Homework :

I. Isothermal solidification of a liquid mixture.

Domain 2D $r \in [0; R_e] \times \theta \in [0; 2\pi[$ → decomposed in two phases:

- solid binary alloy
- melted ———

Material : Binary alloy that presents the two aforementionned phases.

Equation/Hypotheses : no coupling elasticity / diffusivity → diffusivity-driven system.

- $\omega_\ell(\mu_0) = \omega_s(\mu_0)$ with ℓ = liquid phase and s = solid phase.
  continuity at interface
- we consider $\omega_i(\mu) = \omega_i(\mu_0) + \mu_0 \Gamma_i u$ as our potential.

1.1 : General theory: we don't especially take into account the domain's specificities. We have,

$\Omega = \Omega_\ell \cup \Omega_s$ with $\Omega_\ell \cap \Omega_s$ with Ω_α the domain occupied by a phase; $(\alpha = \ell, s)$

Considering mass balance with no inertia effects (slow phase-transformation) we get

$\nabla^2 u$ such that $\boxed{\nabla^2 u_\alpha = 0}$ in $\Omega_\alpha(t)$ $(\alpha = s, \ell)$

Considering mass balance with no inertia effects (slow phase-transformation) we get

$$c_\alpha \frac{\partial u}{\partial t} = \lambda_\alpha . \nabla^2 u_\alpha \quad \text{such that} \quad \boxed{\nabla^2 u_\alpha = 0} \quad \text{in } \Omega_\alpha(t) \quad (\alpha = s, \ell)$$

→ To this eq° of equilibrium one can add state at the interface ~~solid/liquid~~ ~~where~~ of the two phases where:

$$[\![\rho]\!] v_n^* = [\![j]\!] . \underline{n} \quad \text{with} \quad \left[\frac{\partial \omega}{\partial \mu} = -\rho\right] \text{and} \left[\underline{j} = -\Pi \underline{\nabla} \mu\right] \quad \text{It then comes:}$$

by construction of potential $\omega(\mu)$; ↳ Fick's law.

$$[\![-\frac{\partial \omega}{\partial \mu}]\!] v_n^* = [\![-\Pi . \underline{\nabla} \mu]\!] \underline{n}^* \implies \left(\frac{\partial \omega^+}{\partial \mu} - \frac{\partial \omega^-}{\partial \mu}\right) v_n^* = \left(\Pi^+ \underline{\nabla} u^+ - \Pi^- \underline{\nabla} u^-\right) \underline{n}^*$$

$$\implies \left(\cancel{\omega^+|_{\mu_e} - \omega^-|_{\mu_e}} + \mu_0 u(\Pi^+ - \Pi^-)\right) v_n^* = (\Pi^+ \underline{\nabla} u^+ - \Pi^- \underline{\nabla} u^-) \underline{n}^*$$

° by continuity

Eventually : $v_n^* = \frac{-1}{\mu_0 u(\Pi^- - \Pi^+)} \times [\Pi^+ \underline{\nabla} u^+ - \Pi^- \underline{\nabla} u^-] \underline{n}^*$ Yet with $\bar{\Pi} . = \frac{\Pi}{\mu_0(\Pi^- - \Pi^+)}$

this translates as :

$$\boxed{v_n^* = -\left[\bar{\Pi}^+ \underline{\nabla} u^+ - \bar{\Pi}^- \underline{\nabla} u^-\right] \underline{n}^*} \quad \text{interfacial mass balance.}$$

→ Gibbs-Thompson Rela° ~~neglecting HOT in f~~ : $v_n^* = \beta^* \left([\![\omega]\!] + \psi^* \kappa^*\right)$ if no elasticity. Then knowing $\omega(\mu)$ we get $[\![\omega]\!] = \left(\cancel{\omega^+|_{\mu_e} - \omega^-|_{\mu_e}}^{\,\circ} + \mu_0 u(\Pi^+ - \Pi^-)\right)$

° by continuity.

$$\implies v_n^* = \beta^* \left(\mu_0 u (\Pi^+ - \Pi^-) + \psi^* \kappa^*\right) \quad \text{such that :}$$

$$u = \left[\frac{v_n^*}{\beta^*} - \psi^* \kappa^*\right] \times \frac{1}{\mu_0(\Pi^+ - \Pi^-)} = \left[\psi^* \kappa^* - \frac{v_n^*}{\beta^*}\right] \times \frac{1}{\mu_0(\Pi^- - \Pi^+)}$$

. Introducing rescaled variables $\psi = \frac{\psi^*}{\mu_0(\Pi^- - \Pi^+)}$ and $\beta = \mu_0(\Pi^- - \Pi^+)\beta^*$ one gets expected relations $\boxed{u = \psi \kappa^* - \frac{v_n^*}{\beta}}$ (generalized) Gibbs-Thompson rela°.

We then see melting (or solidification) won't take place at μ_0 ~~but~~. Melting temperature will be influenced by speed of phase transformation and curvature of interface (the flattest, the closest to μ_0).

