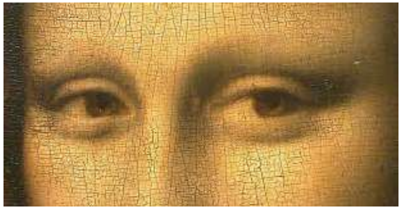
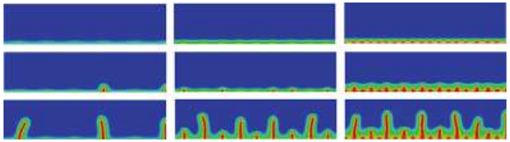
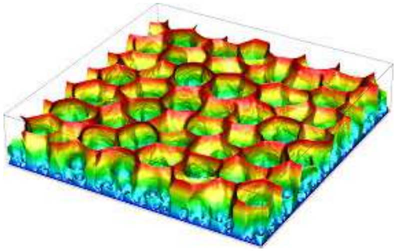
(Brittle) Fracture Mechanics

Corrado Maurini

Institut Jean Le Rond d'Alembert, Sorbonne Université
corrado.maurini@sorbonne-universite.fr
Tour 55-65 414







Acknowledgement: a large part of these slides is taken from the course of J.J. Marigo and K.Danas (Ecole Polytechnique), see the references

Tentative program (to update)

Lecture 1	Intro to Fracture, Stress concentrations, singularities (anti-plane)	21/9
Lecture 2	Stress singularities in plane elasticity, fracture modes, fracture toughness, Irwin criterion	28/9
Lecture 3	Energetic (variational) approach to fracture, Griffith's theory I	5/10
Lecture 4	Energetic (variational) approach to fracture, Griffith's theory II	12/10
Lecture 5	Numerical computation of the stress intensity factors I	19/10
Lecture 6	Numerical computation of the stress intensity factors II	26/10
Lecture 7	Examples	09/11
Lecture 8	Examples/Seminar	23/11
Final Exam (written)		30/11

I will probably give one Homework project at the end of october to do in groups of two students and final note will be calculate as

max (100% final examen, 80% final exam + 20 % homework)

Main references

- Jean-Jacques Marigo, Plasticité et Rupture, Edition Ecole Polytechnique, 2016
  - PDF: https://hal.archives-ouvertes.fr/cel-01374813
  - Chapter 7-9 (Part III)
  - See also Chapter 1-3 (Part I) for linear elasticity
- Pierre Suquet, Plasticité et Rupture, Edition Ecole Polytechnique
  - PDF: https://perso.ensta-paris.fr/~mbonnet/mec551/mec551.pdf
  - Chapter 1-3
- Jean-Baptiste Leblond, Mécanique de la rupture fragile et ductile, Lavoisier/Hermes, 2003

Objectives

At the end of the course you should be able to:

- Describe and model stress concentration around defects
- Describe and model singularities in linear elasticity
- Describe and model crack defects
- Define and evaluate the toughness of the material
- Calculate numerically the energy release rate of a crack
- Identify the conditions for the non-propagation of pre-existing cracks
- Model crack propagation along pre-existing crack path
- Elemental application of the variational approach to fracture

Intro to Fracture, stress concentrations

Content of Lesson 1

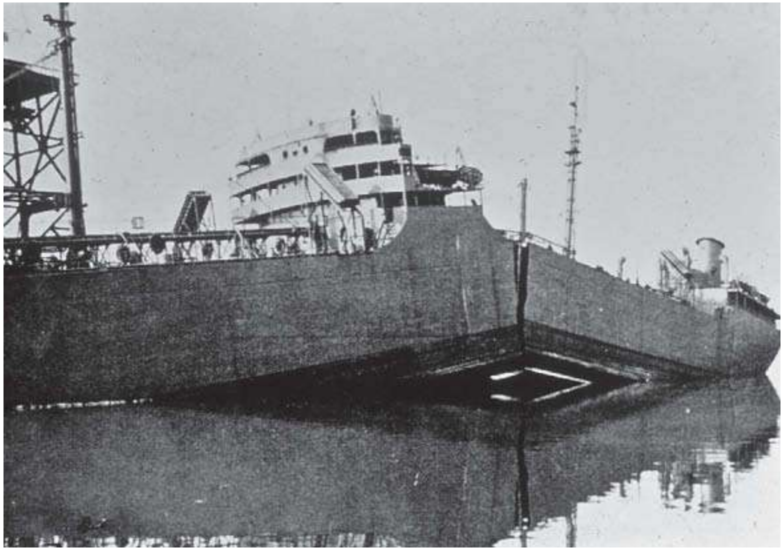
- General introduction to fracture mechanics
- Introduction of brittle fracture approach and review of linear elasticity
- Stress criteria and defects in linear elasticity
- Introduction to singularities in linear elasticity: the case of anti-plane elasticity
- Mode III cracks

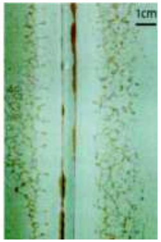
At the end of Lesson 1 you should be able to

- Distinguish between brittle and ductile fracture
- Apply the stress criterion for plates with elliptical holes
- Determine singular solutions for notches and bi-material interfaces in anti-plane elasticity

