SCIENTIFIC CALCULUS: PROJECT 2020
SHALLOW WATER MODEL

Contents

1. 1 D Shallow Water Model

The 1 D Shallow water equations are

$$
\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \\
&\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h}\right) = -gh\frac{\partial \eta}{\partial x} - \frac{1}{\rho}\tau(f)
\end{aligned}
\tag{1}
$$

where h and q are the principal variables, h is the water height, and $q = \int_f^{\eta(x,t)} U_x dy$ the flux in the x direction, U_x being the x component of the flow velocity. The topography $f = f(x)$ defines the bottom side, ρ the density and g is the gravity field. The shear stress at the bottom is $\tau(f)$. The total height is $\eta = h + f$, the the stationary solution ($q = 0$) is the given by η constant. The reference pressure is p_0. Figure 1 presents the configuration, the x scale length is greater than the typical scale length in the y direction, $h \ll L$. The boundary conditions are $\frac{\partial h}{\partial x} = 0$ and $q = 0$ at both sides of the tank, for $x = 0, L$. The bottom side is impermeable then all velocities are zero. The initial condition depend on each configuration studied.

We derive the Shallow Water equation from the incompressible Navier-Stokes equations. Starting from the incompressible 2D Navier-Stokes equations (mass and momentum conservation) in a non conservative form for the vector velocity (U_x, U_y),

$$
\begin{aligned}
&\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = 0 \\
&\rho\left(\frac{\partial U_x}{\partial t} + U_x\frac{\partial U_x}{\partial x} + U_y\frac{\partial U_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 U_x}{\partial x^2} + \mu\frac{\partial^2 U_x}{\partial y^2} \\
&\rho\left(\frac{\partial U_y}{\partial t} + U_x\frac{\partial U_y}{\partial x} + U_y\frac{\partial U_y}{\partial y}\right) = -\frac{\partial p}{\partial y} - \rho g + \mu\frac{\partial^2 U_y}{\partial x^2} + \mu\frac{\partial^2 U_y}{\partial y^2}
\end{aligned}
\tag{2}
$$

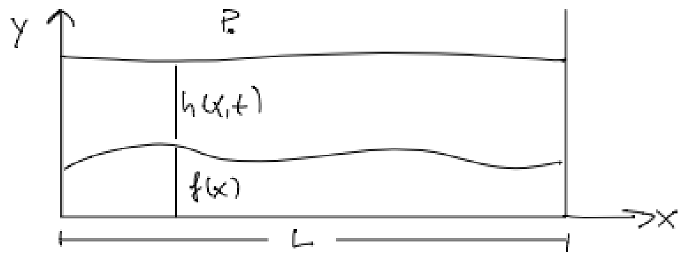

FIGURE 1. Shallow water : scales and description.

where μ is the viscosity. Using the ratio between the vertical scale h and the horizontal scale L, $\epsilon = \frac{h}{L} \ll 1$ we found from mass conservation that

$$U_y \approx \epsilon U_x$$

Keeping the principal terms of momentum equations we have

$$\begin{aligned} \rho\left(\frac{\partial U_x}{\partial t} + U_x \frac{\partial U_x}{\partial x}\right) &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 U_x}{\partial y^2} \\ 0 &= -\frac{\partial p}{\partial y} - \rho g \end{aligned} \tag{3}$$

Mass conservation. We integrate the mass conservation between the topography $f = f(x)$ and the free surface $\eta = \eta(x,t)$ (see Figure 1)

$$\begin{aligned} \int_f^\eta \left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}\right) dy = 0 \\ \int_f^\eta \frac{\partial U_x}{\partial x} dy + U_y(\eta) - U_y(f) = 0 \end{aligned}$$

As the limits of the integral are variable we use the Leibniz rule

$$\frac{\partial}{\partial x}\int_f^\eta U_x dy - U_x(\eta)\frac{\partial h}{\partial x} + U_x(f)\frac{\partial f}{\partial x} + U_y(\eta) - U_y(f) = 0$$

Over the topography the velocity is zero ($U_x(f) = U_y(f) = 0$) then

$$\frac{\partial}{\partial x}\int_f^\eta U_x dy + U_x(f)\frac{\partial f}{\partial x} + U_y(\eta) = 0$$

For the interface $y = \eta(x,t)$, the kinematic condition at the free surface is given by an implicit equation $F(x,t) = \eta(x,t) - y = 0$ then

$$\frac{dF}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} U_x(\eta) - U_y(\eta) = 0$$

