Sorbonne University

MU5MES09 Composite conception and optimisation Course - CompMech - 2021/2022

Optimisation of stratified structures - application to the aerodynamic field

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Introduction

The conception and optimisation of composite structures is a branch of the solid structure's field which consists into creating structure made or reinforced with fibers or tissues. [1]

As one builds this structure we as well want to be able to study it both analytically for simpler models and numerically for more complex ones. In this report we will derive analytical theory in some simple application cases. We will then extend this study to numerical optimization of an aerodynamic structure.

I. Part 1 - Polar tensors for stratified's behavior law

In this part we will derive several points of the theory [1] and apply it in some general applications.

I.1 CLPT

For one to study composite structure (made of several layers) it's very useful to consider the CLPT, then working with equivalent homogeneous tensor components.

We here assume:

- 2D structure
- plane stress (low thickness in third direction) and plane strain (given how we'll consider displacement)
- linear elastic behavior law under assumption of low perturbation

With this in mind, one can express force applying on structure as:

$$\{N\} = \int_{-h/2}^{+h/2} \{\sigma\}\ dz \qquad \{M\} = \int_{-h/2}^{+h/2} z\{\sigma\}\ dz$$

Introducing $\{\epsilon\} = \{\epsilon^0\} + z\{K\}$ [1][2] we get in 2D ($S \equiv Q$):

$$\{N\} = \int_{-h/2}^{+h/2} [Q]\,.\left(\{\epsilon^0\} + z\{K\}\right)\ dz \qquad \{M\} = \int_{-h/2}^{+h/2} z\,[Q]\,.\left(\{\epsilon^0\} + z\{K\}\right)\ dz$$

from which one eventually deduce

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \cdot \begin{Bmatrix} \{\epsilon^0\} \\ \{K\} \end{Bmatrix} \tag{I.1}$$

with:

$$[A] = \sum_{k=1}^{n} (z_k - z_{k-1}).\,[Q(\delta_k)]$$

$$[B] = \sum_{k=1}^{n} \frac{1}{2}(z_k^2 - z_{k-1}^2).\,[Q(\delta_k)]$$

$$[D] = \sum_{k=1}^{n} \frac{1}{3}(z_k^3 - z_{k-1}^3).\,[Q(\delta_k)]$$

where the terms N and M are the force and moments vectors associated to intern stress while A, B and D are matrices that help building homogeneous equivalent behavior of composite structure.

Each of the above tensors A, B abd D are polar tensors that one can express as a function of polar invariants: T_0, T_1, R_0, R_1, $\phi_0 - \phi_1$.

Identical layers

In the case of identical layers (same material and thickness of each layer) above relations can be recast using basic layer properties T_0^{cb}, T_1^{cb}, R_0^{cb}, R_1^{cb}, ϕ_0^{cb}, ϕ_1^{cb}.

$$A\ :\ \bar{T}_0 = h.T_0^{cb};\ \ \bar{T}_1 = h.T_1^{cb};\ \ \bar{R}_0.e^{4i\bar{\phi}_0} = R_0^{cb}.\frac{h}{n}e^{4i\phi_0^{cb}}\sum_{k=1}^{n} e^{4i\delta_k};\ \ \bar{R}_1.e^{2i\bar{\phi}_1} = R_1^{cb}.\frac{h}{n}e^{2i\phi_1^{cb}}\sum_{k=1}^{n} e^{2i\delta_k}$$

$$B\ :\ \hat{T}_0 = 0;\ \hat{T}_1 = 0;\ \hat{R}_0.e^{4i\hat{\phi}_0} = R_0^{cb}.\frac{h^2}{2.n^2}e^{4i\phi_0^{cb}}\sum_{k=1}^{n} e^{4i\delta_k}.b_k;\ \hat{R}_1.e^{2i\hat{\phi}_1} = R_1^{cb}.\frac{h^2}{2.n^2}e^{2i\phi_1^{cb}}\sum_{k=1}^{n} e^{2i\delta_k}.b_k$$

$$D\ :\ \tilde{T}_0 = \frac{h^3}{12}.T_0^{cb};\ \ \tilde{T}_1 = \frac{h^3}{12}.T_1^{cb};\ \ \tilde{R}_0.e^{4i\tilde{\phi}_0} = R_0^{cb}.\frac{h^3}{12.n^3}e^{4i\phi_0^{cb}}\sum_{k=1}^{n} e^{4i\delta_k}.d_k;$$

$$\tilde{R}_1.e^{2i\tilde{\phi}_1} = R_1^{cb}.\frac{h^3}{12.n^3}e^{2i\phi_1^{cb}}\sum_{k=1}^{n} e^{2i\delta_k}.d_k$$

with $b_k = 2k - n - 1$ and $d_k = 12k(k - n - 1) + 4 + 3n(n + 2)$ (see wolframe mathematica code)

We can then build tensors A, B and D from components listed here above, which are characterized by the material one uses (cb meaning basic layer properties) and the sequence of fibers δ_k.

I.2 CLPT polar - non-dimensional

We here introduce non-dimensionalized tensors $A^* = \frac{1}{h}A$, $B^* = \frac{2}{h^2}B$, $D^* = \frac{12}{h^3}D$. Applying this change to above formulation we get that $\bar{T}_0 = \tilde{T}_0 = T_0^{cb}$ and $\bar{T}_1 = \tilde{T}_1 = T_1^{cb}$ (we remove * terms in polar invariant for simplification's sake).
We can then conclude on the fact components of membrane and flexion T_0 and T_1 are equals, translating a part of quasi-homogeneous behavior in flexion and traction is met.

