Numerical solutions of differential equations

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Finite Volumes Schemes of Higher Order

The Lax-Wendroff Scheme

Finite Differences of Higher Order

Goal:

Construct higher order methods using Taylor expansion:

As usual: let $x_j = \frac{\Delta x}{2} + j\Delta x$ for $j \in \mathbb{Z}$ and $t_n = n\Delta t$ for $n \in \mathbb{N}_0$.

Let u be a smooth solution to $\partial_t u + \partial_x f(u) = 0$. Set

$$u_j^n := u(x_j, t_n).$$

Then Taylor expansion yields:

$$u_j^{n+1} \overset{\text{Taylor}}{=} u_j^n + \Delta t\, \partial_t u_j^n + \frac{\Delta t^2}{2} \partial_{tt} u_j^n + \mathcal{O}(\Delta t^3)$$

Idea: Replace time derivatives by space derivates using the conservation law.

Taylor Expansion

We consider

$$u_j^{n+1} = u_j^n + \Delta t\, \partial_t u_j^n + \frac{\Delta t^2}{2} \partial_{tt} u_j^n + \mathcal{O}(\Delta t^3).$$

It holds

$$\partial_t u_j^n = -\partial_x f(u_j^n) = -\frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

and it also holds

$$\begin{aligned}
\partial_t \partial_t u_j^n &= -\partial_t \partial_x f(u_j^n) = -\partial_x (f'(u_j^n) \partial_t u_j^n) \\
&= -\partial_x \left(f'(u_j^n)(-(\partial_x f(u_j^n))) \right) = \partial_x \left(f'(u_j^n)^2 \partial_x u_j^n \right) \\
&= \frac{f'(u_{j+\frac{1}{2}}^n)^2 \partial_x u_{j+\frac{1}{2}}^n - f'(u_{j-\frac{1}{2}}^n)^2 \partial_x u_{j-\frac{1}{2}}{}^n}{\Delta x} + \mathcal{O}(\Delta x) \\
&= f'(u_{j+\frac{1}{2}}^n)^2 \frac{u_{j+1}^n - u_j^n}{\Delta x^2} - f'(u_{j-\frac{1}{2}}^n)^2 \frac{u_j^n - u_{j-1}^n}{\Delta x^2} + \mathcal{O}(\Delta x)
\end{aligned}$$

Taylor Expansion

We have

$$u_j^{n+1} = u_j^n + \Delta t\, \partial_t u_j^n + \frac{\Delta t^2}{2} \partial_{tt} u_j^n + \mathcal{O}(\Delta t^3),$$

where we approximate

$$\partial_t u_j^n = -\frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

and

$$\partial_{tt} u_j^n = f'(u_{j+\frac{1}{2}}^n)^2 \frac{u_{j+1}^n - u_j^n}{\Delta x^2} - f'(u_{j-\frac{1}{2}}^n)^2 \frac{u_j^n - u_{j-1}^n}{\Delta x^2} + \mathcal{O}(\Delta x).$$

Question: How do we chose $f'(u_{j\pm\frac{1}{2}}^n)$?

Taylor Expansion

Question: How do we chose $f'(u^n_{j\pm\frac{1}{2}})$? It holds

$$f'(u^n_{j+\frac{1}{2}}) = \frac{f(u^n_{j+1}) - f(u^n_j)}{u^n_{j+1} - u^n_j} + \underbrace{\mathcal{O}(u^n_{j+1} - u^n_j)}_{\mathcal{O}(\Delta x)}$$

Hence

$$f'(u^n_{j+\frac{1}{2}})^2\,(u^n_{j+1} - u^n_j) = \frac{\left(f(u^n_{j+1}) - f(u^n_j)\right)^2}{u^n_{j+1} - u^n_j} + \mathcal{O}(\Delta x^3).$$

We obtain

$$\begin{aligned}\partial_{tt} u^n_j &= f'(u^n_{j+\frac{1}{2}})^2 \frac{u^n_{j+1} - u^n_j}{\Delta x^2} - f'(u^n_{j-\frac{1}{2}})^2 \frac{u^n_j - u^n_{j-1}}{\Delta x^2} + \mathcal{O}(\Delta x) \\ &= \frac{1}{\Delta x^2}\frac{\left(f(u^n_{j+1}) - f(u^n_j)\right)^2}{u^n_{j+1} - u^n_j} - \frac{1}{\Delta x^2}\frac{\left(f(u^n_j) - f(u^n_{j-1})\right)^2}{u^n_j - u^n_{j-1}} + \mathcal{O}(\Delta x)\end{aligned}$$

