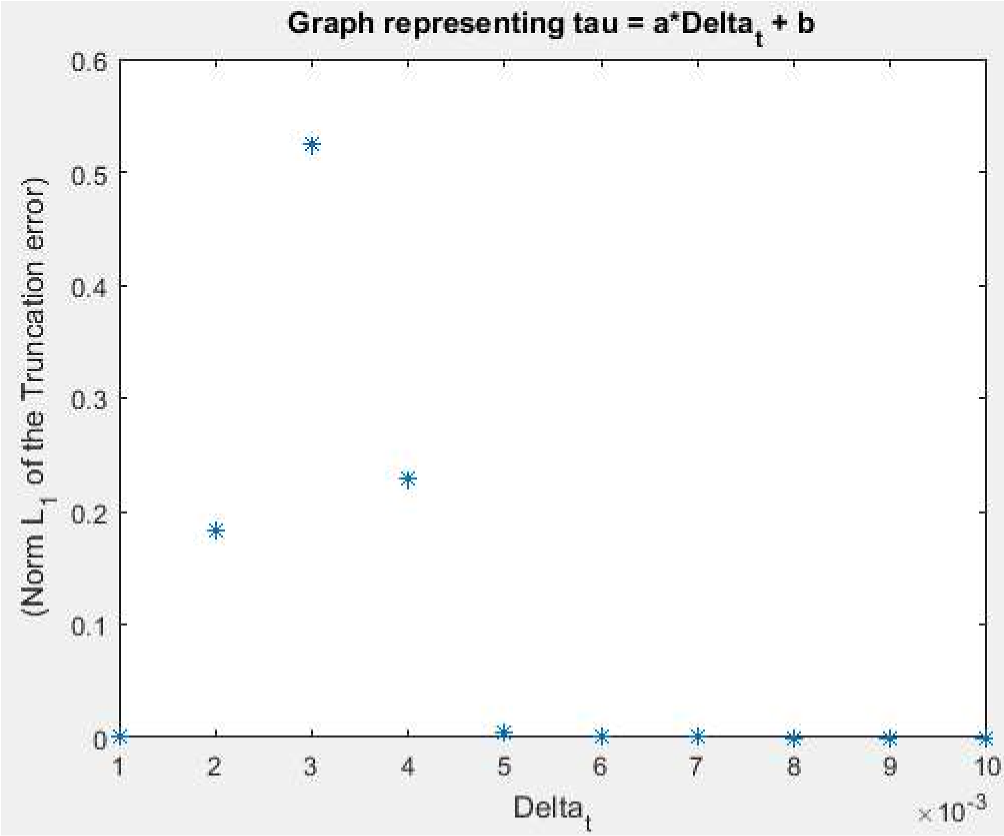
We consider $(h_0, v_0) = (u(n/2, t)[1], u(n/2, t)[2])$ as our constant point. n is the number of point along x, and so we consider the middle of the mesh as the initial point. We apply the "3 points method" and we get :

$$\tau^n = \frac{u_2^{n+1} - u_2^n}{\Delta_t} + \frac{f(u_1) - f(u_{-1}) - \alpha(u_1 - 2 * u_0 + u_{-1})}{2 * \Delta_x}$$

Thus, by plotting τ for different Δ we will check if the consistency is fulfilled. Below, we are displaying a graph showing these dependencies :



We took $\Delta_x = [0.11\ 0.22\ 0.33\ 0.44\ 0.55\ 0.66\ 0.77\ 0.88\ 0.99\ 0.1010]$.
The trending of the function is clear and it converges rapidly toward zero for $\Delta_x - > 0$



We took Δ_t = [1 2 3 4 5 6 7 8 9 10]*10^{-3}

The trending of the function is clear from the third iteration. Before that, we have τ that's increasing for Δ_t decreasing. This little bump can be associated to the fact that the Δ_t isn't the best for stability.
Anyway, here too the function then converges rapidly toward zero for $\Delta_t -> 0$.

4.2 Linear wave

Now that we have shown the consistency for the considered constant state $[h_0; v_0]$, we want to know if this new function $\tilde{u}$ indeed represents waves or not.