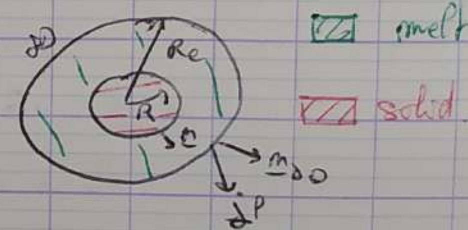
$\underline{1.2}$: We have $\psi_\ell > \psi_s$. As a consequency we have $\psi^+ - \psi^- = \dfrac{\psi^{*+} - \psi^{*-}}{\mu_0(\Gamma^- - \Gamma^+)} > 0$.

→ We then must ensure $\Gamma^- - \Gamma^+ > 0 \iff \underline{\underline{\Gamma^- > \Gamma^+}}$. It finally comes that

$$\bar{\Pi}_\alpha = \frac{\Pi_\alpha}{\mu_0(\Gamma^- - \Gamma^+)} > 0 \text{ as } (\mu_0, \Pi_\alpha, \Gamma^- - \Gamma^+) > 0, \quad \forall\, \Pi_\alpha > 0, \ (\alpha = s, \ell)$$

→ If solid occupies a convex region, then we have $\kappa^* < 0$ and so $u = \psi^* \kappa^* - \dfrac{v_n^*}{\beta^*} < 0$ s.t.
$u < 0 \iff \mu_\alpha < \mu_0$ → we melt (or solidify) at a lowest temperature (lower than μ_0 then).

$\underline{1.3}$ We now consider the aforementioned polar domain



We consider normal $\underline{n}$ the normal from solid to liquid. Given interface
/liquid is given by $r = R(t)$ we get $S = r - R(t) = 0$ the color function:

$$\left[\underline{n} = \frac{\nabla S}{\|\nabla S\|} = \frac{\frac{\partial r}{\partial r} e_r + 0\, e_\theta}{1} = e_r\right]$$ which indeed points towards liquid phase. It then comes

→ $\underline{\underline{\kappa}} = -\nabla \cdot \underline{n} = -\dfrac{1}{r}\dfrac{\partial r}{\partial r} = -\dfrac{1}{r} = -\underline{\underline{\dfrac{1}{R(t)}}}$. Similarly $v_n^* = \dfrac{\partial \underline{x}}{\partial t} \cdot \underline{n}$ and so as $\underline{x} = r\, e_r$ gives information of a particle in space, we have

$$v_n^* = \frac{\partial r}{\partial t} e_r \cdot e_r = \dot{r} = \dot{R}(t)$$

solid/liquid is given by $r = R(t)$ we get $S = r - R(t) = 0$ the color function:

∂P

$$\left[\underline{n} = \frac{\underline{\nabla} S}{\|\underline{\nabla} S\|} = \frac{\frac{\partial r}{\partial r}\underline{e_r} + 0\,\underline{e_\theta}}{1} = \underline{e_r}\right]$$ which indeed points towards liquid phase. It then comes

$$\rightarrow \quad \underline{\underline{K}} = -\underline{\nabla}\cdot\underline{n} = -\frac{1}{r}\frac{\partial r}{\partial r} = -\frac{1}{r} = \underline{\underline{-\frac{1}{R(t)}}}$$

Similarly $V_n^* = \frac{\partial \underset{\sim}{x}}{\partial t}\cdot\underset{\sim}{n}$ and so as $\underset{\sim}{x} = r\,\underline{e_r}$ gives information of a particule in space, we have

$$\rightarrow \quad \underline{\underline{V_n^*}} = \frac{\partial r}{\partial t}\underline{e_r}\cdot\underline{e_r} = \dot{r} = \underline{\underline{\dot{R}(t)}}$$

1_4 We now have some insight into the framework and some variables we work with. Let's apply the equations we derived so far ~~t~~ to this 2D case:

$\underline{\text{solid}}$: $\nabla^2 v_S = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_S}{\partial r}\right) = 0$ By integration: $r\frac{\partial v_S}{\partial r} = C_1$, $C_1 \in \mathbb{R}$ and:

$$\frac{\partial v_S}{\partial r} = \frac{C_1}{r} \rightarrow \left\{ v_S = C_1 \ln|r| + C_2, \ (C_1, C_2) \in \mathbb{R} \quad \forall r \in [0; R(t)] \right.$$

and same goes for $\left\{ v_P = C_3 \ln|r| + C_4, \ (C_3, C_4) \in \mathbb{R} \quad \forall r \in [R(t), Re] \right.$

We now apply BCs + interface conditions:

$$(9): \quad \underline{j}^P\cdot\underline{n}_{\partial\Omega} = j_e \implies -\mu_0 \Pi_P \nabla v_P\big|_{Re}\underline{e_r}\cdot\underline{e_r} = j_e \implies \frac{C_3}{Re} = -\frac{j_e}{\mu_0 \Pi_P} \rightarrow \left\{ C_3 = -\frac{j_e}{\mu_0 \Pi_P}\cdot Re \right.$$

