3. Two dimensionnal flows: Linear systems:

In 1D phase spaces, all trajectories are forced to move monotically and are gathered in a single curve.

Now 2D dynamical systems are written in the typical form:

(1) $$\underline{\dot{X}}(t) = \underline{f}(\underline{X}(t)) \text{ with } \underline{X} \in \mathbb{R}^2$$

=> Linear systems:

(2) $$\underline{\dot{X}}(t) = [A]\,\underline{X}(t) \text{ with } \underline{X}(0) = \underline{X}_0$$

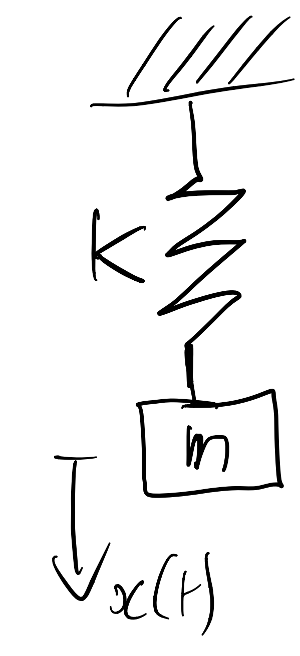
$\underline{X}(t)$ is the vector of state variables.

(2) is linear in the sense that if $\underline{X}_1(t)$ and $\underline{X}_2(t)$ are solutions, then so is any linear combination $c_1 \underline{X}_1(t) + c_2 \underline{X}_2(t)$.

$\underline{X}^* = \underline{0}$ is always a fixed point for any choice of $[A]$.

Solutions of $\underline{\dot{X}} = [A]\underline{X}$ can be vizualized as trajectories moving in the phase plane $\underline{X} = [x, y]$

Example: Simple harmonic oscillator:



(x) $m\ddot{x}(t) + kx(t) = 0$ (physical space)

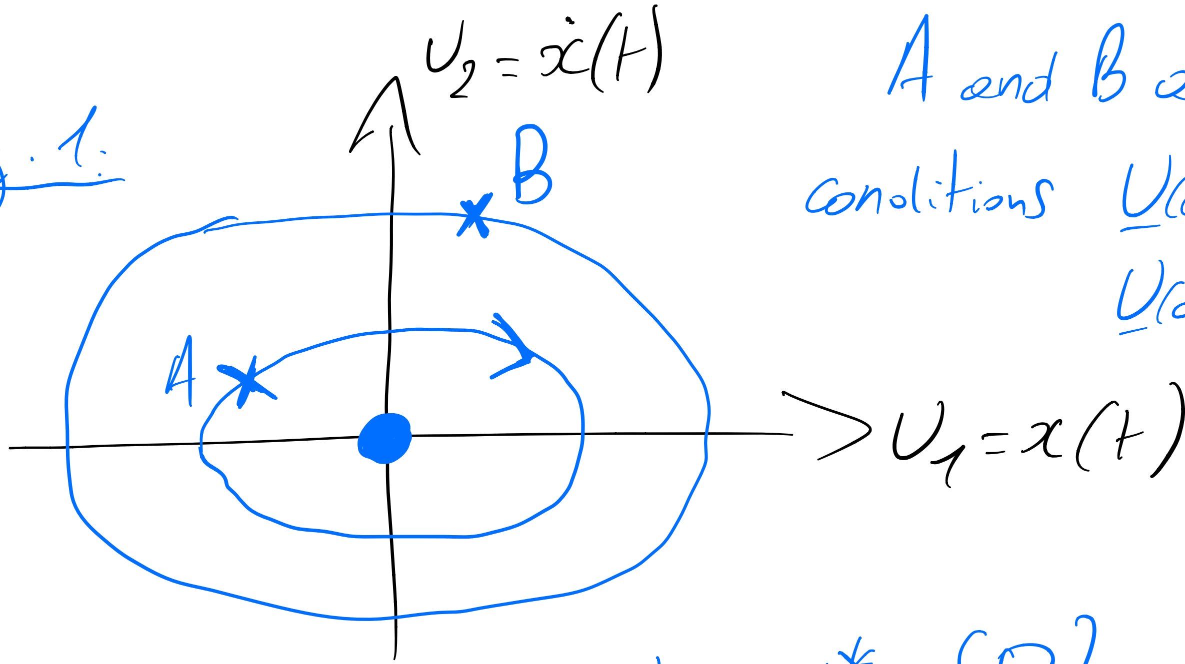
The state space is characterized by position $x(t)$ and velocity $v(t) = \dot{x}(t)$ of the mass.

$$(x) \rightarrow \begin{Bmatrix} \dot{x} \\ v \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{Bmatrix} x \\ v \end{Bmatrix} \quad \text{with } \omega = \sqrt{\frac{k}{m}}$$

$$\underline{\dot{U}} = [A]\,\underline{U}$$

Trajectories in the state plane:

Fig. 1:



A and B are initial conditions $\underline{U}(0) = \underline{A}$
$\underline{U}(0) = \underline{B}$

● is the fixed point at $\underline{U}^* = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Fig. 1 is called a phase portrait, it shows "all" the trajectories in the phase plane.