I.3 Orthotropic layers

In the case of an orthotropic structure we have the relation $\phi_0 - \phi_1 = K\frac{\pi}{4}$, $K = 0, 1$.

Considering we work on the orthotropic framework, we have $\phi_1 = 0$ and so we can rewrite comp. of polar tensors as ($T_{0,1}$ comp not written as unchanged) as:

$$
\begin{aligned}
A^* &: \bar{R}_0.e^{4i\bar{\phi}_0} = R_K^{cb}\frac{1}{n}\sum_{k=1}^{n} e^{4i\delta_k}; \quad \bar{R}_1.e^{2i\bar{\phi}_1} = R_1^{cb}.\frac{1}{n}e^{2i\phi_1^{cb}}\sum_{k=1}^{n} e^{2i\delta_k} \\
B^* &: \hat{R}_0.e^{4i\hat{\phi}_0} = R_K^{cb}.\frac{1}{n^2}\sum_{k=1}^{n} e^{4i\delta_k}.b_k; \quad \hat{R}_1.e^{2i\hat{\phi}_1} = R_1^{cb}.\frac{1}{n^2}e^{2i\phi_1^{cb}}\sum_{k=1}^{n} e^{2i\delta_k}.b_k \\
D^* &: \tilde{R}_0.e^{4i\tilde{\phi}_0} = R_K^{cb}.\frac{1}{n^3}\sum_{k=1}^{n} e^{4i\delta_k}.d_k; \quad \tilde{R}_1.e^{2i\tilde{\phi}_1} = R_1^{cb}.\frac{1}{n^3}e^{2i\phi_1^{cb}}\sum_{k=1}^{n} e^{2i\delta_k}.d_k
\end{aligned}
$$

where $R_0.e^{4i\phi_0} = R_0.cos(K\pi) = (-1)^K.R_0 = R_K$.

With these normalized variables the point will now be to work more on the sequence of stratification.

I.4 Specific stratification

We still consider tensors A^*, B^*, D^* and we want to use them on specific stratifications in order to get associated form of polar tensors.

I.4.1 UD stratifiaction

We consider layers aligned in 0° such that $e^{\delta_k \cdots} = 1$ and so it comes (using partly wolframe mathematica) (T_α, $\alpha = 0, 1$ still unchanged):

$$
\begin{aligned}
A^* &: \bar{R}_0.e^{4i\bar{\phi}_0} = R_K^{cb}; \quad \bar{R}_1.e^{2i\bar{\phi}_1} = R_1^{cb}.e^{2i\phi_1^{cb}} \\
B^* &: \hat{R}_0.e^{4i\hat{\phi}_0} = 0; \quad \hat{R}_1.e^{2i\hat{\phi}_1} = 0 \\
D^* &: \tilde{R}_0.e^{4i\tilde{\phi}_0} = R_K^{cb}; \quad \tilde{R}_1.e^{2i\tilde{\phi}_1} = R_1^{cb}.e^{2i\phi_1^{cb}}
\end{aligned}
$$

In this case we indeed see components T (already verified) and R are expressed directly from layer's characteristics. Indeed, as basic layers are orthotropic, we get $\phi_1^{cb} = 0$ (frame ortho.) and $\phi_0^{cb} = K\frac{\pi}{4}$ such that:

$$
\begin{aligned}
A^* &: \bar{R}_0.e^{4i\bar{\phi}_0} = R_K^{cb}; \quad \bar{R}_1.e^{2i\bar{\phi}_1} = R_1^{cb} \\
B^* &: \hat{R}_0.e^{4i\hat{\phi}_0} = 0; \quad \hat{R}_1.e^{2i\hat{\phi}_1} = 0 \\
D^* &: \tilde{R}_0.e^{4i\tilde{\phi}_0} = R_K^{cb}; \quad \tilde{R}_1.e^{2i\tilde{\phi}_1} = R_1^{cb}
\end{aligned}
$$

and so in an orthotropic frame ($\tilde{\bar{\phi}}_1 = 0$) we get :

$$
\bar{R}_1.e^{2i\bar{\phi}_1} = \bar{R}_1 = \bar{R}_1^{cb} \tag{I.2}
$$

$$
\bar{R}_0.e^{4i\bar{\phi}_0} = \bar{R}_0.(cos(4\bar{\phi}_0) + i.sin(4\bar{\phi}_0)) = R_K^{cb} \quad \Rightarrow \quad 4\bar{\phi}_0 = K.\pi \quad \Rightarrow \quad \bar{\phi}_0 = K.\frac{\pi}{4} \tag{I.3}
$$

and equivalently for D. One can then rewrite above equalities as : $\tilde{C}_0 = \bar{C}_0 = C_0^{cb}$ with $C = R, T$ and recalling we removed * terms. Moreover $\hat{C}_0 = 0$ such that system is decoupled in flexion/-traction.

We therefore get same generalized polar components than for basic orthotropic layer.