For solid phase we must ensure solution doesn't explode in $r \to 0$ and so one must consider $C_1 = 0$ such that

$$\left[v_S = C_2\right]; \quad \left[v_P = -\frac{j_e}{\mu_0 \Pi_P}\cdot Re \cdot \ln|r| + C_4\right] \quad (C_2, C_4) \in \mathbb{R}$$

Applying Gibbs - Thompson Relation to both phases leads to an explicit formulation of C_2 and C_4 constants:

$$\longrightarrow \quad \cdot\ u_\ell\big|_R = \psi_\ell K^* - \frac{V_m^*}{\beta_\ell} \implies -\frac{j_e Re}{\mu_0 \Pi_\ell} \ln(R(t)) + C_4 = \psi_\ell K^* - \frac{V_m^*}{\beta_\ell}$$

$$\implies \left\{ C_4 = \psi_\ell K^* - \frac{V_m^*}{\beta_\ell} + \frac{j_e Re}{\mu_0 \Pi_\ell} \cdot \ln(R). \implies \right\{ u_\ell = \frac{j_e Re}{\mu_0 \Pi_\ell} \ln\left(\frac{R}{r}\right) + \psi_\ell K^* - \frac{V_m^*}{\beta_\ell}$$

$$\longrightarrow \quad \cdot\ u_s\big|_R = \psi_s K^* - \frac{V_m^*}{\beta_s} \implies C_2 = \psi_s K^* - \frac{V_m^*}{\beta_s} \implies \left\{ u_s(r,t) = \psi_s K^* - \frac{V_m^*}{\beta_s}\right.$$

~~1.5 By checking continuity at the interface we get that~~ $\cancel{\left[-\psi_s \frac{1}{R} - \frac{\dot R}{\beta_s}\right] = \frac{j_e Re}{\mu_0 \Pi_\ell} \ln\left(\frac{R}{r}\right) + \frac{\psi_s}{R} - \frac{\dot R}{\beta_\ell}}$

~~We then deduce an equation for $R(t)$:~~ $\cancel{-\psi_s - \frac{R\dot R}{\beta_s} + \frac{R\dot R}{\beta_\ell} + \frac{\psi_s}{1} = \frac{j_e Re}{\mu_0 \Pi_\ell} R \ln\left(\frac{R}{r}\right)}$

1.5 We use interfacial mass balance telling $V_m^* = \dot R = -\left[\Pi_\ell \cdot \nabla u^+ - \Pi_s \cdot \nabla u^-\right]$. We then have

↳ along $S(t)$, i.e. in $r = R$

$$\dot R = -\left[\Pi_\ell \times \left(-\frac{j_e Re}{\mu_0 \Pi_\ell} \cdot \frac{1}{R}\right)\right] \quad \text{such that} \quad R\dot R = \frac{\Pi_\ell\, j_e Re}{\mu_0 \Pi_\ell} \quad \text{and by integration it comes:}$$

$$\frac{R^2}{2}(t) = j_e \times \frac{\Pi_\ell Re}{\mu_0 \Pi_\ell} \cdot t + C_5, \quad C_5 \in \mathbb{R} \implies \left\{ R(t) = \sqrt{2 j_e \times \frac{\Pi_\ell}{\mu_0 \Pi_\ell} Re\, t + 2C_5} \right.$$

which provides position front along time.

$\dot{R} = -\left[\Pi_P \times \left(-\frac{j_e R_e}{\mu_0 \Pi_P} \cdot \frac{1}{R}\right)\right]$ such that $R\dot{R} = \frac{\Pi_P \, j_e R_e}{\mu_e \Pi_P}$ and by integration it comes:

$$\frac{R^2}{2}(t) = j_e \times \frac{\Pi_P R_e}{\mu_e \Pi_P} \times t + C_S, \quad C_S \in \mathbb{R} \implies R(t) = \sqrt{2 j_e \times \frac{\Pi_P R_e}{\mu_0 \Pi_P} t + 2C_S}$$ which provides position front along time.

Yet, we know $R(t=0)=0$ from (10) such that $\sqrt{2C_S} = 0 \implies \underline{C_S = 0} \implies \boxed{R(t) = \left(2 \frac{j_e \Pi_P R_e}{\mu_0 \Pi_P} \times t\right)^{1/2}}$

For the solid to grow, we must ensure square root is positive and by definition that $2 j_e \frac{\Pi_P R_e}{\mu_0 \Pi_P} > 0 \implies \underline{j_e > 0}$