General introduction

Cracks

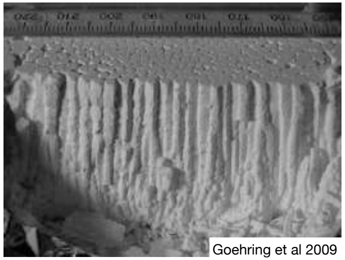


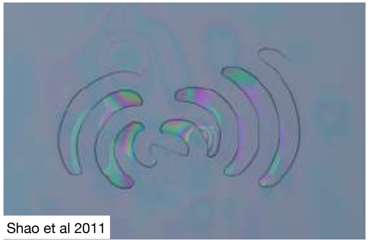


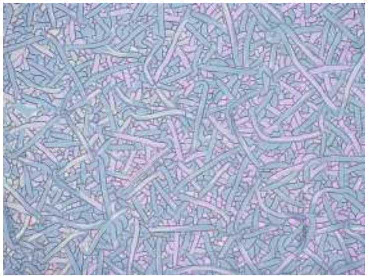
Complex crack patterns


Giant causeway

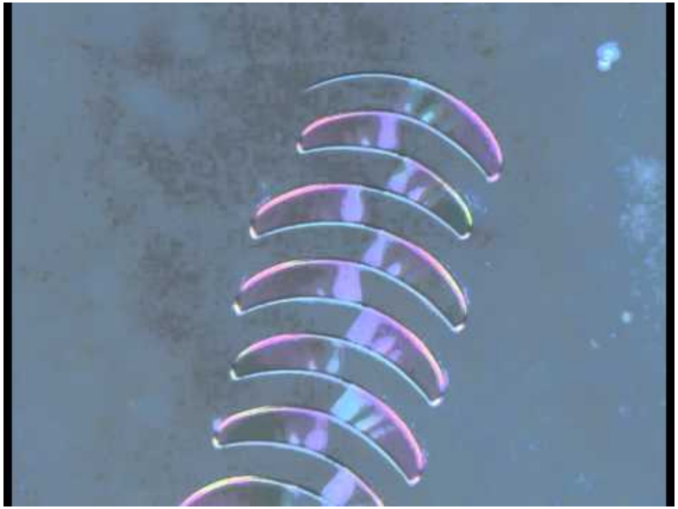
Shao et al 2011

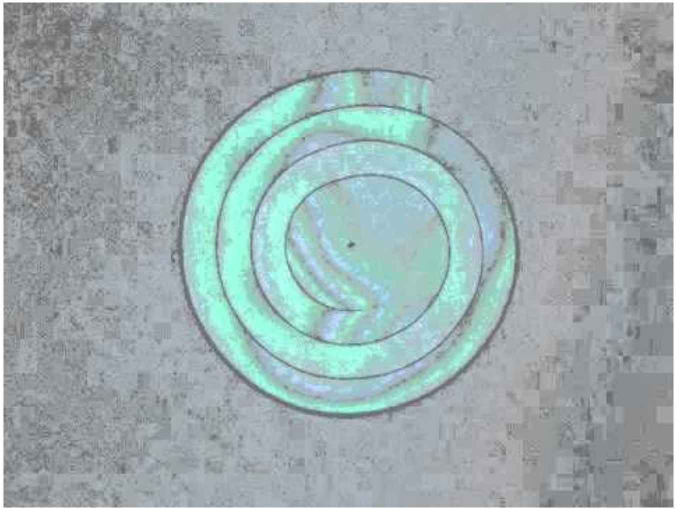

Goehring et al 2009


Shao et al 2011


https://blog.espci.fr/benoitroman/en/tearing-fracture-in-thin-sheets/

Thin film cracks

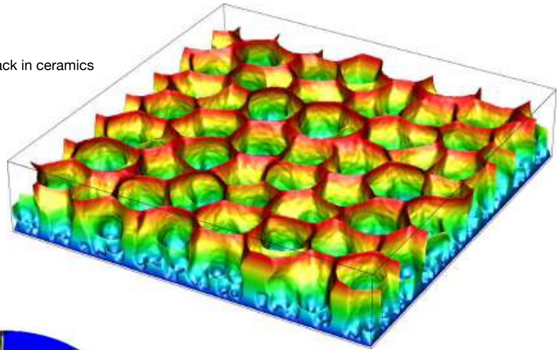


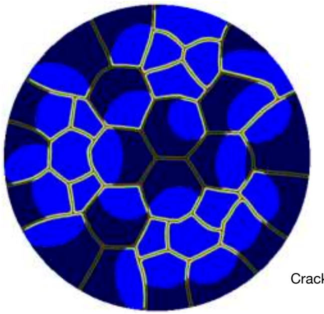


Marthelot et al, PRL 2014

Some recent numerical results using modern variational approach to fracture

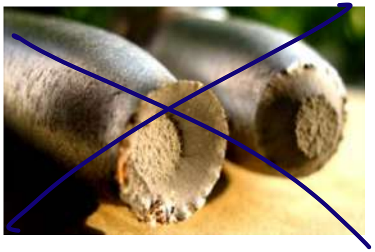
Thermal shock crack in ceramics

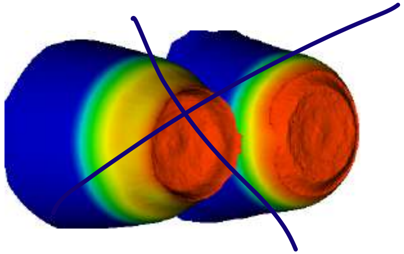


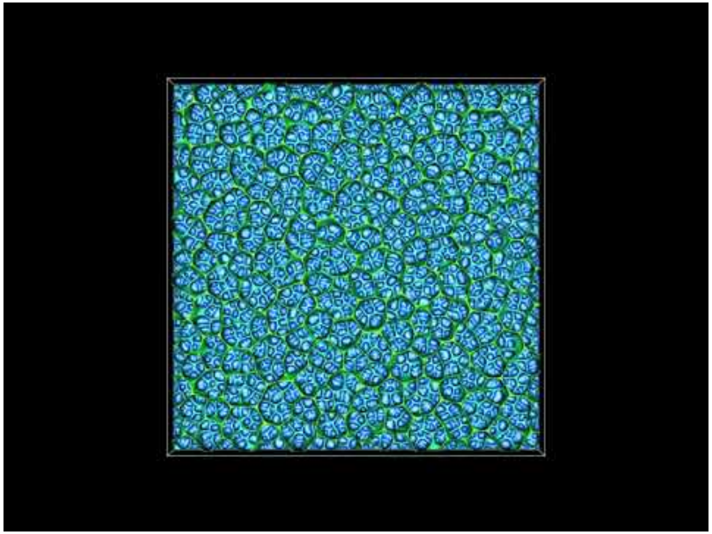


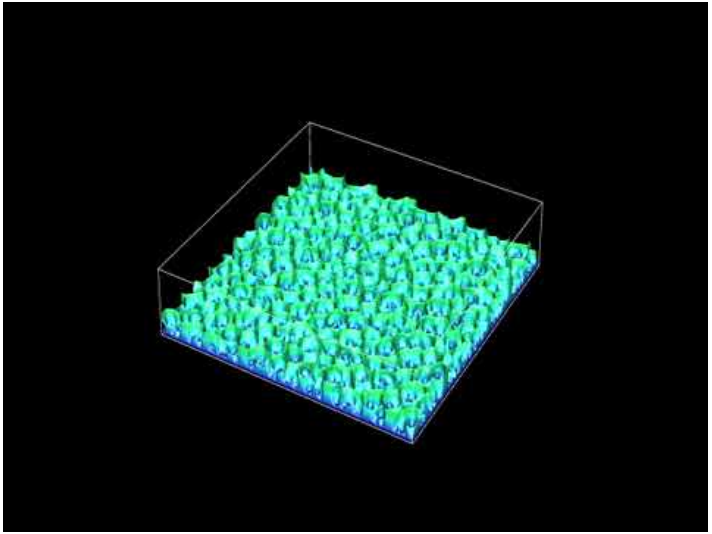
Crack and delimitation in thin film

Cup-cone crack in a ductile material

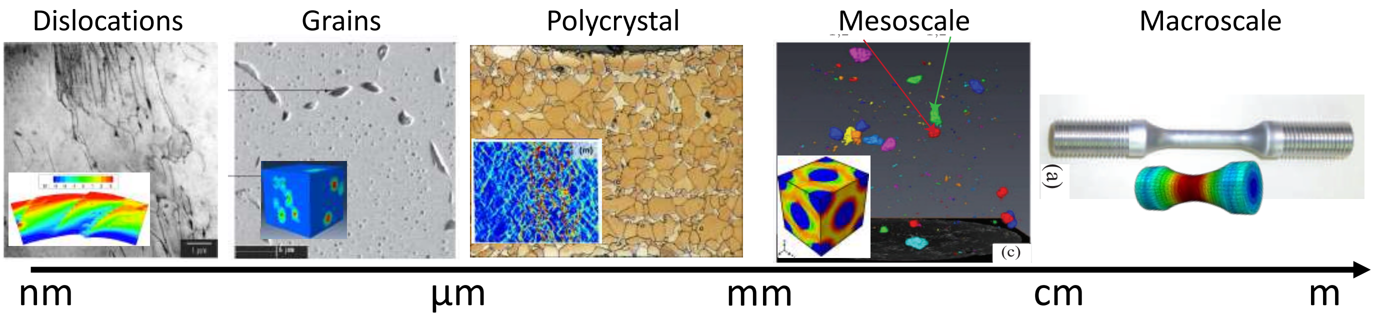


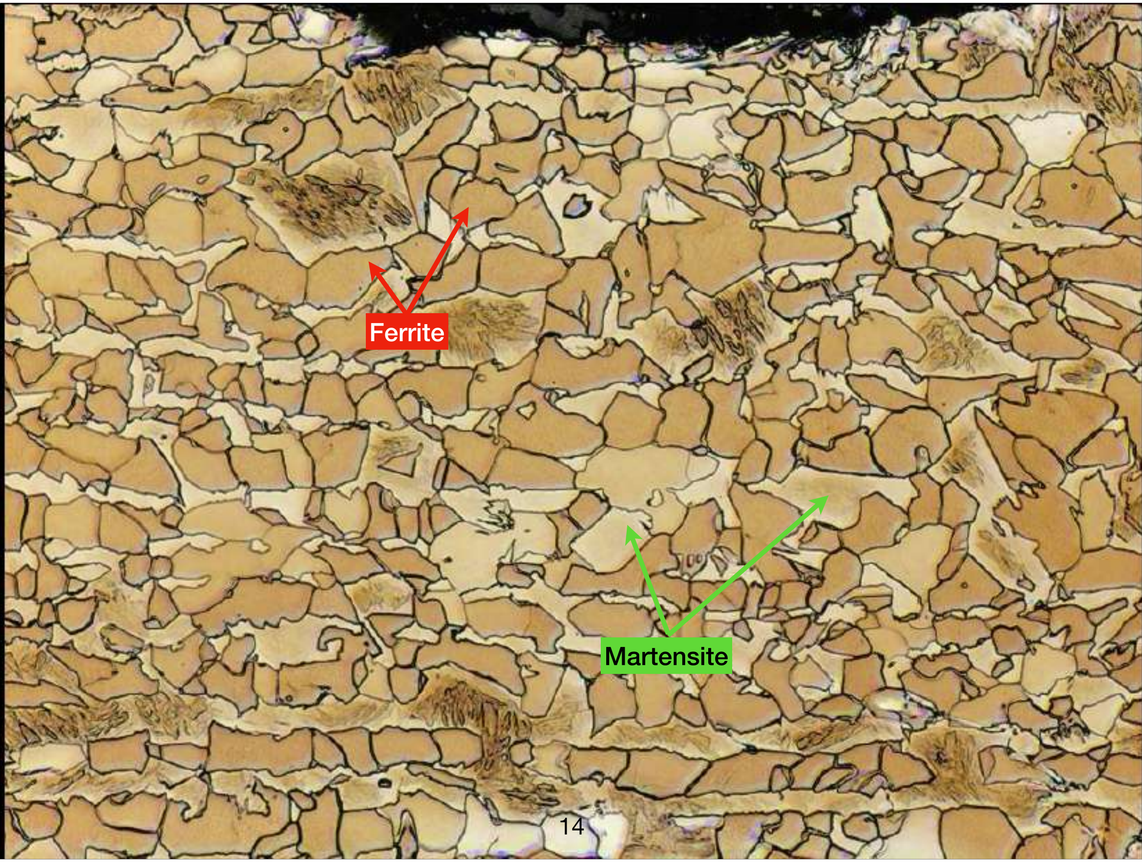


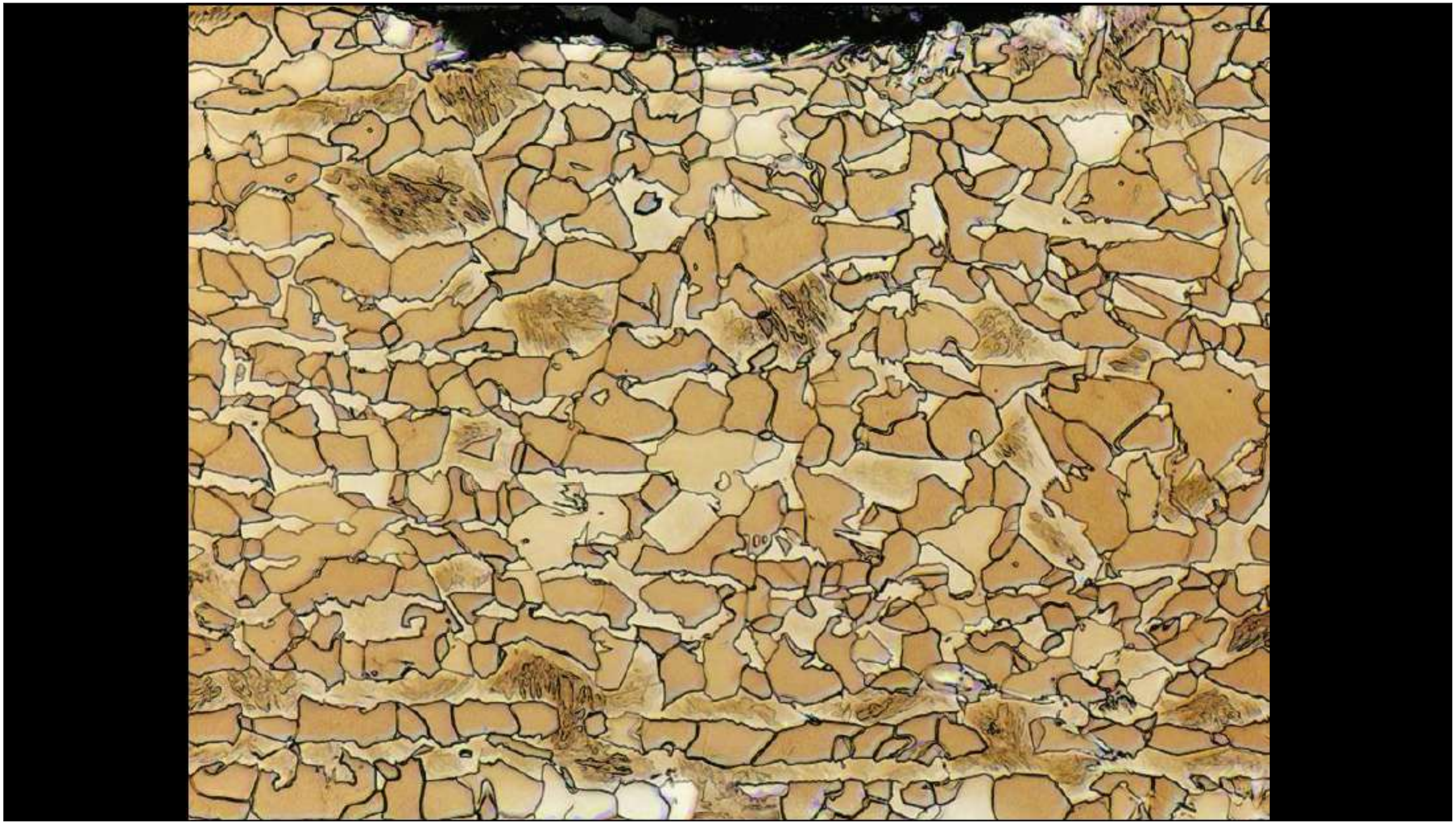


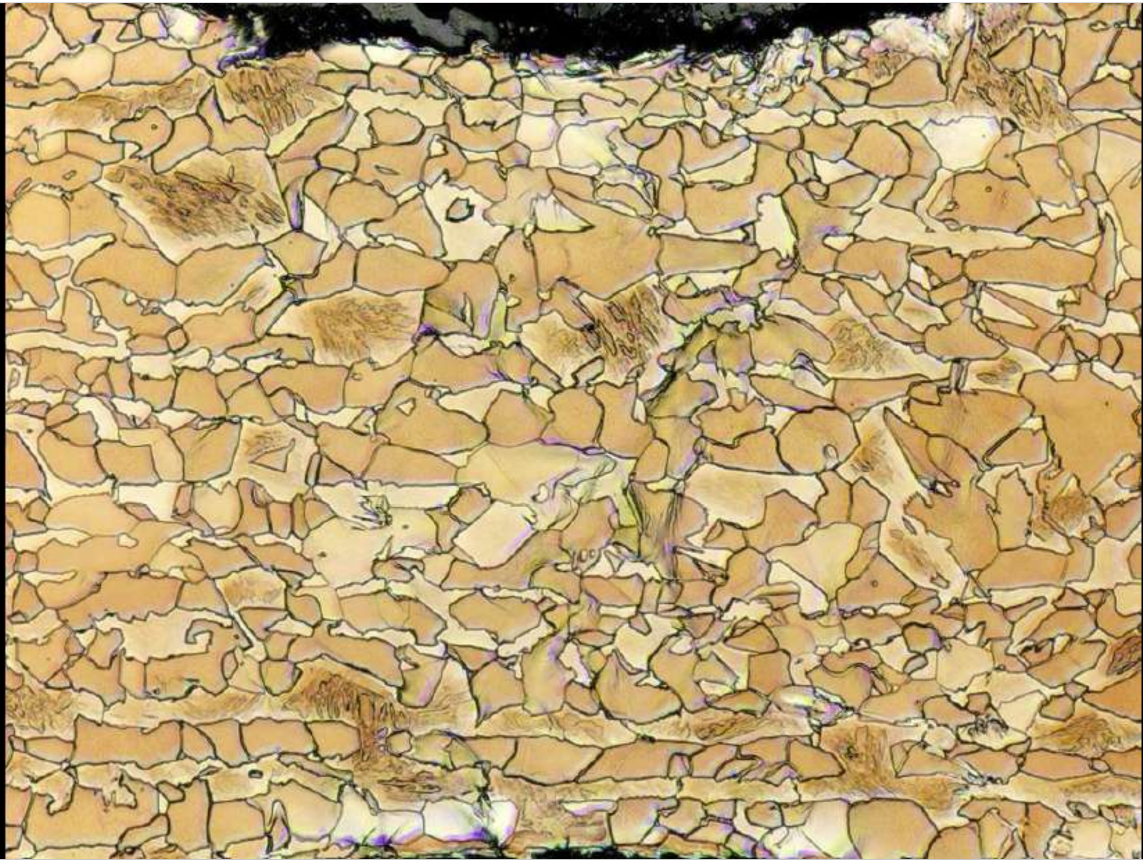


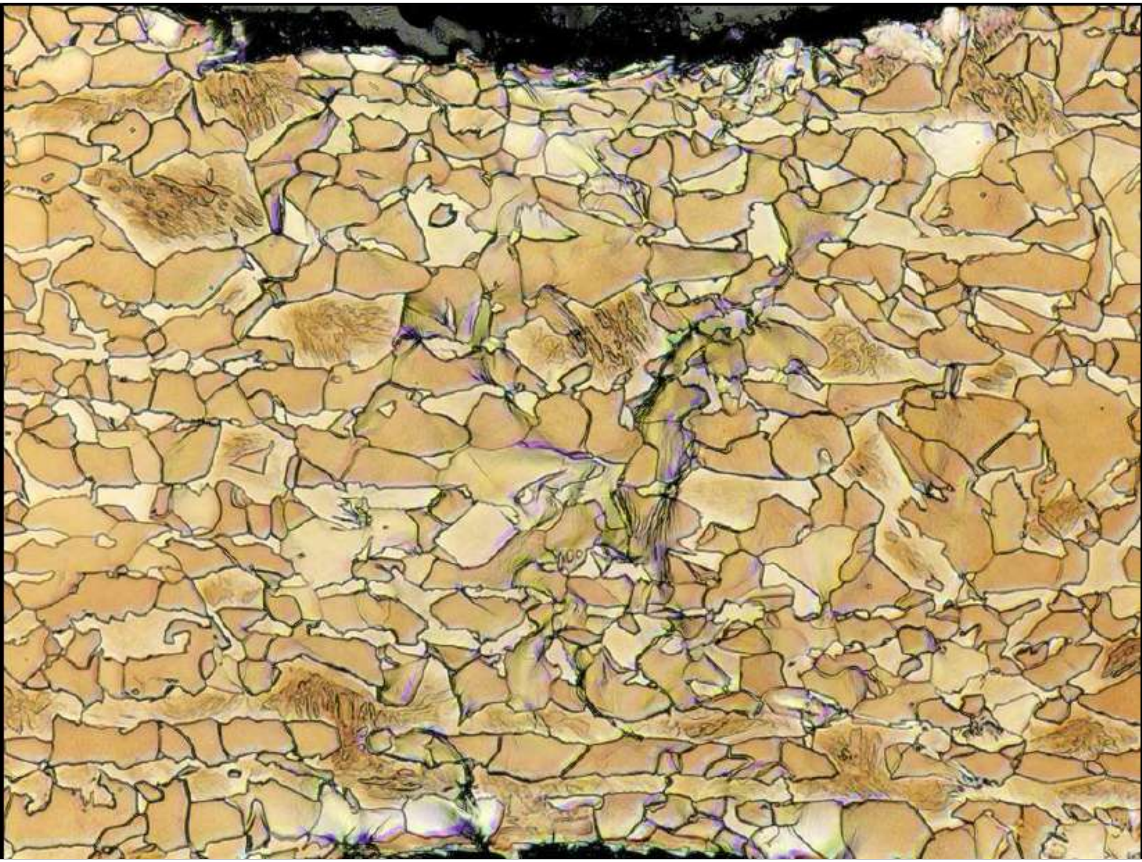
Bourdin et al, PRL 2014

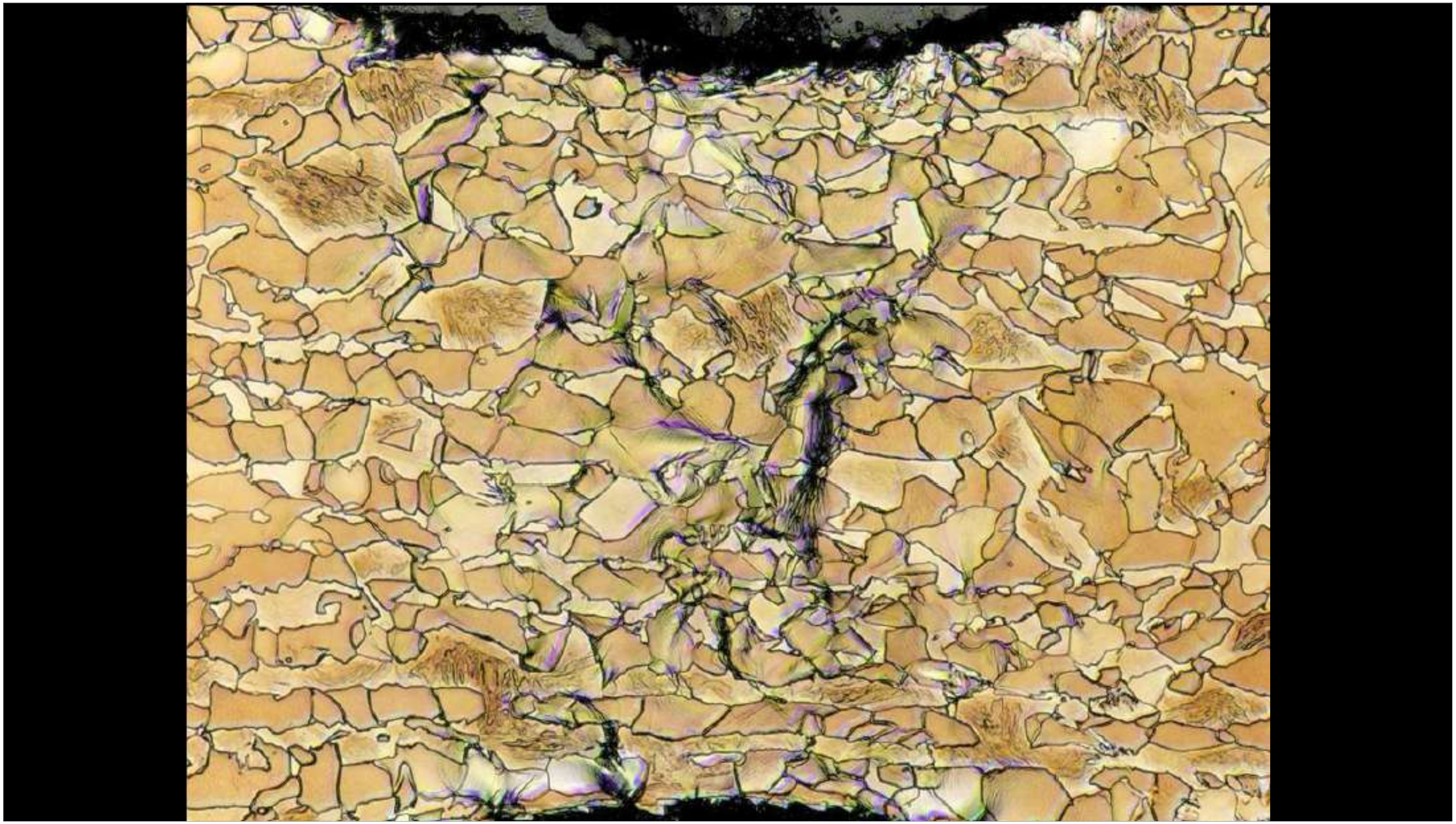

Dislocations
Grains
Polycrystal
Mesoscale
Macroscale
(c)
(a)
nm
µm
mm
cm
m

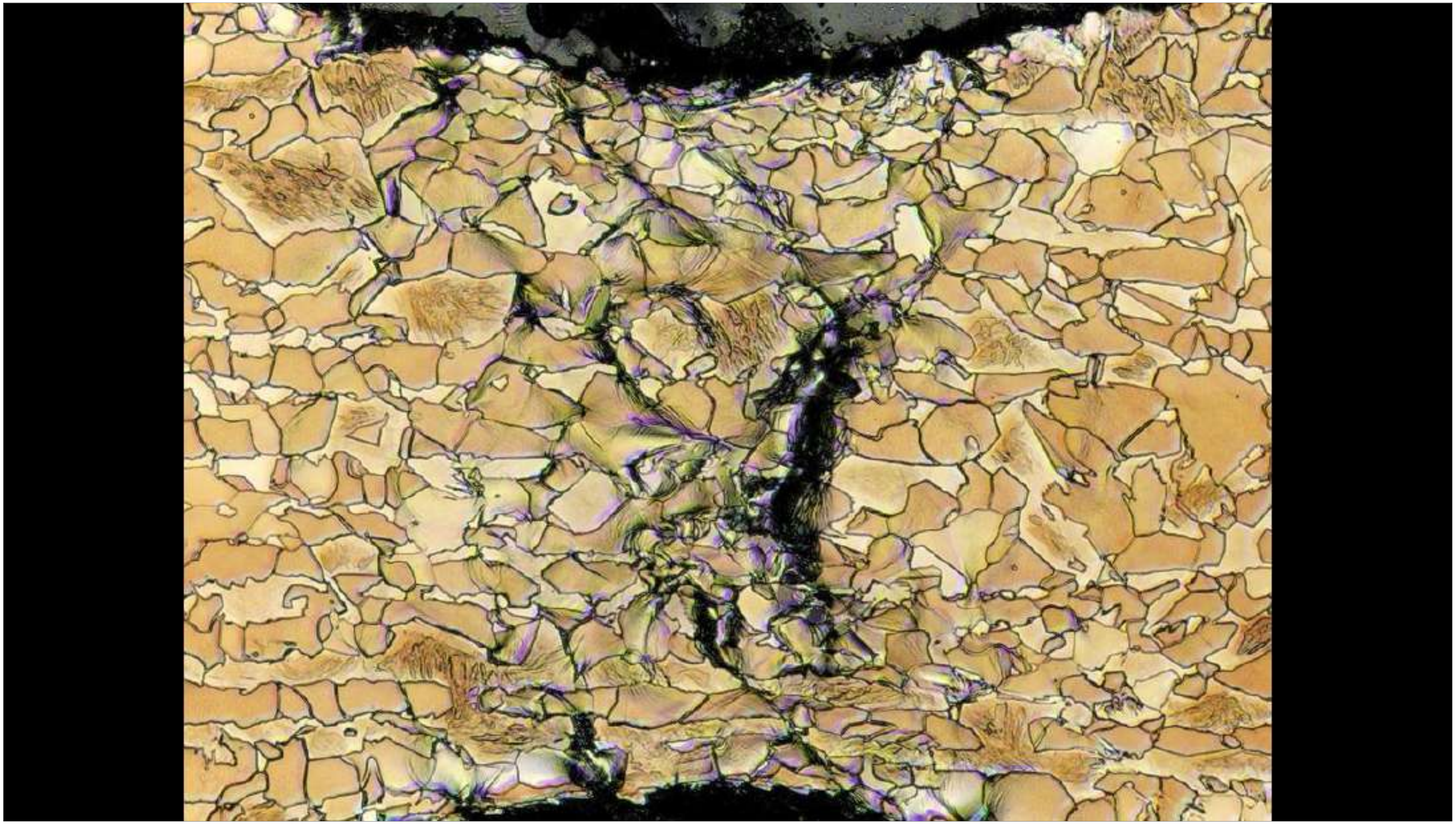

Ferrite
Martensite

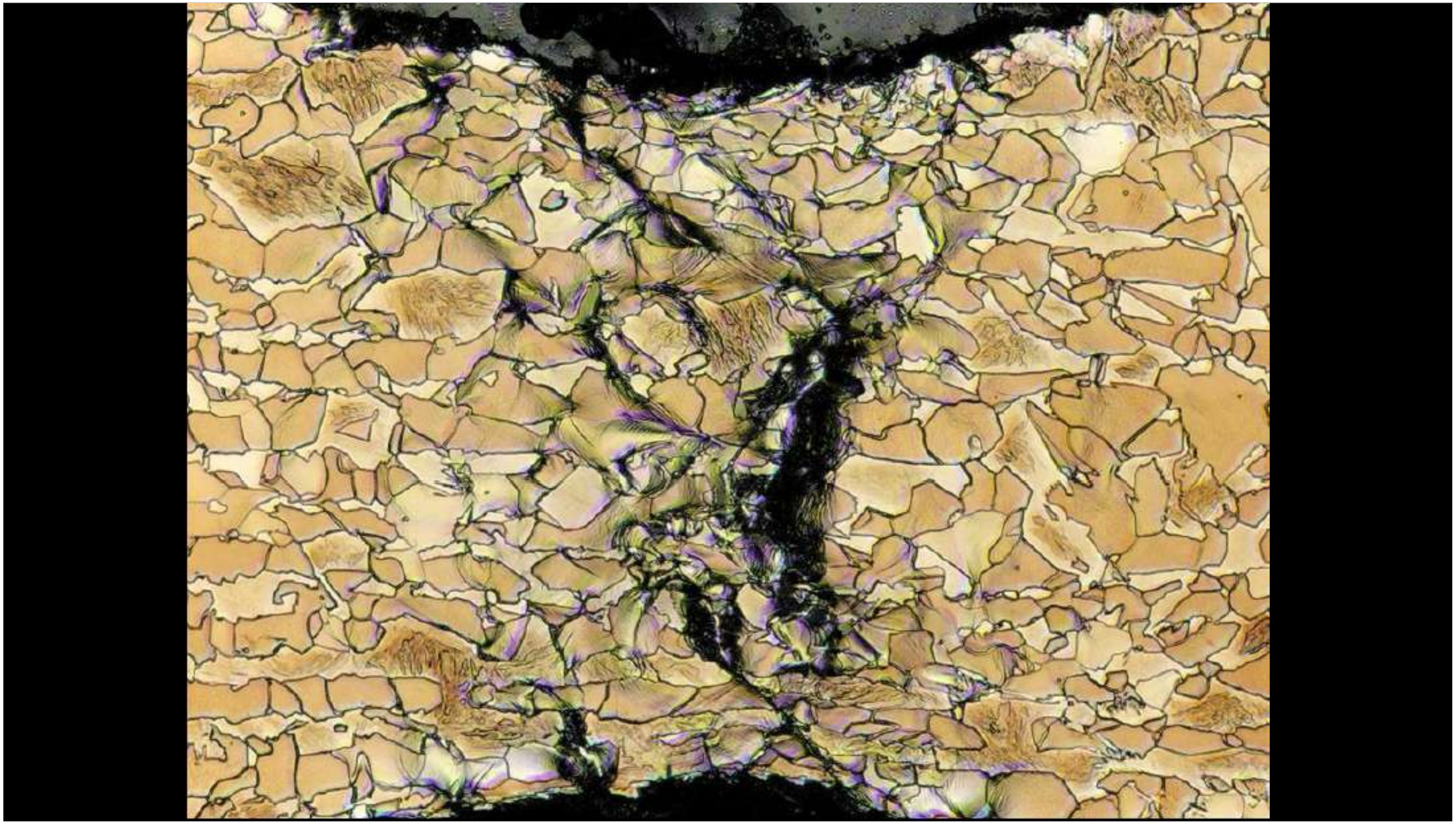


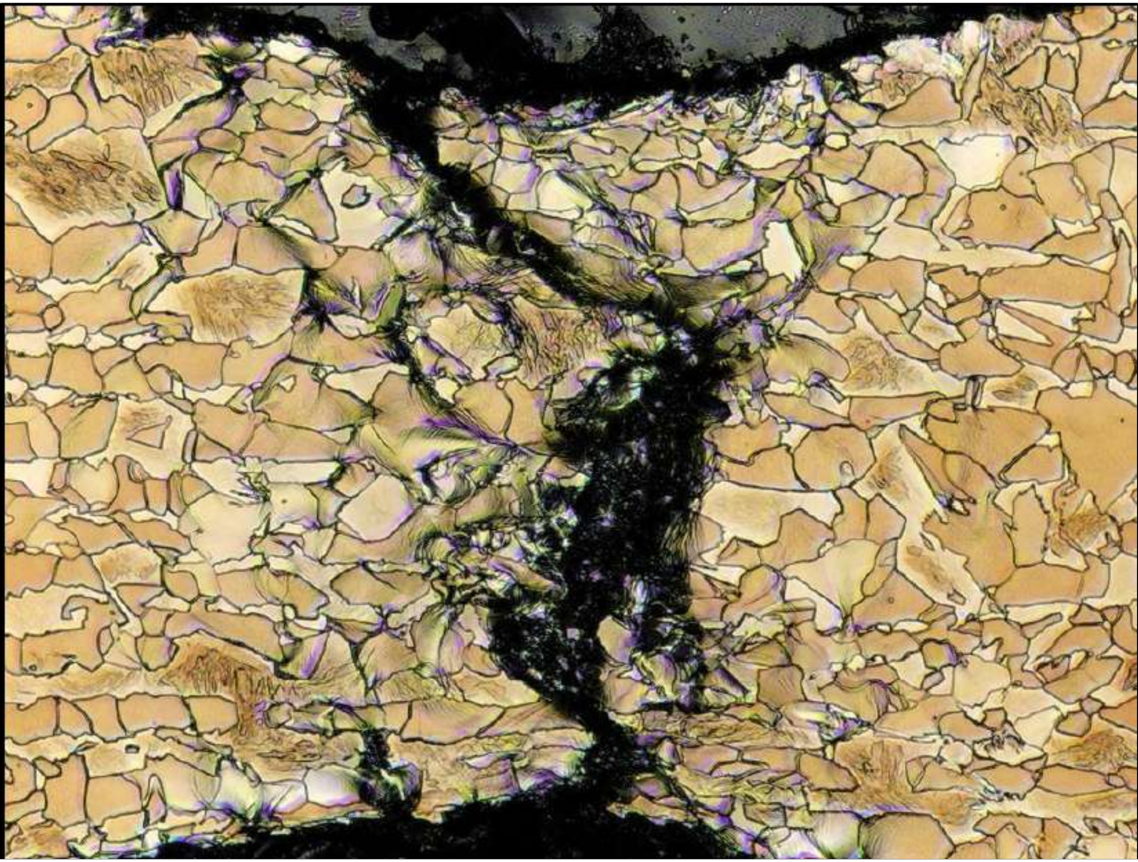




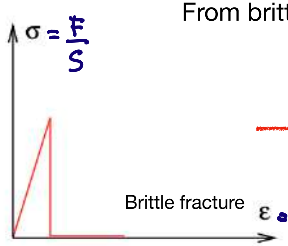


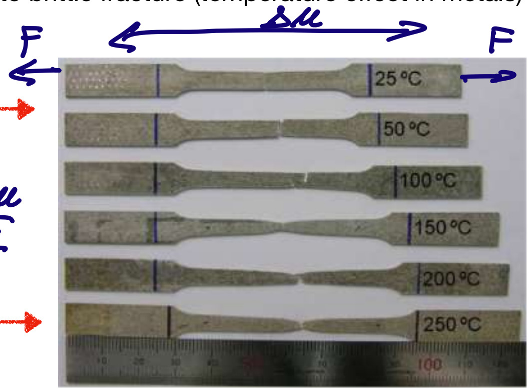


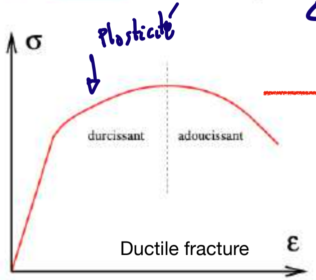


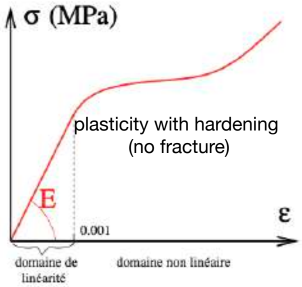


From brittle to brittle fracture (temperature effect in metals)

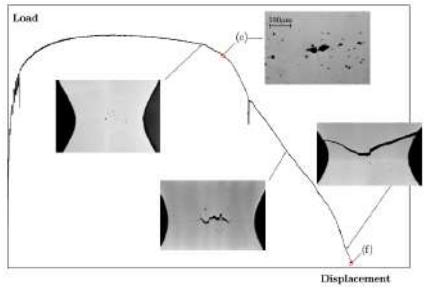






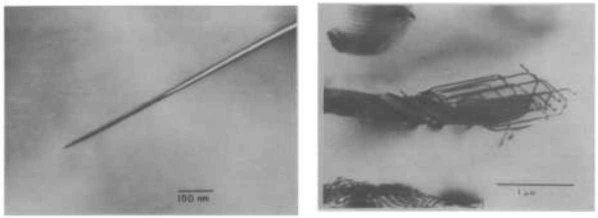


Ductile fracture in metals

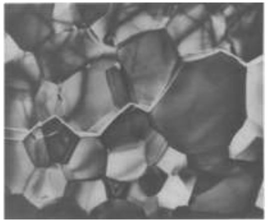


ONLY BRITTLE FRACTURE IN THIS COURSE!

