Numerical solutions of differential equations

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Course SF2521, 7.5 ECTS, VT18

Lecture 5

Hyperbolic Equations of first order - Part 2

Linear Riemann problems

The Riemann-problem - Scalar case

- Riemann-problem: conservation law with discontinuous initial condition.
- In the following we consider the linear Riemann-problem

Scalar case: for $\mathbf{a} \in \mathbb{R}$, find $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with

$$\partial_t u + \mathbf{a} \partial_x u = 0 \qquad \text{and} \qquad u(x, 0) = v(x) := \begin{cases} u_l & \text{for } x \leq 0 \\ u_r & \text{for } x > 0. \end{cases}$$

Formal issue:
v is discontinuous $\Rightarrow$ there exists no continuous (classical) solution u.

However: we know how a physically correct solution looks like:
initial value v moves with speed $\mathbf{a}$, i.e.

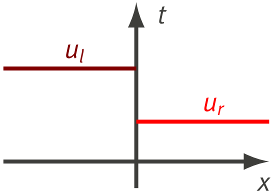
u is constant along the characteristic $\gamma(t) = x_0 + \mathbf{a}\, t$. Hence

$$u(x, t) = v(x - \mathbf{a}t).$$

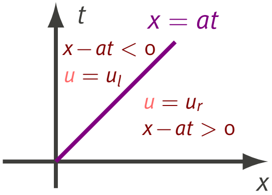
The Riemann-problem - Scalar case

Scalar case: for $\mathbf{a} \in \mathbb{R}$, find $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with

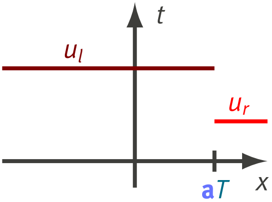
$$\partial_t u + \mathbf{a}\partial_x u = 0 \qquad \text{and} \qquad u(x,0) = v(x) := \begin{cases} u_l & \text{for } x \le 0 \\ u_r & \text{for } x > 0. \end{cases}$$



Initial value v.



Characteristic
$\gamma(t) = x_0 + \mathbf{a}t$
for $x_0 = 0$.



Solution $u(x, T)$.

How does this generalize to systems?

The Riemann-problem - Systems

System of conservation laws:

for diagonalizable $\mathbf{A} \in \mathbb{R}^{m\times m}$, find $\mathbf{u} : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ with

$$\partial_t \mathbf{u} + \mathbf{A}\,\partial_x \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \begin{cases} \mathbf{u}_l & \text{for } x \leq 0 \\ \mathbf{u}_r & \text{for } x > 0 \end{cases}$$

Here: $\mathbf{u}_l, \mathbf{u}_r \in \mathbb{R}^m$. We use the notation from the last Section.

Solution in characteristic variables:

$$\mathbf{z} = \mathbf{R}^{-1}\mathbf{u},$$

which solves

$$\partial_t \mathbf{z} + \mathbf{\Lambda}\,\partial_x \mathbf{z} = 0 \qquad \text{and} \qquad \mathbf{z}(x,0) = \mathbf{R}^{-1}\mathbf{v}(x) := \begin{cases} \mathbf{R}^{-1}\mathbf{u}_l & \text{for } x \leq 0 \\ \mathbf{R}^{-1}\mathbf{u}_r & \text{for } x > 0. \end{cases}$$

The Riemann-problem - Systems

The system

$$\partial_t \mathbf{z} + \mathbf{\Lambda}\,\partial_x \mathbf{z} = 0 \qquad \text{and} \qquad \mathbf{z}(x,0) = \begin{cases} \mathbf{z}_l := \mathbf{R}^{-1}\mathbf{u}_l & \text{for } x \le 0 \\ \mathbf{z}_r := \mathbf{R}^{-1}\mathbf{u}_r & \text{for } x > 0 \end{cases}$$

decouples into $1 \le p \le m$ scalar equations with

$$\partial_t \mathbf{z}_p + \lambda_p\,\partial_x \mathbf{z}_p = 0 \qquad \text{and} \qquad \mathbf{z}_p(x,0) = \begin{cases} \mathbf{z}_l^p & \text{for } x \le 0 \\ \mathbf{z}_r^p & \text{for } x > 0 \end{cases}$$

We have

$$\mathbf{z}_p(x,t) = \mathbf{z}_p(x - \lambda_p t, 0) = \begin{cases} \mathbf{z}_l^p & \text{for } x - \lambda_p t \le 0 \\ \mathbf{z}_r^p & \text{for } x - \lambda_p t > 0 \end{cases}$$

Solution:

$$\mathbf{u}(x,t) = \mathbf{R}\mathbf{z} = \sum_{\substack{p \text{ with} \\ x - \lambda_p t \le 0}} \mathbf{z}_l^p\,\mathbf{r}_p + \sum_{\substack{p \text{ with} \\ x - \lambda_p t > 0}} \mathbf{z}_r^p\,\mathbf{r}_p$$

The Riemann-problem - Systems

$$\mathbf{u}(x,t) = \sum_{\substack{p \text{ with} \\ x-\lambda_p t \le 0}} \mathbf{z}_l^p \, \mathbf{r}_p + \sum_{\substack{p \text{ with} \\ x-\lambda_p t > 0}} \mathbf{z}_r^p \, \mathbf{r}_p$$

We identify $m+1$ regions.