It then comes that $$u_P(r;t) = \frac{j_e R_e}{\mu_0 \Pi_P}\left[\ln R - \ln r\right] + \psi_P \kappa^* - \frac{V_n^*}{\beta_P}$$

$$= \underbrace{\frac{j_e R_e}{\mu_0 \Pi_P}\left[\ln R - \ln r\right]}_{\substack{\leq 0 \\ \forall r \in [R; R_e]}} \underbrace{- \frac{\psi_P}{R}}_{\leq 0} \underbrace{- \frac{\dot{R}}{\beta_P}}_{< 0 \text{ as front propagates}}$$ we deduce that for $j_e > 0$, we have $u_P < 0$ and so

$$\frac{\mu_P - \mu_0}{\mu_0} < 0 \iff \mu_P - \mu_0 < 0 \implies \boxed{\mu_P(r;t) < \mu_0} \quad \forall r \in [R(t); R_e]$$ → for one to solidify we must be at a lower temperature than μ_0 as discussed in the case of a convex curvature.

As we continue to propagate (i.e. as ~~sol~~ fluid phase solidifies) we have at one point a total solidification.

Then $R(t) = R_e$ and $t = t^*$:

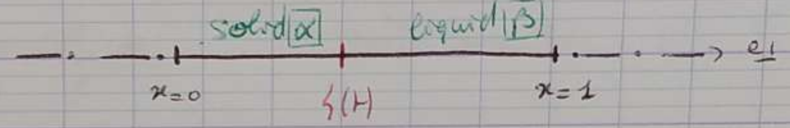
$$R_e = \left[2 \frac{j_e \, \Pi_P}{\mu_0 \Pi_P}\right]^{1/2} \times t^{*\,1/2} \times R_e^{1/2} \implies t^* = R_e \times \frac{\mu_0 \Pi_P}{2 j_e \Pi_P}$$

If one continues to remove solute (i.e. liquid → solid), we expect to lower even more temperature.

II. Diffusion - driven transformation in a two - phase elastic bar.

We now consider a case where phase transformation occurs from both elasticity and diffusion of a solid phase towards a liquid phase.

We consider a 1D case



We look for the propagation of ~~[illegible]~~ substance $\varphi(x,t)$ which translates phase transformation. To this aim, we introduce the potential ω, quadratic in strain and linear affine in diffusion:

$$\omega_i(\mu, \varepsilon) = \frac{1}{2} E_i (\varepsilon - \varepsilon_i^{\circ})^2 - \Lambda_i \varphi \quad (i = \alpha, \beta) \qquad \text{where } \varphi = \mu - \mu_0 \text{ and } \Lambda_i \in \mathbb{R}.$$

One then deduce

$$\left[\sigma_i(\mu, \varepsilon) = \frac{\partial \omega_i}{\partial \varepsilon}\right] \ \& \ \left[-\Lambda_i = \frac{\partial \omega_i}{\partial \mu}\right]$$ by identification $\Lambda_i =$ density of one phase

2.1 : Adapting equation of class to the study of $\varphi(x,t)$ and $u(x,t)$, in a 1D case, one gets:

FBP

equilibrium

$$\operatorname{div} \underline{\sigma}_i = 0 \iff \frac{\partial \sigma_i}{\partial x} = 0$$ → equilibrium under no distributed force, ~~state~~ quasi-static.

$$\nabla^2 \mu_i = 0 \implies \nabla^2 \varphi_i = 0$$

$$[\![u]\!] = 0 \ ; \ [\![\underline{\underline{\sigma}}]\!] \underline{n} = \underline{0} \qquad \forall x \in S(t) \equiv x = \xi(t)$$

$$+ (\Lambda_\alpha - \Lambda_\beta) v_n^* = -(\Pi_\alpha \nabla \mu_\alpha^+ - \Pi_\beta \nabla \mu_\beta^-) \quad \text{in } x = \xi(t)$$ continuity.

$$[\![\omega]\!] = 0 \implies [\![\varphi]\!] = 0 \qquad \text{in } x = \xi(t)$$

2_1 : Adapting equation of class to the study of $\varphi(x,t)$ and $u(x,t)$, in a 1D case, one gets:

FBP

equilibrium:

- $\operatorname{div} \underline{\underline{\sigma}}_i = 0 \iff \dfrac{\partial \sigma_i}{\partial x} = 0$ → equilibrium under no distributed force, ~~static~~ quasi-static.
- $\nabla^2 \mu_i = 0 \implies \nabla^2 \varphi_i = 0$

continuity:

- $[\![u]\!] = 0 \; ; \; [\![\underline{\underline{\sigma}}]\!] \underline{n} = 0 \quad \forall x \in S(t) \equiv x = \zeta(t)$
- $+(\Lambda_\alpha - \Lambda_\beta)\, v_n^* = -(\Pi_\alpha \nabla \mu_\alpha^+ - \Pi_\beta \nabla \mu_\beta^-)$ in $x = \zeta(t)$
- $[\![\mu]\!] = 0 \implies [\![\varphi]\!] = 0$ in $x = \zeta(t)$

linear kinetic relation:

$$-(\Lambda_\alpha - \Lambda_\beta)\varphi = [\![W]\!] - [\![\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}]\!] - \frac{v_n^*}{\beta^*} + \varphi^* \kappa^*$$