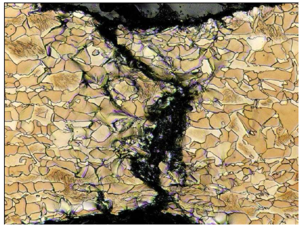
Cracks in Si



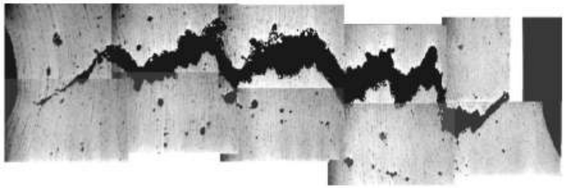
Crack in ceramics



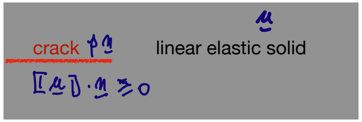
Ductile fracture in polycrystal



Crack in metals



Our **brittle fracture** model:



- a crack is a surface (line in 2d) where displacement may jump
- the solid is linear elastic and in small deformation outside the crack
- this is far from microscopic view, but in most cases gives the good first-order prediction for the crack propagation conditions

Working hypothesis for the course: brittle fracture

- Linear elastic behaviour up to fracture

densité énergie élastique

$$w(\varepsilon) = \frac{1}{2}\varepsilon : \mathbb{C} : \varepsilon, \qquad \boldsymbol{\sigma} = \mathbb{C} : \varepsilon$$

- Small displacements (we do not account for geometrical non-linearities)

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$$

$$\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T + \cancel{\nabla u^T \nabla u}\right)$$

- Isotropic elasticity

$$\boldsymbol{\sigma} = \lambda\, \mathrm{tr}(\varepsilon)\, \mathbf{I} + 2\mu\varepsilon$$

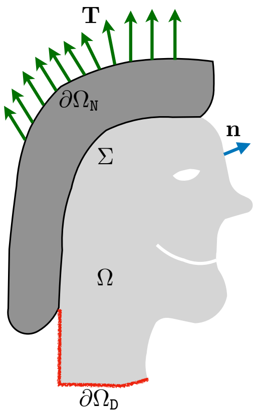
1921

- Griffith-fracture: Fracture energy proportional to the crack surface

$$\text{energy required to create a crack} = \underbrace{\text{fracture toughness}}_{G_c} \times \text{crack area}$$

- Quasi-static behaviour (neglect inertia and viscosity)

Quick review of linear elasticity



$\underline{u} = \underline{u}_d$

- Kinematical compatibility

$$\mathbf{u} = \mathbf{u}^{\mathrm{d}} \text{ on } \partial\Omega_D, \quad \varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \text{ on } \Omega, \quad [\![\mathbf{u}]\!] = \mathbf{0} \text{ on } \Sigma$$

$\underline{u} \longleftrightarrow \underline{\underline{\varepsilon}}$

- Equilibrium

