Faculté des Sciences de Sorbonne Université & Arts et Métiers ParisTech: Master 2 SPI - Model and numerical simulation in aeroelasticity (2021-2022)

Application of Class Note #5:

Solution of the single DOF arbitrary gust problem

Jean-Camille Chassaing, Sorbonne Université (Oct. 2021)

http://www.dalembert.upmc.fr/home/chassaing

Question 1

The operateur of the structural inertial forces is simply given by : $\mathcal{I}_m = m\ddot{z}(t)$ where m denotes the lineic mass and $z(t)$ is the vertical displacement of the aircraft. We remark that the sign of z is the opposite to those considered in Classnote #5.

Defining the non-dimensional time $s = U_\infty t/b$, we apply the derivation chain rule :

$$z^{'} = \frac{dz}{dt}\frac{dt}{ds} = \frac{U_\infty}{b}\dot{z}(s)$$
$$z^{''}(s) = \frac{d\dot{z}}{dt}\frac{dt}{ds} = \left(\frac{U_\infty}{b}\right)^2\ddot{z}(s)$$

which gives

$$\mathcal{I}_m(s) = m(U_\infty/b)^2 z''(s)$$

Question 2

According to Eq. 35 in ClassNote #5, we know that the lift induced by an gust with arbitrary amplitude is given by

$$L_G(s) = 2\pi\rho_\infty U_\infty b\left[w_G(0)\psi^{kus}(s) + \int_0^s \frac{dw_G(\sigma)}{d\sigma}(\sigma)\psi^{kus}(s-\sigma)d\sigma\right] \tag{1}$$

Integrating by parts the integro-differential terms in Eq. 1 equations gives

$$\begin{aligned}\int_0^s \frac{dw_G(\sigma)}{d\sigma}(\sigma)\psi^{kus}(s-\sigma)d\sigma &= \left[w_G(\sigma)\psi^{kus}(s-\sigma)\right]_{\sigma=0}^{\sigma=s} - \int_0^s w_G(\sigma)\left[-\frac{d\psi^{kus}}{d\sigma}(s-\sigma)\right]d\sigma \\ &= 0 - w_G(0)\psi^{kus}(s) + \int_0^s w_G(\sigma)\frac{d\psi^{kus}}{d\sigma}(s-\sigma)d\sigma\end{aligned} \tag{2}$$

Inserting the previous relation in Eq. 2 gives the expected results

$$L_G(s) = 2\pi\rho U_\infty b\int_0^s w_G(\tau)\frac{d\psi^{kus}}{d\tau}(s-\tau)d\tau \tag{3}$$

Question 3

The lift corresponding to an arbitrary motion is described by (see Eq. 36 in ClassNote #5)

$$L_M(s) = L_{\Gamma=0} - 2\pi\rho_\infty U_\infty b\left[w_{3/4}(0)\psi^{wag}(s) + \int_0^s \frac{dw_{3/4}(\sigma)}{d\sigma}(\sigma)\psi^{wag}(s-\sigma)d\sigma\right] \tag{4}$$

Since we consider the vertical motion of a rigid aircraft, we straightfully have the analogy: $z(t) \sim h(t)$ and $w_{3/4}(t) = \dot{h}(t) \sim \dot{z}(t)$.

Moreover, the non-circulatory part of the lift due to the aircraft motion can be deduced from:

$$L_{\Gamma=0}(t) = \pi\rho_\infty b^2(\ddot{h} + U_\infty\dot{\alpha} - ab\ddot{\alpha}) = -\pi\rho_\infty b^2\ddot{z}(t) \tag{5}$$

or equivalently

$$L_{\Gamma=0}(s) = -\pi\rho_\infty U_\infty^2 z^{''}(s) \tag{6}$$

Consequently, we obtain

$$L_M(s) = -\pi\rho_\infty U_\infty^2 z^{''}(s) - 2\pi\rho_\infty U_\infty b\left[z(0)\psi^{wag}(s) + \int_0^s \frac{dz(\sigma)}{d\sigma}(\sigma)\psi^{wag}(s-\sigma)d\sigma\right] \tag{7}$$

Question 4

Assembling Eqs ???, 3 and 7, the equation of the aeroelastic motion given by $I_m = L_G + L_M$ reads

$$\left(\frac{U}{b}\right)^2 mz''(s) = 2\pi\rho U_\infty b\int_0^s w_G(\tau)\frac{d\psi^{kus}}{d\tau}(s-\tau)d\tau - \pi\rho_\infty U_\infty^2\left[z''(s) + 2\int_0^s z''(\tau)\psi^{wag}(s-\tau)d\tau\right] \tag{8}$$

Question 5

Let $\bar{f}(p) = \mathcal{L}[f(s)] = \int_0^\infty f(s)e^{-pt}dt$ with $z(0) = z'(0) = 0$ and $\psi^{kus}(0) = 0$ (see Fig. 6 in Classnote #5).

