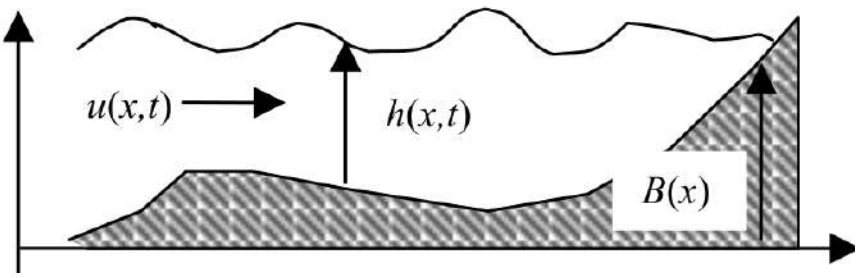
Homework Assignment 4
SF2521, Spring 2019

(max. 5p)

1 Shallow water with non-horizontal bottom (0.25p)

The shallow water model of HW2 is now extended to a non-horizontal bottom "bathymetry" $B(x)$,

$$\begin{pmatrix} h_t \\ hu_t \end{pmatrix} + \begin{pmatrix} h_x u + hu_x \\ huu_x + gh(h_x + B_x) \end{pmatrix} = 0$$



- Show that still water ($u = 0$) must have a horizontal water level.
- Write the equation in conservation form for h and $m = hu$:

$$\begin{pmatrix} h \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ f_2(h, m) \end{pmatrix}_x = \begin{pmatrix} 0 \\ s(h, m, x) \end{pmatrix}$$

  It is important that the source function s be independent of the derivatives of h or m.

In the remaining part of the homework your job is to implement a Roe scheme and extend it to a high-resolution scheme for the model

2 First order Roe scheme

The first order Roe scheme has the numerical flux

$$\boldsymbol{F}^n_{i+1/2} = \frac{1}{2}\left(\boldsymbol{f}(\boldsymbol{Q}^n_i) + \boldsymbol{f}(\boldsymbol{Q}^n_{i+1})\right) - \frac{1}{2}|\tilde{\boldsymbol{A}}_{i+1/2}|(\boldsymbol{Q}^n_{i+1} - \boldsymbol{Q}^n_i)$$

where $\tilde{\boldsymbol{A}}_{i+1/2} = \tilde{\boldsymbol{A}}(\boldsymbol{Q}_i, \boldsymbol{Q}_{i+1})$ in the Roe-average matrix at the $i + 1/2$ interface (Leveque 15.3.3). When you implement the scheme it may be worthwhile to use the form in Leveque eq (15.51) instead,

$$\boldsymbol{F}_{i+1/2} = \frac{1}{2}\left(\boldsymbol{f}(\boldsymbol{Q}_i^n) + \boldsymbol{f}(\boldsymbol{Q}_{i+1}^n)\right) - \frac{1}{2}\sum_{p=1}^{2} |\hat{\lambda}_{j+1/2}^p| \boldsymbol{W}_{j+1/2}^p,$$

which is easier to extend to the high–resolution scheme later on. You will need eigenvalues and eigenvectors of $\tilde{\boldsymbol{A}}_{i+1/2}$ to compute the waves. Formulas can be found in Leveque 15.3.3

2.1 Tests with flat bottom (0.5p)

Implement the first order Roe scheme for the case of a flat bottom, i.e. $s(h, m, x) = 0$. Let the channel be L long, with n cells $\Delta x = L/n$. The bottom is horizontal ($B = 0$) and the water depth is H. Use "wall" boundary conditions at $x = 0$ and L, i.e. $h_0 = h_1$, $m_0 = -m_1$, (and similarly at $x = L$). Choose initial data to give a right running Gaussian pulse with height a and width w starting at $x = L/2$,

$$h(x, 0) = H + ae^{-(x-L/2)^2/w^2}$$

Take $H = 1$, $w = 0.1L$, $a = H/5$ or so, to make the wave almost linear.

- Explain how to choose $m(x, 0)$ to make a single pulse (not two!). Hint: Eigenvectors ...
- Use this initial data and run the Roe solver with $n = 80, 160, 320$, CFL number as large as possible without instability, until the wave has been fully reflected at L. Plot the wave shapes.
- Compare with solutions obtained using your Lax-Friedrichs solver from HW2.
- Comment about order of accuracy, dissipation and dispersion.
- Then try also a higher pulse, say $a = 2H$. Comment on how well the single pulse initial data works.

