Cours de Dynamique et modélisation de la turbulence
Année 2021-2022

TD #1

Question 1

You observe the flow in a small river. Determine by using the phenomenological criteria for turbulence if the flow can be regarded as turbulent.

Question 2

Consider flow of water ($\nu = 10^{-6}m^2/s$) between two parallel plates. The bulk flow velocity is $U = 0.1m/s$. The distance between the plates is $L = 0.1m$.

a Estimate times scales of diffusion and convection. Show that their ratio is the Reynolds number. Is the flow turbulent?

b What are the time scales if the distance is $L = 10\mu m$ instead? Is the flow turbulent?

Question 3

The earth rotates at a rate $\omega_{earth} = \frac{2\pi}{day}$. A characteristic time scale is therefore $T_{earth} \sim 10^4 s$. Estimate a time scale for a hurricane and a time scale for a bath tub vortex. Do you expect rotational bias in each case? What determines the direction of rotation? Is there any difference if the hurricane/bath tub is located in Lund ($\approx 55^o N$) or Port of Spain ($\approx 10^o N$). Hint: think about the Rossby number $Ro = \frac{U}{2 \cdot L \cdot \omega sin(\phi)}$.

Question 4

With a measurement method you are able to generate 2D pictures of the instantaneous velocity field in a transparent pipe with a diameter of 10 cm. The bulk velocity in the pipe is 10 m/s and you expect a turbulence intensity of 5%. Since your hard disk is getting full you can save only a limited number of pictures.

a Assess the accuracy of the estimated average bulk flow velocity if 11 samples are collected.

b Repeat the same exercise assuming that 51 samples were used for the statistics.

c How would you asses the accuracy of the rms velocity fluctuations measured?

d Assuming normal distributions, estimate the minimum number of images needed to assure a confidence interval of maximum 0.01 m/s with a confidence of 95% accuracy.

Data :

Mean squared error of statistical quantities:

	Formula	Mean Squared Error (MSE)
Average	$\bar{x} = \frac{1}{n}\Sigma x_i$	$MSE(\bar{x}) = E((\bar{x}-\mu)^2) = \left(\frac{\sigma}{\sqrt{n}}\right)^2$
Population variance	$S_n^2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2$	$MSE(S_n^2) = E((S_n^2 - \sigma^2)^2) = \frac{2n-1}{n^2}\sigma^4$

Coefficients for confidence intervals:

t	80%	90%	95%
Normal	1.281	1.644	1.959

Exercice 1 : propriétés des fluctuations turbulentes

1 The conservation equations of mass and momentum can be written as:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j}$$

What assumptions have to be made to obtain these equations? Think of examples where the assumptions are questionable.

2 Show that time averaging has the following properties:

$\overline{\overline{u}} = \overline{u}$

$\overline{u'} = 0$

$\overline{u+v} = \overline{u} + \overline{v}$

$\overline{uv} = \overline{u}\ \overline{v} + \overline{u'v'}$

$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \overline{u}}{\partial x}$

3 Show using Reynolds decomposition (assume taking the mean and taking the derivative commute):

a $\frac{\partial \overline{u_i}}{\partial x_i} = 0$

b $\frac{\partial u_i'}{\partial x_i} = 0$

4 a Derive the transport equations for the velocity fluctuations (subtract the average from the instantaneous momentum equations).

$\frac{\partial u_i'}{\partial t} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i'}{\partial x_j} - u_i'u_j' + \overline{u_i'u_j'}\right)$

b Take the mean of the velocity fluctuation transport equations.

5 What is meant by the closure problem of the equations governing turbulent flows?

Exercice 2 (Pope Ch 4) : Equation de Poisson pour la pression moyennée

Like $p(\mathbf{x}, t)$, the mean pressure field $\langle p(\mathbf{x}, t)\rangle$ satisfies a Poisson equation. This may be obtained either by taking the mean of $\nabla^2 p$ (Eq. (2.42)), or by taking the divergence of the Reynolds equations:

$$
\begin{aligned}
-\frac{1}{\rho}\nabla^2\langle p\rangle &= \left\langle \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}\right\rangle \\
&= \frac{\partial \langle U_i\rangle}{\partial x_j}\frac{\partial \langle U_j\rangle}{\partial x_i} + \frac{\partial^2 \langle u_i u_j\rangle}{\partial x_i \partial x_j}. \qquad (4.13)
\end{aligned}
$$