Example: If $m = 3$, we have 4 regions (cases):

1. $x - \lambda_1 t \le 0$; $x - \lambda_2 t \le 0$ and $x - \lambda_3 t \le 0$.
2. $x - \lambda_1 t > 0$; $x - \lambda_2 t \le 0$ and $x - \lambda_3 t \le 0$.
3. $x - \lambda_1 t > 0$; $x - \lambda_2 t > 0$ and $x - \lambda_3 t \le 0$.
4. $x - \lambda_1 t > 0$; $x - \lambda_2 t > 0$ and $x - \lambda_3 t > 0$.

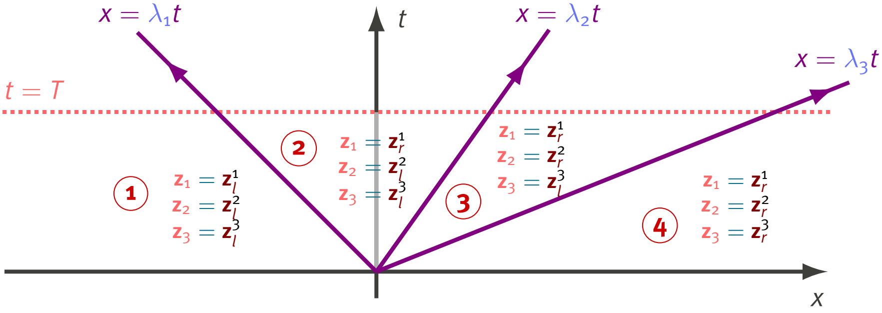
(∗) Since $\lambda_1 \le \lambda_2 \le \lambda_3$, we have

$$x - \lambda_1 t \ge x - \lambda_2 t \ge x - \lambda_3 t$$

and other cases cannot happen.

The Riemann-problem - Systems

Example: $m = 3$, where $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \geq 0$.



Region 1: $x - \lambda_1 t \leq 0$; $x - \lambda_2 t \leq 0$ and $x - \lambda_3 t \leq 0$.

$$\left.\begin{aligned} \mathbf{z}_1 &= \mathbf{z}_l^1 \\ \mathbf{z}_2 &= \mathbf{z}_l^2 \\ \mathbf{z}_3 &= \mathbf{z}_l^3 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{u}(x,t) = \mathbf{z}_l^1\,\mathbf{r}_1 + \mathbf{z}_l^2\,\mathbf{r}_2 + \mathbf{z}_l^3\,\mathbf{r}_3.$$

The Riemann-problem - Systems

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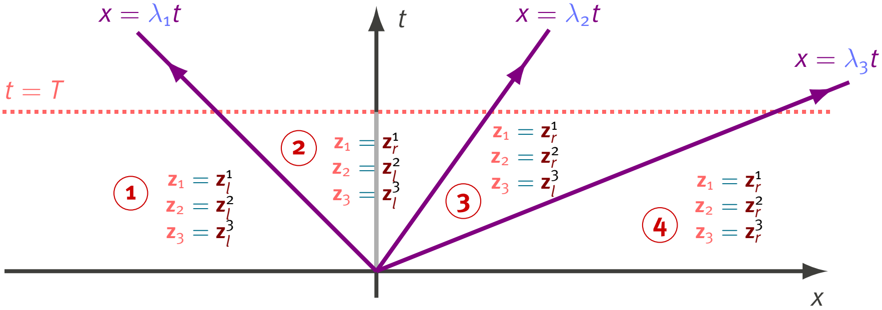
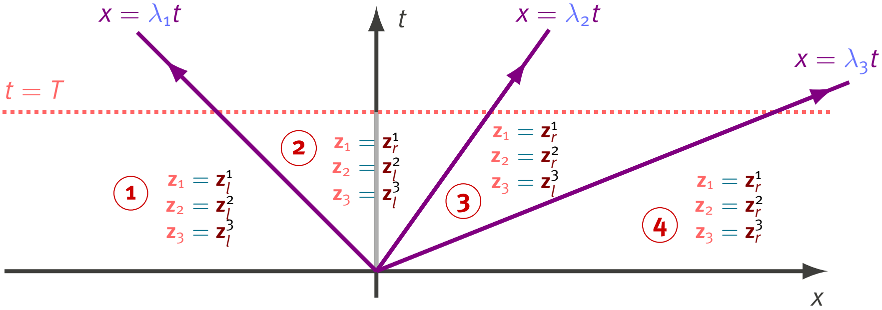


Region 2: $x - \lambda_1 t > 0$; $x - \lambda_2 t \leq 0$ and $x - \lambda_3 t \leq 0$.

$$\left.\begin{aligned} z_1 &= \mathbf{z}_r^1 \\ z_2 &= \mathbf{z}_l^2 \\ z_3 &= \mathbf{z}_l^3 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{u}(x,t) = \mathbf{z}_r^1\, \mathbf{r}_1 + \mathbf{z}_l^2\, \mathbf{r}_2 + \mathbf{z}_l^3\, \mathbf{r}_3.$$

The Riemann-problem - Systems

Example: $m = 3$, where $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \geq 0$.