Taylor Expansion

Combining $u_j^{n+1} = u_j^n + \Delta t\, \partial_t u_j^n + \frac{\Delta t^2}{2}\, \partial_{tt} u_j^n + \mathcal{O}(\Delta t^3)$ with

$$\partial_t u_j^n = -\frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

and

$$\partial_{tt} u_j^n = \frac{1}{\Delta x^2} \frac{\left(f(u_{j+1}^n) - f(u_j^n)\right)^2}{u_{j+1}^n - u_j^n} - \frac{1}{\Delta x^2} \frac{\left(f(u_j^n) - f(u_{j-1}^n)\right)^2}{u_j^n - u_{j-1}^n} + \mathcal{O}(\Delta x)$$

we obtain

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} \left[f(u_{j+1}^n) - f(u_{j-1}^n)\right]$$
$$+ \frac{\Delta t^2}{2\Delta x^2} \left[\frac{\left(f(u_{j+1}^n) - f(u_j^n)\right)^2}{u_{j+1}^n - u_j^n} - \frac{\left(f(u_j^n) - f(u_{j-1}^n)\right)^2}{u_j^n - u_{j-1}^n} \right] + \Delta t\, \mathcal{O}(\Delta x^2)$$

Taylor Expansion

Based on

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left[f(u_{j+1}^n) - f(u_{j-1}^n)\right]$$
$$+ \frac{\Delta t^2}{2\Delta x^2}\left[\frac{\left(f(u_{j+1}^n) - f(u_j^n)\right)^2}{u_{j+1}^n - u_j^n} - \frac{\left(f(u_j^n) - f(u_{j-1}^n)\right)^2}{u_j^n - u_{j-1}^n}\right] + \Delta t\,\mathcal{O}(\Delta x^2)$$

we can formulate a scheme with consistency order 2. Note that the term $\mathcal{O}(\Delta x^2)$ is just the local truncation error.

The resulting scheme is called Lax-Wendroff Scheme with numerical flux

$$g(v,w) := \frac{1}{2}\left[f(v) + f(w) - \lambda\frac{(f(w) - f(v))^2}{w - v}\right], \qquad \lambda = \frac{\Delta t}{\Delta x}.$$

Lax-Wendroff Scheme

Summary:

With Lax-Wendroff flux

$$g(v,w) := \frac{1}{2}\left[f(v)+f(w)-\lambda\frac{(f(w)-f(v))^2}{w-v}\right], \qquad \lambda = \frac{\Delta t}{\Delta x},$$

the Lax-Wendroff Scheme is given by

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(g(Q_j^n, Q_{j+1}^n) - g(Q_{j-1}^n, Q_j^n)\right).$$

Lax-Wendroff Scheme

The Lax-Wendroff Scheme

- is a scheme in conservation form,
- is consistent of order 2,
- is not monotone,
- yields typically no converge to the entropy solution,
- produces typically strong oscillations close to the discontinuities.

- Idea by Sweby: "Erase" the higher order contributions close to the shocks.

Modified equations - a comparison

Consider the linear problem

$$\partial_t u + a\partial_x u = 0, \qquad \text{for } a > 0.$$

We saw for monotone schemes that they better approximate modified equations with an artificial diffusion term. In the Lax-Wendroff scheme, we “killed” this term artificially to obtain a higher order.

Comparison:

- Lax-Friedrichs Scheme. Modified equation:

$$\partial_t u + a\partial_x u = \frac{\triangle x}{2\lambda}\partial_{xx} u \qquad \text{diffusive - everything smeared out}$$

- Lax-Wendroff Scheme. Modified equation:

$$\partial_t u + a\partial_x u = \frac{\triangle x^2}{6} a(1-(a\lambda)^2)\partial_{xxx} u \qquad \text{dispersion - artificial oscillations}$$

Remark: Godunov Theorem

Question: Can we find a “better” second order scheme without oscillations?

Answer: No. The so called Godunov Theorem says:

Any linear scheme for solving conservation laws with the property that it does not create new extrema is <u>at most of order 1</u>.

Hence: there is no scheme of consistency order 2 that is free from the artificial oscillations.

Lax-Wendroff Scheme - Example

We consider the linear problem

$$\partial_t u + a\partial_x u = 0, \qquad \text{for } a > 0.$$

In this case, the Lax-Wendroff flux reads

$$g(v,w) = \frac{1}{2} a \left[v + w - \lambda a(v - w)\right]$$

and the scheme becomes

$$\begin{aligned} Q_j^{n+1} &= Q_j^n - \lambda a(Q_j^n - Q_{j-1}^n) - \frac{\lambda a}{2}(1 - \lambda a)(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n) \\ &= Q_j^n - \underbrace{\lambda a\, \Delta_- Q_j^n}_{\text{“upwind”-part 1. order}} - \underbrace{\frac{\lambda a}{2}(1 - \lambda a)\, \Delta_- \Delta_+ Q_j^n}_{\text{higher order corrector}} \end{aligned}$$