We get $\tilde{u}$ using the previous equality :

$[u(x,t)[1]; u(x,t)[2]]^T = [v(x)[1]; v(x)[2]]^T + \epsilon * [\tilde{u}(x,t)[1]; \tilde{u}(x,t)[2]]^T$

$$\rightarrow u = v_c + \epsilon * \tilde{u}$$

$$\rightarrow \tilde{u} = \frac{(u - v_c)}{\epsilon}$$

This implies that, because we already have u, by getting v_c we should be able to get $\tilde{u}$.

Regarding our function, we have the initial constant state considered $v_c = [v(x)[1], v(x)[2]]^T$, that is equal to $[1; 0]^T$. Indeed, we see that the initial condition on v(x)[1] is being defined as a constant term to which we add a term of small variations :

$$h(x,0) = H + \epsilon \exp\left(-((x-L/2)^2)/\omega^2\right) = v(x)[1] + \epsilon\tilde{v}(x)$$

As we get the same form than for the linearized function, we easily identify $v(x)[1] = H = 1$ and by the same way the fact that $\tilde{h} = \exp -((x-L/2)^2)/\omega^2$. This last term will appear in the initial function v(x) later on.

To finish on this point, $v(x)[2]$ denotes $u_0[2]$. Furthermore, we have $u_0[2] = h_0 * v_0 = 1 * 0$ because we have an initial condition for u[2] that's zero. Moreover it indicates that u[2] isn't gonna vary much from 0 and we then put $v(x)[2] = 0$.

Now, we want to make sure that this linear solution is indeed representing an hyperbolic function displaying waves. To do so, we reconsider our initial equation :

$$[u(x,0)[1], u(x,0)[2]]^T = [v(x)[1], v(x)[2]^T + \epsilon * [\tilde{u}(x)[1], \tilde{u}(x)[2]]^T$$

To simplify the following mathematical manipulation we will only consider u(x,0)[1] and re-write the above equation :

$$u(x,t) = h_0 + \epsilon * \tilde{u}(x)$$

Because we are approximating a non-linear equation by a linear one, we are willing to see if they indeed solve the same problem. We then implement the linearized equation into the differential equation $\partial_t u + \partial_x f(u) = 0$:

$$\partial_t(h_0 + \epsilon * \tilde{u}(x)) + \partial_x f(h_0 + \epsilon * \tilde{u}(x)) = 0$$
$$\rightarrow \quad \epsilon * \partial_t \tilde{u}(x) + \epsilon * \partial_x \tilde{u}(x) * f'(h_0 + \epsilon * \tilde{u}(x)) = 0$$

We then, by applying Taylor expression, get :

$$f'(h_0 + \epsilon * \tilde{u}(x)) = f'(h_0) + \epsilon * \tilde{u}(x) * f''(h_0) + 0(\epsilon^2)\,;$$
$$\rightarrow \quad \partial_t \tilde{u}(x) + \partial_x \tilde{u}(x) * (f'(h_0) + \epsilon * \tilde{u}(x) * f''(h_0)) = 0$$

As we take ϵ low enough to apply the linearization, we have $\epsilon << 1$ and thus :

$$\partial_t \tilde{u}(x) + f'(h_0) * \partial_x \tilde{u}(x) = 0$$

We note that the same relation is fulfilled for the other component of u(x,t).

