3A005 - Ecrit 1 du 8 novembre 2017

Exercice I : $f(x) = x^3 - 7x + 12 \rightarrow r_1 = 3$ et $r_2 = 4$

1. $x_{k+1} = \varphi_1(x_k) = \frac{x_k^2+12}{7}$. On a $x = \varphi_1(x) = \frac{x^2+12}{7} \Rightarrow x^2 - 7x + 12 = 0 \rightarrow$ suite valide

   $\varphi_1(x) = \frac{x^2+12}{7} \rightarrow \varphi_1'(x) = \frac{2x}{7}$, $|\varphi_1'(r)| < 1 \Rightarrow \frac{2r}{7} < 1$ ou $r < \frac{7}{2} \rightarrow$ converge vers $r_1 = 3$

2. $x_{k+1} = \varphi_2(x_k) = \frac{7x_k - 12}{x_k}$. On a $x = \varphi_2(x) = \frac{7x-12}{x} \Rightarrow x^2 - 7x + 12 = 0 \rightarrow$ suite valide

   $\varphi_2(x) = \frac{7x-12}{x} \rightarrow \varphi_2'(x) = \frac{12}{x^2}$, $|\varphi_2'(r)| < 1 \Rightarrow \frac{12}{r^2} < 1$ ou $r > \sqrt{12} \rightarrow$ converge vers $r_2 = 4$

3. $x_{k+1} = \varphi_3(x_k) = \frac{-12}{x_k - 7}$. On a $x = \varphi_3(x) = \frac{-12}{x-7} \Rightarrow x^2 - 7x + 12 = 0 \rightarrow$ suite valide

   $\varphi_3(x) = \frac{-12}{x-7} \rightarrow \varphi_3'(x) = \frac{12}{(x-7)^2}$, $|\varphi_3'(r)| < 1 \Rightarrow \frac{12}{(r-7)^2} < 1$, $\varphi'(r_1) = \frac{12}{16}$, $\varphi'(r_2) = \frac{12}{9} \rightarrow$ converge vers r_1

4. $\varphi_1(x_k) = \frac{x_k^2+12}{7} = x_k + (x_k^2 - 7x_k + 12) \times \frac{1}{7} \Longrightarrow \psi_1(x_k) = -\frac{1}{7}$

   $\varphi_2(x_k) = \frac{7x_k+12}{x_k} = x_k - (x_k^2 - 7x_k + 12) \times \frac{1}{x_k} \Rightarrow \psi_2(x_k) = \frac{1}{x_k}$

   $\varphi_3(x_k) = \frac{-12}{x_k - 7} = x_k - (x_k^2 - 7x_k + 12) \times \frac{1}{x_k - 7} \Rightarrow \psi_3(x_k) = \frac{1}{x_k - 7}$

Exercice II

1- $A = \begin{vmatrix} \alpha & 0 & 1 \\ 0 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix}$, $x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$, $y = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix}$, $b = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$

1.1 $y = Ax \Longrightarrow \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} \alpha & 0 & 1 \\ 0 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} \alpha x_1 + x_3 \\ \alpha x_2 + \beta x_3 \\ x_1 + \beta x_2 + \alpha x_3 \end{vmatrix}$

1-2 $c = y^t x = (\alpha x_1 + x_3)x_1 + (\alpha x_2 + \beta x_3)x_2 + (x_1 + \beta x_2 + \alpha x_3)x_3$

$= \alpha x_1^2 + \alpha x_2^2 + \alpha x_3^2 + 2x_1x_3 + 2\beta x_2 x_3$

$= (x_1 + x_3)^2 + (x_2 + \beta x_3)^2 + (\alpha - 1)x_1^2 + (\alpha - 1)x_2^2 + (\alpha - 1 - \beta^2)x_3^2 > 0 \;\forall x \neq 0$

1-3 $c > 0 \Rightarrow \alpha > 1$ et $\alpha \geq 1 + \beta^2$

2. $\alpha = 2$, $\beta = 1 \longrightarrow \alpha = 1 + \beta^2$

2.1 $A = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{vmatrix} \begin{vmatrix} l_{11} & l_{21} & l_{31} \\ & l_{22} & l_{32} \\ & & l_{33} \end{vmatrix} = \begin{vmatrix} l_{11}^2 & & \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{vmatrix}$

$\left.\begin{matrix} l_{11} = \sqrt{2} & & \\ l_{21} = 0 & l_{22} = \sqrt{2} & \\ l_{31} = 1/\sqrt{2} & l_{32} = 1/\sqrt{2} & l_{31} = 1 \end{matrix}\right\} \Rightarrow L = \begin{vmatrix} \sqrt{2} & & \\ 0 & \sqrt{2} & \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{vmatrix}$

2-2. $\begin{vmatrix} \sqrt{2} & & \\ 0 & \sqrt{2} & \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{vmatrix} \begin{vmatrix} \sqrt{2} & 0 & 1/\sqrt{2} \\ & \sqrt{2} & 1/\sqrt{2} \\ & & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ On retrouve bien $LL^t = A$

2-3 descente $\begin{vmatrix} \sqrt{2} & & \\ 0 & \sqrt{2} & \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 3 \\ 1 \end{vmatrix} \Rightarrow \begin{vmatrix} y_1 = 1/\sqrt{2} \\ y_2 = 3/\sqrt{2} \\ y_3 = (1 - \frac{1}{2} - \frac{3}{2}) = 1 \end{vmatrix} \Longrightarrow y = \begin{vmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \\ -1 \end{vmatrix}$

remontée $\begin{vmatrix} \sqrt{2} & 0 & 1/\sqrt{2} \\ & \sqrt{2} & 1/\sqrt{2} \\ & & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \\ -1 \end{vmatrix} \Rightarrow \begin{vmatrix} x_3 = -1 \\ x_2 = (3/\sqrt{2} + 1/\sqrt{2})/\sqrt{2} = 2 \\ x_1 = (1/\sqrt{2} + 1/\sqrt{2})/\sqrt{2} = 1 \end{vmatrix} \Rightarrow x = \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix}$