Exercice 3 : Equations pour les tensions de Reynolds

Démontrer que les tensions de Reynolds satisfont une équation de transport de la forme ci-dessous et rappeler la signification physique des différents termes.

$$
\frac{\partial \tau_{ij}^R}{\partial t} + \overline{u_k}\frac{\partial \tau_{ij}^R}{\partial x_k} = -\tau_{ik}^R\frac{\partial \overline{u_j}}{\partial x_k} - \tau_{jk}^R\frac{\partial \overline{u_i}}{\partial x_k} + 2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{u_i'\frac{\partial p'}{\partial x_j} + u_j'\frac{\partial p'}{\partial x_i}} + \frac{\partial}{\partial x_k}\left[\nu\frac{\partial \tau_{ij}^R}{\partial x_k} + \rho\overline{u_i' u_j' u_k'}\right]
$$

Exercice 4 (Pope Ch 4) : propriétés de l'approximation de Boussinesq

4.9 Show that, in order for the turbulent-viscosity hypothesis (Eq. (4.45)) to yield non-negative normal stresses, it is necessary and sufficient for the turbulent viscosity to satisfy

$$
\nu_T \le \frac{k}{3\mathcal{S}_\lambda}, \qquad (4.52)
$$

where $\mathcal{S}_\lambda$ is the largest eigenvalue of the mean rate-of-strain tensor.

Exercice 5 : Moyennes de Favre

Un écoulement compressible unidimensionnel est caractérisé par les champs de masse volumique et de vitesse suivants:

$$
\rho(x,t) = 2 - \sin(x+\alpha t) \qquad u(x,t) = \frac{\alpha \sin(x+\alpha t)}{2-\sin(x+\alpha t)} \qquad \text{avec } \alpha>0
$$

1. Vérifier que l'équation de conservation de la masse est bien respectée.
2. Calculer la moyenne temporelle de ρ (notée $\bar{\rho}$), ainsi que le champ fluctuant associé.
3. Calculer la moyenne temporelle et la moyenne temporelle de Favre de u, ainsi que les champs fluctuants associés u' et u''. On notera les moyennes classique avec une barre et les moyennes de Favre avec un tilde.
4. Vérifier que les fluctuations de Favre satisfont la relation:

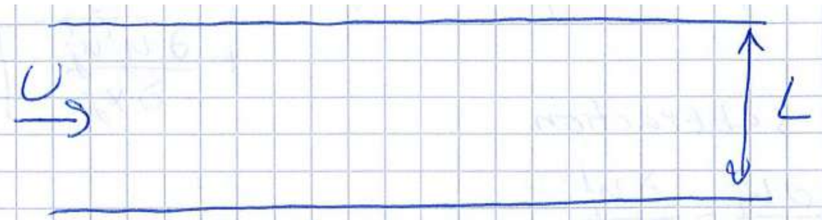
$$
u'' = u' - \frac{\overline{\rho' u'}}{\bar{\rho}}
$$

Aide : Nous rappelons l'intégrale indéfinie suivante : $\displaystyle\int \frac{\sin(ax+b)}{c-\sin(ax+b)}dx = -\frac{2c\tan^{-1}\left[\dfrac{1-c\tan\dfrac{ax+b}{2}}{\sqrt{c^2-1}}\right]}{a\sqrt{c^2-1}} - xc.$

Corrigé

Q1 : Comportement chaotique, fluctuations de vitesse importantes, comportement 3D, transport accrue.