We can compute

$$\mathcal{L}\left[\int_0^s w_G(\tau)\frac{d\psi^{kus}}{d\tau}(s-\tau)d\tau\right] = \mathcal{L}(w_G(s))L(d\psi^{kus}/ds)(s) = \bar{w}_G(p)\left[p\bar{\psi}^{kus}(p) - \underbrace{\psi^{kus}(0)}_{=0}\right] \tag{9}$$

and

$$\mathcal{L}\left[\int_0^s z''(\tau)\psi^{wag}(s-\tau)d\tau\right] = p^2\bar{z}(p)\bar{\psi}^{wag}(p) \tag{10}$$

Then, the dynamic equation of motion written in the Laplace domain are

$$\underbrace{\left(\frac{U}{b}\right)^2 mp^2\bar{z}(p)}_{inertial\ structural\ effect} = \underbrace{2\pi\rho_\infty U_\infty b\bar{w}_G p\bar{\psi}^{kus}}_{gust\ effect} - \underbrace{\pi\rho_\infty U_\infty^2 p^2\bar{z}}_{added\ mass\ effect} - \underbrace{2\pi\rho_\infty U_\infty^2 p^2\bar{z}(p)\bar{\psi}^{wag}}_{arbitrary\ motion} \tag{11}$$

Dividing each member of Eq. (\ref{EqLaplace1} by $ 2 |pi |rho\infty U_\infty^2$, we obtain

$$\frac{m}{2\pi\rho_\infty b_\infty^2}p^2\bar{z}(p) = bp\frac{\bar{w}_G}{U_\infty}\bar{\psi}^{kus} - \frac{p^2}{2}\bar{z} - p^2\bar{z}\bar{\psi}^{wag} \tag{12}$$

Introducing the mass ratio $\lambda = m/(2\pi\rho_\infty b_\infty^2)$, the solution in the Laplace domain can be formely derived as

$$\bar{z}(p) = b\frac{(\bar{w}_G/U_\infty)\bar{\psi}^{kus}}{p\left(\lambda + \frac{1}{2} + \bar{\psi}^{wag}\right)} \tag{13}$$

where the expressions of $\bar{w}_G(s)$, $\bar{\psi}^{kus}(s)$ and $\bar{\psi}^{wag}(s)$, which are the Laplace transforms of $w_G(s)$, $\bar{\psi}^{kus}(s)$ and $\bar{\psi}^{wag}(s)$ must be prescribed to complete the computation of $\bar{z}(s)$.

Question 6

Because taking the Laplace transform of exact expressions of $\bar{\psi}^{wag}(s)$ and $\bar{\psi}^{kus}(s)$ is impractical, we consider the following approximations

$$\begin{aligned}\psi^{kus}(s) &= 1 - 0.5\mathrm{e}^{-\alpha_1 s} - 0.5\mathrm{e}^{-\alpha_2 s} \\ \psi^{wag}(s) &= 1 - b_1\mathrm{e}^{-\beta_1 s} - b_2\mathrm{e}^{-\beta_2 s}\end{aligned} \tag{14}$$

where $\alpha_{1,2}$, $\beta_{1,2}$ et $b_{1,2}$ are positive constants.

Then we compute easily

$$\begin{aligned}\bar{\psi}^{kus}(p) &= \mathcal{L}[\psi^{kus}(s)] = \frac{1}{p} - \frac{0.5}{p+\alpha_1} - \frac{0.5}{p+\alpha_2} \\ \bar{\psi}^{wag}(p) &= \mathcal{L}[\psi^{wag}(s)] = \frac{1}{p} - \frac{b_1}{p+\beta_1} - \frac{b_2}{p+\beta_2}\end{aligned} \tag{15}$$

Consequently, Eq. (13) becomes

$$\bar{z}(p) = \frac{b}{U}\bar{w}_G\underbrace{\frac{1}{p}\left(\frac{1}{p} - \frac{1/2}{p+\alpha_1} - \frac{1/2}{p+\alpha_2}\right)}_{f_1(p)}\left(\underbrace{\lambda + \frac{1}{2} + \frac{1}{p} - \frac{b_1}{p+\beta_1} - \frac{b_2}{p+\beta_2}}_{f_2(p)}\right)^{-1} \tag{16}$$

with $f_1(p) = M_1(p)/N_1(p)$ where $M_1(p)$ and $N_1(p)$ represent polynomials of degree 2 and 4 respectively. Similary, $f_2(p) = M_2(p)/N_2(p)$ where $M_2(p)$ and $N_2(p)$ represent polynomials of degree 3.

A compact form of Eq. 16 may be written as

$$\bar{z}(p) = \frac{b}{U}\bar{w}_G(p)\frac{M(p)}{N(p)} \tag{17}$$

where $M(p)$ et $N(p)$ are polynomials of degree 5 and 7 respectively.

Question 7

Finally, the solution can be obtained in the physical domain as

$$\bar{z}(t) = \int_0^s w_G(u)\left(\sum_{k=1}^{7}\frac{\mathcal{L}^{-1}(M(p_k))}{\mathcal{L}^{-1}(N(p_k)')}e^{p_k(s-u)}\right)du \tag{18}$$

where p_k denote the roots of the polynomial $N(p)$.

End of this notebook

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