Classification of linear systems - Stability analysis:

To understand preferential directions of motion, we seek trajectories in the form:

$$\underline{X}(t) = e^{\lambda t}\,\underline{V}$$

where $\underline{V} \neq \underline{0}$ is some fixed vector.

λ is a growth rate.

Substituting $\underline{x}(t) = e^{\lambda t}\underline{v}$ in $\dot{\underline{x}} = [A]\underline{x}$

yields $[A]\underline{v} = \lambda \underline{v}$

$\underline{v}$ is an eigenvector and λ eigenvalues of $[A]$.

Quick recall:

Eigenvalues of $[A]$ obtained by solving:

$$\det([A] - \lambda[I]) = 0$$

$$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \lambda_{1,2} = \frac{\tau \mp \sqrt{\tau^2 - 4\Delta}}{2}$$

with $\tau = \text{trace}([A]) = a + d = \lambda_1 + \lambda_2$

$\Delta = \det([A]) = ad - bc = \lambda_1 \times \lambda_2$

If the eigenvalues are distinct: $\lambda_1 \neq \lambda_2$

$$\Rightarrow \underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2$$

With $\underline{v}_i$ the eigenvectors of $[A]$.

* Model projection:

$$\underline{X}(t) = [\varphi]\underline{\eta}(t) \quad \text{with } [\varphi] = [\underline{V}_1, \underline{V}_2]$$

and $$\underline{\eta}(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \begin{Bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{Bmatrix}$$

Replacing $\underline{X}(t) = [\varphi]\underline{\eta}(t)$ in $\dot{\underline{X}} = [A]\underline{X}$

$$\Rightarrow \dot{\underline{\eta}}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underline{\eta}(t)$$

in the modal space.

$\underline{\eta}(t)$ is the vector of modal coordinates.

Example: $$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \quad \text{with } \underline{U}(0) = \begin{Bmatrix} x(0) \\ y(0) \end{Bmatrix}$$

