# Numerical solutions of differential equations

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**Course SF2521**, 7.5 ECTS, VT18

# Lecture 6

# Hyperbolic Equations of first order - Part 3

# Boundary conditions for hyperbolic equations

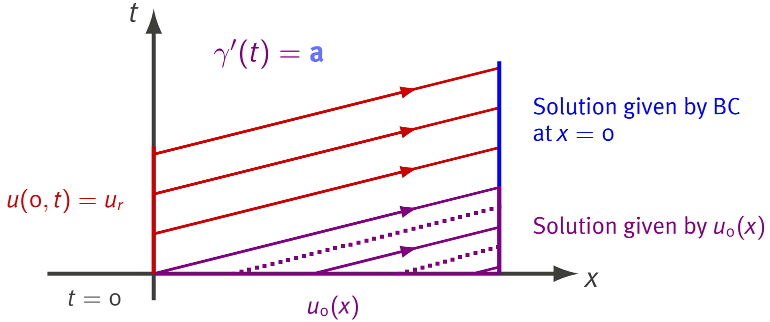
# Boundary conditions for hyperbolic equations

- Hyperbolic problems are posed on whole $\mathbb{R}$ (no boundary conditions) and are typically well-posed.
- Hyperbolic problems on an interval $[a, b]$ are only well-posed for suitable boundary conditions.
- Where and how to pose boundary conditions depends on the characteristics of the problem.

# Boundary conditions for hyperbolic equations

**Example:** Let $a > 0$. We seek $u = u(x,t) : [0,1] \times [0,\infty) \to \mathbb{R}$

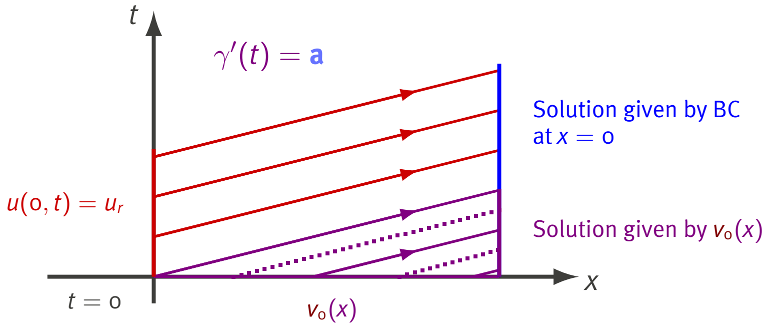
$$\partial_t u(x,t) + a\partial_x u(x,t) = 0 \quad \text{and} \quad u(x,0) = v_0(x) \quad \text{for } 0 \le x \le 1.$$




At $x = 0$: Characteristics going into domain $\Rightarrow$ impose BC.

At $x = 1$: Characteristics going out of domain $\Rightarrow$ no BC.

# Boundary conditions for hyperbolic equations



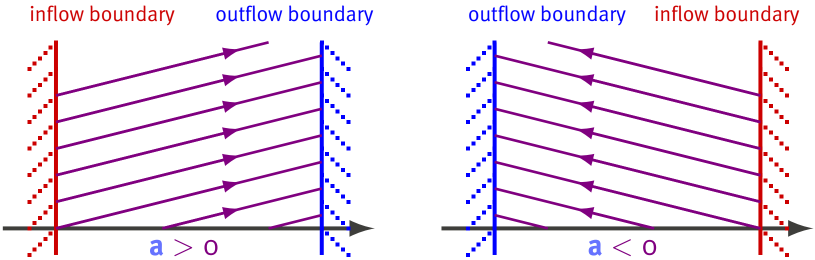

Naturally, values at $x = 1$ are given by

$$u(1,t) = \begin{cases} v_0(1 - at) & \text{for } t \leq \frac{1}{a} \\ u_r & \text{for } t > \frac{1}{a}. \end{cases}$$

$\Rightarrow$ well-posed problem.