Exercice III

$A = \begin{vmatrix} \alpha & 0 & 1 \\ 0 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix}, \ x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}, \ b = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$

1- Jacobi

1-1. $M = D \rightarrow N = E + F \Rightarrow \Omega_J = M^{-1}N = D^{-1}(E+F)$

1-2. $\Omega_J = \begin{vmatrix} 1/\alpha & & \\ & 1/\alpha & \\ & & 1/\alpha \end{vmatrix} \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -\beta \\ -1 & -\beta & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1/\alpha \\ 0 & 0 & -\beta/\alpha \\ -1/\alpha & -\beta/\alpha & 0 \end{vmatrix}$

1-3. $\det(\Omega_J - \lambda I) = \det \begin{vmatrix} -\lambda & 0 & -1/\alpha \\ 0 & -\lambda & -\beta/\alpha \\ -1/\alpha & -\beta/\alpha & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2 - \frac{\beta^2}{\alpha^2}\right) - \frac{1}{\alpha}\left(-\frac{\lambda}{\alpha}\right) = -\lambda\left[\lambda^2 - \frac{1}{\alpha^2}(\beta^2+1)\right]$

$\lambda_1 = 0, \ \lambda_2 = \frac{\sqrt{1+\beta^2}}{\alpha}, \ \lambda_3 = -\frac{\sqrt{1+\beta^2}}{\alpha} \Longrightarrow \rho(\Omega_J) = \frac{1}{\alpha}\sqrt{1+\beta^2} < 1 \Rightarrow \alpha^2 > 1+\beta^2$

2- Gauss-Seidel

2-1. $M = D - E \rightarrow N = F \Rightarrow \Omega_{GS} = M^{-1}N = (D-E)^{-1}F$

2-2 $\begin{vmatrix} \alpha & & \\ 0 & \alpha & \\ 1 & \beta & \alpha \end{vmatrix} \begin{vmatrix} e_1^1 \\ e_2^1 \\ e_3^1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \begin{vmatrix} e_1^1 = 1/\alpha \\ e_2^1 = 0 \\ e_3^1 = -\frac{e_1^1}{\alpha} = -1/\alpha^2 \end{vmatrix} \begin{vmatrix} \alpha & & \\ 0 & \alpha & \\ 1 & \beta & \alpha \end{vmatrix} \begin{vmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \Rightarrow e^2 = \begin{vmatrix} 0 \\ 1/\alpha \\ -\beta/\alpha^2 \end{vmatrix}, M \begin{vmatrix} e_1^3 \\ e_2^3 \\ e_3^3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \Rightarrow e^3 = \begin{vmatrix} 0 \\ 0 \\ 1/\alpha \end{vmatrix}$

2-3 donc $M^{-1} = \begin{vmatrix} 1/\alpha & & \\ 0 & 1/\alpha & \\ -1/\alpha^2 & -\beta/\alpha^2 & 1/\alpha \end{vmatrix}, \ \Omega_{GS} = \begin{vmatrix} 1/\alpha & & \\ 0 & 1/\alpha & \\ -1/\alpha^2 & -\beta/\alpha^2 & 1/\alpha \end{vmatrix} \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -\beta \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1/\alpha \\ 0 & 0 & -\beta/\alpha \\ 0 & 0 & \frac{1+\beta^2}{\alpha^2} \end{vmatrix}$

2-4 $\det(\Omega_{GS} - \lambda I) = \det \begin{vmatrix} -\lambda & 0 & -1/\alpha \\ 0 & -\lambda & -\beta/\alpha \\ 0 & 0 & -\lambda + \frac{1+\beta^2}{\alpha^2} \end{vmatrix} = -\lambda(-\lambda)\left[-\lambda + \frac{(1+\beta^2)}{\alpha^2}\right] \Rightarrow \lambda_1 = \lambda_2 = 0, \ \lambda_3 = \frac{1+\beta^2}{\alpha^2}$

$\rho(\Omega_{GS}) = \frac{1+\beta^2}{\alpha^2} < 1 \Rightarrow \alpha^2 \ 1+\beta^2$

3- $\rho(\Omega_J) = \frac{\sqrt{1+\beta^2}}{\alpha}, \ \rho(\Omega_{GS}) = \frac{1+\beta^2}{\alpha^2}$. On a $\rho(\Omega_{GS}) = \rho^2(\Omega_J) \Rightarrow$ GS converge 2 fois plus vite que J

4- La relation II-1-3 est $\alpha \geq 1+\beta^2$ est contenue dans la relation $\alpha^2 > 1+\beta^2$.

Comme on a $\alpha > 1 \Rightarrow \alpha^2 > \alpha$ donc $\alpha^2 > \alpha \geq 1+\beta^2$ qui respecte $\alpha^2 > 1+\beta^2$.

Dans l'exercice II, on a $\alpha = 2$ et $\beta = 1 \Rightarrow \alpha = 1+\beta^2 = 2$ respecte $\alpha \geq 1+\beta^2$
et respecte surtout $\alpha^2 > 1+\beta^2$ car $4 > 2$.