Numerical solutions of differential equations

Patrick Henning

pathe@kth.se

Division of Numerical Analysis, KTH, Stockholm

Course SF2521, 7.5 ECTS, VT18

Lecture 9

Entropy solutions

Lax Entropy Condition

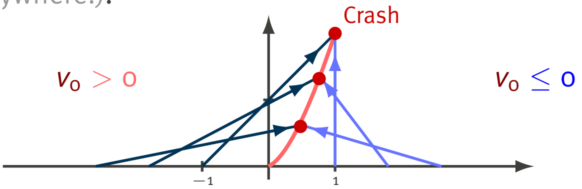
Recap: Characteristics

Recall the general properties of characteristics $\gamma(t)$:

- The map $t \mapsto u(\gamma(t), t)$ is <u>always</u> constant on $[0, T]$.
- The function γ has the form

$$\gamma(t) = f'(v(x_0))\, t + x_0 .$$

- Here, $f'(v(x_0))$ is the propagation speed (possibly different everywhere!).



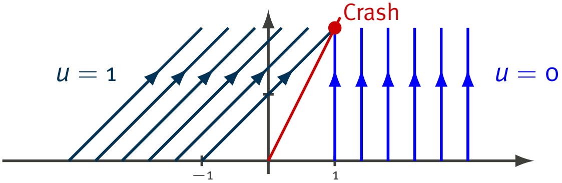
Motivation: Lax entropy condition

We consider the *Riemann problem*: find u with

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{and} \qquad u(x,0) = v_0(x) = \begin{cases} u_l & \text{for } x \le 0 \\ u_r & \text{for } x > 0 \end{cases}$$

for a convex flux $f'' > 0$ (i.e. f' is strictly increasing).

Two characteristic speeds:
$\gamma'(t) = f'(u_l)$ for $x_0 \le 0$ and $\gamma'(t) = f'(u_r)$ for $x_0 > 0$



Experience/heuristically: we have $f'(u_r) \le \sigma' \le f'(u_l)$

Motivation: Lax entropy condition

Lemma

We consider the *Riemann problem* for convex flux from the previous slide. Let u be weak solution with discontinuity along the curve $S = \{(\sigma(t), t), t > 0\}$.

Let $u_\varepsilon \in C^2(\mathbb{R} \times \mathbb{R}^+)$ be solution to

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon\, \Delta u_\varepsilon \qquad \text{in } \mathbb{R} \times \mathbb{R}^+$$

with $u_\varepsilon(x, t) = v_\varepsilon(x - st)$, $s = \sigma'(t)$ ("travelling-wave").

Further, let $u_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} u$ a.e. in $\mathbb{R} \times \mathbb{R}^+$, where u is weak solution to $\partial_t u + \partial_x f(u) = 0$. Assume for $t = t_0$:

$$\lim_{\delta \to 0} u(\sigma(t_0) + \delta, t_0) = u_r \qquad \text{and} \qquad \lim_{\delta \to 0} u(\sigma(t_0) - \delta, t_0) = u_l$$

and $u_\varepsilon(x, 0) \to u(x, 0)$ a.e. in $\mathbb{R}$. Then:

$$f'(u_r) \le s \le f'(u_l) \qquad \text{in } (\sigma(t_0), t_0).$$

The Lax Entropy Condition

Let: u is weak solution to $\partial_t u + \partial_x f(u) = 0$ with some initial value; S is smooth curve in $\mathbb{R} \times \mathbb{R}^+$ along which u is discontinuous.

Let $(x_0, t_0) \in S$, $\quad u_l := \lim_{\delta \to 0} u(x_0 - \delta, t_0)$, $\quad u_r := \lim_{\delta \to 0} u(x_0 + \delta, t_0)$

and $s := \frac{f(u_l) - f(u_r)}{u_l - u_r}$.

Then u fulfills the Lax Entropy Condition in (x_0, t_0) <u>if and only if</u>

$$f'(u_r) < s < f'(u_l).$$

A discontinuity that fulfills both the Lax Entropy Condition and the Rankine-Hugoniot Jump Condition is called shock, and s is the shock speed.

The Lax Entropy Condition - Example 1

We consider a convex flux, i.e. $f'' > 0$.

Recall: u is discontinuous on a smooth curve S; taking the value u_l left from the discontinuity and the value u_r right from it. Speed:

$$s := \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Then u fulfills the Lax Entropy Condition if and only if

$$f'(u_r) < s < f'(u_l).$$

If $u_l < u_r \overset{f''>0}{\Rightarrow} f'(u_l) < f'(u_r) \Rightarrow$ Lax Entropy Condition cannot be fulfilled.

Hence: no discontinuous solutions that fulfill both entropy condition and $u_l < u_r$.

The Lax Entropy Condition - Example 1 B

We consider the convex flux with $f(u) = \frac{1}{2}u^2$ (Burgers' equation).

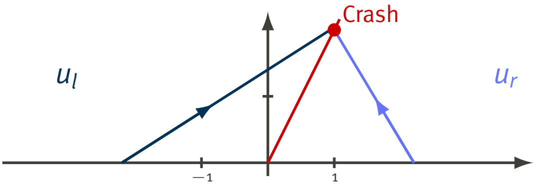
If the initial value is

$$v_0 = \begin{cases} u_l & \text{for } x < 0 \\ u_r & \text{for } x \geq 0. \end{cases}$$

If $u_l < u_r$, a weak solution that fulfills the entropy condition cannot be discontinuous for $t > 0$.

The Lax Entropy Condition - Example 2

We consider a convex flux, i.e. $f'' > 0$, and $u_l > u_r$ on the discontinuity curve.



Lax-Entropy condition

$$f'(u_r) < s = \frac{f(u_l) - f(u_r)}{u_l - u_r} < f'(u_l)$$

is fulfilled (mean value theorem!).

The Lax Entropy Condition - Example 3

Recall: The Burgers' equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0 \quad \text{and} \quad u(x,0) = v_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases},$$

has at least two weak solutions which are given by

$$u_1(x,t) = \begin{cases} 0, & x < \frac{t}{2} \\ 1, & x > \frac{t}{2} \end{cases}$$

and

$$u_2(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \leq x < t \\ 1, & t \leq x \end{cases}$$

Which one is the right one (in terms of viscosity limits)?

Only u_2 fulfills the entropy condition!

The Lax Entropy Condition

Question: Are weak solutions that fulfill the Lax entropy condition unique?

Recall: Weak solutions are in general not unique
(that was the reason why we introduced the viscosity limit, which itself led us to the entropy condition).

Theorem (Uniqueness of entropy solutions)

Let $f \in C^2(\mathbb{R})$ with $f'' > 0$ on $\mathbb{R}$.

Let u_1 and u_2 denote two weak solutions of the conservation law with the same initial value.

If u_1 and u_2 fulfill the Lax entropy condition along all discontinuities.

Then $u_1 = u_2$ (a.e.).

(without proof)