# Boundary conditions for hyperbolic equations

## General rule



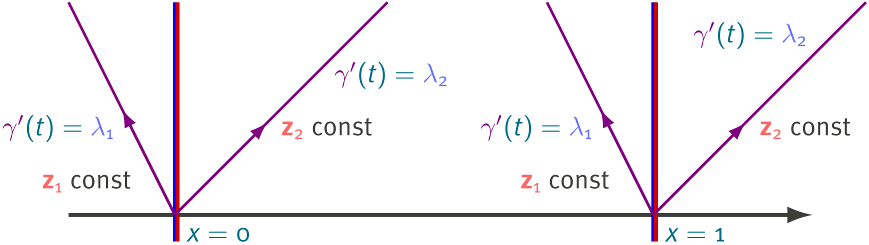

- Boundary conditions **must** be posed at inflow boundaries (ingoing characteristics)
- Boundary conditions **cannot** be posed at outflow boundaries (outgoing characteristics)

# Systems of hyperbolic equations

Boundary conditions for systems:

- same rule applies BUT to the characteristic variables
- more than one characteristic at each boundary

Example: $\lambda_1 < 0$ and $\lambda_2 > 0$




At
- $x = 0$ boundary conditions on $\mathbf{z}_2$, no conditions on $\mathbf{z}_1$.
- $x = 1$ boundary conditions on $\mathbf{z}_1$, no conditions on $\mathbf{z}_2$.

# Systems of hyperbolic equations

Given a problem with BC in terms of “physical” variables must make sure that the problem is well-posed.

**Example.** $\mathbf{u}_2$ could be a flow velocity. We consider:

$$\partial_t \mathbf{u}(x,t) + \mathbf{A}\partial_x \mathbf{u}(x,t) = 0, \qquad \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \qquad x \in [0,1]; \quad t \geq 0;$$

$$\mathbf{u}(x,0) = \mathbf{v}_0(x) \qquad \text{(initial condition)}$$

$$\mathbf{u}_2(0,t) = \mathbf{u}_2(1,t) = 0 \qquad \text{(boundary condition: “solid walls”).}$$

Well-posed?

Assume the eigenvalues of $\mathbf{A}$ fulfill $\lambda_1 < 0$ and $\lambda_2 > 0$. As before with $\mathbf{u} = \mathbf{R}\mathbf{z}$:

$$\left.\begin{array}{ll} x = 0: & \text{BC for } \mathbf{z}_2 \\ x = 1: & \text{BC for } \mathbf{z}_1 \end{array}\right\} \quad \Rightarrow \quad \text{well-posed problem.}$$

Does this match with $\mathbf{u}_2(0,t) = \mathbf{u}_2(1,t) = 0$?

# Systems of hyperbolic equations

Relation between $\mathbf{u}$ and $\mathbf{z}$ on the boundary:

At $x = 0$ we have $\mathbf{u}_2(0,t) = 0$ (fixed) and $\mathbf{u}_1(0,t) = *$ (free, no BC posed):

$$\begin{pmatrix} * \\ 0 \end{pmatrix} = \mathbf{u}(0,t) = \mathbf{R}\mathbf{z}(0,t) = \begin{pmatrix} \mathbf{R}_{11}\mathbf{z}_1 + \mathbf{R}_{12}\mathbf{z}_2 \\ \mathbf{R}_{21}\mathbf{z}_1 + \mathbf{R}_{22}\mathbf{z}_2 \end{pmatrix} \overset{\text{second relation}}{\Rightarrow} \mathbf{z}_2(0,t) = -\frac{\mathbf{R}_{21}}{\mathbf{R}_{22}}\mathbf{z}_1(0,t).$$

At $x = 1$ we have $\mathbf{u}_2(1,t) = 0$ (fixed) and $\mathbf{u}_1(1,t) = *$ (free, no BC posed):

$$\begin{pmatrix} * \\ 0 \end{pmatrix} = \mathbf{u}(1,t) = \mathbf{R}\mathbf{z}(1,t) = \begin{pmatrix} \mathbf{R}_{11}\mathbf{z}_1 + \mathbf{R}_{12}\mathbf{z}_2 \\ \mathbf{R}_{21}\mathbf{z}_1 + \mathbf{R}_{22}\mathbf{z}_2 \end{pmatrix} \overset{\text{second relation}}{\Rightarrow} \mathbf{z}_1(1,t) = -\frac{\mathbf{R}_{22}}{\mathbf{R}_{21}}\mathbf{z}_2(1,t).$$