Q2 :



$\nu = 10^{-6}\,m^2/s \qquad U = 0.1\,m$

a) $L = 0.1\,m$

Diffusive time scale $\tau_{diff} = \frac{L^2}{\nu} = 10^4\,s$

Convective time scale $\tau_{conv} = \frac{L}{U} = 1\,s$

$$\frac{\tau_{diff}}{\tau_{conv}} = \frac{\frac{L^2}{\nu}}{\frac{L}{U}} = \frac{UL}{\nu} = Re = 10000 \text{ turbulent}$$

b) $L = 10^{-5}\,m$

$\tau_{diff} = 10^{-4}\,s \qquad \tau_{conv} = 10^{-4}\,s$

~~[illegible]~~ laminar since $Re = 1$

Q3 :

$$W_{earth} = \frac{2\pi}{86400} \simeq 7.27 \cdot 10^{-5} \text{ rad/s}$$

Assume

Hurricane: $L = 1000$ m $U = 35$ m/s

Bathtubvortex $L = 0{,}05$ m $U = 0{,}3$ m/s

At Lund $\phi = 55°N$

	Hurricane	Bathtub
Rossby number	290	52500

At Port of Spain

Rossby numbe	1370	247500

The Rossby number $R_o = \frac{U}{2L\omega\sin\phi}$

relates inertial forces to the Coriolis force

The lower the Rossby number the more important is the Coriolis force.

Conclusion. The Coriolis force will affect the hurricane but not the bathtub vortex. Also, the closer to the equator one gets the less important is the Coriolis force

Q4 :

1) $U = 10\ m/s$

turbulent kinetic energy

Turbulence intensity $I = \frac{\sqrt{k}}{U} = 0{,}05$

Hence $\sigma \approx \sqrt{k} \simeq u'$

a) $n = 11$ samples

$$MSE(\bar{x}) = \left(\frac{\sigma}{\sqrt{n}}\right) = 2{,}27 \cdot 10^{-2} \Rightarrow \bar{U} = 10 \pm 0{,}15\ m/s$$

b) $n = 51$ $\quad \bar{U} = 10 \pm 0{,}07\ m/s$

c)

$$MSE(S_n^2) = \frac{2n-1}{n^2}\sigma^4$$

$n = 11: \quad u' = 0{,}5 \pm 0{,}1\ m/s$

$n = 51: \quad u' = 0{,}5 \pm 0{,}05\ m/s$

d)

If the standard deviation is known then $U \pm t\frac{\sigma}{\sqrt{n}} = 10 \pm 1{,}959\ \frac{0{,}5}{\sqrt{n}}$

$$1{,}959\ \frac{0{,}5}{\sqrt{n}} = 0{,}01 \Rightarrow n = 9594$$

Exercice 1 :

1) Les hypothèses sont : écoulement incompressible avec masse volumique constante, fluide Newtonien.
2)

$\overline{\overline{u}} = \overline{u}$: $\overline{\overline{u}} = \frac{1}{T}\int_t^{t+T} \overline{u}\,dt$ since $\overline{u} = \frac{1}{T}\int_t^{t+T} u\,dt$

is constant in the interval t, T then

$$\overline{\overline{u}} = \frac{\overline{u}}{T}\int_t^{t+T} dT = \overline{u}$$

$\overline{u'} = 0$: $\overline{u'} = \overline{u - \overline{u}} = \frac{1}{T}\int_t^{t+T}(u - \overline{u})\,dt = \frac{1}{T}\left[\int_t^{t+T} u\,dt - \int_t^{t+T} \overline{u}\,dt\right]$

$$= \overline{u} - \overline{\overline{u}} = \overline{u} - \overline{u} = 0$$

$\overline{u+v}$: $\overline{u+v} = \frac{1}{T}\int_t^{t+T}(u+v)\,dt = \frac{1}{T}\int_t^{t+T} u\,dt + \frac{1}{T}\int_t^{t+T} v\,dt =$

$$= \overline{u} + \overline{v}$$

$\overline{uv}$: $\overline{uv} = \frac{1}{T}\int_t^{t+T}(\bar{u}+u')(\bar{v}+v')dt = \frac{1}{T}\left[\int_t^{t+T}\bar{u}\bar{v}dt + \int_t^{t+T}\bar{u}v'dt + \right.$

$\left. + \int_t^{t+T}\bar{v}u'dt + \int_t^{t+T}u'v'dt\right] =$ next page

cont.