2.2 Steady solutions with non-horizontal bottom

Now consider the case with a bump on the bottom, say

$$B(x) = \begin{cases} B_0 \cos^2\left(\frac{\pi(x-L/2)}{2r}\right) & |x - L/2| < r \\ 0 & |x - L/2| \geq r \end{cases}$$

Variable B introduces a source term, so a conservative first order scheme is

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\hat{F}_{i+1/2}^n - \hat{F}_{i-1/2}^n\right) + \Delta t S(Q^n)$$

where the $\hat{F}_{i+1/2}^n$ are the numerical flux functions for $S = 0$.

2.2.1 Boundary conditions (0.25p)

Since the problem is non-linear, the number of boundary conditions that should be prescribed at each boundary depends on the solution itself. The flow is called *sub–critical* if the water velocity is less than the characteristic speed, i.e. if $u(x) < \sqrt{gh(x)}$. Otherwise it is called *super–critical*.

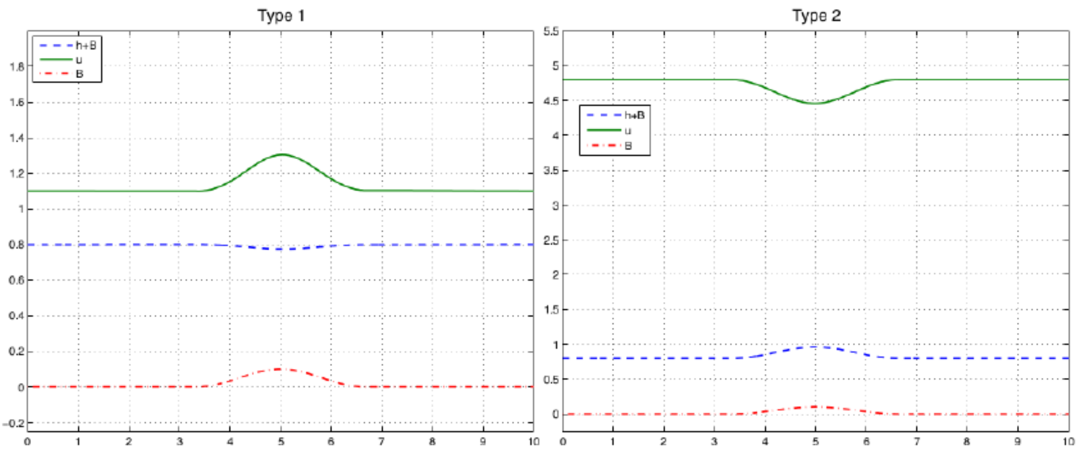
- We want to use Dirichlet boundary conditions for $u(x)$ and/or $h(x)$. Determine where we should put them when the flow is sub- and super–critical respectively in a neighbourhood of the two boundaries $x = 0$ and $x = L$.

2.2.2 Smooth steady state solutions (0.75p)

Now compute steady solutions for the non–horizontal bottom by running a time–accurate simulation for a LONG time. *The outflow in one direction will be constant, and the wave will take a fixed shape in time, as shown in the figures.* Use for instance $B_0 = H/10$, $r = L/6$. Implement inflow by prescribing the value in the ghost cell and outflow by extrapolation.

- Find initial data, inflow conditions and outflow conditions which give the following types of solutions (cf. figures below):

  1. All sub–critical flow: water accelerates on the bump (but remains sub-critical), and then slows down again.
  2. All super–critical flow: water decelerates on the bump (but remains super-critical), then accelerates again to super-critical outflow.



2.2.3 Entropy fix for steady states with transonic rarefaction (1.0p)

You shall now find steady solutions where part of the solution is sub–critical and part of it is super–critical. The two parts will be connected by shocks and rarefactions, which will then be transonic. These rarefactions will not be captured correctly with the vanilla Roe scheme. To get it right one must add an entropy fix.

