Lecture 8

Convergence Theory for Linear Methods - Part 2

Checking stability

von Neumann Analysis

von Neumann analysis

Goal: derive sufficient condition for stability.

- We use von Neumann analysis;
- based on Fourier analysis;
- stability is shown similarly as to how well-posedness is shown for the continuous problem and hence
- von Neumann analysis requires a constant coefficient **a**.
- More precisely, scheme should have same form at all grid points, i.e. general form reduces to

$$Q_j^{n+1} = \sum_{\ell=-m}^{M} b_\ell(\Delta t, \Delta x)\; Q_{j+\ell}^n, \qquad (\mathbf{S})$$

  where b_ℓ does not depend on j.
- Moreover: either no boundaries or periodic boundary conditions.

von Neumann analysis

Example for admissible scheme and problem:

Consider the constant coefficient advection problem

$$\partial_t u + \mathbf{a}\,\partial_x u = 0.$$

For the upwind scheme applied to it we have

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\mathbf{a}(Q_j^n - Q_{j-1}^n),$$

and hence $m = 1$, $M = 0$ and

$$b_0 = 1 - \mathbf{a}\lambda_{\text{CFL}}, \quad b_{-1} = \mathbf{a}\lambda_{\text{CFL}}, \quad \lambda_{\text{CFL}} = \frac{\Delta t}{\Delta x}.$$

von Neumann analysis

Remark before we start:

- von Neumann analysis works for any equation, not just hyperbolic problems.
- However, natural relation between Δt and Δx may then differ.
- E.g. $\Delta t/\Delta x^2 = \mathcal{O}(1)$ for explicit methods for parabolic problems (compare Lecture 3!)

Checking stability

von Neumann Analysis - Periodic boundary conditions

von Neumann analysis - Periodic BC

We consider

$$\partial_t u + \mathbf{a}\,\partial_x u = 0.$$

and assume periodic boundary conditions, i.e. $u(t,0) = u(t,2\pi)$.

The space discretization is

$$x_j = j\Delta x, \qquad \Delta x = \frac{2\pi}{N},$$

and the approximation Q_j^n satisfies discrete periodicity

$$Q_j^n = Q_{j+N}^n, \qquad \forall j, \quad \forall n \geq 0.$$

Hence, we only compute the Q_j^n values for $j = 0, \ldots, N-1$, but then define Q_j^n for all j by periodicity.

von Neumann analysis - Periodic BC

Also assume that N is even.

Let $\hat{\mathbf{Q}}^n \in \mathbb{R}^N$ be the discrete Fourier transform of $\mathbf{Q}^n \in \mathbb{R}^N$, so that

$$Q_j^n = \sum_{k=-N/2}^{N/2-1} \hat{Q}_k^n \, e^{\mathrm{i}kx_j}.$$

The Fourier coefficients can be obtained by the transform

$$\hat{Q}_k^n = \frac{1}{N} \sum_{j=0}^{N-1} Q_j^n \, e^{-\mathrm{i}kx_j}.$$

Using the scheme $\mathbf{\Phi}$ as in **(S)** we can derive an expression for $\hat{Q}_k^{n+1}$ in terms of $\hat{Q}_k^n$ as follows.

von Neumann analysis - Periodic BC

We use $Q_j^n = \sum_{k=-N/2}^{N/2-1} \hat{Q}_k^n e^{ikx_j}$ and $\hat{Q}_k^n = \frac{1}{N}\sum_{j=0}^{N-1} Q_j^n e^{-ikx_j}$ to obtain

$$\hat{Q}_k^{n+1} = \frac{1}{N}\sum_{j=0}^{N-1} Q_j^{n+1} e^{-ikx_j} \overset{\text{scheme}}{=} \frac{1}{N}\sum_{\ell=-m}^{M}\sum_{j=0}^{N-1} b_\ell Q_{j+\ell}^n e^{-ikx_j}$$

$$\overset{\text{periodicity}}{=} \frac{1}{N}\sum_{\ell=-m}^{M}\sum_{j=0}^{N-1} b_\ell Q_j^n e^{-ikx_{j-\ell}}$$

$$\overset{\{x_{j-\ell}=x_j-\ell\Delta x\}}{=} \frac{1}{N}\sum_{\ell=-m}^{M}\sum_{j=0}^{N-1} b_\ell e^{ik\ell\Delta x} Q_j^n e^{-ikx_j}$$

$$= \hat{Q}_k^n \sum_{\ell=-m}^{M} b_\ell e^{ik\ell\Delta x}.$$

von Neumann analysis - Periodic BC

We obtained

$$\hat{Q}_k^{n+1} = \hat{Q}_k^n \sum_{\ell=-m}^{M} b_\ell \, e^{ik\ell\Delta x}.$$

Hence, we can write

$$\hat{Q}_k^{n+1} = g_k(\Delta t, \Delta x)\, \hat{Q}_k^n, \qquad g_k(\Delta t, \Delta x) := \sum_{\ell=-m}^{M} b_\ell(\Delta t, \Delta x)\, e^{ik\ell\Delta x}.$$

The factor g_k is called

amplification factor.