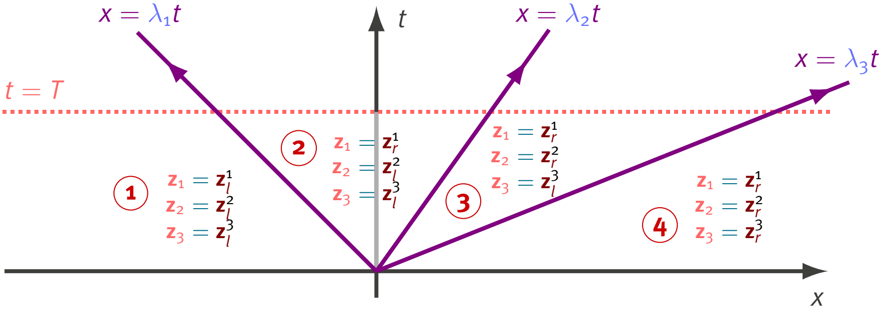


Region 3: $x - \lambda_1 t > 0$; $x - \lambda_2 t > 0$ and $x - \lambda_3 t \leq 0$.

$$\left.\begin{array}{l} \mathbf{z}_1 = \mathbf{z}_r^1 \\ \mathbf{z}_2 = \mathbf{z}_r^2 \\ \mathbf{z}_3 = \mathbf{z}_l^3 \end{array}\right\} \quad \Rightarrow \quad \mathbf{u}(x,t) = \mathbf{z}_r^1\, \mathbf{r}_1 + \mathbf{z}_r^2\, \mathbf{r}_2 + \mathbf{z}_l^3\, \mathbf{r}_3.$$

The Riemann-problem - Systems

Example: $m = 3$, where $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \geq 0$.

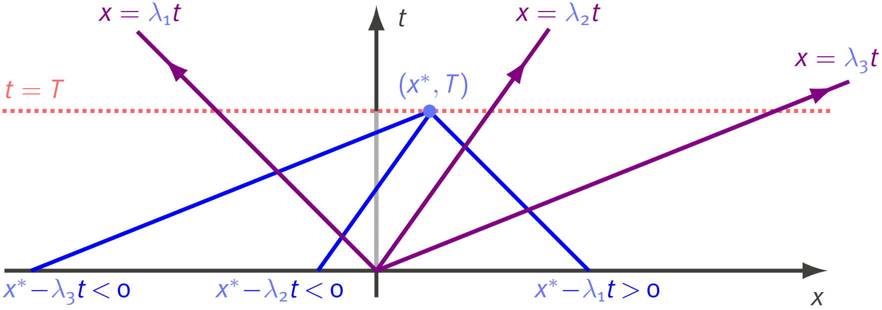


Region 4: $x - \lambda_1 t > 0$; $x - \lambda_2 t > 0$ and $x - \lambda_3 t \leq 0$.

$$\left.\begin{aligned} \mathbf{z}_1 &= \mathbf{z}_r^1 \\ \mathbf{z}_2 &= \mathbf{z}_r^2 \\ \mathbf{z}_3 &= \mathbf{z}_r^3 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{u}(x,t) = \mathbf{z}_r^1\,\mathbf{r}_1 + \mathbf{z}_r^2\,\mathbf{r}_2 + \mathbf{z}_r^3\,\mathbf{r}_3.$$

The Riemann-problem - Systems

Example: $m = 3$, where $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \geq 0$.

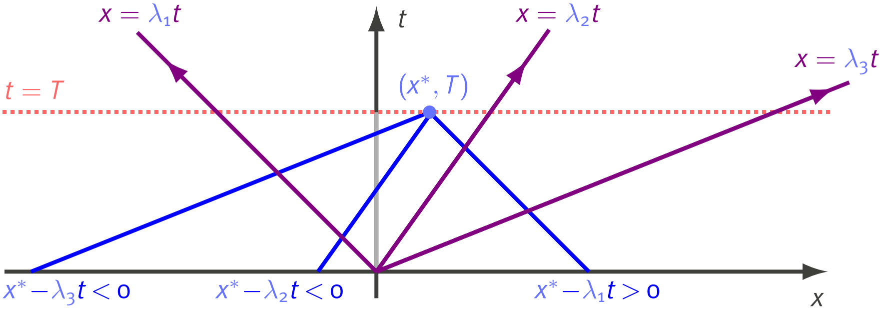


Region 2: $x - \lambda_1 t > 0$; $x - \lambda_2 t \leq 0$ and $x - \lambda_3 t \leq 0$.

$$(x^*, T) \text{ in Region } 2 \quad \Rightarrow \quad \mathbf{u}(x^*, T) = \mathbf{z}_r^1\, \mathbf{r}_1 + \mathbf{z}_l^2\, \mathbf{r}_2 + \mathbf{z}_l^3\, \mathbf{r}_3.$$

The Riemann-problem - Systems

Example: $m = 3$, where $\lambda_1 < 0$ and $\lambda_2, \lambda_3 \geq 0$.



