Lecture 7

Convergence Theory for Linear Methods - Part 1

Convergence Theory

Motivation

- Convergence: usually established using Lax Equivalence Theorem.
- It says:

  scheme is consistent + stable = scheme is convergent.

- To check convergence for a scheme Φ we must thus verify consistency and stability, then apply the theorem.
- In the following, we talk individually about consistency, stability and convergence.

Consistency - Definition

A scheme is consistent if the exact solution fits the scheme well.

More precisely, we define the local truncation error $\boldsymbol{\tau}^n$ such that

$$\mathbf{u}^{n+1} = \mathbf{\Phi}(\mathbf{u}^n) + \Delta t\, \boldsymbol{\tau}^n, \qquad \text{where } u_j^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u(t_n, x)dx$$

- Local truncation error $\simeq$ residual when exact solution $\mathbf{u}^n$ (instead of $\mathbf{Q}^n$) is entered into the scheme, scaled by Δt.
- Alternatively, think of it as error performed in one time step, scaled by Δt:

$$\frac{\mathbf{u}^{n+1} - \mathbf{\Phi}(\mathbf{u}^n)}{\Delta t} = \boldsymbol{\tau}^n.$$

Consistency - Definition

- For convergence we need a small $\boldsymbol{\tau}^n$.
- We say that the method is consistent if

$$\max_{0\leq n\Delta t\leq T} \|\boldsymbol{\tau}^n\|_{\Delta x} \to 0 \qquad \text{as } \Delta t, \Delta x \to 0, \text{ for a fixed } T.$$

- If there is a number C independent of Δt and Δx such that

$$\max_{0\leq n\Delta t\leq T} \|\boldsymbol{\tau}^n\|_{\Delta x} \leq C(\Delta x^p + \Delta t^r)$$

  we say that the method is of order p in space and r in time.
- If $\lambda_{\text{CFL}} = \Delta t/\Delta x$ is constant, with $\lambda_{\text{CFL}} = \mathcal{O}(1)$, then

$$\|\boldsymbol{\tau}^n\|_{\Delta x} = \mathcal{O}(\Delta x^p + \Delta x^r) = \mathcal{O}(\Delta x^q), \qquad \text{where } q = \min(p, r)$$

  and we simply say the method is of order q.
- A consistency order can usually be checked by Taylor expansion of the exact solution and using the fact that it satisfies the PDE.

Consistency, Example - Part 1

Consider the Upwind Scheme for $\partial_t u + \mathbf{a}\partial_x u = 0$.

The local truncation error τ_j^n is defined by

$$u_j^{n+1} = u_j^n - \mathbf{a}\frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) + \Delta t\, \tau_j^n,$$

where u_j^n is the exact local average (see previous slides).

We can rewrite this as

$$\begin{aligned} \tau_j^n &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + \mathbf{a}\frac{u_j^n - u_{j-1}^n}{\Delta x} \\ &= \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \frac{u(t_{n+1},x) - u(t_n,x)}{\Delta t} + \mathbf{a}\frac{u(t_n,x) - u(t_n,x-\Delta x)}{\Delta x} dx. \end{aligned}$$

Consistency, Example - Part 2

We have

$$\tau_j^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \frac{u(t_{n+1},x)-u(t_n,x)}{\Delta t} + \mathbf{a}\frac{u(t_n,x)-u(t_n,x-\Delta x)}{\Delta x}dx.$$

Next, we apply Taylor expressions inside the integral:

$$\frac{u(t_{n+1},x)-u(t_n,x)}{\Delta t} = \partial_t u(t_n,x) + \frac{\Delta t}{2}\partial_{tt}u(t_n,x) + \mathcal{O}(\Delta t^2),$$

and

$$\mathbf{a}\frac{u(t_n,x)-u(t_n,x-\Delta x)}{\Delta x} = \mathbf{a}\,\partial_x u(t_n,x) - \frac{\mathbf{a}\,\Delta x}{2}\partial_{xx}u(t_n,x) + \mathcal{O}(\Delta x^2).$$

Consistency, Example - Part 3

Then, since $\partial_t u + a\partial_x u = 0$,

$$
\begin{aligned}
\tau_j^n &= \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \partial_t u(t_n,x) + \frac{\Delta t}{2}\partial_{tt} u(t_n,x) + \mathcal{O}(\Delta x^2)\, dx \\
&\quad + \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} a\,\partial_x u(t_n,x) - \frac{a\,\Delta x}{2}\partial_{xx} u(t_n,x) + \mathcal{O}(\Delta t^2)\, dx \\
&= \frac{1}{2\Delta x}\int_{x_j}^{x_{j+1}} \Delta t\,\partial_{tt} u(t_n,x) - a\,\Delta x\,\partial_{xx} u(t_n,x) dx + \mathcal{O}(\Delta x^2 + \Delta t^2) \\
&= \frac{\Delta t}{2}\cdot\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \partial_{tt} u(t_n,x)\, dx - \frac{a\,\Delta x}{2}\cdot\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \partial_{xx} u(t_n,x)\, dx \\
&\qquad + \mathcal{O}(\Delta x^2 + \Delta t^2).
\end{aligned}
$$