Use the same setup as in the previous exercise but now find initial data, inflow conditions and outflow conditions which give the following types of steady solutions:

3. Sub–critical inflow which accelerates to super–critical on the bump and then reverts to sub–critical through a shock.

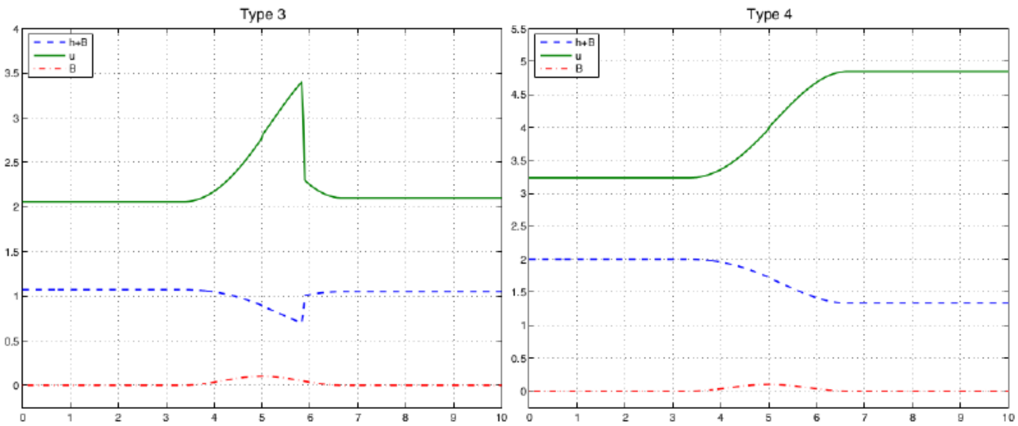
4. Sub–critical inflow which accelerates to super–critical on the bump and stays super–critical.

See figures below for examples.

- First try to find the solutions without an entropy fix. Provide plots showing what goes wrong.

- Implement Harten's entropy fix (Leveque p 326) and compute the steady solutions. Indicate in your plots where the solutions are sub- and super–critical. Try varying the parameter d. How should it scale with Δx?

- Try solving the same problem with Lax–Friedrichs. Show plots that demonstrate the difference in dissipation between the two schemes. How many grid points would you need to get a reasonable looking solution?



Note the small entropy–violating "shocklet" in type 3... the entropy fix should be strengthened

3 High–resolution Roe scheme

Extend to a Roe high–resolution scheme as explained in Leveque 6.13-6.15, 15.4. You need to add a correction to the flux, $\boldsymbol{F}^n_{i+1/2} + \frac{\Delta t}{\Delta x}\boldsymbol{F}^{\tilde{n}}_{i+1/2}$, as given in eq (15.63), where $\boldsymbol{W}^{\tilde{p}}_{j+1/2}$ is a limited version of $\boldsymbol{W}^{p}_{j+1/2}$. Use the *minmod*[1] limiter and note the comments (Leveque pp 121-122) on how to compare the waves from a cell interface to its upstream neighbor: define $\theta^n_{i+1/2}$ using the scalar product to compare the waves. It is sufficient to apply the limiter to the waves in the interior. Just use $\boldsymbol{W}^{p}_{j+1/2}$ itself at the boundary. You can find some references (but not all) in LeVeque [1] §6.15 and §15.4.

[1] The *minmod* function is defined as follows:

$$\varphi(\theta) = \begin{cases} \max(0,\theta) & |\theta| < 1 \\ 1 & |\theta| \geq 1 \end{cases}$$

3.1 High-resolution tests (2.25p)

- Run the flat bottom tests in Section 2.1. Check that the dissipation is much smaller than for the Lax–Friedrichs and the standard Roe scheme. Show plots comparing the solutions after short and long time.
- Check that the timestep restriction is still $\frac{\Delta t c_{\max}}{\Delta x} \leq s$ where $c_{\max}$ is the maximal wave speed and s is a constant $O(1)$. (Numerical experiment!)
- Try the four different types of solution above. Compare with the standard Roe scheme with the same number of grid points.

References

[1] Leveque R.J. - *Finite-Volume Methods for Hyperbolic Problems*. Cambridge University Press (2002).

[2] Gustafsson B., Kreiss H.-O. - *Time-Dependent Problems and Difference Methods (2nd ed.)*. John Wiley & Sons (2013).

[3] Kreiss H.-O., Lorenz J. - *Initial-Boundary Value Problems and the Navier-Stokes Equations*. SIAM Society for Industrial and Applied Mathematics.