Lecture 4

Hyperbolic Equations of first order - Part 1

Characteristics: Example 1, Burgers equation.

Characteristics - Understanding solutions

Example: Understanding Burger's equation, where $f(u) = \frac{1}{2}u^2$:

$$\partial_t u + u\, \partial_x u = 0.$$

For the initial value we assume $v \in C^\infty(\mathbb{R})$ and there are x_0 and x_1 with

$$x_1 < x_2 \qquad \text{and} \qquad v(x_1) > v(x_2).$$

Since $f(u) = \frac{1}{2}u^2$ we have $f'(u) = u$ and hence

$$\gamma_1(t) = v(x_1) \cdot t + x_1 \qquad \text{and} \qquad \gamma_2(t) = v(x_2) \cdot t + x_2.$$

For $t_0 = -\frac{x_1 - x_2}{v(x_1) - v(x_2)} > 0$ we have $\gamma_1(t_0) = \gamma_2(t_0)$ and hence

$$v(x_1) = u(\gamma_1(t_0), t_0) = u(\gamma_2(t_0), t_0) = v(x_2) < v(x_1) \quad \Rightarrow \text{Contradiction.}$$

What does that imply? For $t \geq t_0$ there exists no classical solution u.

Characteristics - Understanding solutions

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Crash of conservation laws. Observe

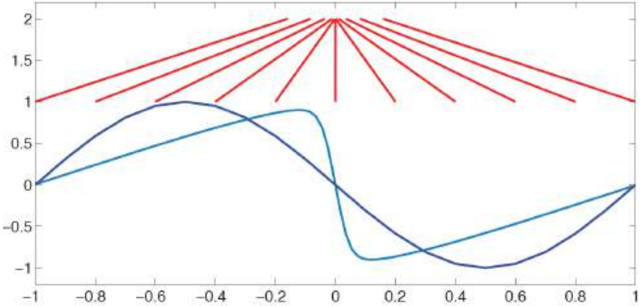
- Physical interpretation: $\gamma_1(t) = \underbrace{v(x_1)}_{\text{Wave speed}}\ t + x_1$

  Wave speed higher the larger the value of $v(x_1)$.
- Particles in the wave peak are fastest
- wave must eventually develop a discontinuity $\Rightarrow$ “Shock waves”.
- if $v(x_1) \leq v(x_2)$ we have $t_0 < 0$ and no contradiction;

Characteristics - Understanding solutions

Example: $v(x) = \sin(\pi x)$. It holds

$$\gamma(t) = \sin(\pi x_0) \cdot t + x_0.$$



Characteristics will cross $\Rightarrow$ discontinuous solution for $t \geq t_0$.

Characteristics - Understanding solutions

The examples shows the following result:

Theorem

Conservation laws do not necessarily have classical solutions for all times $t \in (0, \infty)$. Not even for very regular initial data.