Equilibrium can be seen ~~as~~ in terms on u and φ variables:

- $\nabla^2 \mu_i = 0 \implies \nabla^2(\mu_i - \mu_0) = 0 \implies \boxed{\nabla^2 \varphi_i = 0}$
- $\nabla \underline{\underline{\sigma}} = 0 \implies \dfrac{\partial \sigma}{\partial x} = 0 \implies E_i \dfrac{\partial^2 u_i}{\partial x^2} = 0 \implies \boxed{\nabla^2 u_i = 0}$

We finally complete the FBP by adding BCs:

$$\begin{cases} u_\alpha \big|_{x=0} = 0 \\ \dfrac{\partial \varphi_\alpha}{\partial x}\Big|_{x=0} = 0 \end{cases} \qquad \begin{cases} \sigma_\beta \big|_{x=1} = \sigma^* & \to \text{impose stress state} \\ \varphi_\beta \big|_{x=1} = \varphi^* & \to \text{impose } T^\circ \text{ state} \end{cases}$$

One of the objectives of this part will be to provide a framework in which solid front $\zeta(t)$ propagates rightwards, and ~~deduce~~ derive its formula for some physical post-process.

2.2 We see for FBP ~~that~~ that equilibrium is decoupled for u and φ such that we can solve these fields separately.

Let's compute stress, strain and displacement field for both phases:

$l = \beta;\ s = \alpha$

solid

- $\dfrac{\partial \sigma_s}{\partial x} = 0 \implies \sigma_s = C_1,\ C_1 \in \mathbb{R}$.

By continuity at interface: $\sigma_l\big|_h - \sigma_s\big|_h = 0 \implies \boxed{\sigma_s = \sigma^*}$

- $\sigma_i = \dfrac{\partial w}{\partial \varepsilon_i} = E_i(\varepsilon_i - \varepsilon_i^0) \implies \varepsilon_s = \dfrac{\sigma_s}{E_s} + \varepsilon_s^0$

$\rightarrow \boxed{\varepsilon_\alpha = \dfrac{\sigma^*}{E_\alpha} + \varepsilon_\alpha^0}$

- $\varepsilon_i = \dfrac{\partial u_i}{\partial x} \implies u_i = \left(\dfrac{\sigma^*}{E_i} + \varepsilon_i^0\right)x + C_{3/4},\ C_{3/4} \in \mathbb{R}$

Yet $u_\alpha\big|_{x=0} = C_3 = 0$.

$\implies \boxed{u_\alpha = \left(\dfrac{\sigma^*}{E_\alpha} + \varepsilon_\alpha^0\right)x}$

liquid

- $\dfrac{\partial \sigma_l}{\partial x} = 0 \implies \sigma_l = C_2,\ C_2 \in \mathbb{R}$

Yet $\sigma_l\big|_{x=1} = C_2 = \sigma^* \implies \boxed{\sigma_l = \sigma^*}$

- $\varepsilon_l = \dfrac{\sigma_l}{E_l} + \varepsilon_l^0 = \dfrac{\sigma^*}{E_l} + \varepsilon_l$.

$\rightarrow \boxed{\varepsilon_\beta = \dfrac{\sigma^*}{E_\beta} + \varepsilon_\beta^0}$

- $u_\beta = \left(\dfrac{\sigma^*}{E_\beta} + \varepsilon_\beta^0\right)x + C_4,\ C_4 \in \mathbb{R}$

- By continuity $u_\beta\big|_h - u_\alpha\big|_h = 0$

$\rightarrow \left[\left(\dfrac{\sigma^*}{E_\beta} + \varepsilon_\beta^0\right) - \left(\dfrac{\sigma^*}{E_\alpha} + \varepsilon_\alpha^0\right)\right] h + C_4 = 0 \quad \forall t; \forall s(t)$.

We then must have $C_4 = 0 \implies \boxed{u_\beta = \left(\dfrac{\sigma^*}{E_\beta} + \varepsilon_\beta^0\right) \times x}$.

to $U_x\big|_{x=0} = C_3 = 0$

- - -

$$\Rightarrow \boxed{u_\alpha = \left(\frac{\sigma^*}{E_\alpha} + \varepsilon_\alpha^0\right) x}$$

$\big|_h \quad \big|_h$

$$\rightarrow \left[\left(\frac{\sigma^*}{E_\beta} + \varepsilon_\beta^0\right) - \left(\frac{\sigma^*}{E_\alpha} + \varepsilon_\alpha^0\right)\right] \zeta + C_4 = 0 \quad \forall t, \forall \zeta(t).$$

We then must have $C_4 = 0 \Rightarrow \boxed{u_\beta = \left(\frac{\sigma^*}{E_\beta} + \varepsilon_\beta^0\right) \times x.}$

2.3 Let's now use FBP and question 2.2 in order to derive the formula of $\varphi_\alpha(x,t)$ and $\varphi_\beta(x,t)$, as well as a differential equation $\zeta(t)$ will have to satisfy.

