Numerical solutions of differential equations

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General Finite Volumes Schemes of First Order

Monotone schemes

Consistent Methods

Consistent numerical flux

Definition

Let $f \in C^1(\mathbb{R})$ be a physical flux and $g \in C^{0,1}(\mathbb{R} \times \mathbb{R})$ be a Lipschitz continuous numerical flux.

We say that g is consistent with f if and only if

$$g(u, u) = f(u) \qquad \text{for all } u \in \mathbb{R}.$$

Consistent numerical scheme

Definition (Consistent Numerical Scheme)

Let $f \in C^1(\mathbb{R})$ and $g \in C^{0,1}(\mathbb{R} \times \mathbb{R})$ a numerical flux.

Let $x_j = \frac{\Delta x}{2} + j\Delta x$ for $j \in \mathbb{Z}$ define a spatial mesh and $t_n = n\Delta t$ for $n \in \mathbb{N}_0$ define a time mesh.

The discrete initial value is given by $v_0(x_j) \approx Q_j^0 \in \mathbb{R}$. The scheme

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n\right)$$

with

$$g_{j+\frac{1}{2}}^n := g(Q_j^n, Q_{j+1}^n), \qquad g_{j-\frac{1}{2}}^n := g(Q_{j-1}^n, Q_j^n)$$

is an (explicit) scheme in conservation form with numerical flux g.

The scheme is called consistent if g is consistent with f.

Reminder - Consistency

A scheme is consistent if the exact solution fits the scheme well.

More precisely, we define the local truncation error $\boldsymbol{\tau}^n$ such that

$$\mathbf{u}^{n+1} = \boldsymbol{\Phi}(\mathbf{u}^n) + \Delta t\, \boldsymbol{\tau}^n, \qquad \text{where } u_j^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u(t_n, x)dx$$

- Local truncation error $\simeq$ error performed in one time step, scaled by Δt:

$$\frac{\mathbf{u}^{n+1} - \boldsymbol{\Phi}(\mathbf{u}^n)}{\Delta t} = \boldsymbol{\tau}^n.$$

Reminder - Consistency

- For convergence we need a small $\boldsymbol{\tau}^n$.
- We say that the method is consistent if

$$\max_{0\le n\Delta t\le T} \|\boldsymbol{\tau}^n\|_{\Delta x} \to 0 \qquad \text{as } \Delta t, \Delta x \to 0, \text{ for a fixed } T.$$

- If there is a number C independent of Δt and Δx such that

$$\max_{0\le n\Delta t\le T} \|\boldsymbol{\tau}^n\|_{\Delta x} \le C(\Delta x^p + \Delta t^r)$$

  we say that the method is of order p in space and r in time.
- If $\lambda_{\text{CFL}} = \Delta t/\Delta x$ is constant, with $\lambda_{\text{CFL}} = \mathcal{O}(1)$, then

$$\|\boldsymbol{\tau}^n\|_{\Delta x} = \mathcal{O}(\Delta x^p + \Delta x^r) = \mathcal{O}(\Delta x^q), \qquad \text{where } q = \min(p, r)$$

  and we simply say the method is of order q.

Consistency in our case

Definition (Consistency order)

For a numerical flux $g \in C^{0,1}(\mathbb{R} \times \mathbb{R})$, the scheme is characterized by

$$\Phi(v, w, z) := w - \frac{\Delta t}{\Delta x}[g(w, z) - g(v, w)].$$

For the cell averages of the exact solution u_j^n the local truncation error $\boldsymbol{\tau}^n$ is defined by

$$\tau_j^n := \frac{u_j^{n+1} - \Phi(u_{j-1}^n, u_j^n, u_{j+1}^n)}{\Delta t} \qquad \text{for } j \in \mathbb{Z},\ n \in \mathbb{N}.$$

The scheme is consistent of order p if

$$\tau_j^n \leq C\left(\Delta x^p + \Delta t^p\right) \qquad \text{for } j \in \mathbb{Z},\ n \in \mathbb{N}.$$

Remark on scheme

For a numerical flux $g \in C^{0,1}(\mathbb{R} \times \mathbb{R})$ and with

$$\Phi(v, w, z) := w - \frac{\Delta t}{\Delta x}[g(w, z) - g(v, w)],$$

we can write the scheme in conservation form as:

$$Q_j^{n+1} = \Phi(Q_{j-1}^n, Q_j^n, Q_{j+1}^n).$$

Recall that

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n)$$

with

$$g_{i+\frac{1}{2}}^n := g(Q_j^n, Q_{j+1}^n), \qquad g_{i-\frac{1}{2}}^n := g(Q_{j-1}^n, Q_j^n)$$

Consistency

Consistent numerical schemes have always at least consistency order 1.

Theorem

Let $f \in C^2(\mathbb{R})$ and $u \in C^2(\mathbb{R} \times \mathbb{R}^+)$ a classical solution to

$$\partial_t u + \partial_x f(u) = 0.$$

If $g \in C^2(\mathbb{R} \times \mathbb{R})$ is a numerical flux that is consistent with f. Then for fixed $\frac{\Delta t}{\Delta x} = \text{const}$ the scheme

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right)$$

is consistent of order 1, i.e. $\tau_j^n \leq C\,(\Delta x + \Delta t)$ for $j \in \mathbb{Z},\ n \in \mathbb{N}$.