Observe: over each p-characteristics $x = \gamma_p(t) = \lambda_p t$, there is a jump in $\mathbf{u}$.

For example: jump over $\gamma_1(t) = \lambda_1 t$:

$$\mathbf{u}_{\text{region 1}} - \mathbf{u}_{\text{region 2}} = \mathbf{z}_l^1\, \mathbf{r}_1 + \mathbf{z}_l^2\, \mathbf{r}_2 + \mathbf{z}_l^3\, \mathbf{r}_3 - \mathbf{z}_r^1\, \mathbf{r}_1 - \mathbf{z}_l^2\, \mathbf{r}_2 - \mathbf{z}_l^3\, \mathbf{r}_3 = \underbrace{(\mathbf{z}_l^1 - \mathbf{z}_r^1)}_{\text{jump } \neq 0}\, \mathbf{r}_1.$$

The Riemann-problem - Systems

Explicit example:

Find $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^2$ with

$$\partial_t \mathbf{u} + \mathbf{A}\,\partial_x \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

Here, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)^\top$

$$\mathbf{A} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}(x) = \begin{cases} \mathbf{u}_l = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } x \leq 0 \\ \mathbf{u}_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{for } x > 0. \end{cases}.$$

Eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = 2$.

Eigenvectors: $\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The Riemann-problem - Systems

We have

$$\Lambda := \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathbf{R} := \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{R}^{-1} := \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Look at the system in characteristic variables $\mathbf{z} = \mathbf{R}^{-1}\mathbf{u}$:

$$\partial_t \mathbf{z}_1 + 2\,\partial_x \mathbf{z}_1 = 0 \qquad \text{and} \qquad \partial_t \mathbf{z}_2 - 2\,\partial_x \mathbf{z}_2 = 0.$$

We have:

$$\mathbf{z}(0,x) = \mathbf{R}^{-1}\mathbf{v}(x) = \begin{cases} \mathbf{R}^{-1}\mathbf{u}_l = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{for } x \le 0 \\ \mathbf{R}^{-1}\mathbf{u}_r = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} & \text{for } x > 0. \end{cases}$$

The Riemann-problem - Systems

From

$$\mathbf{z}(0,x) = \mathbf{R}^{-1}\mathbf{v}(x) = \begin{cases} \mathbf{R}^{-1}\mathbf{u}_l = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{for } x \leq 0 \\ \mathbf{R}^{-1}\mathbf{u}_r = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} & \text{for } x > 0. \end{cases}$$

we conclude that the solution in characteristic variables is

$$\mathbf{z}_1(0,x) = \begin{cases} \frac{1}{2} & \text{for } x - 2t \leq 0 \\ \frac{1}{4} & \text{for } x - 2t > 0 \end{cases} \quad \text{and} \quad \mathbf{z}_2(0,x) = \begin{cases} \frac{1}{2} & \text{for } x + 2t \leq 0 \\ \frac{3}{4} & \text{for } x + 2t > 0 \end{cases}$$

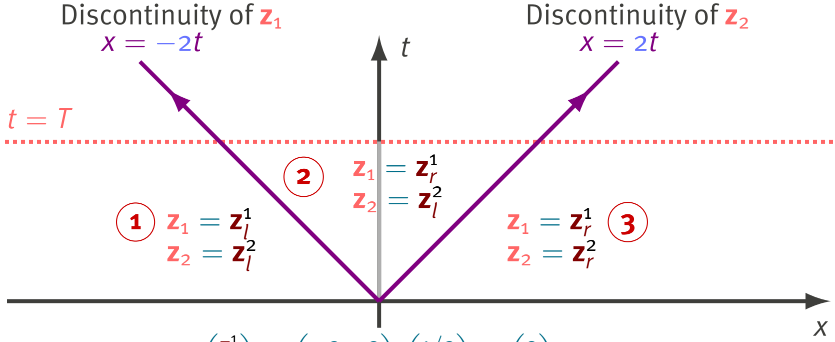
From that we can conclude the solution $\mathbf{u}$.

Since there are 2 characteristics, we know that $\mathbf{u}$ has 3 stats (3 regions with different behavior).