I.4.2 cross-ply

We consider a sequence made of n_0 terms at 0° and n_{90} terms at 90°. Recalling that we work with orthotropic basic layers in the orthotropic frame (i.e. $\phi_1^{cb} = 0$ and $\phi_0^{cb} = \frac{K\pi}{4}$) ; and introducing $\rho_{90} = \frac{n_{90}}{n}$ and $\rho_0 = 1 - \rho_{90}$ we can then express non-dimensional comps as (T unchanged again):

$$A^* \;:\; \begin{cases} \bar{R}_0.e^{4i\bar{\phi}_0} = R_K^{cb}\frac{1}{n}\left(\sum_{k=1}^{n_0} 1 + \sum_{k=1}^{n_{90}} 1\right) = R_K^{cb}; \\ \bar{R}_1.e^{2i\bar{\phi}_1} = R_1^{cb}.\frac{1}{n}\left(\sum_{k=1}^{n_0} 1 - \sum_{k=1}^{n_{90}} 1\right) = R_1^{cb}.\left(\frac{n_0}{n} - \frac{n_{90}}{n}\right) = R_1^{cb}.(1-2.\rho_{90}) \end{cases} \tag{I.4}$$

$$\tag{I.5}$$

$$D^* \;:\; \begin{cases} \tilde{R}_0.e^{4i\tilde{\phi}_0} = R_K^{cb}.\frac{1}{n^3}\sum_{k=1}^{n} d_k = R_K^{cb}; \\ \tilde{R}_1.e^{2i\tilde{\phi}_1} = R_1^{cb}.\frac{1}{n^3}\left(\sum_{k=1}^{n_0} d_k - \sum_{k=1}^{n_{90}} d_k\right) = R_1^{cb}.\left(\frac{n_0^3}{n^3} - \frac{n_{90}^3}{n^3}\right) = R_1^{cb}.\left[(1-\rho_{90})^3 - \rho_{90}^3\right] \end{cases} \tag{I.6}$$

with $\rho_{90} \in [0, 0.5]$ to have a unique principal axis of orthotropy to consider.

Then :

- $\rho_{90} = 0 \Rightarrow$ we have only layers at $0°$ and so we are back to the unidirectional stratification case ;
- $\rho_{90} = 0.5 \Rightarrow$ we have half the layers in one direction and half in other perpendicular-to-first one such that then $\bar{R}_1 = \tilde{R}_1 = 0$ which is characteristic of a composite structure with 2 planes of symmetries (i.e. square symmetry) ;

Orthotropy

Assuming once again we define polar components in orthotropic frame (i.e. $\phi_1^{cb} = 0$) we get same condition than I.2 on $\phi_0^{cb} = K\frac{\pi}{4}$. It then comes that A^* components must be orthotropic to respect above relations I.4.

We deduce subsequently the following relations for A^*:

- $\bar{R}_0 = R_0^{cb} \Rightarrow \bar{R}_K = (-1)^K.R_0^{cb}$
- $\bar{R}_1 = R_1^{cb}.(1-2.\rho_{90})$

I.4.3 angle-ply

We here consider a sequence made of as much α inclinations as $-\alpha$ inclinations. We furthermore consider $\alpha \in [0, 15]$. Re-writing A^* and D^* terms again it comes:

$$A^* \;:\; \begin{cases} \bar{R}_0.e^{4i\bar{\phi}_0} = R_K^{cb}\frac{1}{n}\sum_{k=1}^{n} e^{4i\delta_k} = R_K^{cb}\frac{1}{n}\left(\sum_{k=1}^{n/2} e^{4i\alpha} + \sum_{k=1}^{n/2} e^{-4i\alpha}\right) = R_K^{cb}cos(4\alpha); \\ \bar{R}_1.e^{2i\bar{\phi}_1} = R_1^{cb}\frac{1}{n}\left(\sum_{k=1}^{n/2} e^{2i\alpha} + \sum_{k=1}^{n/2} e^{-2i\alpha}\right) = R_1^{cb}cos(2\alpha) \end{cases}$$

$$D^* \;:\; \begin{cases} \tilde{R}_0.e^{4i\tilde{\phi}_0} = R_K^{cb}\frac{1}{n^3}\left(\sum_{k=1}^{n/2} d_k.e^{4i\alpha} + \sum_{k=1}^{n/2} d_k.e^{-4i\alpha}\right) = R_K^{cb}.cos(4\alpha); \qquad \left(\text{as} \sum_{k=1}^{n} d_k = \frac{1}{n^3}\right) \\ \tilde{R}_1.e^{2i\tilde{\phi}_1} = R_1^{cb}\frac{1}{n^3}\left(\sum_{k=1}^{n/2} d_k.e^{2i\alpha} + \sum_{k=1}^{n/2} d_k.e^{-2i\alpha}\right) = R_1^{cb}.cos(2\alpha); \end{cases}$$

Then, depending on the values α takes we describe different physical states:

- $\alpha = 0° \Rightarrow$ uni-directional stratification case;
- $\alpha = 45° \Rightarrow$ two-directional stratification case which translates physically in a 2 planes-symmetry (i.e. square symmetry).

Orthotropy

Once again if we consider the orthotropic frame (i.e. $\phi_1^{cb} = 0°$) it comes conditions I.2 which tell us the homogeneous equivalent tensor A^* is orthotropic (so is D^*).

Let's conclude on this section by looking at where these 3 cases (UD, cross-ply and angle-ply) stand in the limit orthotropic domain.