Therefore the mass conservation results in

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\int_z^\eta U_x dy = 0 \tag{4}$$

Momentum conservation. We work on the momentum equations

$$\rho\left(\frac{\partial U_x}{\partial t}+U_x\frac{\partial U_x}{\partial x}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 U_x}{\partial y^2}$$
$$0=-\frac{\partial p}{\partial y}-\rho g$$

We recall that the free surface is $f+h=\eta$, integrating the pressure in the y direction and using the boundary condition for the pressure, $p(y=\eta)=p_0$, we have

$$p=\rho g(\eta-y)+p_0 \tag{5}$$

then $\frac{\partial p}{\partial x}=\rho g\frac{\partial \eta}{\partial x}$. In a second time we integrate the x momentum equation

$$\int_f^\eta\left(\rho\left(\frac{\partial U_x}{\partial t}+\frac{\partial (U_x U_x)}{\partial x}\right)\right)dy=\int_f^\eta\left(-\rho g\frac{\partial \eta}{\partial x}+\mu\frac{\partial^2 U_x}{\partial y^2}\right)dy$$

We can extract the derivatives (temporal and spatial) from the integrals by using the Leibniz theorem. We found finally

$$\rho\frac{\partial}{\partial t}\int_f^\eta U_x dy+\rho\frac{\partial}{\partial x}\int_f^\eta U_x^2 dy=-\rho g h\frac{\partial \eta}{\partial x}+\tau(\eta)-\tau(f)$$

where $\tau=\mu\frac{\partial U_x}{\partial y}$ is the shear stress at the topology (f) and at the free surface (η). By defining the flow through a given section

$$q=\int_f^\eta U_x dy \tag{6}$$

we can write, by doing some hypothesis about the velocity profile,

$$\rho\frac{\partial q}{\partial t}+\rho\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)=-\rho g h\frac{\partial \eta}{\partial x}+\tau(h)-\tau(f)$$

We suppose that the shear at the top surface in not important for the dynamics (shear due to the wind is neglected) then $\tau(h)=0$.

Shallow water equations. The Shallow water equations are then

$$\begin{aligned}\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}&=0\\ \frac{\partial q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)&=-gh\frac{\partial \eta}{\partial x}-\frac{1}{\rho}\tau(f)\end{aligned} \tag{7}$$

2. Numerical approaches

We want to solve

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}=\mathbf{S} \tag{8}$$

where $\mathbf{U}=(h,q)^T$ is the vector of conserved variables and $\mathbf{F}(\mathbf{U})=(q,\frac{q^2}{h}+\frac{1}{2}gh^2)^T$ the flux vector and $\mathbf{S}$ is the source term which depend on the problem. We have used $\eta=h+f$ and write the gravity term in a conservative form.

2.1. **Finite differences.** The best numerical scheme for hyperbolic equations is the McCormak scheme. This second-order finite difference method was introduced by Robert W. MacCormack in 1969. The MacCormack method is the simplest and elegant approach adapted to hyperbolic equations. This a predictor-corrector approach, i.e. it predicts in the 1st step an intermediate solution $\mathbf{U}_i^*$ which is corrected in a 2nd step.

The algorithm is given in two steps

(1) Predictor step

$$\mathbf{U}_i^* = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1}^n - \mathbf{F}_i^n) + \Delta t \mathbf{S}(\mathbf{U}_i^n)$$

(2) Corrector step

$$\mathbf{U}_i^{n+1} = \frac{1}{2}(\mathbf{U}_i^n + \mathbf{U}_i^*) - \frac{1}{2}\frac{\Delta t}{\Delta x}(\mathbf{F}_i^* - \mathbf{F}_{i-1}^*) + \Delta t \mathbf{S}(\mathbf{U}_i^*)$$

The MacCormack algorithm does not introduce diffusive errors but it is known to be dispersive (Gibbs phenomenon) in high gradient regions.