Consistency, Example - Part 4

We have

$$\tau_j^n = \frac{\Delta t}{2} \cdot \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_{tt} u(t_n, x)\, dx - \frac{a\, \Delta x}{2} \cdot \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_{xx} u(t_n, x)\, dx$$

$$+ \mathcal{O}(\Delta x^2 + \Delta t^2).$$

Noting that

$$\left| \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_{tt} u(t_n, x)\, dx \right| \leq \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \max_{x_j \leq y \leq x_{j+1}} |\partial_{tt} u(t_n, y)|\, dx$$

$$= \max_{x_j \leq y \leq x_{j+1}} |\partial_{tt} u(t_n, y)| \frac{\Delta x}{\Delta x} = \max_{x_j \leq y \leq x_{j+1}} |\partial_{tt} u(t_n, y)| = \mathcal{O}(1)$$

and analogously

$$\left| \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_{xx} u(t_n, x)\, dx \right| \leq \max_{x_j \leq y \leq x_{j+1}} |\partial_{xx} u(t_n, y)| = \mathcal{O}(1)$$

we conclude that $\tau_j^n = \mathcal{O}(\Delta x + \Delta t)$.

Consistency, Example - Part 5

We have

$$\tau_j^n = \mathcal{O}(\Delta x + \Delta t)$$

This shows that the Upwind Scheme is consistent and its is of first order in time and space.

More precise characterization of local truncation error possible by differentiating the equation once in time and space:

$$\partial_{tt} u + \mathbf{a}\partial_{xt} u = 0, \qquad \partial_{tx} u + \mathbf{a}\partial_{xx} u = 0.$$

Together this shows that $\partial_{tt} u = \mathbf{a}^2 \partial_{xx} u$. Therefore

$$\tau_j^n = \frac{\mathbf{a}(\mathbf{a}\,\Delta t - \Delta x)}{2} \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_{tt} u(t_n, x)\, dx + \mathcal{O}(\Delta x^2 + \Delta t^2).$$

Consistency, Example - Part 6

Characterization

$$\tau_j^n = \frac{a(a\,\Delta t - \Delta x)}{2} \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \partial_{tt} u(t_n, x)\, dx + \mathcal{O}(\Delta x^2 + \Delta t^2).$$

- useful when deriving modified equations (see Leveque 8.6).
- It also shows: if one chooses the "magic time step" $\Delta t = \Delta x / a$ the method is more accurate.
- More precisely: if $\Delta t = \Delta x / a$ the numerical scheme is exact and $\tau_j^n \equiv 0$.
- However: very special case for the constant coefficient advection equation, and does not happen in general.

Stability

The scheme is called Lax-Richtmyer stable if

$$\|\boldsymbol{\Phi}(\mathbf{Q})\|_{\Delta x} \leq (1 + \alpha \Delta t)\|\mathbf{Q}\|_{\Delta x}$$

for all $\mathbf{Q}$ and with α independent of $\mathbf{Q}$, Δt and Δx.

- Later we get back on how to show this for a scheme.
- For nonlinear schemes $\boldsymbol{\Phi}$, we use instead the “almost contraction” property,

$$\|\boldsymbol{\Phi}(\mathbf{Q}) - \boldsymbol{\Phi}(\mathbf{Q}')\|_{\Delta x} \leq (1 + \alpha \Delta t)\|\mathbf{Q} - \mathbf{Q}'\|_{\Delta x}$$

for all $\mathbf{Q}$, $\mathbf{Q}'$ and with α independent of $\mathbf{Q}$, $\mathbf{Q}'$, Δt and Δx.
See Leveque 8.3.

Convergence

Lax Equivalence Theorem

"*stability* + *consistency* $\Leftrightarrow$ *convergence*".

More precisely:

- if the method is stable and consistent with order p in space and r in time we have

$$\max_{0\le n\Delta t\le T} \|\mathbf{Q}^n - \mathbf{u}^n\|_{\Delta x} \le C(\Delta x^p + \Delta t^r), \qquad (*)$$

with C independent of Δx and Δt, but in general depending on T and $u(t,x)$.

- The error estimate $(*)$ obviously implies convergence.

Proof: stability + consistency $\Rightarrow$ convergence / 1

Assume

- Stability

$$\|\mathbf{\Phi}(\mathbf{Q})\|_{\Delta x} \leq (1+\alpha\Delta t)\|\mathbf{Q}\|_{\Delta x}, \qquad \forall \mathbf{Q},$$

- Consistency such that

$$\tau := \max_{0\leq n\Delta t\leq T} \|\boldsymbol{\tau}^n\|_{\Delta x} \leq C(\Delta x^p + \Delta t^r),$$

- Exact initial data,

$$Q_j^0 = u_j^0, \qquad \|\mathbf{Q}^0 - \mathbf{u}^0\|_{\Delta x} = 0.$$

We define the error

$e_j^n := u_j^n - Q_j^n$ and in vector form $\mathbf{e}^n = \mathbf{u}^n - \mathbf{Q}^n$.

