Hydrodynamique

Écoulements visqueux

1 Le couteau à enduire de Taylor (1962)

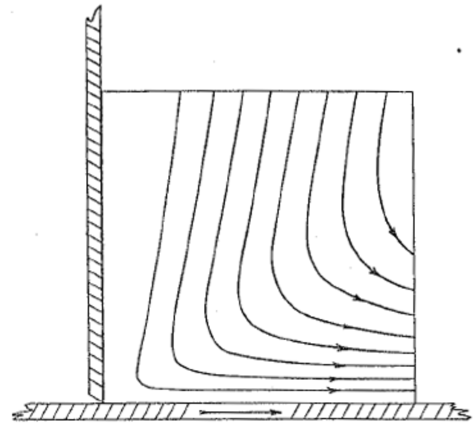

Fig. 1 : **Écoulement forcé dans un coin d'angle** α**.** Lignes de courant de l'écoulement visqueux induit par le déplacement de la plaque du bas à vitesse U (ici $\alpha = \frac{\pi}{2}$).

1. Écrire les équations et conditions limites décrivant ce problème en variables natives (pression, vitesse).
2. On recherche la solution de cet écoulement en introduisant la fonction de courant ψ telle que $\boldsymbol{u} = -\nabla \times \big(\psi(r,\theta)\boldsymbol{e}_z\big)$. Montrer que les équations de Stokes se ramènent à :

$$\nabla^4 \psi = 0 \tag{1}$$

3. Montrer que les conditions limites pour ψ s'écrivent, en supposant que la plaque avance à vitesse $-U$:

$$\begin{cases} \dfrac{\partial \psi}{\partial r} = 0 \quad \text{et} \quad \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = U & \text{en } \theta = 0 \quad \text{(2a)} \\[2ex] \dfrac{\partial \psi}{\partial r} = 0 \quad \text{et} \quad \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = 0 & \text{en } \theta = \alpha \quad \text{(2b)} \end{cases}$$

4. On recherche ψ sous la forme :

$$\psi(r,\theta) = r f(\theta) \tag{3}$$

   Justifier ce choix.
5. En introduisant l'opérateur différentiel $\mathscr{D} = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \mathscr{I}$, où $\mathscr{I}$ est l'opérateur identité, montrer :

$$\mathscr{D}^2 f = 0, \tag{4}$$

   Et en déduire la forme générale de $f(\theta)$:

$$f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta \tag{5}$$

6. Expliciter les conditions limites pour $f(\theta)$.
7. Par applications des conditions limites, montrer :

$$\big(A,B,C,D\big) = \big(-\alpha^2, 0, \theta_0 - \sin\alpha\cos\alpha, \sin^2\alpha\big) \times \frac{U}{\alpha^2 - \sin^2\alpha}. \tag{6}$$

8. Évaluer en ordre de grandeur le terme d'accélération négligé dans l'approximation de Stokes en fonction de ρ, U et r. Évaluer de la même façon l'ordre de grandeur de la force visqueuse. En déduire une région de validité de cette approximation en fonction de μ, ρ et U.
9. Retrouver l'expression de la contrainte normale exercée sur le couteau trouvée par Taylor.
10. Discuter sa conclusion sur la façon dont les peintres se servent de leur couteau pour nettoyer leur palette.

Appendice : les opérateurs différentiels en coordonnées cylindriques

Soit $f(r,\theta,z)$ une fonction scalaire de l'espace et $\boldsymbol{u}(r,\theta,z) = (u_r(r,\theta,z), u_\theta(r,\theta,z), u_z(r,\theta,z))$ un champ vectoriel, on définit :

$$\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} \\ \dfrac{\partial f}{\partial z} \end{pmatrix}, \quad \nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}, \quad \Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

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ON SCRAPING VISCOUS FLUID FROM A PLANE SURFACE

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The stream function ψ, representing two-dimensional fluid flow, satisfies $\nabla^4\psi = 0$ when the flow is so slow that inertia is negligible. Simple solutions of this equation expressed in polar co-ordinates can be found in the form $\psi = r^n f(\Theta)$ but they seldom have much physical interest except as terms in series expansions. When n is greater than 1 they usually represent flow in a corner produced by agents which can only be described by other types of solution. When n is less than 1 the flow velocity near the origin is infinite. The case when $n = 1$ has physical significance because it can represent flow of a viscous fluid when a flat scraper moves over a flat sheet pushing fluid before it.

