Introduction to PDEs

Well-posedness of PDEs

Well-Posed Problems

Discussed problems follow a pattern:

- elliptic PDEs are coupled with boundary conditions;
- parabolic PDEs require initial conditions and boundary conditions for all times.
- hyperbolic PDEs require always initial conditions and (depending on their order) *sometimes* boundary conditions.
- This is part of a more general pattern.
- Certain types of PDEs go naturally with certain side conditions.

Well-Posed Problems

A problem is called well-posed if

a.) it has a solution

b.) the solution is unique

c.) the solution depends continuously on data and parameters.

- meaning of a) is clear.
- "uniqueness" typically means "unique within a certain class of functions".
  Exp.: a problem might have several solutions, only one of which is bounded.
  We'd say: solution is unique in the space of bounded functions.
- A solution depends continuously on data and parameters if "small" changes in initial or boundary values (in appropriate norms) and in parameter values result in "small" changes in the solution (in some appropriate norm).

Well-Posed Problems

- Notion of well-posedness is important in applied math.
- If you were using an initial-boundary value problem **(P)** to make predictions about some physical process, you'd obviously like **(P)** to have solution.
- You'd also want the solution to be unique.
- If solution depends continuously on data and parameters, you don't have to worry about small errors in measurement producing large errors in your predictions.

Well-Posed Problems

- Theory of PDEs deals mainly with the well-posedness of problem
- still, ill-posed problems can be mathematically and scientifically interesting.

Notation

- Let $\|\cdot\|$ denote the norm on a linear space V.
- For a function $u(x,t)$ (in space x and time t) we write

$$\|u(t)\| := \|u(\cdot,t)\|$$

  for the norm at fixed time t.
- Example:
  Let $B \subset \mathbb{R}^d$ be a domain. The $L^1(B)$-norm of u at time t is

$$\|u(t)\| := \int_B |u(x,t)|\, dx.$$

Well-Posed Problems - Example 1

Let $0 < \varepsilon \ll 1$ and $\|\cdot\|_\infty$ be the maximum norm on $C^0(\mathbb{R})$.

We seek $u_\varepsilon(x,t)$ with

$$\partial_{tt}u_\varepsilon + \partial_{xx}u_\varepsilon = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

and initial conditions

$$u_\varepsilon(x,0) = 0 \qquad \text{and} \qquad \partial_t u_\varepsilon(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right).$$

A solution is given by:

$$u_\varepsilon(x,t) = \varepsilon^2 \sin\left(\frac{x}{\varepsilon}\right)\sinh\left(\frac{t}{\varepsilon}\right).$$

We have $u_0 \equiv 0$.

The solutions u_0 and u_ε only differ in the choice of the initial value with

$$\|\partial_t u_\varepsilon(0) - \partial_t u_0(0)\|_\infty = \varepsilon.$$

Well-Posed Problems - Example 1

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and initial conditions $u_\varepsilon(x,0) = 0$ and $\partial_t u_\varepsilon(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right)$. Solution:

$$u_\varepsilon(x,t) = \varepsilon^2 \sin\left(\frac{x}{\varepsilon}\right) \sinh\left(\frac{t}{\varepsilon}\right).$$

The solutions u_0 and u_ε only differ in the choice of the initial value with

$$\|\partial_t u_\varepsilon(0) - \partial_t u_0(0)\|_\infty = \varepsilon.$$

But:

$$\|u_\varepsilon(t) - u_0(t)\|_\infty = \varepsilon^2 \left|\sinh\left(\frac{t}{\varepsilon}\right)\right| \to \infty \qquad \text{for } \varepsilon \to 0.$$

Since $\sinh(t/\varepsilon) = \frac{1}{2}(e^{t/\varepsilon} - e^{-t/\varepsilon})$, the error is exponentially large.