- solid $\nabla^2 \varphi_\alpha = 0 \rightarrow \varphi_\alpha = ax + b, \quad x = x(t), (a,b) \in \mathbb{R}$

BC : $\frac{\partial \varphi_\alpha}{\partial x} = a = 0 \rightarrow \underline{\underline{\varphi_\alpha = b}}$

liquid $\nabla^2 \varphi_\beta = 0 \rightarrow \varphi_\beta = cx + d, (c,d) \in \mathbb{R}$.

BC : $\varphi_\beta\big|_{x=1} = c + d = \varphi^* \rightarrow c = \varphi^* - d$.

$$\Rightarrow \underline{\underline{\varphi_\beta = (\varphi^* - d)x + d}}$$

where $(d, \varphi^*) \in \mathbb{R}$.

We look for the two last unknowns using :

- continuity $[\![\varphi]\!] = 0$ in $x = \zeta(t)$
- $-(\Lambda_\alpha - \Lambda_\beta) v_n^+ = -(\Pi_\alpha \nabla u_\alpha^+ - \Pi_\beta \nabla u_\beta^-)$ in $x = \zeta(t)$.

$$V_n^* = \frac{\partial x}{\partial t} \cdot n = \dot\zeta \rightarrow -(\Lambda_\alpha - \Lambda_\beta)\dot\zeta = -(\Pi_\alpha \cancel{\nabla \varphi_\alpha^+} - \Pi_\beta \nabla\varphi_\beta).$$

$\overset{0}{}$ as $\underline{\underline{\varphi_\alpha = c}}$

$$\Rightarrow \varphi_a = b$$

$$\Rightarrow \nabla\varphi_\beta\big|_{x=\zeta} = -\frac{1}{\Pi_\beta}(\Lambda_\alpha - \Lambda_\beta)\dot\zeta = \varphi^* - d$$

$$\rightarrow d = \varphi^* + \frac{1}{\Pi_\beta}(\Lambda_\alpha - \Lambda_\beta)\dot\zeta$$

$$\Rightarrow \varphi_\beta(x,t) = \left[-\frac{1}{\Pi_\beta}(\Lambda_\alpha - \Lambda_\beta)\dot\zeta\right] x + \left[\varphi^* + \frac{1}{\Pi_\beta}(\Lambda_\alpha - \Lambda_\beta)\dot\zeta\right]$$

Rewriting these equations and applying continuity at interface (i.e. $[\![\varphi]\!](x=\xi(t))=0$) it comes

$$\left\{\begin{array}{ll} \varphi_\alpha = \varphi^* + \dfrac{\dot\xi(t)}{\Pi_\beta}\cdot(\Lambda_\alpha-\Lambda_\beta)\cdot(\xi(t)-1), & \forall\, x(t)\in[0;\xi(t)] \\ \\ \varphi_\beta = \varphi^* + \dfrac{\dot\xi(t)}{\Pi_\beta}\cdot(\Lambda_\alpha-\Lambda_\beta)\cdot(x(t)-1), & \forall\, x(t)\in[\xi(t);1] \end{array}\right.$$

The differential equation on ξ is obtained by use of the linear kinetic relation:

$$\rightarrow\; -(\Lambda_\alpha-\Lambda_\beta)\,\varphi = [\![W]\!] - [\![\underline{\underline{\sigma}}:\underline{\underline{\varepsilon}}]\!] \underbrace{-\frac{V_m^*}{\beta^*}}_{=-\dot\xi/\beta^*} + \cancel{\varphi^*\kappa^*}$$

no curvature as 1D.

$$-\; [\![W]\!] = \frac{1}{2}E_i(\varepsilon_i-\varepsilon_i^0)^2 \Longrightarrow [\![W]\!] = \frac{1}{2}E_\beta(\varepsilon_\beta-\varepsilon_\beta^0)^2 - \frac{1}{2}E_\alpha(\varepsilon_\alpha-\varepsilon_\alpha^0)^2$$

$$-\; [\![\underline{\underline{\sigma}}:\underline{\underline{\varepsilon}}]\!] = \cancel{\sigma_\beta(\varepsilon_\beta-\varepsilon_\beta^0)}\,\sigma_\beta\varepsilon_\beta - \sigma_\alpha\varepsilon_\alpha = \sigma^*\cdot\left(\frac{\sigma^*}{E_\beta}+\varepsilon_\beta^0\right) - \sigma^*\left(\frac{\sigma^*}{E_\alpha}+\varepsilon_\alpha^0\right)$$

$$= \sigma^*\left[\sigma^*\left(\frac{1}{E_\beta}-\frac{1}{E_\alpha}\right) + (\varepsilon_\beta^0-\varepsilon_\alpha^0)\right]$$