The general solution of $\nabla^4\psi = 0$ when $n = 1$ is

$$\psi = r(A\cos\Theta + B\sin\Theta + C\Theta\cos\Theta + D\Theta\sin\Theta). \tag{1}$$

If (1) represents the flow in front of a scraper moving with velocity U over a flat surface (taken as $\Theta = 0$) it is convenient to superpose a velocity $-U$ parallel to $\Theta = 0$ so as to reduce the motion to steady flow. The radial and tangential velocities are

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial\Theta} \quad \text{and} \quad v = \frac{\partial\psi}{\partial r}.$$

If the scraper is inclined at angle α to the flat plate the boundary conditions are

$$\psi = 0, \ \frac{1}{r}\frac{\partial\psi}{\partial\Theta} = U \quad \text{at} \quad \Theta = 0 \quad \text{and} \quad \psi = \frac{\partial\psi}{\partial\Theta} = 0 \quad \text{at} \quad \Theta = \alpha.$$

Using these conditions (1) becomes

$$\psi = \frac{Ur}{\alpha^2 - \sin^2\alpha}\left\{\alpha^2\sin\Theta - \sin^2\alpha\,\Theta\cos\Theta - \left(\frac{\alpha\sin^3\alpha + \alpha^2\cos\alpha - \cos\alpha\sin^2\alpha}{\sin\alpha + \alpha\cos\alpha}\right)\Theta\sin\Theta\right\}. \tag{2*}$$

* *Editor's note.* Fig. 1, which shows the streamlines for the case $\alpha = \frac{1}{2}\pi$, is taken from an article by Sir Geoffrey Taylor entitled 'Similarity solutions of hydrodynamic problems' published in *Aeronautics and Astronautics* (Pergamon Press, 1960) and not reprinted in these volumes.

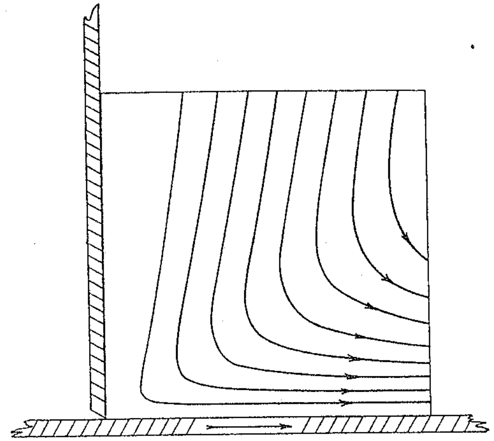

Fig. 1. Streamlines of flow of viscous fluid for $\alpha = \frac{1}{2}\pi$.

If P and S are the stresses normal and tangential to the scraper at distance r from the point of contact and μ is the viscosity

$$P = \frac{2\mu U}{r}\left(\frac{\alpha\sin\alpha}{\alpha^2 - \sin^2\alpha}\right), \tag{3}$$

$$S = \frac{2\mu U}{r}\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2 - \sin^2\alpha}. \tag{4}$$

When α is small these tend to $P = 6\mu U/r\alpha^2$ and $S = 2\mu U/r\alpha$ which are the values that can be found using the approximate equations of lubrication theory.