It shows how the different frequencies in the solution are amplified in each time step.

Amplification factor - Example

For the upwind scheme we have

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x} \mathbf{a} (Q_j^n - Q_{j-1}^n),$$

and hence $m = 1$, $M = 0$ and

$$b_0 = 1 - \mathbf{a}\lambda_{\text{CFL}}, \quad b_{-1} = \mathbf{a}\lambda_{\text{CFL}}, \quad \lambda_{\text{CFL}} = \frac{\Delta t}{\Delta x}.$$

We conclude that the amplification factor for upwind is

$$g_k(\Delta t, \Delta x) = b_0 + b_{-1}\, e^{-i\ell\Delta x} = 1 - \mathbf{a}\frac{\Delta t}{\Delta x} + \mathbf{a}\frac{\Delta t}{\Delta x} e^{-ik\Delta x}.$$

von Neumann analysis - Periodic BC

We have $\hat{Q}_k^{n+1} = g_k(\Delta t, \Delta x)\, \hat{Q}_k^n$. For simplicity, we write $g_k = g_k(\Delta t, \Delta x)$.

We now use Parseval's theorem, which says that

$$||\mathbf{Q}^n||^2_{2,\Delta x} = \sum_{j=0}^{N-1} |Q_j^n|^2 \Delta x = \sum_{k=-N/2}^{N/2-1} |\hat{Q}_k^n|^2.$$

We get

$$||\mathbf{\Phi}(\mathbf{Q}^n)||^2_{2,\Delta x} = ||\mathbf{Q}^{n+1}||^2_{2,\Delta x} = \sum_{k=-N/2}^{N/2-1} |\hat{Q}_k^{n+1}|^2 = \sum_{k=-N/2}^{N/2-1} |g_k \hat{Q}_k^n|^2$$

$$\leq \max_{-\frac{N}{2}\leq k\leq \frac{N}{2}-1} |g_k|^2 \sum_{k=-N/2}^{N/2-1} |\hat{Q}_k^n|^2 = \max_{-\frac{N}{2}\leq k\leq \frac{N}{2}-1} |g_k|^2 ||\mathbf{Q}^n||^2_{2,\Delta x}.$$

von Neumann analysis - Periodic BC

Hence,

$$||\mathbf{\Phi}(\mathbf{Q}^n)||_{2,\Delta x} \leq \max_{-\frac{N}{2}\leq k\leq \frac{N}{2}-1} |g_k| ||\mathbf{Q}^n||_{2,\Delta x},$$

and we see that a sufficient condition for stability is

$$\max_{-\frac{N}{2}\leq k\leq \frac{N}{2}-1} |g_k| \leq 1 + \alpha\Delta t.$$

Remark:

- In most cases when the exact solution does not grow exponentially we can actually show the stronger version

$$\max_{-\frac{N}{2}\leq k\leq \frac{N}{2}-1} |g_k| \leq 1.$$

von Neumann analysis - Periodic BC

Condition for amplification factor:

$$\max_{-\frac{N}{2}\leq k\leq \frac{N}{2}-1} |g_k(\Delta t, \Delta x)| \leq 1 + \alpha \Delta t.$$

Example ($\partial_t u + a\partial_x u = 0$ - Advection equation)

Assume that it holds the CFL condition

$$|\mathbf{a}|\frac{\Delta t}{\Delta x} \leq 1.$$

For the upwind scheme and $\mathbf{a} > 0$ we have

$$|g_k(\Delta t, \Delta x)| \leq \left|1 - \mathbf{a}\frac{\Delta t}{\Delta x}\right| + \left|\mathbf{a}\frac{\Delta t}{\Delta x}\right| = 1 - \mathbf{a}\frac{\Delta t}{\Delta x} + \mathbf{a}\frac{\Delta t}{\Delta x} = 1.$$

CFL condition is hence both necessary and sufficient for upwind scheme.