$$= \frac{1}{T}\left[\bar{u}\bar{v}\int_t^{t+T}dt + \bar{u}\cancel{\int_t^{t+T}v'dt}^{\to 0} + \bar{v}\cancel{\int_t^{t+T}u'dt}^{\to 0} + \int_t^{t+T}u'v'dt\right] =$$

$$= \bar{u}\bar{v} + \overline{u'v'}$$

$\overline{\frac{\partial u}{\partial t}} = \frac{\partial \bar{u}}{\partial t}$: $\overline{\frac{\partial u}{\partial t}} = \frac{1}{T}\int_t^{t+T}\frac{\partial u}{\partial t}dt$

Use definition of derivative

$$\frac{\partial u}{\partial t} = \frac{u(t+\tau)-u(t)}{\tau} \qquad \tau \to 0$$

$$\overline{\frac{u(t+\tau)-u(t)}{\tau}} = \frac{1}{T}\left[\frac{1}{\tau}\int_t^{t+T}u(t+\tau)dt - \frac{1}{\tau}\int_t^{t+T}u(t)dt\right] =$$

$$= \frac{\overline{u(t+\tau)} - \overline{u(t)}}{\tau}$$

3)

a) $\overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial}{\partial x_i}\left(\overline{\bar{u}_i + u_i'}\right) = \frac{\partial \bar{u}_i}{\partial x_i} + \cancel{\frac{\partial \overline{u_i'}}{\partial x_i}}^{\to 0}$

b) $\frac{\partial u_i'}{\partial x_i} = \underbrace{\frac{\partial u_i}{\partial x_i}}_{0} - \underbrace{\frac{\partial \bar{u}_i}{\partial x_i}}_{0}$

4)

Subtract the average momentum equation from the instantaneous one.

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} - \left[\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} - \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right]$$

Termwise subtraction

$$\frac{\partial u_i}{\partial t} - \frac{\partial \bar{u}_i}{\partial t} = \frac{\partial u_i'}{\partial t}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} = \frac{1}{\rho}\frac{\partial p'}{\partial x_i}$$

$$-\nu \frac{\partial^2 u_i}{\partial x_j^2} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} = -\frac{\partial u_i'}{\partial x_j^2}$$

$$\frac{\partial u_i u_j}{\partial x_j} - \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left((\bar{u}_i + u_i')(\bar{u}_j + u_j')\right) - \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} =$$

$$= \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} + \frac{\partial \bar{u}_i u_j'}{\partial x_j} + \frac{\partial \bar{u}_j u_i'}{\partial x_j} + \frac{\partial u_i' u_j'}{\partial x_j} - \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j}$$

rewrite $$\frac{\partial \bar{u}_i u_j'}{\partial x_j} = u_j' \frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_i \underbrace{\frac{\partial u_j'}{\partial x_j}}_{\substack{=0 \text{ due to} \\ \text{mass conservation}}} = u_j' \frac{\partial \bar{u}_i}{\partial x_j}$$

Hence

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \bar{u}_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \nu \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial u_i' u_j'}{\partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} = 0$$

5) Le problème de la fermeture des équations moyennées à la Reynolds est l’apparition de termes supplémentaires, fonction des moments statistiques de produits de quantités fluctuantes, qui introduisent plus d’inconnues que d’équations. Ces termes doivent être

modélisés et exprimés en fonction des variables moyennes (vitesse et pression moyennes) afin que le système d'équation soit fermé.

Exercice 2 :

Equations de Navier-Stokes :

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}$$

En notation vectorielle :

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}$$

On prend la divergence de la deuxièmeéquation :

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k}\right) \Rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right) + \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} + \nu \frac{\partial^2}{\partial x_k \partial x_k}\left(\frac{\partial u_i}{\partial x_i}\right)$$

$$\Rightarrow -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \Rightarrow -\frac{1}{\rho}\nabla^2 p = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

On injecte la décomposition de Reynolds : $\vec{u} = \vec{\bar{u}} + \vec{u}'$; $p = \bar{p} + p'$

et ou moyenne.