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ on } \Omega, \quad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{T} \text{ on } \partial\Omega_{\mathrm{N}}, \quad [\![\boldsymbol{\sigma}\cdot\mathbf{n}]\!] = \mathbf{0} \text{ on } \Sigma$$

$\underline{\underline{\varepsilon}}$

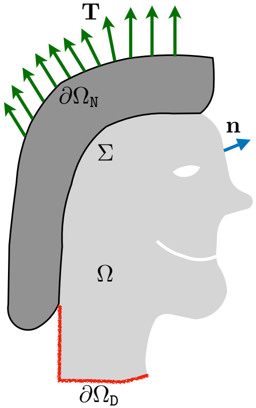
- Constitutive equations

$$w(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon}, \quad \boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{C} : \boldsymbol{\varepsilon} \qquad \begin{cases} \boldsymbol{\sigma} = \lambda\, \mathrm{Tr}\boldsymbol{\varepsilon}\, \mathbb{I} + 2\mu\, \boldsymbol{\varepsilon}, & \mu > 0,\ k := \lambda + \frac{2}{3}\mu > 0 \\ \boldsymbol{\varepsilon} = -\frac{\nu}{E}\, \mathrm{Tr}\boldsymbol{\sigma}\, \mathbb{I} + \frac{1+\nu}{E}\boldsymbol{\sigma}, & E > 0,\ -1 < \nu < \frac{1}{2} \end{cases}$$

The Boundary Value problem (BVP)

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\boxed{\text{Given} \quad (\boldsymbol{u}^{\mathrm{d}}, \mathbf{f}, \mathbf{T}, \mathbf{C}) \quad \text{find} \quad (\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})}$$



- Space of kinematically admissible displacement

$$\mathcal{C} \equiv \{\boldsymbol{u} \in H^1(\Omega) : \quad \boldsymbol{u} = \boldsymbol{u}^{\mathrm{d}} \text{ on } \partial\Omega_{\mathrm{D}}\}$$

$$\mathcal{C}_0 \equiv \{\underline{u} \in H^1(\Omega) : \quad \underline{u} = \underline{0} \text{ on } \partial\Omega_D\} \rightarrow \text{vector space}$$

- Potential energy

$$\mathcal{E}(\underline{u}) = \int_\Omega \frac{1}{2} \underline{\underline{\varepsilon}}(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u}) \, d\Omega - \int_\Omega \underline{f} \cdot \underline{u} \, d\Omega - \int_{\partial_N \Omega} \underline{T} \cdot \underline{u} \, ds$$

- Minimum principle and stationarity condition