The Riemann-problem - Systems

Solution in characteristic variables:

$$\mathbf{z}_1(0,x) = \begin{cases} \frac{1}{2} & \text{for } x - 2t \le 0 \\ \frac{1}{4} & \text{for } x - 2t > 0 \end{cases} \quad \text{and} \quad \mathbf{z}_2(0,x) = \begin{cases} \frac{1}{2} & \text{for } x + 2t \le 0 \\ \frac{3}{4} & \text{for } x + 2t > 0 \end{cases}$$



1 $$\mathbf{u}(x,t) = \mathbf{R}\begin{pmatrix} \mathbf{z}_l^1 \\ \mathbf{z}_l^2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{u}_l \qquad \text{for } x + 2t < 0.$$

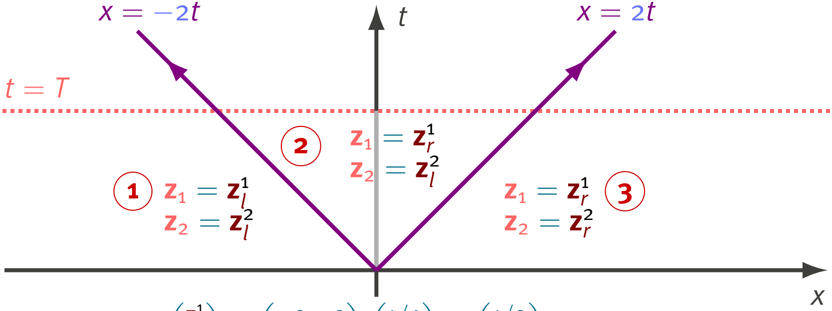
The Riemann-problem - Systems

Solution in characteristic variables:

$$\mathbf{z}_1(t,x) = \begin{cases} \frac{1}{2} & \text{for } x - 2t \le 0 \\ \frac{1}{4} & \text{for } x - 2t > 0 \end{cases} \quad \text{and} \quad \mathbf{z}_2(t,x) = \begin{cases} \frac{1}{2} & \text{for } x + 2t \le 0 \\ \frac{3}{4} & \text{for } x + 2t > 0 \end{cases}$$

Discontinuity of $\mathbf{z}_1$ $x = -2t$

Discontinuity of $\mathbf{z}_2$ $x = 2t$



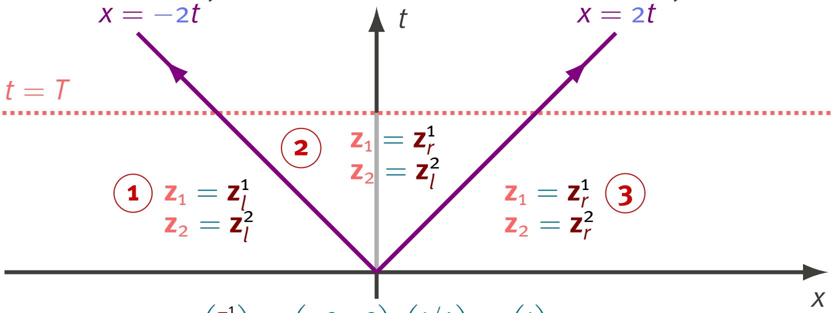
2 $\mathbf{u}(x,t) = \mathbf{R}\begin{pmatrix} \mathbf{z}_r^1 \\ \mathbf{z}_l^2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix}$ for $x + 2t \ge 0$ and $x - 2t < 0$.

The Riemann-problem - Systems

Solution in characteristic variables:

$$\mathbf{z}_1(t,x) = \begin{cases} \frac{1}{2} & \text{for } x - 2t \leq 0 \\ \frac{1}{4} & \text{for } x - 2t > 0 \end{cases} \quad \text{and} \quad \mathbf{z}_2(t,x) = \begin{cases} \frac{1}{2} & \text{for } x + 2t \leq 0 \\ \frac{3}{4} & \text{for } x + 2t > 0 \end{cases}$$

Discontinuity of $\mathbf{z}_1$ Discontinuity of $\mathbf{z}_2$

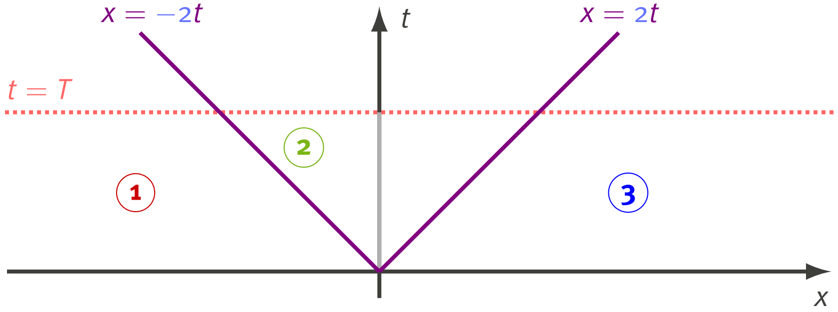


3 $\mathbf{u}(x,t) = \mathbf{R}\begin{pmatrix} \mathbf{z}_r^1 \\ \mathbf{z}_r^2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{u}_r \qquad \text{for } x - 2t \geq 0.$

The Riemann-problem - Systems

We have

$$\mathbf{u}_1(x,t) = \begin{cases} 0 & \text{for } x + 2t < 0 \\ \frac{1}{2} & \text{for } x + 2t \geq 0 \text{ and } x - 2t < 0 \\ 1 & \text{for } x - 2t \geq 0 \end{cases}.$$