En utilisant les propriétés de la moyenne (les termes en rouge sont nuls):

$$-\frac{1}{\rho}(\nabla^2 \bar{p} + \nabla^2 \bar{p'}) = \left\langle \frac{\partial(\bar{u}_i + u_i')}{\partial x_j}\frac{\partial(\bar{u}_j + u_j')}{\partial x_i} \right\rangle \Rightarrow$$

$$-\frac{1}{\rho}\nabla^2 \bar{p} = \left\langle \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i} \right\rangle + \left\langle \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial u_j'}{\partial x_i} \right\rangle + \left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i} \right\rangle + \left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i} \right\rangle \Rightarrow$$

$$-\frac{1}{\rho}\nabla^2 \bar{p} = \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i} + \left\langle \frac{\partial}{\partial x_j}\left(u_i' \frac{\partial u_j'}{\partial x_i}\right) - u_i{}'\frac{\partial}{\partial x_i}\left(\frac{\partial u_j'}{\partial x_j}\right) \right\rangle \Rightarrow$$

$$-\frac{1}{\rho}\nabla^2 \bar{p} = \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i} + \left\langle \frac{\partial^2}{\partial x_i \partial x_j}(u_i' u_j') - \frac{\partial}{\partial x_j}\left(u_j' \frac{\partial u_i'}{\partial x_i}\right) \right\rangle \Rightarrow$$

$$-\frac{1}{\rho}\nabla^2 \bar{p} = \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial^2 \langle u_i' u_j' \rangle}{\partial x_i \partial x_j}$$

Exercice3

On pose d'abord :

$$NS(u_i) = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \times \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = 0$$

On construit l'opérateur :

$\overline{u_j' NS(u_i) + u_i' NS(u_j)} = 0$

On remplace la décomposition de Reynolds pour la vitesse et la pression :

$$\boxed{\overline{u_j' \frac{\partial(\bar u_i + u_i')}{\partial t}}} + \boxed{\overline{u_j'(\bar u_k + u_k')\frac{\partial(\bar u_i+u_i')}{\partial x_k}}}$$
$$+ \boxed{\frac{1}{\rho}\overline{u_j'\frac{\partial(\bar p + p')}{\partial x_i}}} - \boxed{\nu \overline{u_j' \frac{\partial^2(\bar u_i+u_i')}{\partial x_k \partial x_k}}}$$
$$+ \boxed{\overline{u_i' \frac{\partial(\bar u_j + u_j')}{\partial t}}} + \boxed{\overline{u_i'(\bar u_k + u_k')\frac{\partial(\bar u_j+u_j')}{\partial x_k}}}$$
$$+ \boxed{\frac{1}{\rho}\overline{u_i'\frac{\partial(\bar p + p')}{\partial x_j}}} - \boxed{\nu \overline{u_i' \frac{\partial^2(\bar u_j+u_j')}{\partial x_k \partial x_k}}}$$

On combine les termes de même couleur :

$$A = \overline{u_j'\frac{\partial(\bar u_i+u_i')}{\partial t}} + \overline{u_i'\frac{\partial(\bar u_j + u_j')}{\partial t}} =$$
$$= \underbrace{\cancel{\overline{u_j'}}}_{0} \frac{\partial \bar{\bar u}_i}{\partial t} + \overline{u_j'\frac{\partial u_i'}{\partial t}} + \underbrace{\cancel{\overline{u_i'}}}_{0}\frac{\partial\bar{\bar u}_j}{\partial t} + \overline{u_i'\frac{\partial u_j'}{\partial t}} =$$
$$= \overline{u_j'\frac{\partial u_i'}{\partial t} + u_i'\frac{\partial u_j'}{\partial t}} = \frac{\partial \overline{(u_i'u_j')}}{\partial t} = \boxed{-\frac{1}{\rho}\frac{\partial \tau_{ij}^R}{\partial t}}$$