I.5 Graphic representation orthotropic frame

In this last section of part 1 we translate relations of section I.4 in a graphical representation:

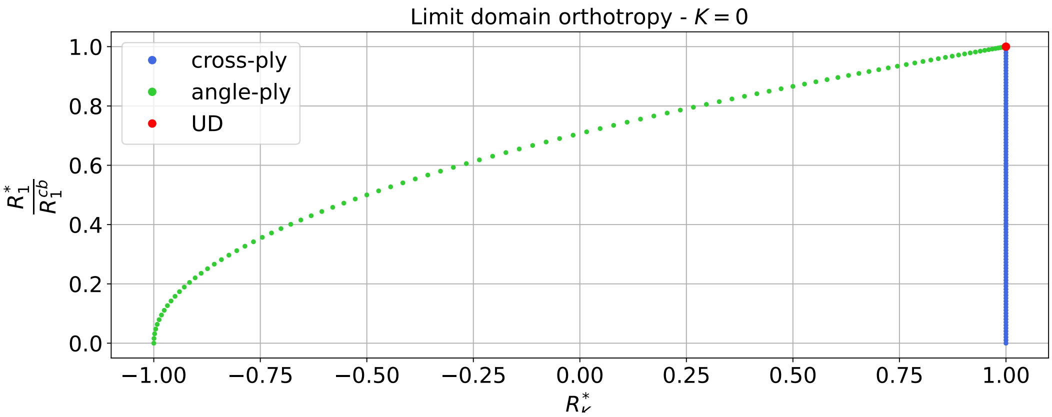

Figure I.1: Limit graph for orthotropic case. Each points of the inner domain between these lines should be a good choice for one to build an orthotropic homogeneous equivalent case.

In figure II.1 we get the three cases devoped in section I.4. It limits where one can pick a couple $(R_1^*,\ R_K^*)$ that ensures A^* and D^* are orthotropic and system is decoupled. Eventually, to get optimal case it depends on application one wants to do with it ; and one must go through an optimization process.

II. Part 2 - Maximisation of global stiffness and buckling of a composite panel

This part of the report takes roots in a real case application : aeronautic structures. We here develop the homogeneous equivalent state of the composite structure and introduce a real application case of structure's optimization.

The structure we will consider is a square plate defined by :

- Domain : 2D $(x, y) \equiv [-L/2, +L/2] \times [-L/2, +L/2]$ with $L = 1\ m$.

- Material : stiff composite plate that may present buckling. All layers are made of same material which is carbon/epoxy T300/5208 whose elastic properties are : $E_1 = 181\ GPa$, $E_2 = 10.3\ GPa$, $G_{12} = 7.17\ GPa$, $\nu_{12} = 181\ GPa$

- Equations/Hypotheses : we assume linear elasticity of each layers together with stress plane (i.e. $h = 4\ mm \ll 1m$) & strain plane theory. We consider simple support on the structure's edges as well as a bi-axial loading characterized by $N_x = 2.N_y = -100\ N.m^{-1}$

- Composite structure : the plate is made of 32 layers that will be assembled through a sequence δ_k, $k = [1, 2, ..., N]$.

With these informations in mind, let's adapt Part 1 study to this case.

II.1 Basic layer

Knowing the structure's elastic properties, one can deduce the associated 2D flexibility tensor Q^{cb} [1]:

$$Q^{cb} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \tag{II.1}$$

from which one may deduce polar components of basic layer using following relations:

$$\begin{aligned}
8T_0 &= Q_{11}- \quad 2.Q_{1,2}+ \quad 4.Q_{3,3}+ \quad Q_{2,2} \\
8T_1 &= Q_{11}+ \quad 2.Q_{1,2} \quad + \quad Q_{2,2} \\
8R_0.e^{4i\phi_0} &= Q_{11}- \quad 2.Q_{1,2}- \quad 4.Q_{3,3}+ \quad Q_{2,2}+ \quad 4i\left(Q_{1,3} - Q_{2,3}\right) \\
8R_1.e^{2i\phi_1} &= Q_{11} \quad - \quad Q_{2,2}+ \quad 4i\left(Q_{1,3} + Q_{2,3}\right)
\end{aligned}$$

These relations, once combined with values of Q^{cb} lead to :

$$\phi_0^{cb} = \phi_1^{cb} = 0 \ \text{ (in principal axis of orthotropy, } \textbf{see CODE - Part 2}\text{)} \quad \Rightarrow \quad R_K^{cb} = R_0^{cb}$$

For the remaining components, we have $T_0^{cb} \approx 8.94e^{-11}$, $T_1^{cb} \approx 6.03e^{-12}$, $R_0^{cb} \approx 5.01e^{-11}$, $R_1^{cb} \approx 1.14e^{-11}$.

II.2 CLPT

From I.2 we know that $T_0^* = T_0^{cb}$ and $T_1^* = T_1^{cb}$ in the case of identical layers (material and thickness), for both A^* and D^*.

II.3 Graphic representation orthotropic frame - numerical application

In this section we derive a similar graph to the one of section I.5, in an orthotropic case with $K = 0$:

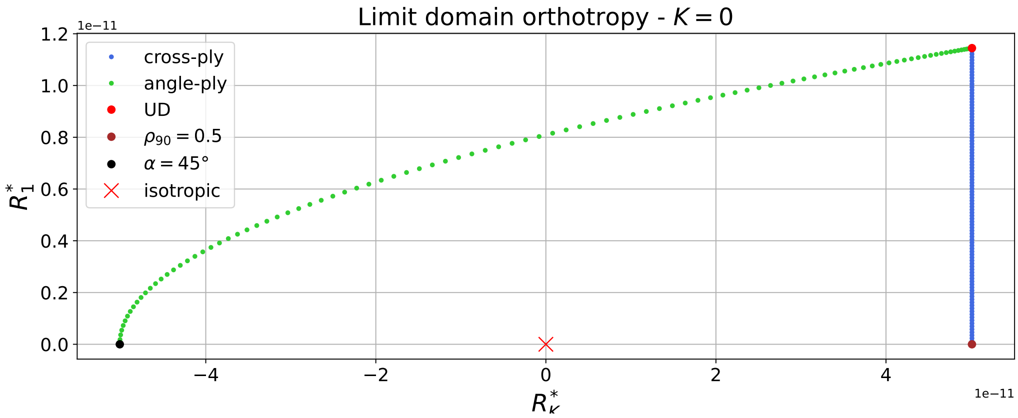

Figure II.1: Limit graph for orthotropic case. i. cross-ply stratification. ii. angle-ply stratification. iii. UD. iv. cross-ply - $\rho_{90} = 0.5$ v. angle-ply - $\alpha = 45°$ vi. isotropic case. See CODE Part 2 for more details on computation undergone.