Find $\underline{u}^* \in \mathcal{C}$: $\quad \mathcal{E}(\underline{u}^*) \leq \mathcal{E}(\underline{u}) \quad , \forall \underline{u} \in \mathcal{C}$

$\leq \mathcal{E}(\underline{u}^* + \underline{v}) \quad , \forall \underline{v} \in \mathcal{C}_0$

First order optimality condition

$$\mathcal{E}'(\underline{u})(\underline{v}) := \left. \frac{d}{dh} \mathcal{E}(\underline{u} + h\underline{v}) \right|_{h=0} = 0 \Rightarrow$$

$$\mathcal{E}(\underline{u}) = \int_\Omega \left(\frac{1}{2}\underline{\underline{\varepsilon}}(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u}) - \underline{f}\cdot\underline{u}\right)dx - \int_{\partial\Omega_N} \underline{T}\cdot\underline{u}\, ds$$

$$\mathcal{E}(\underline{u}+h\underline{v}) = \int_\Omega \frac{1}{2}\underline{\underline{\varepsilon}}(\underline{u}+h\underline{v}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u}+h\underline{v})\, dx - \int_\Omega \underline{f}\cdot(\underline{u}+h\underline{v})\,dx - \int_{\partial\Omega_N} \underline{T}\cdot(\underline{u}+h\underline{v})\,ds$$

$$\left.\frac{d}{dh}\mathcal{E}(\underline{u}+h\underline{v})\right|_{h=0} = \left.\int_\Omega \underline{\underline{\varepsilon}}(\underline{u}+h\underline{v}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{v})\,dx\right|_{h=0} - \int_\Omega \underline{f}\cdot\underline{v}\,dx - \int_{\partial\Omega_N}\underline{T}\cdot\underline{v}\,ds$$

$$\mathcal{E}'(\underline{u})(\underline{v}) = \overbrace{\int_\Omega \overbrace{\underline{\underline{\varepsilon}}(\underline{u}) : \mathbb{C}}^{\underline{\underline{\sigma}}} : \underline{\underline{\varepsilon}}(\underline{v})\,dx}^{a(\underline{u},\underline{v}) \text{ bilinear form}} - \underbrace{\left(\int_\Omega \underline{f}\cdot\underline{v}\,dx + \int_{\partial\Omega_N}\underline{T}\cdot\underline{v}\,dS\right)}_{\ell(\underline{v}) \text{ linear form}}$$