$$B = \overline{u_j'(\bar u_k+u_k')\frac{\partial(\bar u_i+u_i')}{\partial x_k}} + \overline{u_i'(\bar u_k+u_k')\frac{\partial(\bar u_j+u_j')}{\partial x_k}} =$$
$$= \underbrace{\cancel{\overline{u_j'}}}_{=0}\,\bar{\bar u}_k \frac{\partial\bar{\bar u}_i}{\partial x_k} + \underbrace{\cancel{\overline{u_i'}}}_{=0}\,\bar{\bar u}_k\frac{\partial \bar{\bar u}_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \bar u_i}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial\bar u_j}{\partial x_k}$$
$$+ \bar u_k \overline{u_j'\frac{\partial u_i'}{\partial x_k}} + \bar u_k \overline{u_i'\frac{\partial u_j'}{\partial x_k}} + \overline{u_j'u_k'\frac{\partial u_i'}{\partial x_k}} + \overline{u_i'u_k'\frac{\partial u_j'}{\partial x_k}} =$$
$$= \overline{u_j'u_k'}\frac{\partial\bar u_i}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial \bar u_j}{\partial x_k} + \boxed{\bar u_k \frac{\partial\overline{u_i'u_j'}}{\partial x_k}}$$
$$+ \boxed{\frac{\partial \overline{(u_i'u_j'u_k')}}{\partial x_k}} = \boxed{-\frac{1}{\rho}\bar u_k\frac{\partial\tau_{ij}^R}{\partial x_k} - \frac{1}{\rho}\tau_{jk}^R\frac{\partial\bar u_i}{\partial x_k} - \frac{1}{\rho}\tau_{ik}^R\frac{\partial \bar u_j}{\partial x_k} + \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'}}$$

Nota : $\overline{u_j'u_k'\frac{\partial u_i'}{\partial x_k}} = \overline{u_j'\frac{\partial(u_i'u_k')}{\partial x_k}} - \overline{u_j'u_i'\underbrace{\cancel{\frac{\partial u_k'}{\partial x_k}}}_{0}} \Rightarrow$

$$\overline{u_j'\frac{\partial(u_i'u_k')}{\partial x_k}} + \overline{u_i'u_k'\frac{\partial u_j'}{\partial x_k}} = \frac{\partial}{\partial x_k}\overline{(u_i'u_j'u_k')}$$

$$C = \overline{\frac{1}{\rho} u'_j \frac{\partial(\bar{p}+p')}{\partial x_i}} + \overline{\frac{1}{\rho} u'_i \frac{\partial(\bar{p}+p')}{\partial x_j}} =$$

$$= \frac{1}{\rho} \underbrace{\cancel{\overline{u'_j}}}_{=0} \frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i}} + \frac{1}{\rho} \underbrace{\cancel{\overline{u'_i}}}_{=0} \frac{\partial \bar{p}}{\partial x_j} + \frac{1}{\rho} \overline{u'_i \frac{\partial p'}{\partial x_j}} =$$

$$= \frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i}} + \frac{1}{\rho} \overline{u'_i \frac{\partial p'}{\partial x_j}} = \boxed{\frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i} + u'_i \frac{\partial p'}{\partial x_j}}}$$

$$D = \nu \overline{u'_j \frac{\partial^2(\bar{u}_i + u'_i)}{\partial x_k \partial x_k}} + \nu \overline{u'_i \frac{\partial^2(\bar{u}_j + u'_j)}{\partial x_k \partial x_k}} =$$

$$= \nu \underbrace{\cancel{\overline{u'_j}}}_{=0} \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} + \nu \overline{u'_j \frac{\partial^2 u'_i}{\partial x_k \partial x_k}} + \nu \underbrace{\cancel{\overline{u'_i}}}_{=0} \frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_k} + \nu \overline{u'_i \frac{\partial^2 u'_j}{\partial x_k \partial x_k}} =$$

$$= \nu \frac{\partial}{\partial x_k} \left[\overline{u'_j \frac{\partial u'_i}{\partial x_k}} \right] - \nu \overline{\frac{\partial u'_j}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} + \nu \frac{\partial}{\partial x_k} \left[\overline{u'_i \frac{\partial u'_j}{\partial x_k}} \right]$$

$$- \nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} =$$

$$= \nu \frac{\partial}{\partial x_k} \left[\overline{u'_j \frac{\partial u'_i}{\partial x_k} + u'_i \frac{\partial u'_j}{\partial x_k}} \right] - 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} =$$

$$= \nu \frac{\partial}{\partial x_k} \frac{\partial \overline{u'_i u'_j}}{\partial x_k} - 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} =$$

$$= \boxed{-\frac{\nu}{\rho} \frac{\partial}{\partial x_k} \frac{\partial \tau^R_{ij}}{\partial x_k} - 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}}$$