It then comes

$$[\![W]\!] - [\![\underline{\underline{\sigma}}:\underline{\underline{\varepsilon}}]\!] = \left(\frac{1}{2}E_\beta\frac{\sigma^{*2}}{E_\beta^2} - \frac{1}{2}E_\alpha\frac{\sigma^{*2}}{E_\alpha^2}\right) - \sigma^*\left[\cancel{\frac{\sigma^*}{E_\alpha E_\beta}(E_\alpha-E_\beta)}\ \sigma^*\cdot\left(\frac{1}{E_\beta}-\frac{1}{E_\alpha}\right) + (\varepsilon_\beta^0-\varepsilon_\alpha^0)\right]$$

$$" = \frac{\sigma^{*2}}{2}\left[\frac{1}{E_\beta}-\frac{1}{E_\alpha}\right] - \frac{\sigma^*}{2}\left[\sigma^*\left(\frac{2}{E_\beta}-\frac{2}{E_\alpha}\right) + 2(\varepsilon_\beta^0-\varepsilon_\alpha^0)\right]$$

$$" = \frac{\sigma^*}{2}\left[\sigma^*\left(\frac{1}{E_\alpha}-\frac{1}{E_\beta}\right) - 2(\varepsilon_\beta^0-\varepsilon_\alpha^0)\right]$$

$$= \sigma^* \left[\sigma^* \left(\frac{1}{E_\beta} - \frac{1}{E_\alpha} \right) + (\varepsilon_\beta^\circ - \varepsilon_\alpha^\circ) \right]$$

It then comes $[\![W]\!]^* - [\![\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}]\!] = \left(\frac{1}{2} E_\beta \frac{\sigma^{*2}}{E_\beta^2} - \frac{1}{2} E_\alpha \frac{\sigma^{*2}}{E_\alpha^2} \right) - \sigma^* \left[\cancel{\frac{\sigma^*}{E_\alpha E_\beta} (E_\alpha - E_\beta)} + (\varepsilon_\beta^\circ - \varepsilon_\alpha^\circ) \right]$ $+ \sigma^* \cdot \left(\frac{1}{E_\beta} - \frac{1}{E_\alpha} \right)$

$$" = \frac{\sigma^{*2}}{2} \left[\frac{1}{E_\beta} - \frac{1}{E_\alpha} \right] - \frac{\sigma^*}{2} \left[\sigma^* \left(\frac{2}{E_\beta} - \frac{2}{E_\alpha} \right) + 2(\varepsilon_\beta^\circ - \varepsilon_\alpha^\circ) \right]$$

$$" = \frac{\sigma^*}{2} \left[\sigma^* \left(\frac{1}{E_\alpha} - \frac{1}{E_\beta} \right) - 2 (\varepsilon_\beta^\circ - \varepsilon_\alpha^\circ) \right]$$

Eventually : $\rightarrow \frac{\dot{\xi}}{\beta^*} = -(\Lambda_\beta - \Lambda_\alpha) + \frac{\sigma^*}{2} \left[\sigma^* \left(\frac{1}{E_\alpha} - \frac{1}{E_\beta} \right) - 2 (\varepsilon_\beta^\circ - \varepsilon_\alpha^\circ) \right]$. If we consider

E, ε° and Λ we have $\rightarrow \frac{\dot{\xi}}{\beta^*} = -\Lambda \varphi + \frac{\sigma^*}{2} \left[\frac{\sigma^*}{E_\alpha E_\beta} \times E - 2 \varepsilon^\circ \right]$

$$\varphi = \varphi^* + \frac{\dot{\xi}}{\Pi_\beta} \times (-\Lambda) \times (\xi - 1)$$

$$\Rightarrow \left\{ \dot{\xi} = \frac{\left[\frac{\sigma^*}{2} \cdot \left[\frac{\sigma^*}{E_\alpha E_\beta} \cdot E - 2 \varepsilon^\circ \right] - \Lambda \varphi^* \right] \times \beta}{\left[1 + \frac{\Lambda^2}{\Pi_\beta} (1 - \xi) \right]} \right.$$

aimed diff. equation.

2.4. We have a phase transformation when $\dot{\xi} > 0$. This positivity however depends on the signs of E; ε°; Λ and their intensity.

We therefore see that denominator is ~~negative~~ positive if $1 - \underbrace{\frac{\Lambda^2}{\Pi_\beta} (1 - \xi)}_{> 0} > 0$. $\Rightarrow \frac{\Pi_\beta}{1 - \xi} > \Lambda^2$

We thus have a limit in intensity Λ^2 can take.

For the numerator, ~~rather "optimal"~~ we have :

- "optimal" case : $E > 0$; $\mathcal{E}^\circ < 0$; $\Lambda < 0$. → solid more dense → good physically.

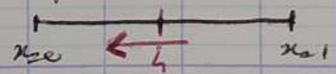
  ↳ solid part ~~more~~ less stiff than liquid part → good physically.