Trouver $\underline{u} \in \mathcal{E}$:

$$\mathcal{E}'(\underline{u})(\underline{v}) = 0 \quad \forall \underline{v} \in \mathcal{E}_0$$

$$a(\underline{u},\underline{v}) = \ell(\underline{v})$$

Stress criteria and defects in linear elasticity

Limit of validity of linear elastic model is usually expressed in the form

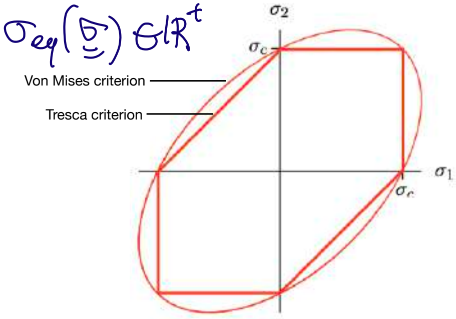
$$\boldsymbol{\sigma} = \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon})\, \mathbf{I} + 2\mu\boldsymbol{\varepsilon}$$

$\sigma_{eq} : \underline{\underline{\sigma}} \in \mathbb{R}^n \times \mathbb{R}^n \to \sigma_{eq}(\underline{\underline{\sigma}}) \in \mathbb{R}^+$

$$\boxed{\sigma_{\mathrm{eq}} \leq \sigma_c}$$

Equivalent stress (scalar)

Maximum allowable stress



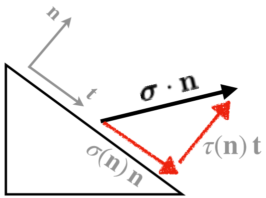
- Von Mises criterion

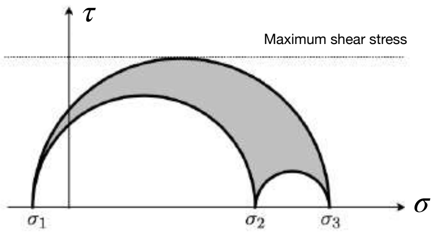
DEVIATORIC PART

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}}, \quad \mathbf{s} = \boldsymbol{\sigma} - \underbrace{\frac{\mathrm{Tr}\boldsymbol{\sigma}}{3}\mathbb{I}}_{\text{ISOTROPIC PART}}$$

- Tresca Criterion

$$\sigma_{\mathrm{eq}} = \sup_{|\mathbf{n}|=1} \tau(\mathbf{n}), \quad \boldsymbol{\sigma} \cdot \mathbf{n} = \sigma(\mathbf{n})\mathbf{n} + \tau(\mathbf{n})\mathbf{t}$$

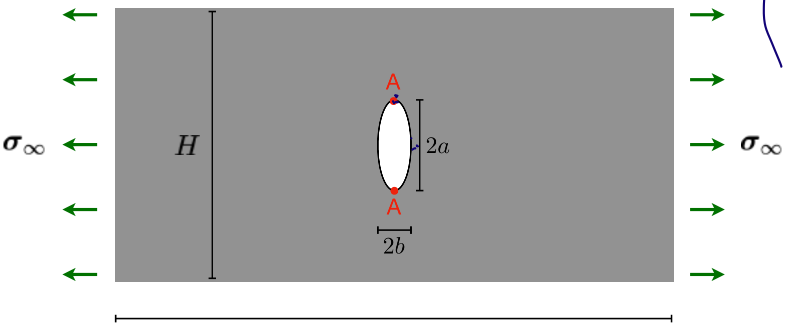




Mohr's circles representation of the stress state (σ_i are the principal stresses)

Other important criteria are Mohr-Coulomb, Drucker-Prager, ….

$$\boldsymbol{\sigma}_\infty = \sigma_\infty\,(\mathbf{e}_1 \otimes \mathbf{e}_1)$$



$$L > H \gg (a, b)$$

- The application of the stress criteria, given the critical stress σ_c gives:

$$\sigma_\infty < \frac{\sigma_c}{1 + 2\frac{a}{b}}$$

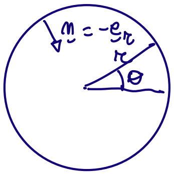
$\sigma_{ey} = \sigma_{\theta\theta}$

$\sigma_{rr} = \sigma_{r\theta} = 0$

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}$

- **Elliptical defect**
  - For an infinite plate at the hole boundary:

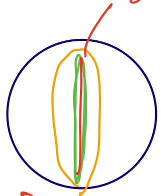
$$\boxed{\sigma_{\theta\theta}(A) = \underbrace{\left(1 + 2\frac{a}{b}\right)}_{R} \sigma_\infty}$$



Stress intensity factor

$R = 1 + 2\dfrac{a}{b}$

$b \to 0$



$R \to \infty$

- For a circular plate

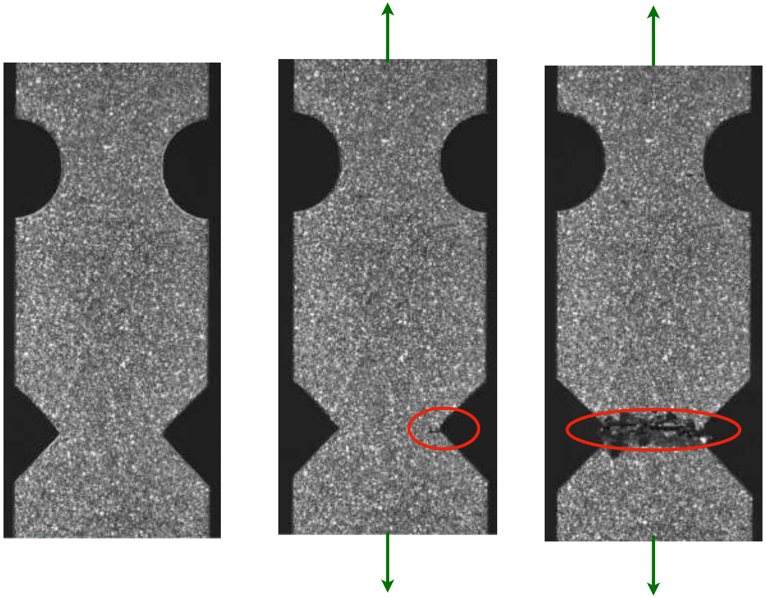
$$\frac{a}{b} = 1 \quad \Rightarrow \quad \sigma_\infty < \frac{\sigma_c}{3}$$

- For a "crack"

$$b \to 0 \quad \Rightarrow \quad \sigma_\infty < 0 \quad (?)$$

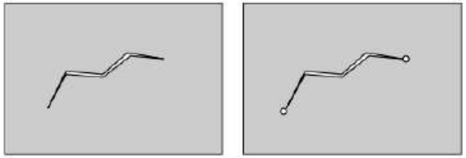
The stress becomes singular
The stress criterion is not good in this case!

Defects and singularities

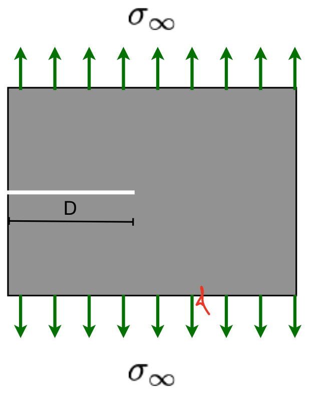


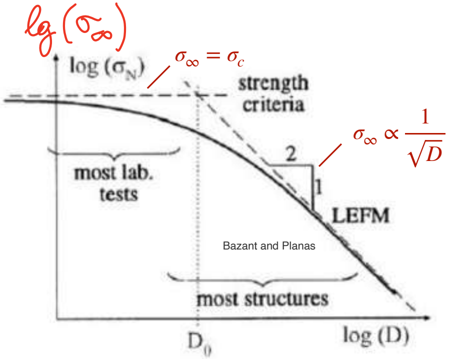
- A V-Notch is worst than a hole
- Why?

A simple technique to mitigate crack risk …



Critical stress as a function of the crack length





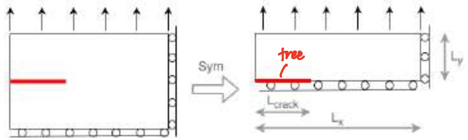
Schematic representation of the experimental observations

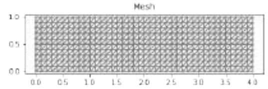
Bazant

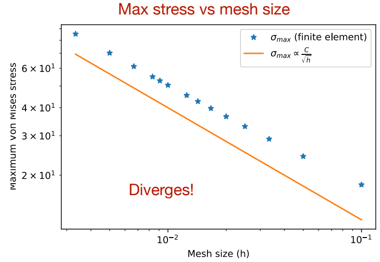
- The experimental observations cannot be explained with the stress criterion only
- There is a size effect (a subtle point from the theoretical point of view)

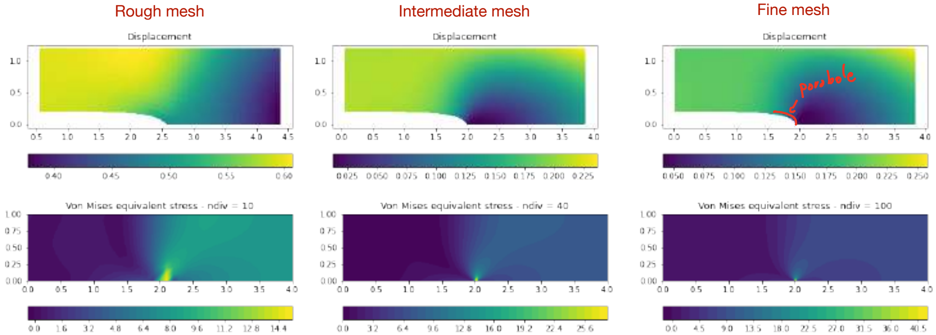
Plate with a crack: finite element results in linear elasticity