This figure adapts figure II.1 in a real case application. We furthermore present more specific points, each corresponding to a certain set (R_1^*, R_K^*) and associated physical meaning (isotropy, etc)

II.4 Mechanical point of view

First, here is a drawing of the system we are studying :

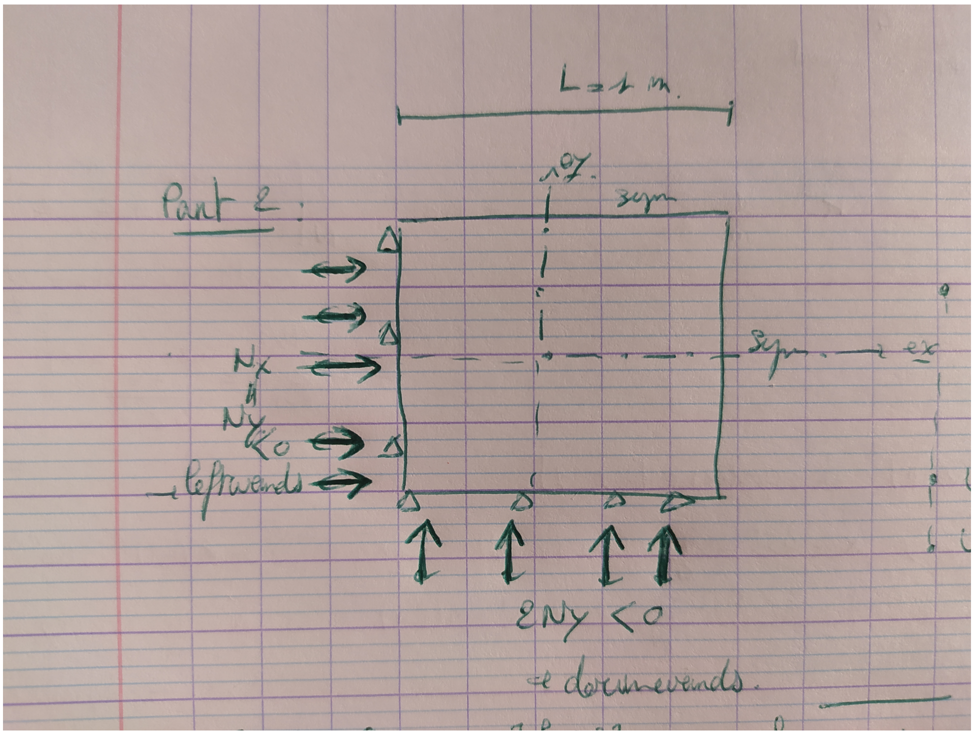

Figure II.2: Drawing of the 2D composite structure and the forces we apply on it. We have a square plate under compressive load at its ends. The structure is furthermore simply supported in each of its 4 sides.

By use of the CLPT we look for components T, R and ϕ associated to laoding state and given by:

$$T = \frac{N_x + N_y}{2}$$
$$R.e^{2i\phi} = \frac{N_x - N_y}{2} + i.N_s$$

Yet, with $N_x = 2.N_y = -100\ N.m^{-1}$ and $N_s = 0\ N.m^{-1}$ it leads to:

$$T = \frac{3.N_y}{2} = -150$$
$$R.e^{2i\phi} = \frac{N_x - N_y}{2} \quad \Rightarrow \quad \phi = 0 \ \ \& \ \ R = \frac{-N_y}{2} = +50$$
$$\Rightarrow \quad X = \frac{R}{|T|} = \frac{1}{3}$$

This last relation brings informations on ratio deviatoric/spherical components in behavior law relations.

At this stage we have an equivalent homogeneous representation of polar components, which allowed us to plot the graph of admissible couples (R_1^*, R_K^*). What we can do now is to apply this theory to an structure's optimisation case.

II.5 Domain variations

The admissible domain of variations for parameters R_K^*, R_1^* and ϕ_1 are:

- R_1^* : this module must be positive such that $R_1^* \geq 0$;
- R_K^* : this module equals $(-1)^K.R_0^*$, $K = 0, 1$ and so it belongs to : $-R_0^* \leq R_K^* \leq +R_0^*$;

- ϕ_1 : this module must be positive such that $\phi_1 \in \left]-\frac{\pi}{2}, +\frac{\pi}{2}\right[$.

where material has a decoupled and quasi-homogeneous behavior.

II.6 Maximisation buckling load

In order to maximize stiffness and buckling load we consider frame $\phi_1 = 0$. Given above physical considerations, we are left with the two parameters R_K^* and R_1^*.

Given the system's geometry, we are looking for a stratified made of a same material (composition+thickness layers). We furthermore consider that half the stratified will be at an angle $+\alpha$ while the other half will be at $-\alpha$ such that we will have a square symmetry (i.e. $R_1 = 0$, R_K).