(proof: Taylor expansion)

Intermezzo

Question: Is a consistent scheme enough for convergence?

Answer: No, it is not enough. Consistency is only a necessary condition. For convergence we require additionally that the scheme is stable.

Examples of consistent numerical fluxes

Recall $g^n_{j+\frac{1}{2}} = g(Q^n_j, Q^n_{j+1})$ and $g^n_{j-\frac{1}{2}} = g(Q^n_{j-1}, Q^n_j)$. Let

$$\partial f^n_j := \frac{g^n_{j+\frac{1}{2}} - g^n_{j-\frac{1}{2}}}{\Delta x} \quad \Rightarrow \quad \partial f^n_j \approx \partial_x f(u(x_j, t_n)).$$

- Backwards differences

$$g(v, w) := f(v) \quad \Rightarrow \quad \partial f^n_j = \frac{f(Q^n_j) - f(Q^n_{j-1})}{\Delta x}.$$

- Forward differences

$$g(v, w) := f(w) \quad \Rightarrow \quad \partial f^n_j = \frac{f(Q^n_{j+1}) - f(Q^n_j)}{\Delta x}.$$

- Central differences

$$g(v, w) := \frac{f(v) + f(w)}{2} \quad \Rightarrow \quad \partial f^n_j = \frac{f(Q^n_{j+1}) - f(Q^n_{j-1})}{2\Delta x}.$$

Examples of consistent numerical fluxes

Let $\partial f_j^n := \dfrac{g(Q_j^n, Q_{j+1}^n) - g(Q_{j-1}^n, Q_j^n)}{\Delta x} \quad \Rightarrow \quad \partial f_j^n \approx \partial_x f(u(x_j, t_n)).$

- Lax-Friedrich flux

$$g(v,w) := \frac{f(v)+f(w)}{2} + \frac{1}{2\lambda}(v-w), \qquad \lambda = \frac{\Delta t}{\Delta x}.$$

Then

$$\partial f_j^n = \underbrace{\frac{f(Q_{j+1}^n) - f(Q_{j-1}^n)}{2\Delta x}}_{\approx \partial_x f(u)} - \underbrace{\frac{Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n}{2\Delta t}}_{\approx (2\lambda)^{-1}\Delta x\, \partial_{xx} u}$$

which gives the Lax-Friedrich scheme:

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{2\Delta x}(f(Q_{j+1}^n) - f(Q_{j-1}^n)) + \frac{1}{2}(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n).$$

Examples of consistent numerical fluxes

Lax-Friedrich flux

$$g(v,w) := \frac{f(v)+f(w)}{2} + \frac{1}{2\lambda}(v-w), \qquad \lambda = \frac{\Delta t}{\Delta x}.$$

Then

$$\partial f_j^n = \underbrace{\frac{f(Q_{j+1}^n) - f(Q_{j-1}^n)}{2\Delta x}}_{\approx \partial_x f(u)} - \underbrace{\frac{Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n}{2\Delta t}}_{\approx (2\lambda)^{-1}\Delta x\, \partial_{xx} u}.$$

The scheme can be considered as an approximation of

$$\partial_t u + \partial_x f(u) = \frac{\Delta x}{2\lambda}\,\partial_{xx} u.$$

Hence, we can interpret $\varepsilon = \frac{\Delta x}{2\lambda}$ as an artifical viscosity term!

The numerical flux g is consistent and Lipschitz continuous, if f is Lipschitz continuous.

Examples of consistent numerical fluxes

- Engquist-Osher flux. **Idea:** following the direction of characteristics, we use

$$\partial f_j^n = \frac{1}{\Delta x}(f(Q_j^n) - f(Q_{j-1}^n)), \qquad \text{if } f' > 0,$$
$$\partial f_j^n = \frac{1}{\Delta x}(f(Q_{j+1}^n) - f(Q_j^n)), \qquad \text{if } f' < 0.$$

We define

$$f^+(v) := f(0) + \int_0^v \max(f'(s), 0)\, ds, \qquad f^-(v) := \int_0^v \min(f'(s), 0)\, ds.$$

Then $f(v) = f^+(v) + f^-(v)$ and we define the Engquist-Osher flux by

$$g(v, w) := f^+(v) + f^-(w).$$

Hence

$$\partial f_j^n := \frac{1}{\Delta x}(f^+(Q_j^n) - f^+(Q_{j-1}^n) + f^-(Q_{j+1}^n) - f^-(Q_j^n))$$

and we obtain the Engquist-Osher scheme

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}(f^+(Q_j^n) - f^+(Q_{j-1}^n) + f^-(Q_{j+1}^n) - f^-(Q_j^n)).$$

Consistent numerical schemes

Issue: How do we ensure convergence to the entropy solution?

Motivation:

Theorem

Let $g \in C^2(\mathbb{R} \times \mathbb{R})$ be consistent with f and let Q_j^n be the corresponding numerical approximation obtained with the scheme in conservation form.

Then, the local truncation error for smooth solutions to

$$\partial_t u + \partial_x f(u) = \frac{\Delta x}{2} \partial_x (b(u) \partial_x u)$$

with $$b(u) = \partial_1 g(u,u) - \partial_2 g(u,u) - \lambda (f'(u))^2, \quad \lambda = \frac{\Delta t}{\Delta x}$$

is of order 2.

Proof: Taylor expansion.