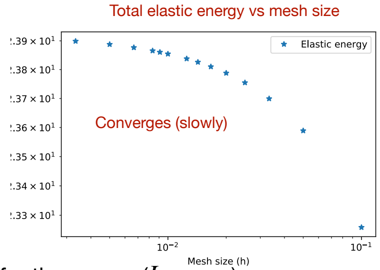
- Finite element solution of the linear elastic problem of a slab with a crack. The crack is a given stress-free surface:











- Warning! The maximum stress in the numerical simulations is mesh-dependent.
  There is not converge for the point-wise value of the stress (L_∞-norm), but there is convergence for the energy (L_2-norm)

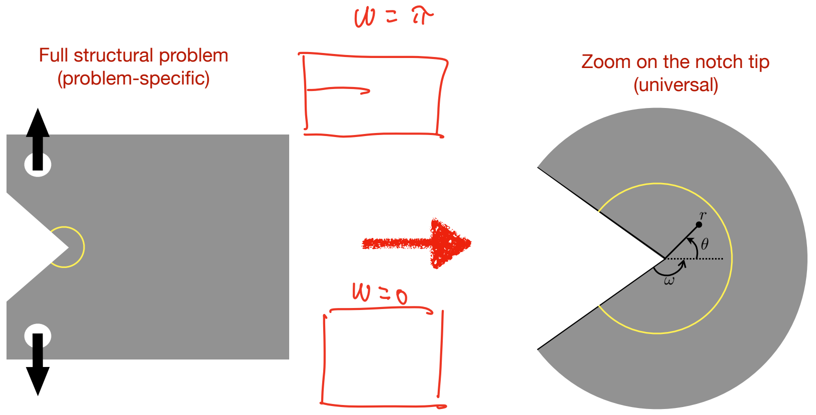
- Indeed, in the limit for the mesh size going to zero, it is infinite, we need a new way to characterise how “big it is”

Comments on stress criteria and singluarities

- Stress criteria are not applicable for V-notches and cracks where stress tends to infinity
- We need to better understand the solution around V-notches and cracks
- We need to define other criteria (not the maximum stress) when the stresses are singular
- How can correctly compute numerically the stress? How to catch and characterise the singular behaviour?
- The hypotheses of small deformations and linear elastic behaviour seem absurd when the strain/stress are singular. Do we need to leave these hypotheses ? The answer (at least for brittle fracture) is NO!

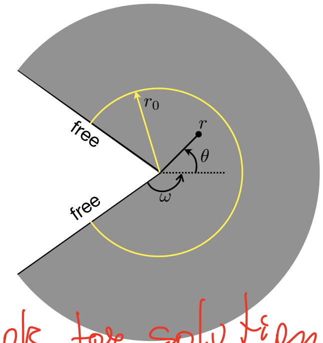
Introduction to singularities in linear elasticity: anti-plane elasticity and mode-III cracks

Main idea of the asymptotic approach



- Instead of solving the full problem, let us focus on the notch tip only
- Study (analytically) the possible solution for V-notch as a function of the notch opening-angle ω
- Link the “outer” structural problem with the “inner” notch problem through as *stress-intensity factor* (SIF)

Form of singularities in plane linear elasticity



- **Form of displacements**

$$\mathbf{u}(r,\theta) = \sum_i^N K_i\, r^{\lambda_i} \mathbf{U}^i(\theta) + \cdots, \quad \text{at the neighborhood of } r = 0$$

$$\begin{cases} N : \text{number of singularities} \\ \lambda_i : \text{strength of singularity} \end{cases} \qquad \begin{cases} \mathbf{U}^i : \text{angular function of i-th singularity} \\ K_i : \text{intensity factor of i-th singularity} \end{cases}$$

- **Form of stresses (and strains)**

$$\boldsymbol{\sigma}(r,\theta) = \sum_i^N K_i\, r^{\lambda_i - 1} \tilde{\sigma}^i(\theta) + \cdots, \quad \text{at the neighborhood of } r = 0$$

- **Constraints on the strength of singularity near the tip** $r_0 \to 0$
  - unbounded stresses: $\lambda_i < 1$ ($\lambda_i > 1$: solution not singular)
  - bounded elastic energy: $\lambda_i > 0$

$$\int_\Omega \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \mathrm{d}\Omega \sim \int_0^{2\pi} \int_0^{r_0} r^{\lambda_i - 1} r^{\lambda_i - 1} r \mathrm{d}r\mathrm{d}\theta \sim \int_0^{r_0} r^{2\lambda_i - 1} \mathrm{d}r = \frac{1}{2\lambda_i} r^{2\lambda_i}$$

We look for solutions in the form

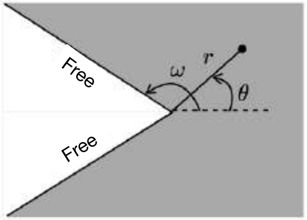
$$\underline{u}(r,\theta) = r^{\lambda}\, \underline{U}(\theta)$$

singularité — angular function

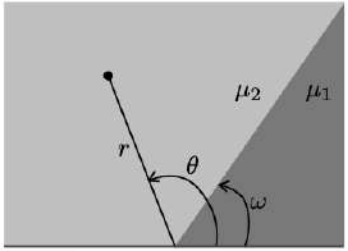
$$0 < \lambda < 1$$

Singulaire si: $\lambda < 1$, $\sigma \sim r^{\lambda-1}$, $\varepsilon \sim r^{\lambda-1}$ $\Rightarrow$ Finite energy $\Leftrightarrow$ $\lambda > 0$

Geometric

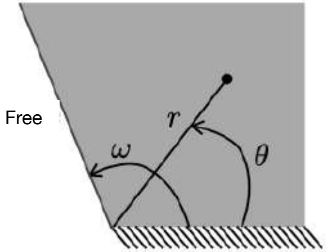


Bimaterial



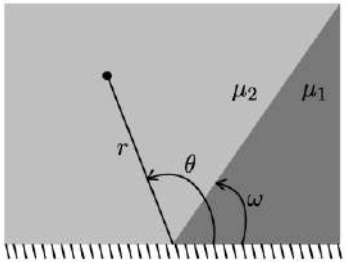
Free boundary

Boundary conditions



Clamped boundary

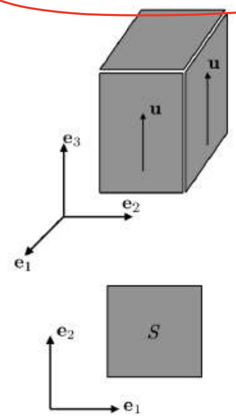
Bimaterial



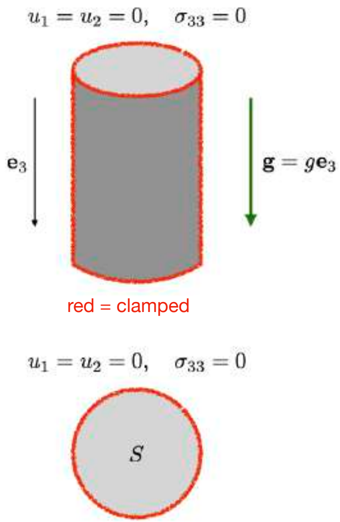
Fixed boundary

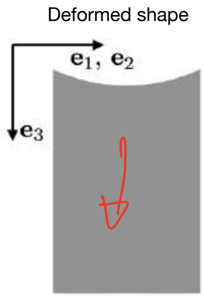
The antiplane displacement *ansatz*

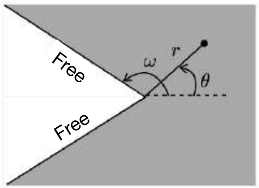
$$\mathbf{u}(x_1, x_2, x_3) = u_3(x_1, x_2)\,\mathbf{e}_3$$



Example: heavy clamped cylinder







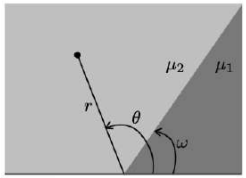
Finite stress

Condition for singularity

$$\frac{\pi}{2} < \omega < \pi$$

Displacement field

$$u_3 = K\, r^{\frac{\pi}{2\omega}} \sin \frac{\pi}{2\omega}\theta + \ldots$$



Free boundary

Condition for singularity

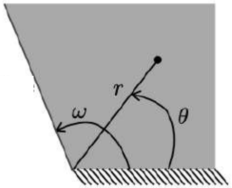
$$0 < \omega < \frac{\pi}{2}, \quad \mu_1 > \mu_2$$

Displacement field

$$u_3 = K\, r^{\lambda}\, U_3(\theta)$$

$$\lambda: \quad \tan \lambda\omega = -\frac{\mu_2}{\mu_1} \tan \lambda(\pi - \omega)$$

$$U_3(\theta) = \begin{cases} \cos \lambda(\pi - \omega)\, \cos \lambda\theta, & 0 \le \theta \le \omega \\ \cos \lambda\omega\, \cos \lambda(\pi - \theta), & \omega \le \theta \le \pi \end{cases}$$



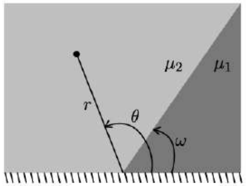
Clamped

Condition for singularity

$$\frac{\pi}{2} < \omega < 2\pi$$

Displacement field

$$u_3 = K\, r^{\frac{\pi}{2\omega}} \sin \frac{\pi}{2\omega}\theta + \ldots$$



Fixed boundary

Condition for singularity

$$0 < \omega < \frac{\pi}{2}, \quad \mu_1 < \mu_2$$

Displacement field

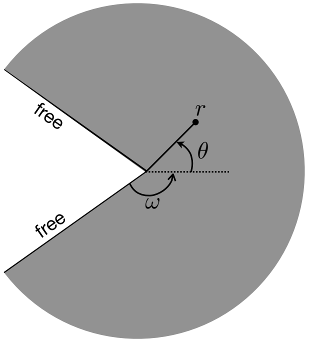
$$u_3 = K\, r^{\lambda}\, U_3(\theta)$$

$$\lambda: \quad \tan \lambda\omega = -\frac{\mu_2}{\mu_1} \tan \lambda(\pi - \omega)$$

$$U_3(\theta) = \begin{cases} \cos \lambda(\pi - \omega)\, \cos \lambda\theta, & 0 \le \theta \le \omega \\ \cos \lambda\omega\, \cos \lambda(\pi - \theta), & \omega \le \theta \le \pi \end{cases}$$