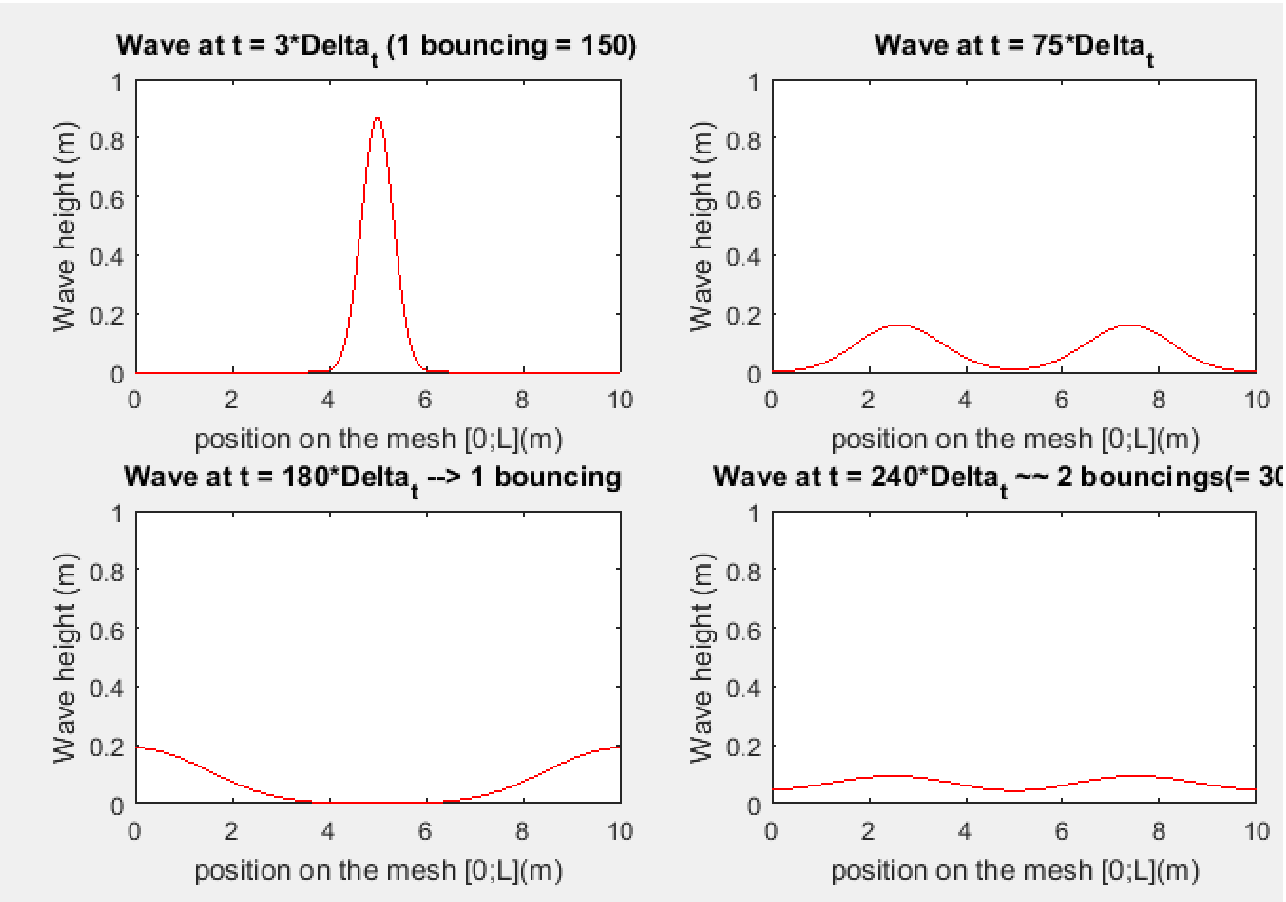
We eventually got the equation for the new linearized function that verifies the equation of a wave. More specifically, if we consider once the derivative with respect to x and then the one with respect to t, and by applying the Schwartz theorem, we finally get :

$$\partial_{tt}\tilde{u}(x) + f'(h_0) * \partial_{xt}\tilde{u}(x) = 0$$
$$\partial_{tx}\tilde{u}(x) + f'(h_0) * \partial_{xx}\tilde{u}(x) = 0$$

$$\rightarrow \quad \partial_{tt}\tilde{u}(x) = (f'(h_0))^2 * \partial_{xx}\tilde{u}(x)$$

Which is indeed an hyperbolic equation, as $\delta(L)(x,y) = b^2 - 4ac = 4*(f'(h_0))^2 > 0$, that compute wave speed, with a speed of $f'(h_0)$.

To conclude on the relevance of the method in term of its capability to plot waves, we displayed this function for different step time :



It is clear, with respect to these graphs or even the drawing on Matlab, that our linearized function is indeed representing waves, for which the amplitude is as expected lower than for u.

