The Heat Equation

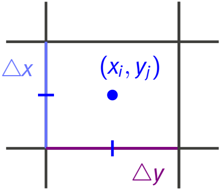
Finite Volume Discretization - $2D$

Finite Volume Discretization - Heat equation in 2d

Generalization to 2d.

- More or less direct generalization: with $C_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$

$$Q_{ij} \approx u_{ij} = \frac{1}{\triangle x \, \triangle y} \int_{C_{ij}} u(x, y, t) \, dx \, dy.$$



- For simplicity let $\triangle x = \triangle y = h$ and consider

$$\partial_t u(x, y, t) = \triangle u(x, y, t) + S(x, y, t) \qquad \text{for } (x, y) \in [0, 1] \times [0, 1]$$
$$\mathbf{n} \cdot \nabla u = 0.$$

Finite Volume Discretization - Heat equation in 2d

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$$\mathbf{n} \cdot \nabla u = 0.$$

- Exact update formula:

$$\underbrace{\frac{1}{h^2} \int_{C_{ij}} \partial_t u(x, y, t) \, dx \, dy}_{\approx \frac{d}{dt} Q_{ij}(t)} = \frac{1}{h^2} \int_{C_{ij}} \triangle u(x, y, t) \, dx \, dy + \underbrace{\frac{1}{h^2} \int_{C_{ij}} S(x, y, t) \, dx \, dy}_{\approx S_{ij}(t)}$$

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Generalization to 2d.

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- Furthermore with $\mathbf{e}_x = (1,0)^\top$ and $\mathbf{e}_y = (0,1)^\top$

$$\frac{1}{h^2}\int_{C_{ij}} \triangle u \overset{\{\mathbf{F}:=-\nabla u\}}{=} -\frac{1}{h^2}\int_{\partial C_{ij}} \mathbf{F}\cdot\mathbf{n}$$

$$= \quad -\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{F}(x, y_{j+\frac{1}{2}}, t)\cdot\mathbf{e}_y\,dx + \frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{F}(x, y_{j-\frac{1}{2}}, t)\cdot\mathbf{e}_y\,dx$$

$$-\frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{F}(x_{i+\frac{1}{2}}, y, t)\cdot\mathbf{e}_x\,dy + \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{F}(x_{i-\frac{1}{2}}, y, t)\cdot\mathbf{e}_x\,dy$$

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Generalization to 2d.

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$$= \quad -\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{F}(x, y_{j+\frac{1}{2}}, t)\cdot\mathbf{e}_y\,dx + \frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{F}(x, y_{j-\frac{1}{2}}, t)\cdot\mathbf{e}_y\,dx$$

$$-\frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{F}(x_{i+\frac{1}{2}}, y, t)\cdot\mathbf{e}_x\,dy + \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{F}(x_{i-\frac{1}{2}}, y, t)\cdot\mathbf{e}_x\,dy$$

- Let with $\mathbf{F} = (\mathbf{F}_x, \mathbf{F}_y)^\top$

$$F_{i,j\pm\frac{1}{2}} := \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{F}_y(x, y_{j\pm\frac{1}{2}}, t)\,dx \quad \text{and} \quad F_{i\pm\frac{1}{2},j} := \frac{1}{h}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{F}_x(x_{i+\frac{1}{2}}, y, t)\,dy.$$

- Hence

$$\frac{1}{h^2}\int_{C_{ij}} \triangle u = -\frac{1}{h}\left(F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}\right).$$

Finite Volume Discretization - Heat equation in 2*d*

Generalization to 2*d*.

- Recall: Exact update formula:

$$\underbrace{\frac{1}{h^2}\int_{C_{ij}} \partial_t u(x,y,t)\,dx\,dy}_{\approx \frac{d}{dt}Q_{ij}(t)} = \frac{1}{h^2}\int_{C_{ij}} \triangle u(x,y,t)\,dx\,dy + \underbrace{\frac{1}{h^2}\int_{C_{ij}} S(x,y,t)\,dx\,dy}_{\approx S_{ij}(t)}$$

- Recall: We have

$$\frac{1}{h^2}\int_{C_{ij}} \triangle u = -\frac{1}{h}\left(F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}\right).$$

- Hence (approximative formula) for $i,j = 1,2,\ldots,N-1$

$$\frac{d}{dt}Q_{ij}(t) = -\frac{1}{h}\left(F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}\right) + S_{ij}(t).$$

Finite Volume Discretization - Heat equation in 2d

Generalization to 2d.

- Approximate the flux. Example:

$$
\begin{aligned}
F_{i,j+\frac{1}{2}} &= -\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_y u(x, y_{j+\frac{1}{2}}, t)\, dx \\
&= -\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{u(x, y_{j+1}, t) - u(x, y_j, t)}{h}\, dx + \mathcal{O}(h^2) \\
&= -\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, y_{j+1}, t)\, dx + \frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, y_j, t)\, dx + \mathcal{O}(h^2) \\
&\overset{\text{const in } y}{=} -\frac{1}{h^3}\int_{C_{ij}} u(x, y_{j+1}, t)\, dx\, dy + \frac{1}{h^3}\int_{C_{ij}} u(x, y_j, t)\, dx\, dy + \mathcal{O}(h^2) \\
&\overset{u_{ij} := \frac{1}{h^2}\int_{C_{ij}} u}{=} \frac{u_{ij} - u_{i,j+1}}{h} + \mathcal{O}(h^2).
\end{aligned}
$$

Drop $\mathcal{O}(h^2)$ and let $Q_{ij} \approx u_{ij}$.

Finite Volume Discretization - Heat equation in 2d

Generalization to 2d.

- Recall: (approximative formula) for $i, j = 1, 2, \ldots, N-1$

$$\frac{d}{dt}Q_{ij}(t) = -\frac{1}{h}\left(F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}\right) + S_{ij}(t).$$

- Approximate the flux. Example:

$$F_{i,j+\frac{1}{2}} = \frac{u_{ij} - u_{i,j+1}}{h} + \mathcal{O}(h^2).$$

  Drop $\mathcal{O}(h^2)$ and let $Q_{ij} \approx u_{ij}$.

- Scheme:

$$\frac{d}{dt}Q_{ij} = \frac{1}{h^2}\left(Q_{i+1,j} - 2Q_{ij} + Q_{i-1,j} + Q_{i,j+1} - 2Q_{ij} + Q_{i,j-1}\right) + S_{ij}.$$

Finite Volume Discretization - Heat equation in $2d$

Generalization to $2d$.

- Scheme:

$$\frac{d}{dt}Q_{ij} = \frac{1}{h^2}\left(Q_{i+1,j} - 2Q_{ij} + Q_{i-1,j} + Q_{i,j+1} - 2Q_{ij} + Q_{i,j-1}\right) + S_{ij}.$$

- Boundary condition $\nabla u \cdot \mathbf{n} = 0$: use ghost points.

$$\frac{d}{dt}Q_{0,0} = \frac{1}{h^2}\left(Q_{1,0} - 2Q_{0,0} + Q_{-1,0} + Q_{0,1} - 2Q_{0,0} + Q_{0,-1}\right) + S_{00}.$$

As in the $1d$-case:

$$Q_{0,-1} = Q_{00} \qquad \text{and} \qquad Q_{-1,0} = Q_{00}.$$

Hence:

$$\frac{d}{dt}Q_{0,0} = \frac{1}{h^2}\left(Q_{1,0} - 2Q_{0,0} + Q_{0,1}\right) + S_{00}.$$

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