2.2. **Finite Volume.** In a finite volume approach the vecteur $\mathbf{U}$ at time n, say $\mathbf{U}_i^n$ is the integral of the function $\mathbf{U}(x, t^n)$ over the cell size between $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ $(x_{i-\frac{1}{2}} + \Delta x = x_{i+\frac{1}{2}})$ then

$$\mathbf{U}_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(x, t^n)\, \mathrm{d}x$$

Doing the same for the vector F we found

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \partial_x \mathbf{F}(\mathbf{U}) dx = \mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n$$

the discretization of the system $\dfrac{\partial \mathbf{U}}{\partial t} + \dfrac{\partial F(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U})$ is

$$\mathbf{U}_i^n = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n) + \Delta t \mathbf{S}(\mathbf{U}_i^n)$$

which is exact if we know the numerical flux at the interfaces left $(L = i - 1/2)$ and right $(R = i + 1/2)$ of the cell.

2.2.1. *Hyperbolic equations.* The system

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0$$

can be expressed as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial x} = 0$$

where $\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}$ is the Jacobian matrix.

Definition 2.1. The system is said hyperbolic in a point x if all eigenvalues of $\mathbf{A}(\mathbf{U})$ are real and strictly hyperbolic if all the real eigenvalues all distinct.

Definition 2.2. A matrix $\mathbf{A}(\mathbf{U})$ is said to be diagonalisable if can be expressed as

$$\mathbf{A} = \mathbf{K}\mathbf{\Lambda}\mathbf{K}^{-1}\text{or } \mathbf{\Lambda} = \mathbf{K}^{-1}\mathbf{A}\mathbf{K}$$

in terms of a diagonal matrix $\mathbf{\Lambda}$ and a matrix $\mathbf{K}$. The diagonal elements of $\mathbf{\Lambda}$ are the eigenvalues λ_i of $\mathbf{A}$

The existence of the inverse matrix $\mathbf{K^{-1}}$ makes it possible to define a new set of dependent variables $\mathbf{W} = (w_1, w_2, ...,)^T$ using the transformation

$$\mathbf{W} = \mathbf{K}^{-1}\mathbf{U} \text{ or } \mathbf{U} = \mathbf{K}\mathbf{W}$$

then

$$\mathbf{U}_t = \mathbf{K}\mathbf{W}_t, \quad \mathbf{U}_x = \mathbf{K}\mathbf{W}_x$$

and finally

$$\mathbf{K}\mathbf{W}_t + \mathbf{A}\mathbf{K}\mathbf{W} = 0$$

Multipliant at the left by $\mathbf{K}^{-1}$ we found a system wich are decoupled (because $\mathbf{\Lambda}$ is diagonal!)

$$\mathbf{W}_t + \mathbf{\Lambda}\mathbf{W} = 0 \tag{9}$$

The system has a simple solution

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, i = 1, \dots, m \tag{10}$$

each w_i traveling a velocity λ_i. This is the solution of a classical initial value of the Riemann problem. For a Shallow Water model we have two eigenvalues $\lambda = \frac{q}{h} \pm \sqrt{gh}$. The solution in terms of the Riemann problem is

$$\frac{\partial w_1}{\partial t} + \left(\frac{q}{h} + \sqrt{gh}\right)\frac{\partial w_1}{\partial x} = 0$$
$$\frac{\partial w_2}{\partial t} + \left(\frac{q}{h} - \sqrt{gh}\right)\frac{\partial w_2}{\partial x} = 0$$

Several approaches use the Riemann structure of the hyperbolic equation to evaluate the numerical flux, we want to propose two.

2.2.2. *HLL solver.* The HLL solver, proposed by Harten, Lax and van Leer compute the interfacial flux by using the fastest signal velocities SL and SR at the left and the right of the interface. These velocities are the eigenvalues of the Jacobian matrix $\frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}$ at the left and the right of the interface.

The algorithm for the numerical flux is

$$\mathbf{F}^{hll}_{num} = \begin{cases} \mathbf{F}_L & \text{if} \quad 0 \le S_L \\ \frac{S_R\mathbf{F}_L - S_L\mathbf{F}_R + S_L S_R(\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} \quad , & \text{if } S_L \le 0 \le S_R \\ \mathbf{F}_R & \text{if} \quad 0 \ge S_R \end{cases}$$

2.2.3. *Rusanov solver.* A particular (and simplification) case of the HLL solver is proposed by Rusanov by setting only one velocity S as the maximum of all interface velocities

$S = max(SL(U_L), SR(U_L), SL(U_R), SR(U_R)$

The algorithm is

$$\mathbf{F}_{num} = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{1}{2}S\left(\mathbf{U}_R - \mathbf{U}_L\right)$$

3. Some questions

The project has

- a numerical side where the group implements some specific features of the Shallow Water model. Some of them are mandatory.
- a physical side where the group, based on the Shallow Water implementation, analyse a particular configuration.