Now that we showed consistency and that the function is relevant regarding the problem, we will see how well the linearized function fit the non-linear one.

4.3 Eigenvalues

The idea from now on will be to check numerically the theory developed here. To do so, we are mainly gonna see if this linear function does or not fit with the non-linear one.

In a first time, we are to do an analytical reflection on the linearized equation :

— We know u ;

— We have been defining v_c ;

— We are solving a problem for which we have a matrix $f'(v_c)$ instead of a single value a.

— We rather do the all mathematical calculations for the global case, and thus we will consider v(x)[2] rather than 0 during the calculations.

These points lead us to the following equation in matrix form :

$$\partial_t \begin{bmatrix} \tilde{u}(x,t)[1] \\ \tilde{u}(x,t)[2] \end{bmatrix} + \begin{bmatrix} \frac{\partial f(x)[1]}{\partial_t v_c[1]} & \frac{\partial f(x)[1]}{\partial_t v_c[2]} \\ \frac{\partial f(x)[2]}{\partial_t v_c[1]} & \frac{\partial f(x)[2]}{\partial_t v_c[2]} \end{bmatrix} * \partial_x \begin{bmatrix} \tilde{u}(x,t)[1] \\ \partial_t \tilde{u}(x,t)[2] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Considering the terms of f defined previously :

$$f(u) = \begin{pmatrix} u2 \\ (u2^2)/u1 + \frac{1}{2} g u1^2 \end{pmatrix}$$

we have :

$$f'(v_c) = \begin{pmatrix} f'(v(x)[1]) \\ f'(v(x)[2]) \end{pmatrix}$$

$$\rightarrow f'(v_c) = \begin{bmatrix} 0 & 1 \\ 2gv(x)[1] - (\frac{v(x)[2]}{v(x)[1])})^2 & 2\frac{v(x)[2]}{v(x)[1])} \end{bmatrix}$$

We now search the eigenvalues from this matrix :

$$det(f'(v_c) - \lambda * I) = 0 \rightarrow \lambda^2 - 2\lambda \frac{v(x)[2]}{v(x)[1])} - 2gv(x)[1] + (\frac{v(x)[2]}{v(x)[1])})^2 = 0$$

Which leads to $\delta(x,y) = b^2 - 4ac = 8gv(x)[1] > 0$

We finally get the two following eigenvalues :

$$\lambda_1 = \frac{2\frac{v(x)[2]}{v(x)[1])} - \sqrt{8gv(x)[1]}}{2} \text{ and } \lambda_2 = \frac{2\frac{v(x)[2]}{v(x)[1])} + \sqrt{8gv(x)[1]}}{2}$$

The associated eigenvectors are then :

$$\nu_1 = \begin{bmatrix} 1 \\ \frac{v(x)[2]}{v(x)[1])} - \sqrt{2gv(x)[1]} \end{bmatrix} \text{ and } \nu_2 = \begin{bmatrix} 1 \\ \frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]} \end{bmatrix}$$

Let's make the point :

- We got the eigenvalues of $f'(v_c)$ which will allow us to diagonalize the matrix;
- We got the eigenvectors which will help us get an expression for the function z(x,t).

That far, we can give the three following matrices that will be used throughout these calculations :

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} ; R = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \text{ and } R^{-1} = \frac{1}{\sqrt{8gv(x)[1]}} * \begin{bmatrix} \frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]} & -1 \\ -\frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]} & 1 \end{bmatrix}$$

We now re-write our equation. As u = Rz, we have :

$$\partial_t z + \Lambda \partial_x z = 0$$

Yet, we have $\tilde{u}(x,0) = \tilde{v}(x) = \begin{pmatrix} \tilde{h} \\ 0 \end{pmatrix}$ from the expression of h(x,t) which we introduced previously. Thus $z(x,0) = R^{-1}\tilde{v}(x)$:

$$z(x,0) = \frac{1}{\sqrt{8gv(x)[1]}} \begin{pmatrix} \frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]} \\ -\frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]} \end{pmatrix}$$