We study this examples in details and prove the formulas (possible exercices for the exam …)

Exercice 1: V-notch in antiplane elasticity



- Assuming linear elasticity, show that if the body forces are regular the solution at a V-notch is in the form

  1. The stress is singular if and only if $\pi/2 < \omega \leq \pi$
  2. The most singular displacement and stress field satisfying the governing equations are in the form

$$u_z = K r^{\frac{\pi}{2\omega}} \sin\left(\frac{\pi}{2\omega}\theta\right) + \cdots \qquad \sigma_{rz} = \frac{K\pi}{2\omega} r^{\frac{\pi}{2\omega}-1} \sin\left(\frac{\pi}{2\omega}\theta\right) + \cdots$$

Antiplane elasticity

ANSATZ : $\underline{u}(x,y,z) = u_3(x,y)\,\underline{e}_3 = u_3(r,\theta)\,\underline{e}_3$

DEFORMATIONS

$$\underline{\underline{\nabla u}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & 0 \end{bmatrix}_{(\underline{e}_1,\underline{e}_2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_3}{\partial r} & \frac{\partial u_3}{\partial \theta} & 0 \end{bmatrix}_{(\underline{e}_r,\underline{e}_\theta)}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla u}} + \underline{\underline{\nabla}}^T u\right) = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial y} \\ \frac{1}{2}\frac{\partial u_3}{\partial x} & \frac{1}{2}\frac{\partial u_3}{\partial y} & 0 \end{bmatrix} = \ldots$$

STRESS

$$\underline{\underline{\sigma}} = \lambda \underbrace{\cancel{\mathrm{tr}\,\underline{\underline{\varepsilon}}}}_{0}\,\underline{\underline{1}} + 2\mu\,\underline{\underline{\varepsilon}} = \mu \begin{bmatrix} 0 & 0 & \frac{\partial u_3}{\partial x} \\ 0 & 0 & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & 0 \end{bmatrix}$$

EQUILIBRE

$$\underline{\mathrm{div}}\,\underline{\underline{\sigma}} + \underline{f} = 0 \quad \text{dans } \Omega \qquad \sigma_{ij,j} + f_i = 0$$

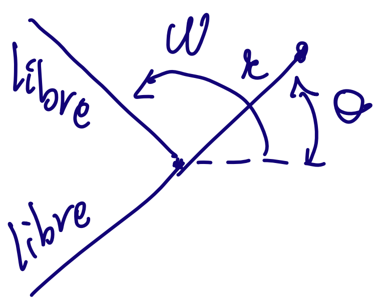
$$\cancel{\sigma_{11,1}} + \cancel{\sigma_{12,2}} + \cancel{\sigma_{13,3}} + f_1 = 0 \qquad f_1 = 0$$

$$\cancel{\sigma_{21,1}} + \cancel{\sigma_{22,2}} + \cancel{\sigma_{23,3}} + f_2 = 0 \qquad f_2 = 0$$

$$\sigma_{31,1} + \sigma_{32,2} + \cancel{\sigma_{33,3}} + f_3 = 0$$

$$\mu\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2}\right) + f_3 = 0 \Leftrightarrow \boxed{\Delta u_3 + \frac{f_3}{\mu} = 0}$$

equilibre



EQUILIBRE

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_3}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2} + \frac{f_3}{\mu} = 0$$

CONDITIONS AUX BORDS

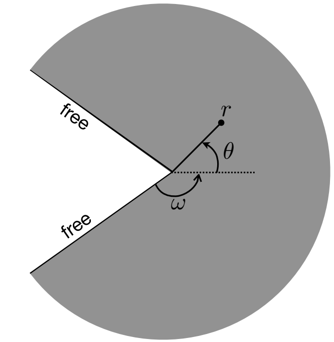
$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \qquad \theta = \omega \Rightarrow \underline{n} = \underline{e}_\theta$$

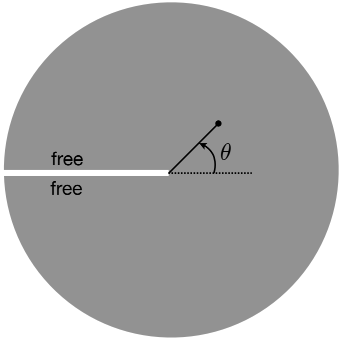
$$\theta = -\omega \Rightarrow \underline{n} = -\underline{e}_\theta$$

$\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ pour $\theta = \pm\omega$

$\hookrightarrow$ $\dfrac{\partial u_3}{\partial \theta} = 0$ pour $\theta = \pm \omega$

V-notch: singular solution in anti-plane linear elasticity





- Single V-notch with free edges
  - Assumptions: linear elasticity, regular body forces

$$\boxed{\frac{\pi}{2} < \omega \leq \pi}$$

$$u_z = K r^{\frac{\pi}{2\omega}} \sin\left(\frac{\pi}{2\omega}\theta\right) + \cdots$$

$$\sigma_{rz} = \frac{K\pi}{2\omega} r^{\frac{\pi}{2\omega}-1} \sin\left(\frac{\pi}{2\omega}\theta\right) + \cdots$$

- Single V-notch with free edges
  - Assumptions: linear elasticity, regular body forces

$$\boxed{\omega = \pi}$$

$$u_z = K\sqrt{r}\sin(2\theta) + \cdots$$

$$\sigma_{rz} = \frac{K}{2\sqrt{r}}\sin(2\theta) + \cdots$$

General Remarks

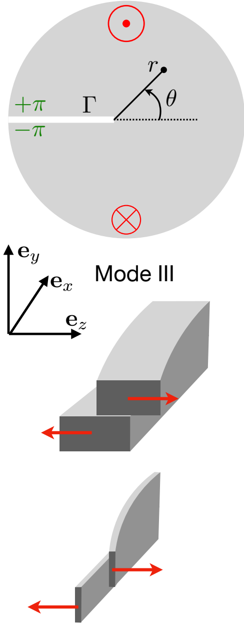
- The multiplicative constant K (the singularity intensity factor) is indeterminate at this stage. It is a global quantity depending on the entire set of data of the BVP (geometry, elasticity, loading conditions)
- The body forces do not affect the results provided that they are not (too) singular (they enter in the regular terms of the solution and in K)
- We obtain the same singular solution if the crack faces are subjected to surface tractions that are not (too) singular
- The crack corresponds to the lowest power (i.e. the strongest singularity)

- Displacement field: $\boxed{\mathbf{u}(r,\theta) = u_z(r,\theta)\mathbf{e}_z}$

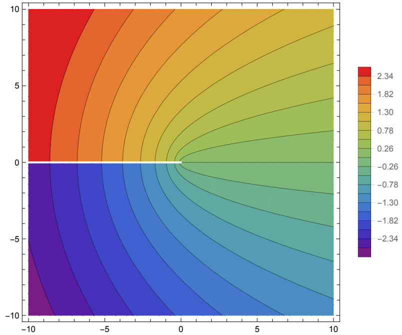
STRESS INTENSITY FACTOR (SIF): $\mathrm{K}_{\mathrm{III}}$

$$u_z = \frac{2\mathrm{K}_{\mathrm{III}}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2} + \mathcal{O}(r), \qquad \begin{pmatrix}\sigma_{rz}(r,\theta)\\ \sigma_{\theta z}(r,\theta)\end{pmatrix} = \boxed{\frac{\mathrm{K}_{\mathrm{III}}}{\sqrt{2\pi r}}}\begin{pmatrix}\sin\dfrac{\theta}{2}\\ \cos\dfrac{\theta}{2}\end{pmatrix} + \mathcal{O}(r^0)$$

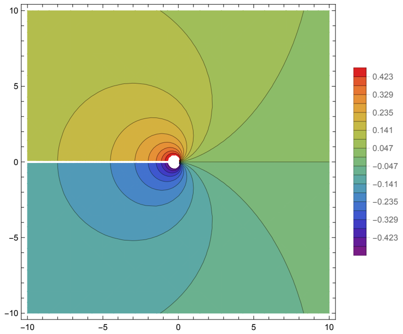
Units of SIF : $\mathrm{MPa}\sqrt{\mathrm{m}}$



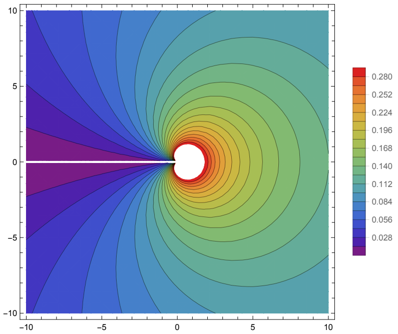
$\mu u_z/\mathrm{K}_{\mathrm{III}}$



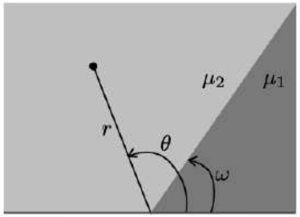
$\sigma_{rz}/\mathrm{K}_{\mathrm{III}}$



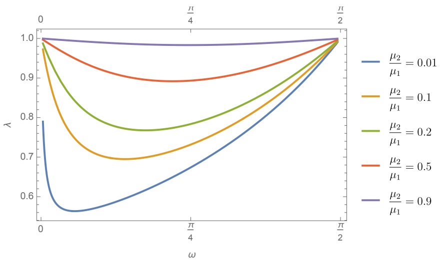
$\sigma_{\theta z}/\mathrm{K}_{\mathrm{III}}$



We will work out this solution in the PC.



Free boundary



- Assuming linear elasticity, show that if the body forces are regular the solution at a bimestrial interface is such that

  1. The stress is singular if and only if $0 < \omega \leq \pi/2$, $\mu_1 > \mu_2$ (acute angle on the stiffer material)

  2. The singular displacement field satisfying the governing equations are in the following form. Give also the corresponding strain and stress fields

$$u_3 = K\, r^\lambda\, U_3(\theta) \qquad \lambda: \quad \tan \lambda\omega = -\frac{\mu_2}{\mu_1} \tan \lambda(\pi - \omega)$$

$$U_3(\theta) = \begin{cases} \cos \lambda(\pi - \omega)\, \cos \lambda\theta, & 0 \leq \theta \leq \omega \\ \cos \lambda\omega\, \cos \lambda(\pi - \theta), & \omega \leq \theta \leq \pi \end{cases}$$

End of Lesson I