Lecture 7

Convergence Theory for Linear Methods - Part 1

Introduction

Setting

Let Q_j^n be the numerical approximation of the exact cell average,

$$Q_j^n \approx u_j^n := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u(t_n, x)dx, \qquad t_n = n\,\Delta t.$$

We want to check

- Convergence $Q_j^n \to u_j^n$ as $\Delta x, \Delta t \to 0$,
- Accuracy and convergence rate

$$Q_j^n = u_j^n + \mathcal{O}(\Delta x^p + \Delta t^r),$$

for some $p, r \geq 1$.

Notation

Consider two cases of the numerical approximation:

- with boundaries: $\mathbf{Q}^n$ is finite length vector

$$\mathbf{Q}^n = (Q_0^n, \dots, Q_N^n)^\top.$$

- without boundaries: $\mathbf{Q}^n$ is infinite length vector

$$\mathbf{Q}^n = (\dots, Q_{-1}^n, Q_0^n, Q_1^n \dots)^\top,$$

Analogous notation for the exact solution $\mathbf{u}^n$.

We write numerical scheme compactly as operator $\mathbf{\Phi}$ acting on $\mathbf{Q}^n$,

$$\mathbf{Q}^{n+1} = \mathbf{\Phi}(\mathbf{Q}^n, \Delta t, \Delta x).$$

When $\mathbf{\Phi}$ only depends on the CFL number $\lambda_{\text{CFL}} := \Delta t / \Delta x$ we simply write $\mathbf{\Phi}(\mathbf{Q}^n, \lambda_{\text{CFL}})$ or just $\mathbf{\Phi}(\mathbf{Q}^n)$ (when there is no risk for confusion).

Linear methods

Assume that $\mathbf{\Phi}$ is a linear method, i.e. if $\alpha, \beta \in \mathbb{R}$ we have

$$\mathbf{\Phi}(\alpha \mathbf{Q} + \beta \mathbf{W}) = \alpha \mathbf{\Phi}(\mathbf{Q}) + \beta \mathbf{\Phi}(\mathbf{W}).$$

Any linear method can be represented by sequences of numbers, $\{b_{j,\ell}\}$, that depend on the mesh and time step size,

$$Q_j^{n+1} = \sum_{\ell=-m}^{M} b_{j,\ell}(\Delta t, \Delta x)\, Q_{j+\ell}^n.$$

- m and M are finite and determines the width of the spatial stencils.
- In general, when the equation is nonlinear the scheme is not linear either.

Linear methods - Example

$$Q_j^{n+1} = \sum_{\ell=-m}^{M} b_{j,\ell}(\Delta t, \Delta x)\, Q_{j+\ell}^n.$$

Example. When $\mathbf{\Phi}$ is the *Upwind Scheme* applied to

$$\partial_t u(x,t) + \mathbf{a}(x)\partial_x u(x,t) = 0, \qquad \mathbf{a}(x) > 0,$$

then

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\mathbf{a}(x_j)(Q_j^n - Q_{j-1}^n).$$

Hence, we have

$$b_{j,0} = 1 - \mathbf{a}(x_j)\frac{\Delta t}{\Delta x}, \qquad b_{j,-1} = \mathbf{a}(x_j)\frac{\Delta t}{\Delta x},$$

and all other $b_{j,\ell}$ are zero, i.e. $m = 1$ and $M = 0$.

Norms

- To measure errors we need norms.
- We use the discrete L^2-norm which mimics a midpoint rule approximation of the continuous L^2-norm (for smooth functions)
- Case with boundaries: use

$$\|\mathbf{Q}\|_{2,\Delta x}^2 := \sum_{j=0}^{N} |\mathbf{Q}_j|^2 \Delta x.$$

  Note: by scaling we have $N\Delta x =$constant. Hence, size of the norm does not explode if we refine the grid.

- Case without boundaries: use

$$\|\mathbf{Q}\|_{2,\Delta x}^2 := \sum_{j=-\infty}^{\infty} |\mathbf{Q}_j|^2 \Delta x.$$

- Note: for $\Delta x \to 0$ the discrete L^2-norm becomes the continuous L^2-norm (for regular $\mathbf{Q}$).

Norms

- Analogously: discrete L^1-norms:

$$\|\mathbf{Q}\|_{1,\Delta x} = \sum_{j=0}^{N} |\mathbf{Q}_j| \Delta x, \qquad \|\mathbf{Q}\|_{1,\Delta x} = \sum_{j=-\infty}^{\infty} |\mathbf{Q}_j| \Delta x.$$

- We just write $\| \cdot \|_{\Delta x}$ when the precise norm type is not important.