Resolving the stress in direction L at right angle to the plate and D parallel with it,

$$L = P\cos\alpha + S\sin\alpha = \frac{2\mu U}{r}\left(\frac{\sin^2\alpha}{\alpha^2 - \sin^2\alpha}\right),$$

$$D = P\sin\alpha - S\cos\alpha = \frac{2\mu U}{r}\left(\frac{\alpha - \sin\alpha\cos\alpha}{\alpha^2 - \sin^2\alpha}\right).$$

Values of L, D, P and S divided by $2\mu U/r$ for various values of α are given in table 1, and are displayed in fig. 2. It will be seen that D decreases as α increases, and attains its least value $2\mu U/\pi r$ when $\alpha = \pi$. The most interesting and perhaps unexpected feature of the calculations is that L does not change sign in the range $0 < \alpha < \pi$. In the range $\frac{1}{2}\pi < \alpha < \pi$ the contribution to L due to normal stress is of opposite sign to that due to tangential stress, but the latter is the greater. The palette knives used by artists for removing paint from their palettes are very flexible scrapers. They can therefore only be used at such an angle that P is small and as will be seen in the figure this occurs only when α is nearly 180°. In fact artists instinctively hold their palette knives in this position.

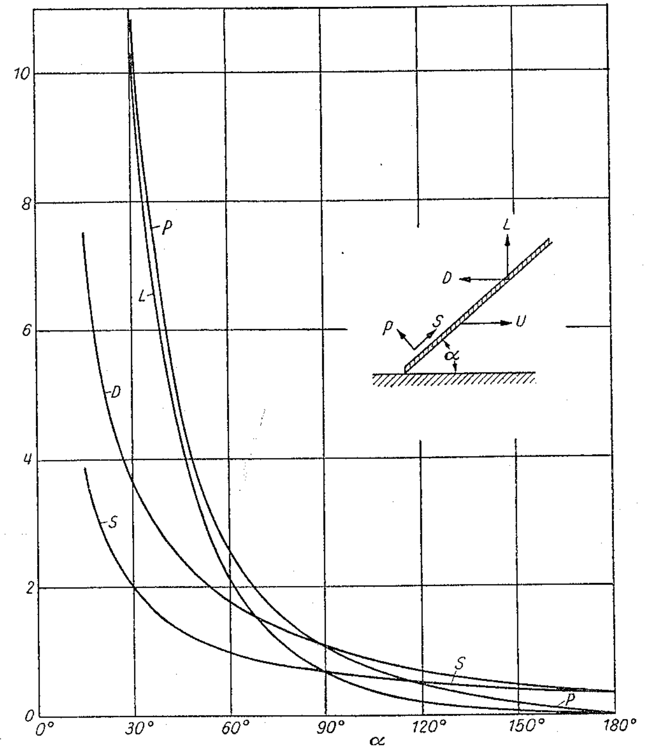

Fig. 2. Values of P, S, L and D divided by $2\mu U/r$.

Table 1

$\alpha°$	$\frac{Pr}{2\mu U}$	$\frac{Sr}{2\mu U}$	$\frac{Lr}{2\mu U}$	$\frac{Dr}{2\mu U}$
0	∞	∞	∞	∞
15	43	3·82	42	7·5
30	10·8	2·03	10·3	3·67
45	4·8	1·31	4·30	2·44
60	2·61	0·98	2·15	1·77
75	1·61	0·80	1·19	1·36
90	1·07	0·68	0·68	1·07
105	0·73	0·60	0·38	0·85
120	0·50	0·53	0·21	0·70
135	0·33	0·47	0·10	0·56
150	0·20	0·42	0·04	0·46
165	0·08	0·37	0·014	0·38
180	0	0·32	0	0·32

A plasterer on the other hand holds a smoothing tool so that α is small. In that way he can get the large values of L/D which are needed in forcing plaster from protuberances to hollows.

Though the fluid velocity is everywhere finite the stress becomes infinite at $r = 0$ in this solution. In fact in any real situation continuous contact between scraper and plate along a line will not occur so that infinite stress at $r = 0$ will be relieved over a region comparable with the width of the gap.