Introduction to PDEs

Classification of PDEs

Diffusion of ink in a sponge

Diffusion process

Example: distribution of ink in a sponge.

Diffusion process

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Let

- $\Omega \subset \mathbb{R}^d$ (for $d = 2, 3$):
  domain occupied by sponge (bounded and connected);
- u: concentration of ink in sponge; $0 \le u \le 1$;
  (where $u(x) = 0 \simeq 0\%$ ink at position x; $u(x) = 1 \simeq 100\%$ ink at position x).

Diffusion process

Goal: Compute spatial and temporal evolution of ink, i.e.

$$\text{find } u(x,t) \text{ with } \qquad u : \Omega \times \mathbb{R}^+ \to [0,1].$$

Question: How do we derive an equation?
Exploit a fundamental principle of continuum mechanics:

Definition (The Conservation Principle)

(i) *Physical principle:* change of an (extensive state) quantity *(e.g. mass, momentum or energy)* in any volume V results from transport of the quantity over the boundary of the volume.

(ii) *Mathematical equivalent:* Let $u(x,t)$ denotes the density distribution of an extensive state quantity. Then, for an arbitrary test volume $V \subset \Omega$ it holds

$$\frac{d}{dt}\int_V u(x,t)\, dx = -\int_{\partial V} \mathbf{q}(x,t)\cdot \mathbf{n}(x)\, d\sigma(x),$$

where $\mathbf{n}$: unit outer normal on ∂V; $\quad \mathbf{q}$: flux density of quantity.

Diffusion process

We have

$$\frac{d}{dt}\int_V u(x,t)\,dx = -\int_{\partial V} \mathbf{q}(x,t)\cdot\mathbf{n}(x)\,\mathrm{d}\sigma(x).$$

To derive mathematical model for diffusion of ink, we require physical relationship between flux density $\mathbf{q}$ and concentration u.

- We can assume: $\mathbf{q} \sim -\nabla u$, $\quad \nabla := (\partial_{x_1},\dots,\partial_{x_d})$ is *gradient* w.r.t. x! flux is proportional to change of concentration.
- We obtain the law

$$\mathbf{q}(x,t) = -\mathbf{a}\nabla u(x,t),$$

where $\mathbf{a} > 0$ is proportionality factor, called *diffusion coefficient*.

- The larger $\mathbf{a}$ the better the ink is transported.

Diffusion process

Inserting $\mathbf{q}(x,t) = -\mathbf{a}\nabla u(x,t)$ into the conservation law yields

$$\frac{d}{dt}\int_V u(x,t)\ dx = \int_{\partial V} \mathbf{a}\nabla u(x,t)\cdot \mathbf{n}(x)\ d\sigma(x).$$

Assuming smoothness of concentration $u(x,t)$ and coefficient $\mathbf{a}$, we can apply Gauss's theorem (divergence theorem) to obtain

$$\frac{d}{dt}\int_V u(x,t)\ dx = \int_{\partial V} \mathbf{a}\nabla u(x,t)\cdot \mathbf{n}(x)\, d\sigma(x)$$

$$\overset{\text{Gauss's theorem}}{=} \int_V \nabla\cdot(\mathbf{a}\nabla u(x,t))\ dx,$$

Permuting differentiation and integration, we get

$$\int_V \partial_t u(x,t)\ dx = \int_V \nabla\cdot(\mathbf{a}\nabla u(x,t))\ dx.$$

Since this holds for arbitrary test volumes $V\subset\Omega$ we conclude that pointwise

$$\partial_t u(x,t) = \nabla\cdot(\mathbf{a}\nabla u(x,t)), \qquad \text{for all } (x,t)\in\Omega\times\mathbb{R}^+.$$

Diffusion process

Equation:

$$\partial_t u(x,t) = \nabla \cdot (\mathbf{a} \nabla u(x,t)), \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}^+.$$

If $\Omega \subset \mathbb{R}$, it reduces to

$$-\partial_t u(x,t) + \mathbf{a} \partial_{xx} u(x,t) = 0.$$

Recalling that

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + g u + f = 0$$

is classified as follows

Elliptic $b^2 - 4ac < 0$

Hyperbolic $b^2 - 4ac > 0$

Parabolic $b^2 - 4ac = 0$

We have $a = \mathbf{a} > 0$ and $b = c = 0$. Hence: the equation is parabolic.

Diffusion process

We obtained

$$\partial_t u(x,t) = \nabla \cdot (\mathbf{a} \nabla u(x,t)), \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}^+.$$

- In considered setting: equation is called time-dependent diffusion equation.
- It is a parabolic equation.
- Well-posedness requires initial value condition and boundary condition.
- Initial condition is of the type

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega,$$

where u_0 describes (known) initial distribution of ink at time $t = 0$.

- For the boundary condition, there are several possibilities.
Example: Neumann boundary condition - prescribe flux over boundary of Ω.
Recalling flux $\mathbf{q} = -\mathbf{a}\nabla u$, a **Neumann boundary condition** is given by

$$\mathbf{a}\nabla u(x,t) \cdot \mathbf{n}(x) = g(x,t) \qquad \text{for } x \in \partial\Omega \qquad \text{and } t \in \mathbb{R}^+,$$

where $g(x,t)$ describes flux of ink in normal direction over the boundary of sponge at $x \in \partial\Omega$ and time t (i.e. $g(x,t)$ describes externally inserted ink).