von Neumann analysis - Periodic BC

Example ($\partial_t u - \triangle u = 0$ - Heat equation)

Consider forward Euler and central differences for the heat equation:

$$Q_j^{n+1} = Q_j^n + \mu(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n), \qquad \mu = \frac{\Delta t}{\Delta x^2}.$$

Here $m = M = 1$ and $b_{-1} = \mu$, $b_0 = 1 - 2\mu$, $b_1 = \mu$. Then

$$\begin{aligned} g_k(\Delta t, \Delta x) &= \mu\, e^{-ik\Delta x} + 1 - 2\,\mu + \mu\, e^{ik\Delta x} \\ &= 1 + 2\,\mu\,(\cos(k\Delta x) - 1) = 1 - 4\,\mu\,\sin^2\left(\frac{k\Delta x}{2}\right). \end{aligned}$$

Since

$$\max_{|k|\leq \frac{N}{2}} \sin^2\left(\frac{k\Delta x}{2}\right) = \max_{|k|\leq \frac{N}{2}} \sin^2\left(\frac{k\pi}{N}\right) = \sin^2\left(\frac{\pi}{2}\right) = 1,$$

g_k takes values in the interval $[1 - 4\mu, 1]$ and the method is stable if $\mu \leq 1/2$.

von Neumann analysis - Periodic BC

Example ($\partial_t u + \partial_x u = 0$ - Advection equation)

Consider forward Euler and central differences for the advection equation:

$$Q_j^{n+1} = Q_j^n + \frac{1}{2}\lambda_{\text{CFL}}(Q_{j+1}^n - Q_{j-1}^n), \qquad \lambda_{\text{CFL}} = \frac{\Delta t}{\Delta x}.$$

Again, here $m = M = 1$, but $\quad b_{-1} = -\frac{\lambda_{\text{CFL}}}{2}, \quad b_0 = 1, \quad b_1 = \frac{\lambda_{\text{CFL}}}{2}$. Then

$$g_k(\Delta t, \Delta x) = -\frac{\lambda_{\text{CFL}}}{2} e^{-ik\Delta x} + 1 + \frac{\lambda_{\text{CFL}}}{2} e^{ik\Delta x} = 1 + \lambda_{\text{CFL}} \mathrm{i} \sin(k\Delta x).$$

Hence,

$$\max_k |g_k(\Delta t, \Delta x)| = \max_k \sqrt{1 + \lambda_{\text{CFL}}^2 \sin(k\Delta x)^2} > 1,$$

and the method is unstable for all fixed λ_{CFL}.

von Neumann analysis - Periodic BC

Example ($\partial_t u + \partial_x u = 0$ - Advection equation)

Consider forward Euler and central differences for the advection equation:

$$Q_j^{n+1} = Q_j^n + \frac{1}{2}\lambda_{\text{CFL}}(Q_{j+1}^n - Q_{j-1}^n), \qquad \lambda_{\text{CFL}} = \frac{\Delta t}{\Delta x}.$$

The method is unstable for all fixed λ_{CFL}.

However, if using the "parabolic" CFL condition

$$\lambda_{\text{CFL}} \sim \frac{\Delta t}{\Delta x^2} \qquad \Rightarrow \qquad \lambda_{\text{CFL}} \sim \Delta x \sim \sqrt{\Delta t}.$$

then

$$|g_k(\Delta t, \Delta x)| = \sqrt{1 + \lambda_{\text{CFL}}^2 \sin(k\Delta x)^2} \sim \sqrt{1 + \Delta t \sin(k\Delta x)^2} \leq 1 + \alpha\Delta t,$$

for some α and small enough Δt. Choice makes the unstable method stable.

Checking stability

von Neumann Analysis - No boundaries

von Neumann analysis - No boundaries

- Case of no boundaries is similar to periodic case.
- Difference:
  instead of discrete Fourier transform we use Fourier series.
- More precisely, we use $\mathbf{Q}^n$ a coefficients in a Fourier series.
- The infinite sequence $\mathbf{Q}^n$ defines a 2π-periodic function $\hat{\mathbf{Q}}^n \in L^2(0, 2\pi)$ via

$$\hat{\mathbf{Q}}^n(x) = \sum_{j=-\infty}^{\infty} Q_j^n \, e^{\mathrm{i}jx}, \qquad Q_j^n = \frac{1}{2\pi} \int_0^{2\pi} \hat{\mathbf{Q}}^n(\xi)\, e^{-\mathrm{i}j\xi}\, d\xi.$$