The Riemann-problem - Systems

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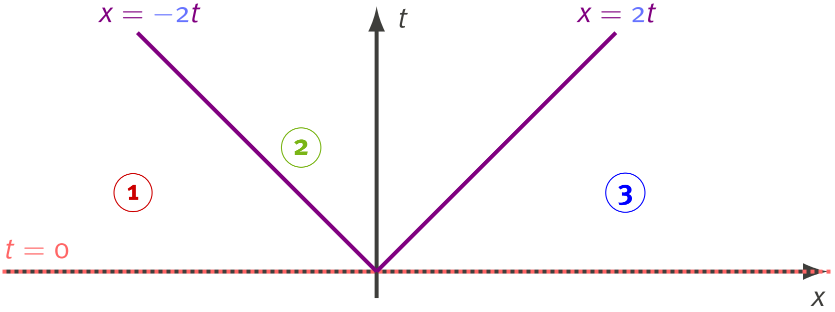
$$\mathbf{u}_1(x,t) = \begin{cases} 0 & \text{for } x+2t<0 \\ \frac{1}{2} & \text{for } x+2t\geq 0 \text{ and } x-2t<0 \\ 1 & \text{for } x-2t\geq 0 \end{cases}.$$



The Riemann-problem - Systems

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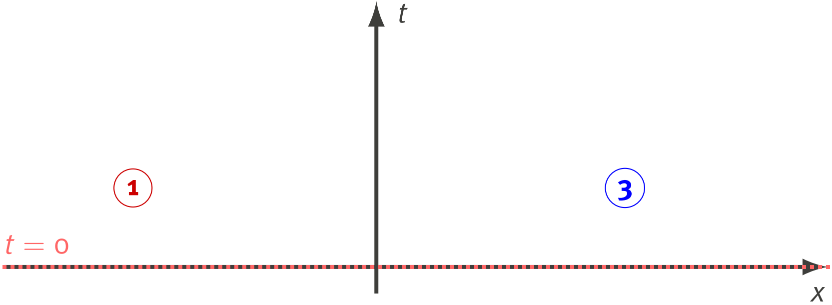

The Riemann-problem - Systems

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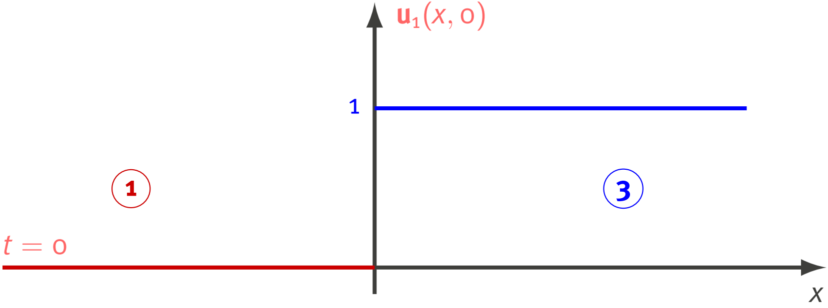

The Riemann-problem - Systems

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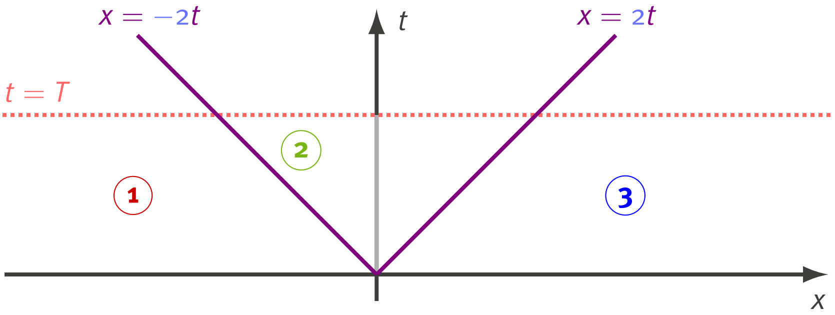

The Riemann-problem - Systems

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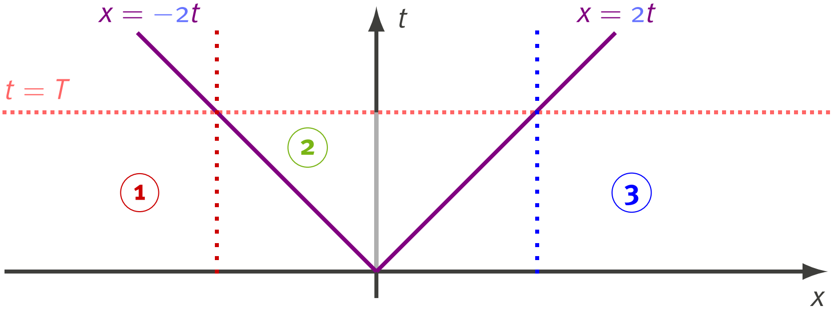