  ↳ greater deformation in solid phase → solid suffers higher deformation as we "break connections".

Provided Λ low enough → no condition over σ^* and φ^*.

- ~~Case 2~~ "worth" case : $E < 0$; $\mathcal{E}^\circ > 0$; $\Lambda > 0$ (+ Λ low enough)

Then we don't respect some physic properties such has $\Lambda < 0 \iff$ solid more dense than liquid phase. This case ~~on~~ then isn't associated to a solid → liquid but to liquid ← solid (namely liquid have properties of solid and inversely).



- Case 3 : $E > 0$; $\Lambda < 0$; $\mathcal{E}^\circ > 0$ (+ Λ low enough)

We then have a condition to get $\xi > 0$, namely $\frac{\sigma^*}{2}\left[\frac{\sigma^*}{E_\alpha E_\beta} \times E - 2\mathcal{E}^\circ\right] - \Lambda\varphi^* > 0$

Let be $p = -\Lambda > 0$, we have :

$$\frac{\sigma^{*2}}{2E_\alpha E_\beta} \times E + p\varphi^* > \sigma^*\mathcal{E}^\circ$$ ≡ energy of ~~first~~ misfit strain.

If we add no misfit strain, ~~this~~ this inequality would simply translate positivity of σ^* and φ^*.

$(+\Lambda$ low enough$)$

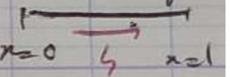
We then have a condition to get $\dot\zeta > 0$, namely $\frac{\sigma^*}{2}\left[\frac{\sigma^*}{E_\alpha E_\beta} \times E - 2\mathcal{E}^\circ\right] - \Lambda \varphi^* > 0$

Let be $p = -\Lambda > 0$, we have:

$$\frac{\sigma^{*2}}{2E_\alpha E_\beta} \times E + p\varphi^* > \sigma^* \mathcal{E}^\circ \equiv \text{energy of ~~first~~ misfit strain}$$

If we add no misfit strain, ~~it~~ this inequality would simply translate positivity of σ^* and φ^*.

Note: multiple different combinations of signs and amplitude of $E, \Lambda, \mathcal{E}^\circ$ exist, but they are not always physical, namely they don't respect the fact we study solidification at the expense of a liquid, i.e. ~~solid~~



In hindsight we can however see that:

- the lower φ^* $\rightarrow$ the more $\dot\zeta$ is eager to be positive $\Rightarrow$ lower "temperature" at rhs.
- the higher σ^* $\rightarrow$ ——— $\Rightarrow$ as we pull at rhs.

2.5 By integration of diff. equation we derived for $\dot\zeta(t)$:

$$\left[1 - \frac{\Lambda^2}{\Pi_\beta} + \frac{\Lambda^2}{\Pi_\beta}\zeta\right]\dot\zeta = C, \quad C = \frac{\sigma^*}{2}\left[\frac{\sigma^*}{E_\alpha E_\beta} \cdots\right] \times \rho^* \in \mathbb{R}.$$

Let be $A = \frac{\Lambda^2}{\Pi_\beta}$, we have

$$[(1-A) + A\zeta]\dot\zeta = C \implies X = (1-A) + A\zeta \implies \dot X = A\dot\zeta \text{ and so}$$

$$X \times \dot\zeta = \frac{X A \dot\zeta}{A} = \frac{X\dot X}{A} \xrightarrow{\text{integration}} \boxed{\frac{X^2}{2A} = Ct + D}, \quad D \in \mathbb{R}$$

Replacing X by $X = (1-A) + A\zeta \Rightarrow \left[(1-A) + A\zeta\right]^2 = 2ACt + D', \; D' \in \mathbb{R}$

$$\Leftrightarrow \zeta = \frac{(2ACt + D')^{1/2} - (1-A)}{A}$$

~~and so replacing A and~~ IC : $\zeta(t=0) = \dfrac{D'^{\,1/2} - (1-A)}{A} = 0 \Rightarrow D' = (1-A)^2$ and so :

$$\zeta(t) = \frac{\left[2ACt + (1-A)^2\right]^{1/2} - (1-A)}{A} \quad \text{with } A = \frac{\Lambda^2}{\Pi_\beta} \text{ and } C = \left[\frac{\sigma^*}{2}\left[\frac{\sigma^*}{\varepsilon_\alpha \varepsilon_\beta} \cdot \varepsilon - 2\varepsilon_\alpha^\circ\right] - \Lambda\varphi^*\right] \times \beta^*$$

Finally, we have ben entirely transformed once $\zeta(t) = 1$, such that then $t = t^*$ and so:

$$1 = \frac{\left[2ACt^* + (1-A)^2\right]^{1/2} - (1-A)}{A} \Rightarrow t^* = \frac{\left[A + (1-A)\right]^2 - (1-A)^2}{2AC}$$

$$t^* = \frac{1 - (1-A)^2}{2AC}$$, ben entirely transformed from β phase to α one.