(simple calculation)

von Neumann analysis - No boundaries

We have

$$\hat{\mathbf{Q}}^n(x) = \sum_{j=-\infty}^{\infty} Q_j^n \, e^{ijx}, \qquad Q_j^n = \frac{1}{2\pi} \int_0^{2\pi} \hat{\mathbf{Q}}^n(\xi) \, e^{-ij\xi} \, d\xi.$$

As before we derive an expression for $\hat{\mathbf{Q}}^{n+1}(x)$ in terms of $\hat{\mathbf{Q}}^n(x)$:

$$\begin{aligned}
\hat{\mathbf{Q}}^{n+1}(x) &= \sum_{j=-\infty}^{\infty} Q_j^{n+1} e^{ijx} = \sum_{\ell=-m}^{M} \sum_{j=-\infty}^{\infty} b_\ell \, Q_{j+\ell}^n \, e^{ijx} \\
&= \sum_{\ell=-m}^{M} \sum_{j=-\infty}^{\infty} b_\ell \, Q_j^n \, e^{i(j-\ell)x} = \sum_{\ell=-m}^{M} \sum_{j=-\infty}^{\infty} b_\ell \, e^{-i\ell x} \, Q_j^n \, e^{ijx} \\
&= \hat{\mathbf{Q}}^n(x) \sum_{\ell=-m}^{M} b_\ell \, e^{-i\ell x}.
\end{aligned}$$

von Neumann analysis - No boundaries

We thus have

$$\hat{\mathbf{Q}}^{n+1}(x) = \hat{g}(x, \Delta t, \Delta x)\,\hat{\mathbf{Q}}^{n}(x), \qquad \hat{g}(x, \Delta t, \Delta x) = \sum_{\ell=-m}^{M} b_\ell(\Delta t, \Delta x)\, e^{-i\ell x}.$$

For this setting Parseval's theorem says

$$\frac{1}{2\pi}\int_0^{2\pi} |\hat{\mathbf{Q}}^n(x)|^2\,dx = \sum_{j=-\infty}^{\infty} |Q_j^n|^2 = \frac{1}{\Delta x}\|\mathbf{Q}^n\|_{2,\Delta x}^2.$$

Therefore,

$$\begin{aligned}\|\mathbf{\Phi}(\mathbf{Q}^n)\|_{2,\Delta x}^2 = \|\mathbf{Q}^{n+1}\|_{2,\Delta x}^2 &= \frac{\Delta x}{2\pi}\int_0^{2\pi} |\hat{\mathbf{Q}}^{n+1}(x)|^2 dx = \frac{\Delta x}{2\pi}\int_0^{2\pi} |\hat{g}(x)\hat{\mathbf{Q}}^n(x)|^2 dx \\ &\le \sup_{x\in[0,2\pi]} |\hat{g}(x)|^2 \frac{\Delta x}{2\pi}\int_0^{2\pi} |\hat{\mathbf{Q}}^n(x)|^2 dx = \sup_{x\in[0,2\pi]} |\hat{g}(x)|^2 \|\mathbf{Q}^n\|_{2,\Delta x}^2.\end{aligned}$$

von Neumann analysis - No boundaries

Hence

$$\|\mathbf{\Phi}(\mathbf{Q}^n)\|_{2,\Delta x} \leq \sup_{x\in[0,2\pi]} |\hat{g}(x)| \|\mathbf{Q}^n\|_{2,\Delta x},$$

and we see that a sufficient condition for stability is

$$\sup_{x\in[0,2\pi]} |\hat{g}(x)| \leq 1 + \alpha\Delta t.$$

We note here the relationship

$$\hat{g}(-k\Delta x, \Delta t, \Delta x) = \sum_{\ell=-m}^{M} b_\ell(\Delta t, \Delta x)\, e^{i\ell k\Delta x} = g_k(\Delta t, \Delta x).$$

Since $\hat{g}$ is 2π-periodic in x this shows that as $\Delta x \to 0$ the two stability conditions (for g_k and $\hat{g}$) are equivalent.

von Neumann analysis - No boundaries

Remark

- For problems with variable coefficients, von Neumann analysis for scheme can only be applied for a *fixed* value of the coefficient.
- Stability for each such *frozen coefficient* problem is a necessary condition for stability of the whole scheme.
- This is often also sufficient.
- **Example:** von Neumann analysis shows that for the upwind scheme we should have $a\Delta t/\Delta x \leq 1$.

  In the variable coefficient case we would then require $a(x)\Delta t/\Delta x \leq 1$.