The Riemann-problem - Systems

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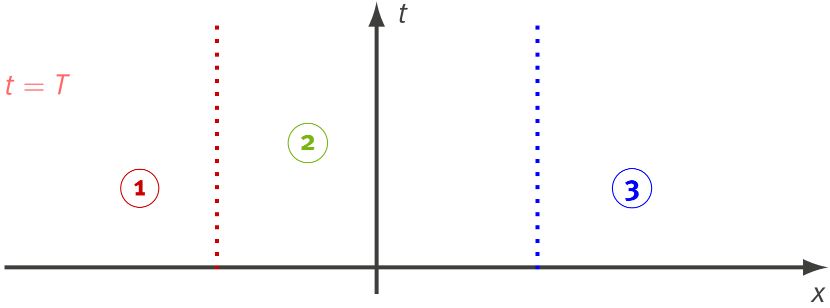

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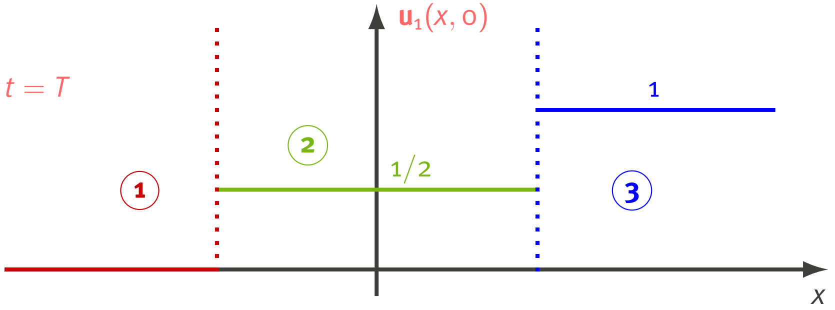

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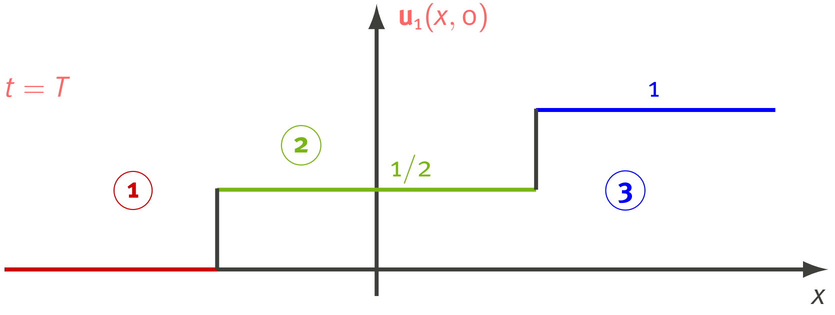
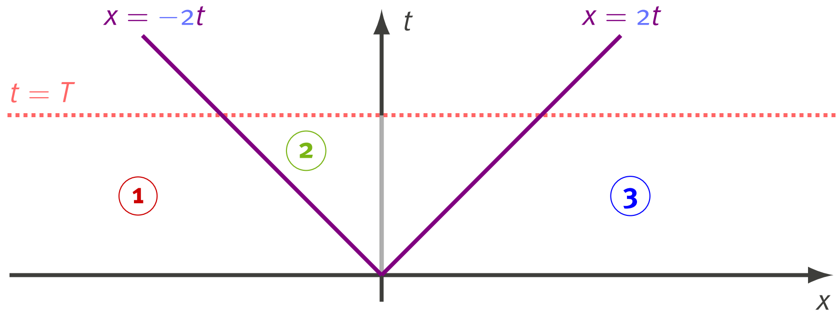


The Riemann-problem - Systems

We have

$$\mathbf{u}_2(x,t) = \begin{cases} 0 & \text{for } x + 2t < 0 \\ \frac{3}{4} & \text{for } x + 2t \geq 0 \text{ and } x - 2t < 0 \\ 1 & \text{for } x - 2t \geq 0 \end{cases}.$$



The Riemann-problem - Systems

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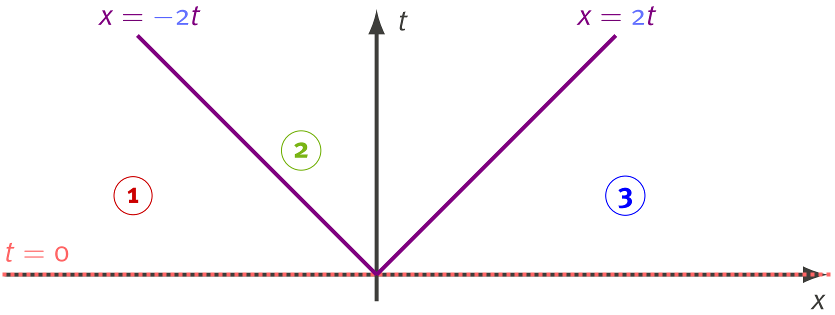