Diffusion process

Find u with $u(x,0) = u_0(x)$ and $\mathbf{a}\nabla u(x,t) \cdot \mathbf{n}(x) = g(x,t)$ for $(x,t) \in \partial\Omega \times \mathbb{R}^+$

$$\text{and} \qquad \partial_t u(x,t) = \nabla \cdot (\mathbf{a}\nabla u(x,t)), \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R}^+.$$

Question: How relates the parabolic problem to an elliptic problem?

Answer: Under suitable assumptions on data (i.e. $\mathbf{a}$, $u_0(x)$, $g(x,t)$ and Ω), it can be shown that there exists a stationary state $\bar{u}$ such that for all $t \geq 0$

$$\|\bar{u} - u(\cdot,t)\|_{L^2(\Omega)} \leq Ce^{-ct} \overset{t\to\infty}{\longrightarrow} 0$$

(i.e. exponential convergence to stationary state). For $\bar{g}(x) := \lim_{t\to\infty} g(x,t)$, the stationary state $\bar{u} : \Omega \to [0,1]$ is characterized by an elliptic equation:

$$\begin{aligned} \nabla \cdot (\mathbf{a}\nabla \bar{u}(x)) &= 0 && \text{for } x \in \Omega, \\ \mathbf{a}\nabla \bar{u}(x) \cdot \mathbf{n}(x) &= \bar{g}(x) && \text{for } x \in \partial\Omega. \end{aligned}$$

Introduction to PDE's

Recall: Linear, scalar second order PDEs of the type

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + g u + f = 0$$

are classified according to

Elliptic $b^2 - 4ac < 0$

Hyperbolic $b^2 - 4ac > 0$

Parabolic $b^2 - 4ac = 0$

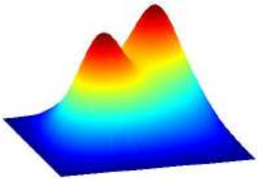
Elliptic equations

Model equation: Laplace/Poisson (1780's)

$$-\triangle u(\mathbf{x}) = f(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \cdots, x_d)$ and

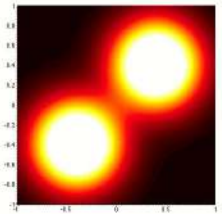
$$\triangle u(\mathbf{x}) := \partial_{x_1 x_1} u(\mathbf{x}) + \cdots + \partial_{x_d x_d} u(\mathbf{x}).$$



Solution given by boundary values and $f(x, y)$

- Properties: stationary sate; describes equilibrium.
- Numerics: solve for all point values simultaneously
  $\Rightarrow$ Can be memory demanding.
- Physics:
  Diffusion processes, electric potentials, structural mechanics, ...

Parabolic equations



Model equation: Heat equation (1800)

$$\partial_t u(\mathbf{x}) - \mathbf{a}\triangle u(\mathbf{x}) = f(\mathbf{x}),$$

- **Properties:** time dependent; convergence to stationary solution; smoothing (rough initial values smoothed out).
- **Numerics:** time stepping possible
  large diffusion constant $\Rightarrow$ small time steps.
- **Physics:**
  heat conduction, instationary diffusion, ...

Hyperbolic equations

Model equation: (Acoustic) wave equation (d'Alember 1740)

$$\partial_{tt} u(\mathbf{x}) - c^2 \triangle u(\mathbf{x}) = f(\mathbf{x}),$$

- Properties: time dependent; transport; wave propagation; no convergence to stationary solution;
- Numerics: time stepping possible;
  energy and mass conservation are important.
- Physics:
  acoustic waves, elastic waves, ...

Hyperbolic equations

Other wave equations:

- 2'nd order wave equation:

$$\partial_{tt} u - c^2 \triangle u = 0.$$

- Linear advection/wave equation:

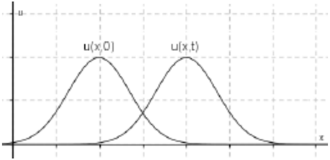
$$\partial_t u + \mathbf{a} \partial_x u = 0.$$

- Burger's equation, non-linear advection:

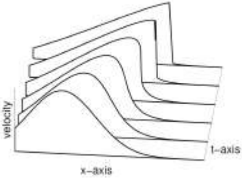
$$\partial_t u + u \, \partial_x u = 0.$$

Hyperbolic equations

- Linear advection: $\partial_t u + \mathbf{a}\partial_x u = 0$ (Sol. $u(x,t) = u_0(x - \mathbf{a}t)$).
  initial profile propagates with (constant) speed:



- Non-linear advection: $\partial_t u + u\,\partial_x u = 0$.
  smooth initial profile can develop discontinuity:



Hyperbolic equations

- Numerics: Have to consider direction of advection; special methods for discontinuous solutions.
- Physics:
  electromagnetics, acoustics, compressible flow, elastic and water waves, . . .

Mixed type

Model equation: Advection-diffusion equation, $\mathbf{a} \ll c$

$$\partial_t u - \mathbf{a}\partial_{xx}u + c\partial_x u = 0,$$

Properties: advection + diffusion.