Proof: stability + consistency $\Rightarrow$ convergence / 2

Using

- $\mathbf{Q}^{n+1} = \mathbf{\Phi}(\mathbf{Q}^n)$,
- the definition of the truncation error $\boldsymbol{\tau}^n$,
- and the linearity of $\mathbf{\Phi}$

we have

$$\begin{aligned} \mathbf{e}^{n+1} &= \mathbf{u}^{n+1} - \mathbf{Q}^{n+1} \\ &= \mathbf{\Phi}(\mathbf{u}^n) + \Delta t\,\boldsymbol{\tau}^n - \mathbf{\Phi}(\mathbf{Q}^n) \\ &= \mathbf{\Phi}(\mathbf{e}^n) + \Delta t\,\boldsymbol{\tau}^n. \end{aligned}$$

Proof: stability + consistency $\Rightarrow$ convergence / 3

From $\mathbf{e}^{n+1} = \boldsymbol{\Phi}(\mathbf{e}^n) + \Delta t\, \boldsymbol{\tau}^n$ we have

$$
\begin{aligned}
\|\mathbf{e}^{n+1}\|_{\Delta x} \\
&\leq && \|\boldsymbol{\Phi}(\mathbf{e}^n)\|_{\Delta x} + \Delta t \|\boldsymbol{\tau}^n\|_{\Delta x} \\
&\overset{\{\text{stability and def. of } \tau\}}{\leq} && (1 + \alpha\Delta t)\|\mathbf{e}^n\|_{\Delta x} + \Delta t\, \tau \\
&\overset{\{\text{same estimate to } \mathbf{e}^n\}}{\leq} && (1 + \alpha\Delta t)^2\|\mathbf{e}^{n-1}\|_{\Delta x} + (1 + \alpha\Delta t)\Delta t\, \tau + \Delta t\, \tau \\
&\overset{\{\text{induction}\}}{\leq} && (1 + \alpha\Delta t)^{n+1}\|\mathbf{e}^0\|_{\Delta x} + \sum_{j=0}^{n}(1 + \alpha\Delta t)^j \Delta t\, \tau \\
&\overset{\{\text{exact initial data}\}}{=} && \Delta t\, \tau \sum_{j=0}^{n}(1 + \alpha\Delta t)^j .
\end{aligned}
$$

Proof: stability + consistency ⇒ convergence / 4

We have

$$\|\mathbf{e}^{n+1}\|_{\Delta x} \leq \Delta t\, \tau \sum_{j=0}^{n} (1 + \alpha \Delta t)^j.$$

The sum is a geometric series,

$$\sum_{j=0}^{n} (1 + \alpha \Delta t)^j = \frac{(1 + \alpha \Delta t)^{n+1} - 1}{(1 + \alpha \Delta t) - 1} = \frac{(1 + \alpha \Delta t)^{n+1} - 1}{\alpha \Delta t}.$$

Hence

$$\|\mathbf{e}^{n}\|_{\Delta x} \leq \tau \frac{(1 + \alpha \Delta t)^{n} - 1}{\alpha}.$$

Proof: stability + consistency $\Rightarrow$ convergence / 5

We have

$$\|\mathbf{e}^n\|_{\Delta x} \leq \tau \frac{(1+\alpha\Delta t)^n - 1}{\alpha}.$$

using the fact that $1 + x \leq e^x$ for all x, we get

$$\max_{0\leq n\Delta t\leq T} \|\mathbf{e}^n\|_{\Delta x} \leq \max_{0\leq n\Delta t\leq T} \tau \frac{e^{\alpha n\Delta t} - 1}{\alpha} \leq \tau \frac{e^{\alpha T} - 1}{\alpha}.$$

Hence, with the consistency assumption we have

$$\max_{0\leq n\Delta t\leq T} \|\mathbf{u}^n - \mathbf{Q}^n\|_{\Delta x} \leq C'(\Delta x^p + \Delta t^r).$$

Here, $C' = C(e^{\alpha T} - 1)/\alpha$, where C is the constant from the consistency assumption.

This proves convergence and the error estimate.

Convergence - Remarks

- In general: boundary conditions can have significant effect on stability, accuracy and convergence.
- Above analysis is not always sharp.
  E.g.: the local truncation error can, sometimes, be allowed to have lower order at the boundaries without ruining the overall convergence rate.
- For higher order approximations wider spatial stencils are needed, which means that more ghost cells are needed. Then also more boundary conditions for these cells are needed. However, the PDE itself has a fixed number of boundary conditions. Hence, the number of numerical boundary conditions is often larger than the number of PDE boundary conditions. Choosing these extra conditions can be a delicate issue.