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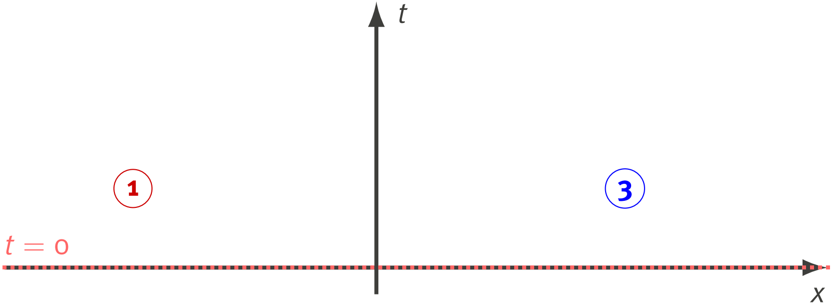

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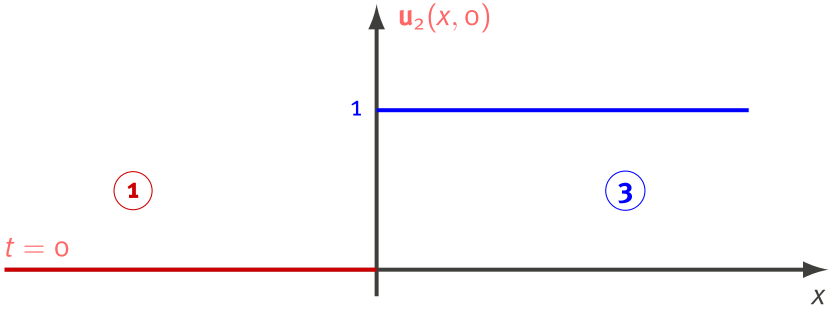

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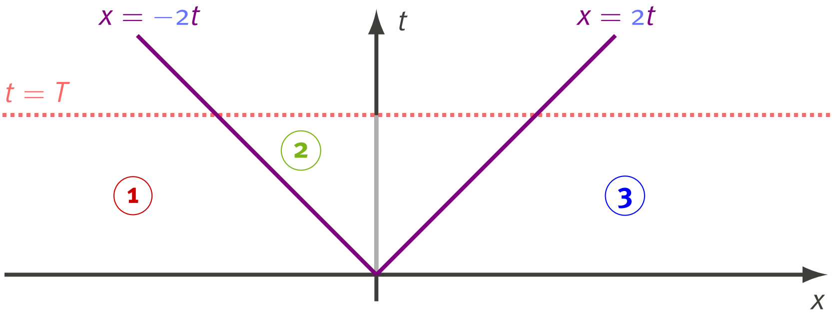

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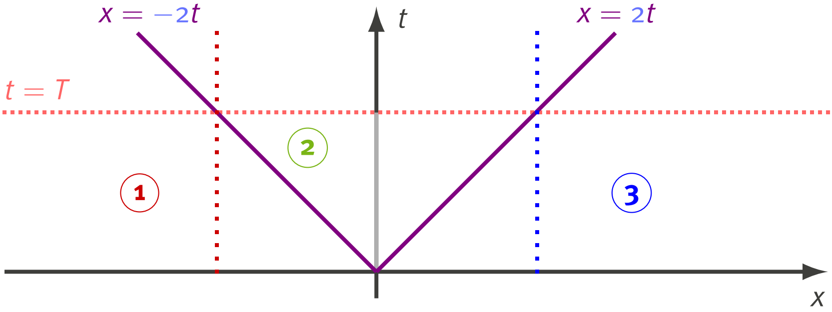

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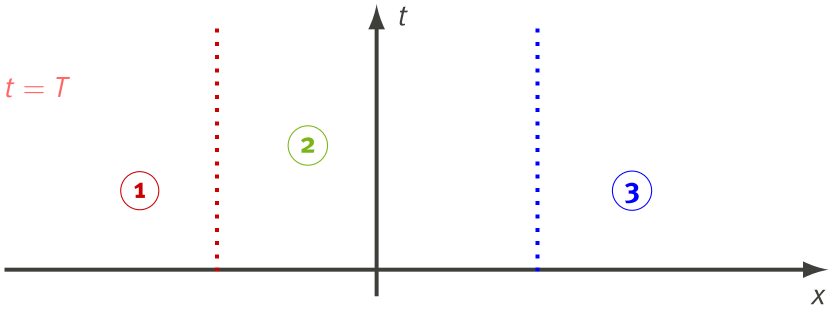

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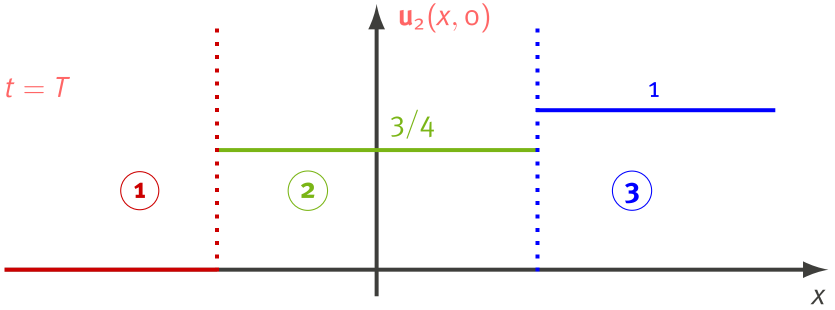

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