3.1. Modeling and implementation.

For the projet some numerical features are mandatory and another others optional.

Mandatory. Use the MacCormak scheme to implement

(1) Friction: We want to model the topology friction, the term $\tau(f)$.
Difficulty : +
The term $\tau = \mu \frac{\partial U_x}{\partial y}$ is the shear stress at the topology (f), its depend on the velocity profile. For a Poiseuille flow we have the theoretical expression of $u(y)$ and then compute τ.

(2) Topology : We want to model the topology, the term $f(x)$
Difficulty : ++
The source term is a function depending on x, $f(x)$, but appears into the model trough its derivative. A complicated point is the implementation of the "rest lake", which is the solution for $q = 0$.

(3) Dry state : We want to model the dry state, when $h = 0$
Difficulty : ++
The dry state is the situation $h = 0$ which gives a indetermination in the momentum equation. This case have to be treated carefully.

(4) 2D geometry
Difficulty : +++
We want to extend the 1D model to a 2D configuration.

Optional.

(1) Riemann solvers : We want to implement the Riemann solvers.
Difficulty : ++

(2) Surface tension : We want to model the surface tension, the boundary condition at the free surface
Difficulty : +++
When we integrate the equation

$$0 = -\frac{\partial p}{\partial y} - \rho g$$

we have to add the tension surface as boundary condition for the pressure.

3.2. Mechanics and physics.

We propose several configuration to study. Pick one, apply the numerical scheme and analyse the results.

(1) Mass damper
In modern buildings a water pool is placed at the top to reduce the building oscillations in case of earthquake. As a simple model we can imagine a spring of constant k coupled to a water tank. We want to study the coupling.

(2) Forcing
We we do an horizontal temporal forcing to a water tank, we observe waves

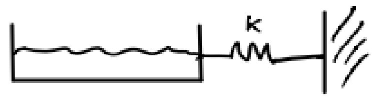

FIGURE 2. Mass damper.

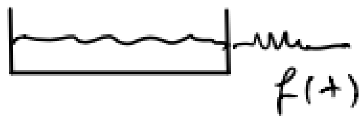

FIGURE 3. Forcing.

at the free surface, we want to study those weaves with and without surface tension.

(3) Super and sub critical states when emptying
If we open a side of a water tank the fluid flows, if we have a smooth

FIGURE 4. Emptying.

topography we can observe the sub-critical to super-critical transition. We want to study such transitions in terms of the Froude number.

(4) Discontinuities
A dam failure is a catastrophic type of failure characterized by the sudden,

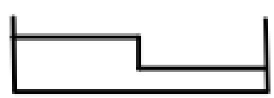

FIGURE 5. Discontinuities.

rapid, and uncontrolled release of impounded water. The dynamics of a dam failure can be modeled by a discontinuity. We want to study such discontinuities.

(5) Vertical forcing
Faraday waves are nonlinear stationary waves that appear on liquids en-

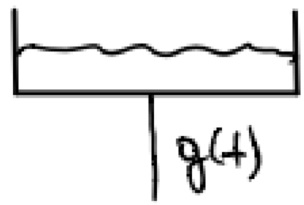

FIGURE 6. Vertical forcing.

closed by a vibrating receptacle. We want to study the Faraday waves. May be that surface tension is necessary.

(6) The Beach
When sea waves arrive to a beach several phenomena appears. We want

FIGURE 7. The Beach

to study these kind of waves. The problem can be done in 2D.

3.3. **2-D modeling.** If you decide to try a 2D modeling several cases are possible. Some cases are also proposed for 1D modeling.

(1) Variable gravity
(2) Surface tension
(3) The Beach
(4) Coriolis (rotation)
(5) Wind over free surface
(6) Importing topologies from files
(7) Sediments transport
(8) Concentration transport

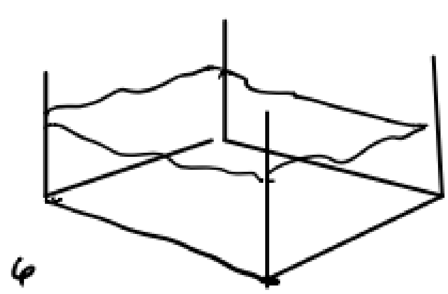

FIGURE 8. 2D