TD 0.1 : Variational formulations and Galërkin approximations in 1D.

Exercise 1

The first and second directional derivatives of a functional $J : f \in \mathcal{C} \to \mathbb{R}$ are defined as:

$$J'(f)(\varphi) = \frac{d}{dh}\left[J(f + h\,\varphi)\right]_{h=0} \quad \text{et} \quad J''(f)(\varphi) = \frac{d^2}{dh^2}\left[J(f + h\,\varphi)\right]_{h=0}, \quad h \in \mathbb{R},$$

Calculate the first and second derivatives of the following functionals, where $f : [0, 1] \to \mathbb{R}$ is a smooth function:

$$J_1(f) = \int_0^1 f'(x)^2 dx + 2f(0) \tag{1}$$

$$J_2(f) = \int_0^1 f'(x)^4 dx + f'(0) + f(1)^2 \tag{2}$$

$$J_3(f) = \int_0^1 \left[\frac{K}{2} f'(x)^2 + F\ell\cos(f(x))\right] dx \tag{3}$$

Exercise 2

Let us consider a bar of length L occupying the domain $\Omega \equiv (0, L)$. Considering a one-dimensional bar model, let be ES the axial stiffness of the bar and $u(x)$ the axial displacement. The bar is clamped at $x = 0$ and loaded by a axial end-force F at $x = L$ and a distributed axial loading $p(x)$. We suppose at the beginning that ES is independent of x (uniform material and cross-sectional area).

Part I

- Write the strong formulation of the problem.
- Write the weak formulation of the problem.
- Write the total potential energy and give the variational formulation of the problem.
- Obtain the weak and the strong formulation as stationarity conditions of the total potential energy.

Part II For $p(x) = p_0 \sin \dfrac{\pi x}{L}$ and $F = 0$:

- Determine the analytical solution of the problem.
- Determine an approximate solution of the problem using the Galërkin approach with the basis functions $\phi_1(x) = \frac{x}{L}$, $\phi_2(x) = \frac{x^2}{L^2}$.
- Evaluate the error between the approximate and the analytical solution with the following norms: L_∞, L_2, H_1, and energy norm.

Exercise 3

Repeat all the steps of Part I of the previous exercise for the case of stepped beam, where ES is given by:

$$ES(x) = \begin{cases} ES, & x \in (0, L/2) \\ ES/2, & x \in (L/2, L) \end{cases}$$