Nous pouvons maintenant rassembler tous les termes :

$$\boxed{-\frac{1}{\rho} \frac{\partial \tau^R_{ij}}{\partial t}} + \boxed{-\frac{1}{\rho} \bar{u}_k \frac{\partial \tau^R_{ij}}{\partial x_k} - \frac{1}{\rho} \tau^R_{jk} \frac{\partial \bar{u}_i}{\partial x_k} - \frac{1}{\rho} \tau^R_{ik} \frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u'_i u'_j u'_k}} - \boxed{\frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i} + u'_i \frac{\partial p'}{\partial x_j}}}$$

$$\boxed{+\frac{\nu}{\rho} \frac{\partial}{\partial x_k} \frac{\partial \tau^R_{ij}}{\partial x_k} + 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}} = 0$$

$$\Rightarrow \frac{\partial \tau^R_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau^R_{ij}}{\partial x_k} = -\tau^R_{ik} \frac{\partial \bar{u}_j}{\partial x_k} - \tau^R_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + 2\mu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}$$

$$- \overline{\left(u'_j \frac{\partial p'}{\partial x_i} + u'_i \frac{\partial p'}{\partial x_j} \right)} + \frac{\partial}{\partial x_k} \left[\nu \frac{\partial \tau^R_{ij}}{\partial x_k} + \rho \overline{u'_i u'_j u'_k} \right]$$

Exercice 4 :

1. Equation de conservation de la masse:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \Rightarrow -\alpha\cos(x+\alpha t) + \frac{\partial}{\partial x}\left(\cancel{\left[2-\sin(x-\alpha t)\right]}\frac{\alpha\sin(x+\alpha t)}{\cancel{2-\sin(x+\alpha t)}}\right) = 0$$

$$\Rightarrow -\alpha\cos(x+\alpha t) + \alpha\cos(x+\alpha t) = 0 \quad \rightarrow \underline{OK}$$

2. $$\bar{\rho} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[2-\sin(x+\alpha t')\right]dt' =$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[2T + \left[\frac{1}{\alpha}\cos(x+\alpha t')\right]_0^T\right] =$$

$$= 2 + \lim_{T\to\infty}\left(\underbrace{\frac{1}{\alpha T}\cos(x+\alpha T)}_{\to 0} - \underbrace{\frac{1}{\alpha T}\cos x}_{\to 0}\right) = 2 \quad \rightarrow \quad \bar{\rho} = 2,\ \rho' = -\sin(x+\alpha t)$$

3. $$\bar{u} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{\alpha\sin(x+\alpha t')}{2-\sin(x+\alpha t')}dt' =$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[\frac{4}{\alpha\sqrt{3}}\tan^{-1}\left(\frac{1-2\tan\frac{\alpha t'+x}{2}}{\sqrt{3}}\right) - \alpha t'\right]_0^T =$$

$$= \lim_{T\to\infty}\left[\underbrace{\frac{4}{\alpha T\sqrt{3}}\tan^{-1}\left(\frac{1-2\tan\frac{\alpha T+x}{2}}{\sqrt{3}}\right)}_{\to 0} - \underbrace{\frac{4}{\alpha T\sqrt{3}}\tan^{-1}\left(\frac{1-2\tan\frac{x}{2}}{\sqrt{3}}\right)}_{\to 0} + \alpha\right] = \alpha$$

$$\tilde{u} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\alpha\sin(x+\alpha t')\,dt' = 0$$

$$u' = u - \bar{u} = \frac{\alpha\sin(x+\alpha t)}{2-\sin(x+\alpha t)} - \alpha = \frac{2\alpha}{2-\sin(x+\alpha t)}$$

$$u'' = u - \tilde{u} = \frac{\alpha\sin(x+\alpha t)}{2-\sin(x+\alpha t)}$$

4) $$\overline{\rho' u'} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{2\alpha\sin(x+\alpha t)}{2-\sin(x+\alpha t)}dt = 2$$

$$\Rightarrow u' - \frac{\overline{\rho' u'}}{\bar{\rho}} = \frac{2}{2-\sin(x+\alpha t)} - 1 = \frac{\alpha\sin(x+\alpha t)}{2-\sin(x+\alpha t)} = u'' \Rightarrow \underline{\underline{OK}}$$