For such a system as in figure II.2 the buckling multiplier in mode 1-1 writes down:

$$\lambda = \frac{\pi^2.h^3}{36.N_y.L^2}\left[D_{11}^* + 2(D_{12}^* + 2.D_{66}^*) + D_{22}^*\right]$$

where $D_{ij} \equiv Q_{ij}$. Expressing these tensor components in the polar frame, we eventually get:

$$\lambda = \frac{\pi^2.h^3}{36.N_y.L^2}\left[4.T_0^* + 8.T_1^* + 0.R_1^* - 4.R_K^*\right]$$

It then comes that for one to maximize λ value it's gonna be necessary to minimize minus term, namely minimize R_K^* (given $T_{0,1}$ are known). Given $R_0^* \geq 0$ and that $R_K^* = (-1)^K.R_0^*$, we get that $R_{K_{min}}^* = -R_0^*$. In figure II.1 it's located in a point that physically corresponds to the case of a square symmetry for a material built from angle-ply stratifications at $\alpha = \frac{\pi}{4}$.

Compared to the foretold behavior, we here refine domain of existence to a unique point ($R_1 = 0$, $R_K = -R_0$), while we provided a whole domain of existence ($R_1 = 0$, R_K) in our hypothesis. We finally note that there is no influence of R_1 value in λ formula.

Value λ

Let's here compare the value of λ for different stratification and symmetry cases:

- isotropic : $R_0^* = R_1^* = 0 \Rightarrow |\lambda_{iso}| \approx 1.42.e^{-10}$;
- unidirectional : $R_K^* = R_0^{cb} \Rightarrow |\lambda_{UD}| \approx 7.20.e^{-11}$;
- square symmetry at $\alpha = \pi/4$: $R_K^* = -R_0^{cb} \Rightarrow |\lambda_{sol}| \approx 2.13.e^{-10}$;

We deduce the following comparisons:

Case	$\lvert\lambda\rvert$	$\frac{\lvert\lambda-\lambda_{sol}\rvert}{\lvert\lambda_{sol}\rvert} \times 100$ (%)
isotropic	$1.42.e^{-10}$	33
unidirectional (UD)	$7.20.e^{-11}$	66
square symmetry at $\alpha = \pi/4$	$\lambda_{sol} = 2.13.e^{-10}$	X

Table II.1: Implemented grid resolutions

As a conclusion we see that even for stratifications sharing some elements of symmetries, the difference on λ go up to a 66% difference.

II.7 Maximisation stiffness

In this section we tackle down the maximisation process of structure's stiffness. For an orthotropic material, in its framework (i.e. $\phi_1 = 0$), we get a condition of maximisation of stiffness looking at elastic energy

$$W = T^2 . \frac{2X^2T_1 - 4XR_1 + T_0 + R_K}{4\left[T_1\left(T_0 + R_K\right) - 2R_1^2\right]}$$

In fact, we will aim at minimizing $W \equiv$ inverse stiffness. In particular, looking to its derivative with respect to both R_1 and R_K one gets:

$$\frac{\partial W}{\partial R_1} = \frac{T^2\left[-16X.\left(T_1\left(T_0+R_K\right)-2R_1^2\right) - \left(2X^2T_1 - 4XR_1 + T_0 + R_K\right).(-8R_1)\right]}{16\left[T_1\left(T_0+R_K\right)-2R_1^2\right]^2} = 0$$
$$\frac{\partial W}{\partial R_K} = \frac{T^2\left[4.\left(T_1\left(T_0+R_K\right)-2R_1^2\right) - \left(2X^2T_1 - 4XR_1 + T_0 + R_K\right).(4R_1)\right]}{16\left[T_1\left(T_0+R_K\right)-2R_1^2\right]^2} = 0$$

We see it equals zero if denominator cancels out, and so:

$$\frac{\partial W}{\partial R_K} = 0 \quad \Longleftrightarrow \quad R_1^2 - 2XT_1R_1 + (XT_1)^2 = 0$$

and one subsequently gets $\Delta = 0 \Rightarrow R_1 = XT_1$ as expected. We note that the second equation doesn't bring a solution on the value R_K as once we set $R_1 = XT_1$ it cancles out without condition on R_K. We thus optimize for $R_1 = X.T_1$, no matter what R_K we have.

Value R_1^{opt}

Given $X = \frac{R}{|T|} = \frac{1}{3}$ and $T_1 = T_1^{cb} \approx 6.03.e^{-12}$ we get $R_1^{opt} \approx 2.e^{-12}$.

Associated R_K domain

As we have $R_1 > 0$ we have $R_K \in]-R_0, +R_0]$. In details, the limiting curves for figure II.1 are:

- $R_1 = 0$, which is the limit in $R_1 \geq 0$;
- $R_K = R_0^{cb}$, which is the limit from cross-ply case ;
- a third non-affine one associated to the following equations:

$$\begin{aligned}
R_1 &= R_1^{cb}.cos(2\alpha) \\
\rightarrow \left(\frac{R_1}{R_1^{cb}}\right)^2 &= cos(2\alpha)^2 \\
R_K &= R_K^{cb}.cos(4\alpha) \\
\frac{R_K}{R_K^{cb}} &= cos(4\alpha) = 2.cos(2\alpha)^2 - 1 \\
\Rightarrow \left(\frac{R_K}{R_K^{cb}}\right) &= 2.\left(\frac{R_1}{R_1^{cb}}\right)^2 - 1
\end{aligned}$$