Finally, we can consider the following characteristic :

$$\gamma_1 = x + \lambda_1 * t \text{ and } \gamma_2 = x + \lambda_2 * t$$

and then the explicit expression of z(x,t) :

$$z(x,t) = \begin{pmatrix} z(x + \lambda_1 * t, 0) \\ z(x + \lambda_2 * t, 0) \end{pmatrix} \rightarrow \begin{pmatrix} (\frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]})\tilde{h}(x + \lambda_1 * t) \\ (-\frac{v(x)[2]}{v(x)[1])} + \sqrt{2gv(x)[1]})\tilde{h}(x + \lambda_2 * t) \end{pmatrix}$$

As u = Rz, we get the analytical solution $\tilde{u}$ of the linearized equation :

$$u(x,t) = \\ \frac{1}{\sqrt{8gv(x)[1]}} \begin{pmatrix} (\frac{v(x)[2]}{v(x)[1]}(\tilde{h}(x+\lambda_1 t)) - \tilde{h}(x+\lambda_2 t))) + \sqrt{2gv(x)[1]}(\tilde{h}(x+\lambda_1 t)) + \tilde{h}(x+\lambda_2 t))) \\ (\frac{v(x)[2]}{v(x)[1]})^2(\tilde{h}(x+\lambda_1 t) - \tilde{h}(x+\lambda_2 t)) + 2gv(x)[1](\tilde{h}(x+\lambda_2 t) - \tilde{h}(x+\lambda_1 t)) \end{pmatrix}$$

We recall $u(x,t)[1] = 1$, $u(x,t)[2] = 0$ and $\tilde{h} = \exp(-((x - L/2)^2)/\omega^2)$ which gives a re-expression of u(x,t) :

$$u(x,t) = \frac{1}{\sqrt{8g}} \begin{pmatrix} \sqrt{2g} * (\tilde{h}(x+\lambda_1 t) + \tilde{h}(x+\lambda_2 t)) \\ (2g) * (\tilde{h}(x+\lambda_2 t) - \tilde{h}(x+\lambda_1 t)) \end{pmatrix}$$

Which can be decomposed as :

$u(x,t)[1] = \frac{\tilde{h}(x+\lambda_1 t)+\tilde{h}(x+\lambda_2 t)}{2}$ and $u(x,t)[2] = -\sqrt{\frac{g}{2}} * (\tilde{h}(x+\lambda_2 t) - \tilde{h}(x+\lambda_1 t))$

We give at last the following values for the eigenvalues, etc :

— g = 9.61 ;

— t = 1 ;

— x = 5m, as L = 10m ;

— $\lambda_1 = -\sqrt{2g}$;

— $\lambda_2 = +\sqrt{2g}$;

— $\tilde{h}(x+\lambda_1 t) = exp^{(-(5-\sqrt{2g}*1-10/2)^2/0.16)} \approx exp^{-52} = 0$

— $\tilde{h}(x+\lambda_2 t) = exp^{(-(2+\sqrt{2g}*1-10/2)^2/0.16)} \approx exp^{-52} \approx 0$

The results obtains are the following one :

— u(2,1)[1] = 0 ;

— u(2,1)[2] = 0 ;

As we took x = 5m, we have the terms $\tilde{h}(x+\lambda_1 t)$ and $\tilde{h}(x+\lambda_2 t)$ that are equal and equal to approximately 0.

It looks like the analytical solution is either not well posed, and so x = 5m isn't an appropriate point, or simply source of an error in the calculations.

When considering the non-linear result, we should have got :

— u(2,1)[1] = 1 ;

— u(2,1)[2] = 0 ;

5 Non-reflecting BC

Lets consider an other type of condition at the edges : a non reflection condition.

5.1

The boundary conditions that do not cause reflections for the linear problem are defined using approximation by extrapolation of ghost points.