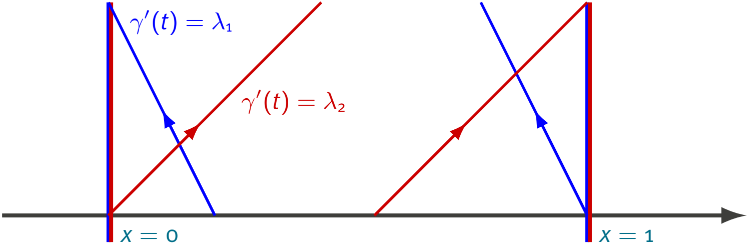
Hence, the setting $\mathbf{u}_2(0,t) = \mathbf{u}_2(1,t) = 0$ corresponds to

$$\left.\begin{aligned} \mathbf{z}_2(0,t) &= -\tfrac{\mathbf{R}_{21}}{\mathbf{R}_{22}}\mathbf{z}_1(0,t) \\ \mathbf{z}_1(1,t) &= -\tfrac{\mathbf{R}_{22}}{\mathbf{R}_{21}}\mathbf{z}_2(1,t) \end{aligned}\right\} \quad \text{Reflective boundary conditions}$$

Why is it called a reflective boundary condition?

# Systems of hyperbolic equations

We have $\mathbf{z}_2(0,t) = -\frac{\mathbf{R}_{21}}{\mathbf{R}_{22}}\mathbf{z}_1(0,t)$ and $\mathbf{z}_1(1,t) = -\frac{\mathbf{R}_{22}}{\mathbf{R}_{21}}\mathbf{z}_2(1,t)$. Recall




- In $(0,t)$ the information "$\mathbf{z}_1(0,t)$" came from the interior.
- In $(1,t)$ the information "$\mathbf{z}_2(1,t)$" came from the interior.

Hence both are known information.

- Value of $\mathbf{z}_1$ is reflected back in $\mathbf{z}_2$ at $x = 0$.
- Value of $\mathbf{z}_2$ is reflected back in $\mathbf{z}_1$ at $x = 1$.

We conclude $\mathbf{u}_2(0,t) = \mathbf{u}_2(1,t) = 0$ is well-posed.

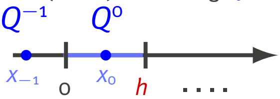
# Systems of hyperbolic equations

How do we implement the boundary conditions?

For $j = 1, \cdots, N-1$ and $n \in \mathbb{N}$ let

$$\mathbf{Q}^{j,n} = (\mathbf{Q}_1^{j,n}, \mathbf{Q}_2^{j,n}) \approx (\mathbf{u}_1(x_j, t^n), \mathbf{u}_2(x_j, t^n)).$$

At $x = 0$: use a ghost cell (for $\mathbf{u}_2$). Denoting $Q^j := Q_2^{j,n}$:




- Boundary condition for $\mathbf{u}_2$ in $x = 0$:

$$\mathbf{u}_2(0,t) = 0 \quad \Rightarrow \quad \frac{Q_2^{-1,n} + Q_2^{0,n}}{2} = 0 \quad \Rightarrow \quad Q_2^{-1,n} = -Q_2^{0,n}, \qquad n \geq 0.$$

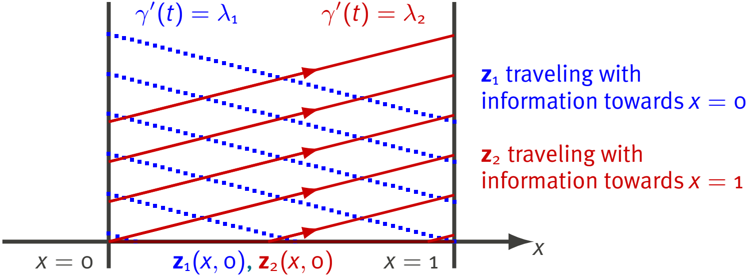
- No boundary condition for $\mathbf{u}_1$ in $x = 0$:

  Extrapolate to have conditions for $Q_1^{-1,n}$. E.g.

$$Q_1^{-1,n} = Q_1^{0,n} \qquad \text{for all } n \geq 0.$$

# Non-reflecting boundary conditions

What boundary condition should we impose in order to avoid reflections at the boundary?




