Numerical solutions of differential equations

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Lecture 3

The Heat Equation - Part 2

Repetition: Stability of linear ODE solvers

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Example. Linear test equation: for $\lambda \in \mathbb{C}$, find $u(t)$ with

$$\frac{d}{dt}u(t) = \lambda u(t) \qquad \text{and} \qquad u(0) = u_0.$$

Solution:

$$u(t) = u_0\, e^{\lambda t}.$$

If λ is real and $\lambda < 0$ we have

$$u(t) = u_0\, e^{\lambda t} \to 0 \qquad \text{for } t \to \infty.$$

<u>This behavior should be reproduced for discretizations.</u>

Repetition: Stability of linear ODE solvers

Example. Linear test equation: for $\lambda < 0$, find $u(t)$ with

$$\frac{d}{dt}u(t) = \lambda u(t) \qquad \text{and} \qquad u(0) = u_0 \quad \Rightarrow \quad u(t) = u_0\, e^{\lambda t}.$$

Explicit Euler discretization with step size $\triangle t > 0$:

$$u_{n+1} = u_n + \triangle t\, \lambda u_n \qquad\qquad (u_{n+1} = u_n + \triangle t\, f(t_n, u_n))$$

Hence:

$$u_n = \underbrace{(1 + \triangle t\, \lambda)}_{=:\Phi(\triangle t \lambda)} u_{n-1} = (1 + \triangle t\, \lambda)^n u_0$$

Case 1: $|\lambda \triangle t| < 2 \quad \Rightarrow \quad -1 < (1 + \triangle t\, \lambda) < 1.$ Physically correct behavior

$$u_n = (1 + \triangle t\, \lambda)^n u_0 \quad \xrightarrow{n \to \infty} \quad 0$$

Case 2: $|\lambda \triangle t| \geq 2 \quad \Rightarrow \quad 1 + \triangle t\, \lambda \leq -1.$ Unphysical behavior

$u_n = (1 + \triangle t\, \lambda)^n u_0$ is strongly oscillating (with blow-up).

In Case 2 the method is not numerically stable!

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$$\frac{d}{dt}u(t) = \lambda u(t) \quad \text{and} \quad u(0) = u_0 \quad \Rightarrow \quad u(t) = u_0\, e^{\lambda t}.$$

General numerical one step scheme with step size $\triangle t$:

$$u_n = \Phi(\triangle t\lambda)u_{n-1} \quad \Rightarrow \quad u_n = \Phi(\triangle t\lambda)^n u_0.$$

The scheme defined through Φ and $\triangle t$ is stable, if

$$\triangle t\lambda \in \{z \in \mathbb{C} |\, |\Phi(z)| < 1\} = \underline{\text{Stability region}}.$$

Repetition: Stability of linear ODE solvers

General case in $\mathbb{R}^N$: Let (as in our case)

- $\mathbf{A} \in \mathbb{R}^{N\times N}$: real, symmetric and invertible matrix;
- hence, $\mathbf{A}$ is diagonalizable, i.e.

$$\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1},$$

where $\boldsymbol{\Lambda}$ is diagonal matrix of eigenvalues $\lambda_1, \cdots, \lambda_N$ of $\mathbf{A}$, i.e.

$$\boldsymbol{\Lambda} := \mathrm{diag}(\lambda_1, \cdots, \lambda_N).$$

Find $u(t) \in \mathbb{R}^N$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \mathbf{A}\,u(t) \qquad \text{and} \qquad u(0) = u_0 \quad \Rightarrow \quad u(t) = e^{\mathbf{A}t}\,u_0.$$

General numerical one step scheme with step size $\triangle t$:

$$u_n = \Phi(\triangle t\mathbf{A})u_{n-1} \qquad \Rightarrow \qquad u_n = \Phi(\triangle t\mathbf{A})^n u_0.$$

Repetition: Stability of linear ODE solvers

Find $u(t) \in \mathbb{R}^N$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \mathbf{A}\, u(t) \qquad \text{and} \qquad u(0) = u_0 \quad \Rightarrow \quad u(t) = e^{\mathbf{A}t}\, u_0 .$$

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$$u_n = \Phi(\triangle t \mathbf{A}) u_{n-1} \qquad \Rightarrow \qquad u_n = \Phi(\triangle t \mathbf{A})^n u_0 .$$

To define stability, we reduce the problem to the case $N = 1$:
using $\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$ we define $z := \mathbf{R}^{-1} u$ and obtain that z solves

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \boldsymbol{\Lambda}\, z(t) \qquad \text{and} \qquad z(0) = \mathbf{R}^{-1} u_0 \quad \Rightarrow \quad u(t) = \mathbf{R} e^{\boldsymbol{\Lambda}t} \mathbf{R}^{-1} u_0 .$$

We recover N scalar ODEs with $1 \le k \le N$

$$\frac{\mathrm{d}}{\mathrm{d}t} z_k(t) = \lambda_k\, z_k(t) \qquad \Rightarrow \qquad \text{define stability as before for each } \lambda_k .$$

Repetition: Stability of linear ODE solvers

Example: Find $u(t) \in \mathbb{R}^N$ with

$$\frac{d}{dt}u(t) = \mathbf{A}\,u(t) \qquad \text{and} \qquad u(0) = u_0 \quad \Rightarrow \quad u(t) = e^{\mathbf{A}t}\,u_0 .$$

Explicit Euler discretization with step size $\triangle t$:

$$\begin{aligned} u_{n+1} &= u_n + \triangle t \mathbf{A} u_n = (I + \triangle t \mathbf{A}) u_n \\ &= \mathbf{R}(I + \triangle t \mathbf{\Lambda})\mathbf{R}^{-1} u_n = \mathbf{R}(I + \triangle t \mathbf{\Lambda})^{n+1}\mathbf{R}^{-1} u_0 . \end{aligned}$$

Since $I + \triangle t \mathbf{\Lambda}$ is a diagonal matrix with entries $1 + \triangle t \lambda_k$, the Explicit Euler discretization is stable if

$$|1 + \triangle t \lambda_k| < 1 \qquad \text{for all } 1 \leq k \leq N.$$