This results numerically by the following conditions for the non-linear case :

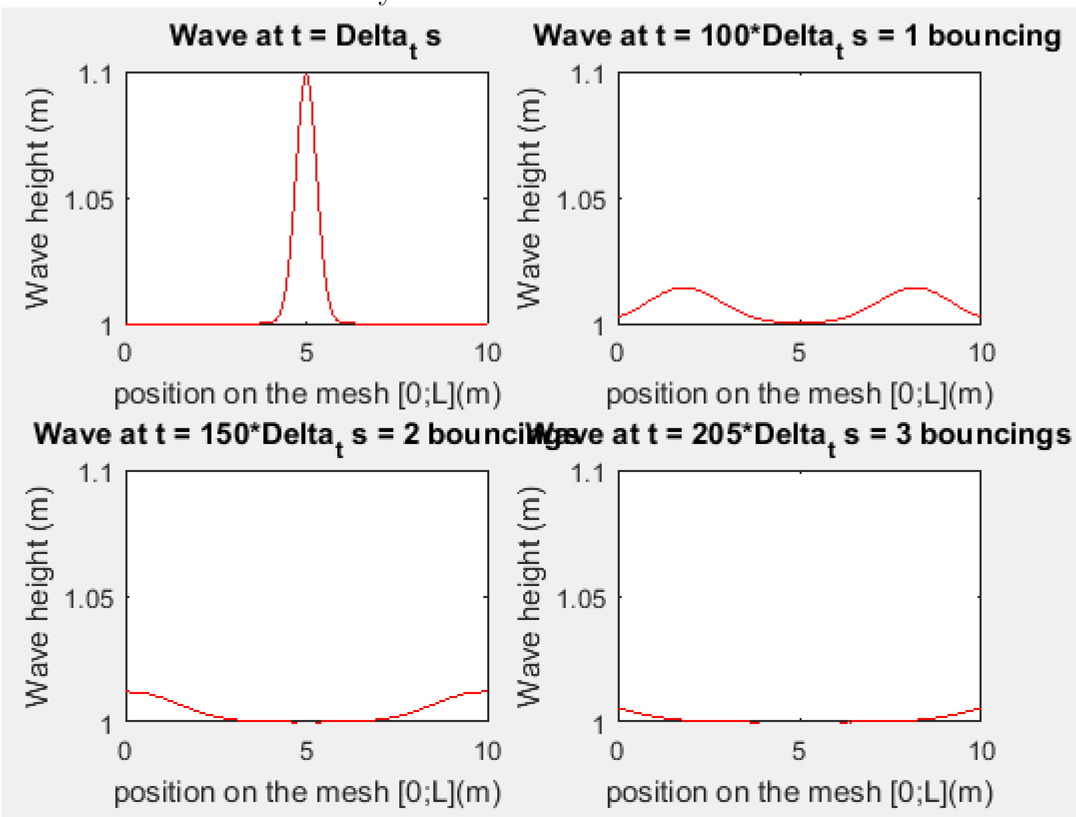
$\frac{Q_0 - Q_{-1}}{2} = 0$ which leads to :

$Q_0 = Q_{-1}$ for the 0^{th} order. In a description in same norm than the one given in subject, we get : $(u_0^m[1], u_0^m[2])^T = (u_1^m[1], u_1^m[2])^T$ and $(u_{N+1}^m[1], u_{N+1}^m[2])^T = (u_N^m[1], u_N^m[2])^T$

We show below the repercussion that were well foreseen.

5.2

As mentioned, we make the implementation considering a direct extrapolation of the variables at the boundary :



5.3

For what is up to the measurement of how well the condition work we have been measuring the flow in the tank, as well as the one living it. Then, the difference between the two of them represents losses due to the reliability of the method.
The results gathered are in the following table :

Quantities / Flux variables	Initial volume	Outgoing flux
	0.7019	0.2416
	100 percent	34 percent

The reflected part amounts to 0.46. We thus have $0.46/0.7 \approx 66$ percent of the flux that isn't going through the boundary.

To conclude on this part, she allowed us to make comparison between theory and numerical analysis regarding the reliability of hyperbolic equations. We have then been able to build several functions and code calculating, measuring and displaying these bounds between linear and non-linear equations.

6 Conclusion

To conclude on this report, we have been taking a look at hyperbolic equation through the example of the waves equation. We then have been studying it considering a linear approach, a non linear approach, by an analytical approach as well as a numerical one.

It comes out that this type of equation is going with strong dependence on its variables, namely that the variables have to be well-posed in order to make sure of stability, consistency and convergence with it.