We do not want information to be reflected back into the domain. Recall

$$\mathbf{u}(0,t) = \begin{pmatrix} \mathbf{R}_{11}\mathbf{z}_1(0,t) + \mathbf{R}_{12}\mathbf{z}_2(0,t) \\ \mathbf{R}_{21}\mathbf{z}_1(0,t) + \mathbf{R}_{22}\mathbf{z}_2(0,t) \end{pmatrix} \overset{\text{if } \mathbf{z}_2(0,t)=0}{\Longrightarrow} \mathbf{u}(0,t) = \begin{pmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{pmatrix} \mathbf{z}_1(0,t)$$

"$\mathbf{z}_2(0,t) = 0$" is admissible choice, because: allowed to prescribe BC for $\mathbf{z}_2$ in $(0,t)$. On the other hand, $\mathbf{z}_1(0,t)$ is information coming from the interior.

# Non-reflecting boundary conditions

We have

$$\mathbf{u}(0,t) = \begin{pmatrix} \mathbf{R}_{11}\mathbf{z}_1(0,t) + \mathbf{R}_{12}\mathbf{z}_2(0,t) \\ \mathbf{R}_{21}\mathbf{z}_1(0,t) + \mathbf{R}_{22}\mathbf{z}_2(0,t) \end{pmatrix} \quad \overset{\text{if } \mathbf{z}_2(0,t)=0}{\Longrightarrow} \quad \mathbf{u}(0,t) = \begin{pmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{pmatrix} \mathbf{z}_1(0,t)$$

Analogously

$$\mathbf{u}(1,t) = \begin{pmatrix} \mathbf{R}_{11}\mathbf{z}_1(1,t) + \mathbf{R}_{12}\mathbf{z}_2(1,t) \\ \mathbf{R}_{21}\mathbf{z}_1(1,t) + \mathbf{R}_{22}\mathbf{z}_2(1,t) \end{pmatrix} \quad \overset{\text{if } \mathbf{z}_1(1,t)=0}{\Longrightarrow} \quad \mathbf{u}(1,t) = \begin{pmatrix} \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} \mathbf{z}_2(1,t)$$

- "$\mathbf{z}_2(0,t) = 0$" and "$\mathbf{z}_1(1,t) = 0$" are admissible choices (allowed to prescribe BCs).
- $\mathbf{z}_1(0,t)$ and $\mathbf{z}_2(1,t)$ are information coming from the interior (natural).
- Solution only given by initial conditions; rest is cancelled at boundaries.
- $\mathbf{u}(0,t)$ and $\mathbf{u}(1,t)$ only depends on information coming from characteristic directions.
- Values $\mathbf{u}(0,t)$ and $\mathbf{u}(1,t)$ are decoupled, because $\mathbf{z}_1$ and $\mathbf{z}_2$ are decoupled.
- $\Rightarrow$ No reflection $\Rightarrow$ Well-posed problem.

# Non-reflecting boundary conditions

Note on the transfer to physical boundary conditions:

- Translate BCs $\mathbf{z}_2(0,t) = 0$ and $\mathbf{z}_1(1,t) = 0$ into explicit BCs for $\mathbf{u}$.
- From $\mathbf{z}_2(0,t) = 0$ and $\mathbf{z}_1(1,t) = 0$ we have

$$\mathbf{u}(0,t) = \begin{pmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{pmatrix} \mathbf{z}_1(0,t) \qquad \text{and} \qquad \mathbf{u}(1,t) = \begin{pmatrix} \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} \mathbf{z}_2(1,t)$$

- Recalling that $\mathbf{u} = \mathbf{R}\mathbf{z}$ and that for $p = 1, 2$

$$\mathbf{z}_p(x,t) = \mathbf{z}_p(x - \lambda_p\, t, 0) = (\mathbf{R}^{-1}\mathbf{v}(x - \lambda_p\, t))_p$$

we conclude the explicit physical boundary conditions

$$\mathbf{u}(0,t) = \begin{pmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{pmatrix} \mathbf{z}_1(0,t) = \begin{pmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} \end{pmatrix} (\mathbf{R}^{-1}\mathbf{v}(-\lambda_1\, t))_1$$

and

$$\mathbf{u}(1,t) = \begin{pmatrix} \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} \mathbf{z}_2(0,t) = \begin{pmatrix} \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} (\mathbf{R}^{-1}\mathbf{v}(1 - \lambda_2\, t))_2 .$$

We can also apply extrapolation to realize the BC for $\mathbf{u}$.