Well-Posed Problems - Example 1

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and initial conditions $u_\varepsilon(x,0) = 0$ and $\partial_t u_\varepsilon(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right)$. Solution:

$$u_\varepsilon(x,t) = \varepsilon^2 \sin\left(\frac{x}{\varepsilon}\right) \sinh\left(\frac{t}{\varepsilon}\right).$$

Exponentially large error for any time $t > 0$

$$\|u_\varepsilon(t) - u_0(t)\|_\infty = \frac{\varepsilon^2}{2}\left|e^{t/\varepsilon} - e^{-t/\varepsilon}\right| \to \infty \qquad \text{for } \varepsilon \to 0.$$

- Very small change in initial value results in large change in solution for positive time.
- Explanation:
  - considered problem is elliptic, which naturally describes stationary equilibriums.
  - Pairing of elliptic equation with initial conditions led to an ill-posed problem.

Well-Posed Problems - Example 2

We change the problem an make it hyperbolic (to make it well-posed):

We seek $u_\varepsilon(x,t)$ with

$$\partial_{tt} u_\varepsilon - \partial_{xx} u_\varepsilon = 0 \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

and initial conditions

$$u_\varepsilon(x,0) = 0 \qquad \text{and} \qquad \partial_t u_\varepsilon(x,0) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right).$$

A solution is given by:

$$u_\varepsilon(x,t) = \varepsilon^2 \sin\left(\frac{x}{\varepsilon}\right) \sin\left(\frac{t}{\varepsilon}\right).$$

We have $u_0 \equiv 0$ and

$$\|u_\varepsilon(t) - u_0(t)\|_\infty = \varepsilon^2 \left|\sin\left(\frac{t}{\varepsilon}\right)\right| \leq \varepsilon^2 \to 0 \qquad \text{for } \varepsilon \to 0.$$

Small change in initial data $\Rightarrow$ small change of solution for $t > 0$.

Well-Posed Problems

- Establishing existence can be quite difficult.
- We will do this only with simple problems for which one can write down a solution.
- Proving uniqueness and continuous dependency is usually easier, especially for linear problems.

Well-Posed Problems - Example 3

Parabolic problem: initial-boundary value problem for the heat equation:

Let

- $B \subset \mathbb{R}^d$ smooth bounded domain;
- ∂B is the smooth boundary (surface) of B;
- continuous function:
  - source term $f(x,t)$;
  - initial value $h(x)$ and
  - boundary values $g(x,t)$.

Well-Posed Problems - Example 3

We seek $u(x,t)$ with

$$\partial_t u - \mathbf{a}\triangle u = f \qquad \text{for } x \in B \text{ and } t > 0,$$

and initial condition $u(x,0) = h(x)$ and boundary condition $u(x,t)|_{\partial B} = g(x,t)$.

We want to verify uniqueness.

Let u_1 and u_2 denote two solutions. Then $w := u_1 - u_2$ solves

$$\partial_t w - \mathbf{a}\triangle w = 0 \qquad \text{for } x \in B \text{ and } t > 0,$$

and initial condition $w(x,0) = 0$ and boundary condition $w(x,t)|_{\partial B} = 0$.

Multiplying by w, integrating by parts and using the boundary condition yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{L^2(B)} = 2\int_B \partial_t w(x,t) w(x,t)\,dx = 2\mathbf{a}\int_B \triangle w(x,t) w(x,t)\,dx = -2\mathbf{a}\int_B |\nabla w(x,t)|^2\,dx \leq 0.$$

Hence

$$\|w(t)\|^2_{L^2(B)} \leq \|w(0)\|^2_{L^2(B)} = 0 \quad \Rightarrow \quad u_1(x,t) - u_2(x,t) = w(x,t) = 0 \quad \Rightarrow \quad \text{unique.}$$

Well-Posed Problems

- In many PDEs the term

$$\|u\|^2_{L^2(B)} = \int_B u(x)^2 \, dx$$

is called the mass.

- The term

$$\int_B \mathbf{a} |\nabla u(x)|^2 \, dx$$

is called the energy.

- The approach (from previous slide) is therefore often called energy method.
- Its use is not restricted to uniqueness arguments for linear parabolic problems.
- It is an important tool in the analysis of PDEs, appearing in all parts of well-posedness proofs for all sorts of problems - linear and nonlinear, parabolic, hyperbolic, elliptic and mixed.