$\alpha \in [0, \pi/4]$. This last expression then is the equation describing third limiting curve. It then comes for $R_1 \approx 2E^{-12}$ that we have two limiting points for R_K:

$$R_K^+ = R_0^{cb}; \quad \text{(II.2)}$$

$$R_K^- = R_K^{cb}.\left[2.\left(\frac{R_1}{R_1^{cb}}\right)^2 - 1\right] = R_0^{cb}.\left[2.\left(\frac{R_1}{R_1^{cb}}\right)^2 - 1\right] \approx -4.7E^{-11} \quad \text{(II.3)}$$

with R_K^- = lower limit and R_K^+ = upper limit. Graphically this leads to:

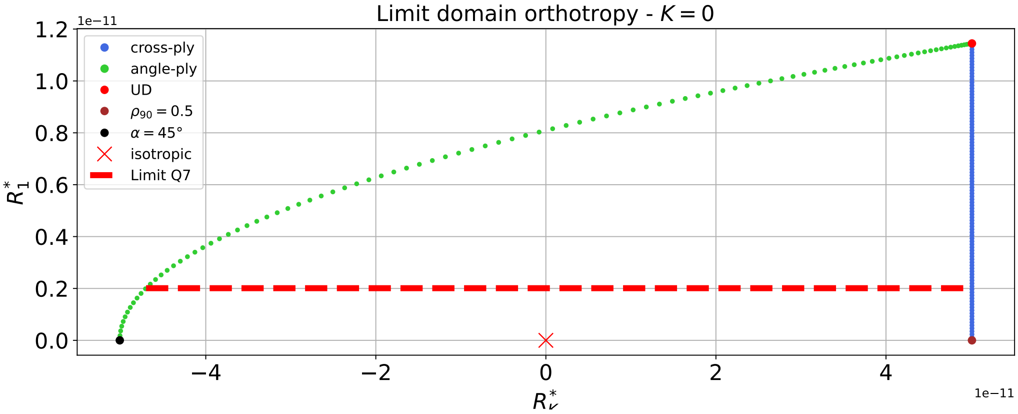

Figure II.3: Limit graph for orthotropic case. i. cross-ply stratification. ii. angle-ply stratification. iii. UD. iv. cross-ply - $\rho_90 = 0.5$ v. angle-ply - $\alpha = 45°$ vi. isotropic case. vii. Range of admissible R_K values for Q7. See code for more details on computation undergone.

Let's now look at 3 specific points of this new line to have more insight on composite-structure's properties then:

- $R_{K_{min}}^{opt} = R_K^- \approx -4.7E^{-11}$. The associated stratification angle is given by $\frac{R_K}{R_K^{cb}} = cos(4\alpha)$ and so $\alpha \approx 39.9°$;

- $R_{K_{max}}^{opt} = R_K^+ = R_0^{cb} \approx 5.0E^{-11}$. The associated stratification parameter ρ_{90}^{opt} is given by $\rho_{90}^{opt} = (1 - R_1/R_1^{cb})/2$ and so $\rho_{90}^{opt} \approx 40.1$. It implies 40% of the stratified will then be at an angle of 90°. Given $N = 32$, it implies **13** layers at 90°;

- We know $R_1^{opt} = R_1^{cb}$ and $R_K^{opt} =\in [-4.7E^{-11}, +5.0E^{-11}]$

For this last case, as we have several available values for R_K, we must go through an optimisation process:

$$\min_{R_K}(I[P(R_K)]) = \min_{R_K}\left(\underbrace{\hat{R}_0^2 + \hat{R}_1^2}_{B=0} + \underbrace{\tilde{\tilde{R}}_0^2 + \tilde{\tilde{R}}_1^2}_{A^*=D^*} + \underbrace{\left(\phi_0 - \phi_1 - K\frac{\pi}{4}\right)^2}_{\text{ortho.}} + \underbrace{\left(R_1^* - R_1^{opt}\right)^2 + \left(R_K^* - R_K^{opt}\right)^2}_{\text{optimal values}}\right)$$

Given we optimize at a fix R_1 and that we meet, by assumptions, most of above points (i.e. they equal zero), to minimize above equation is equivalent to minimize following equation:

$$\min_{R_K}(I[P(R_K)]) = \min_{R_K}\left({R_0^*}^2 + \left(R_K^* - R_K^{opt}\right)^2\right) \equiv \min_{R_K}\left({R_K^*}^2 + \left(R_K^* - R_K^{opt}\right)^2\right) \quad \text{(II.4)}$$

We here optimize as a function of the stratification R_K and so as a function of R_K for R_1 fixed. Yet, we don't have a value for R_K^{opt} with which to compare, we only have a range. What we do here is to consider $R_K^{opt} = R_K^{opt-Q6} = -R_0^{cb}$. It then comes for $R_K^* \in [-4.7E^{-11}, +5.0E^{-11}]$:

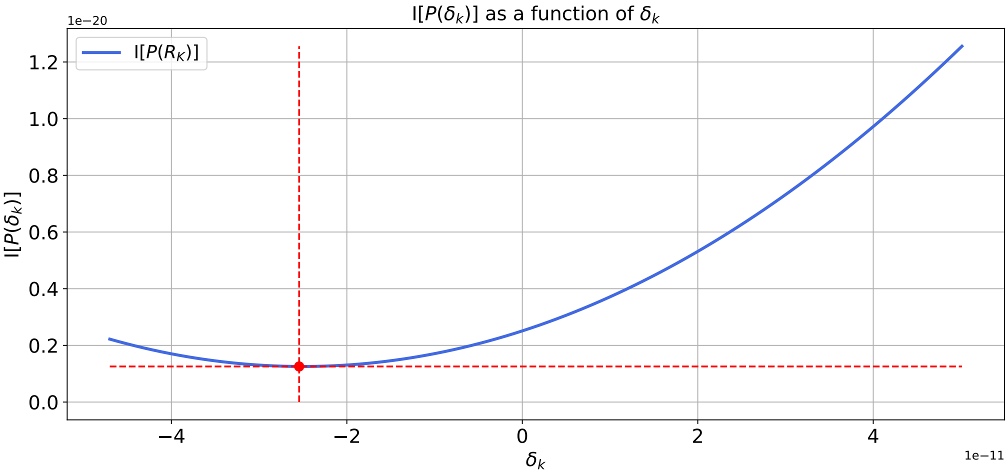

Figure II.4: Qualitative optimization of stiffness-error with respect to sequence R_K. The amplitude taken by $I[P(\delta_k)]$ are to be taken lightly and only tell on the overall qualitative behavior, not on quantitative behavior.

We get $R_k^{opt} \approx -2.54E^{-11}$. This value aims at optimizing partly the stiffness and the buckling load. Let's see how to maximize both buckling load λ and stiffness $\approx \frac{1}{W}$.

II.8 Maximization aerodynamic structure

In order for one to maximize at the same time stiffness and buckling load, we can work graphically. Figure II.4 tells us about minimum R_K^{comp} of compliance while we'll have to plot a graph for the maximization of $\lambda(R_K^{buck})$. In this last case we have $R_1 = 0$ such that $R_K^{buck} \in [-R_0^{cb}, +R_0^{cb}]$.

Numerically we will here think in terms of maximization : we aim at maximizing both buckling load and stiffness. We subsequently want to maximize the sum of these two parameters. Considering both parameters' maximization is as important, we normalize the process, leading to:

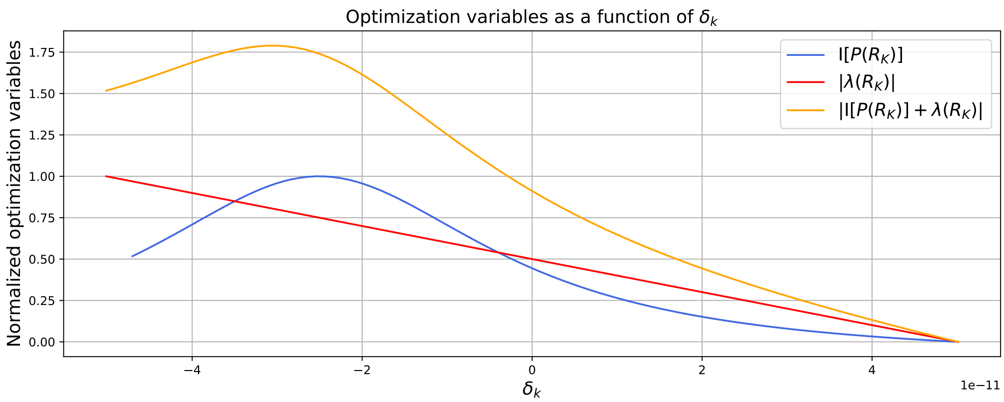

Figure II.5: Optimization process for both first-mode buckling load λ and stiffness optimization error $I[P(\delta_k)]$. The third graph is the addition of the two fundamental parameters we have to optimize ; it tells on where we maximize both parameters. The normalization process must be seen as aiming at judging equivalently-important load and stiffness parameters.

We note that we made maximization process such that along the domain $[-R_K^{cb}, +R_K cb]$ we maximize the sum $\lambda(R_K) + I\left[P\left(R_K\right)\right]$ where λ is first buckling load and $I\left[P\left(R_K\right)\right]$ is error associated to stiffness. Moreover, the maximization process is such that both λ and I take values between 0 and 1.

This method may be subject to discussion as we may not reproduce accurately the error quantities ; however this assumption accounts well for an equality in the matter of both parameters (i.e. we must equivalently optimize buckling load and stiffness).

Eventually, after calculation it comes $R_K^{opt} \approx -3E^{-11}$. At this stage, we still have an unknown on what R_1 one must choose. Indeed, the two different maximization (stiffness and buckling load) are associated to 2 different R_1^{opt}. Given R_K^{opt} we have a range of available R_1 values and what should choose a value between $R_1^{Q6} = 0$ and $R_1^{Q7} = 2e^{-12}$, namely $R_1 \in [0; 2e^{-12}]$. It can potentially depend on what symmetry we want when building the stratified structure (square symmetry, What angles ?, etc)

Conclusion

During this report we went from an analytical approach to numerical applications and modelisation of a physical subject : composite structures. We have then been able to reduce CLPT theory to simple orthotropic cases and deduce values as well as limiting domain for polar variables II.1 II.3. We furthermore looked at some arbitrary points in order to test physical response.

It resulted that an optimization process comes with theoretical assumption that translate a physical behavior. Once we knew what physical behavior we had to model (from geometry, applications, etc) we adapted consequently the CLPT and working with non-dimensional polar variables in polar framework, we concluded on an optimal set of polar parameters, optimizing stiffness and first buckling load.

Eventually, the optimization process provided an optimal case, and it would be a good expansion of this numerical study to have experimental data to compare.

In a more personal conclusions, the last application to composite structures has been the opportunity to have some insight on a real application case as well as the numerical process one must undergo to analyse a structure, combining theory of plates, physical perspective, stratification theory, etc.

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