$$\underline{U}(0) = \begin{Bmatrix} 2 \\ -3 \end{Bmatrix}$$

$\tau = -1$

$\Delta = -6$

$\Rightarrow \lambda_1 = 2$

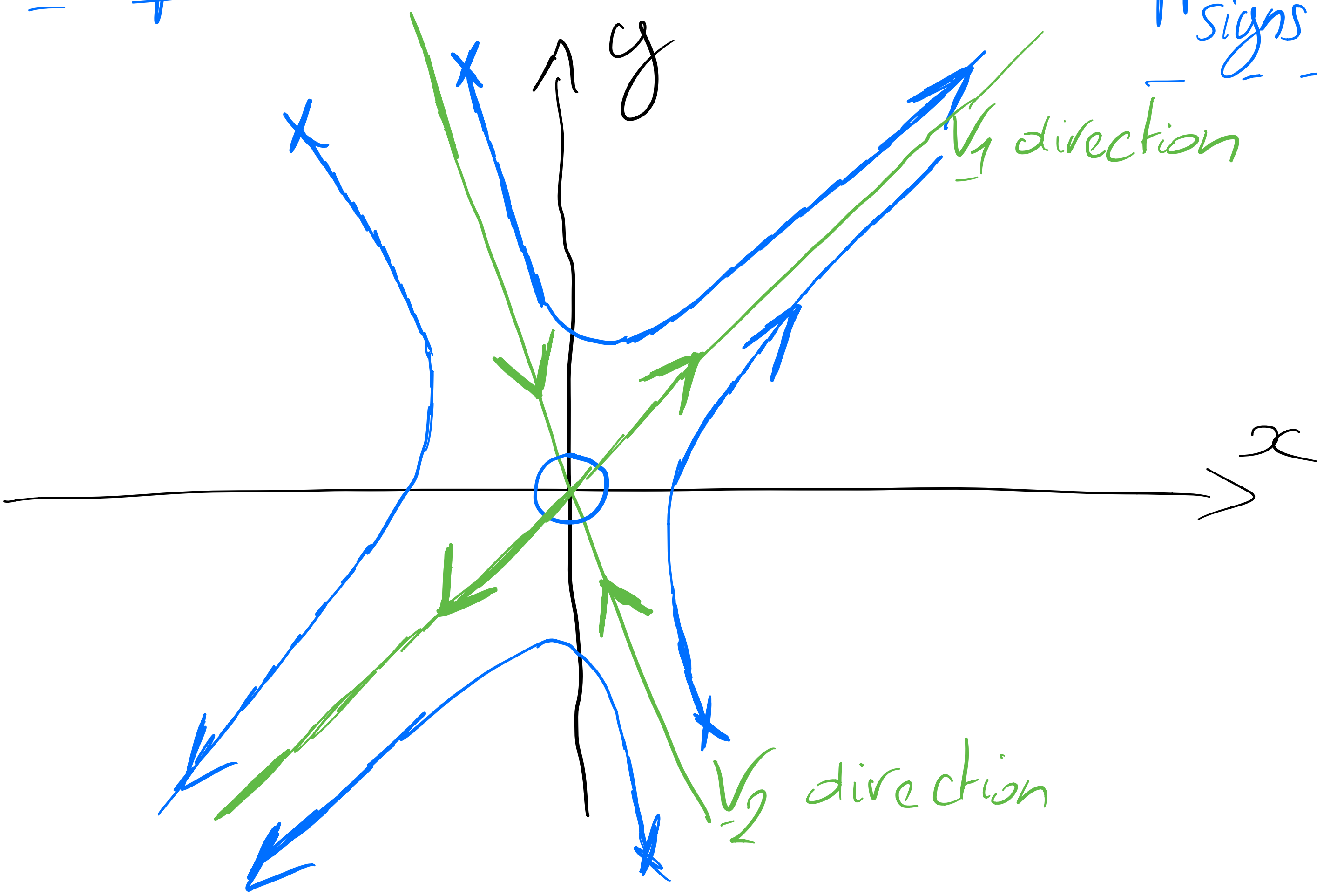
$\lambda_2 = -3$

$\underline{V}_1 = \begin{Bmatrix}1\\1\end{Bmatrix} \qquad \underline{V}_2 = \begin{Bmatrix}1\\-4\end{Bmatrix}$

$$\Rightarrow \underline{U}(t) = c_1 \begin{Bmatrix}1\\1\end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix}1\\-4\end{Bmatrix} e^{-3t}$$

Introducing initial conditions: $c_1 = 1$, $c_2 = 1$.

* Phase portrait when $\lambda_i \in \mathbb{R}$ and have opposite signs:

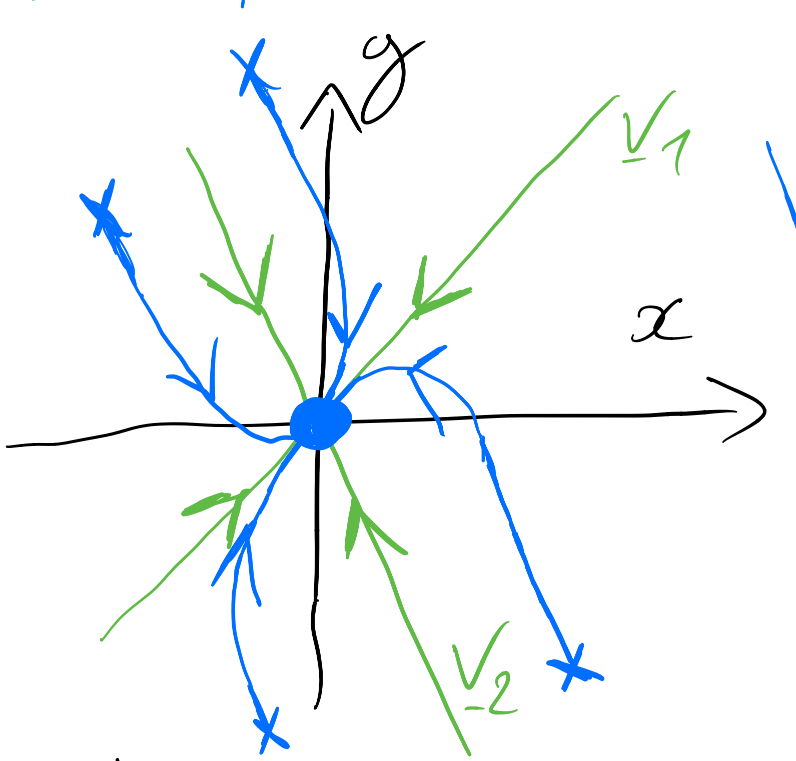


V_1 is associated with a positive real eigenvalue.
We say that V_1 spanned the unstable manifold.

$\underline{V}_2$ associated with a negative eigenvalue so $\underline{V}_2$ spanned the stable manifold.

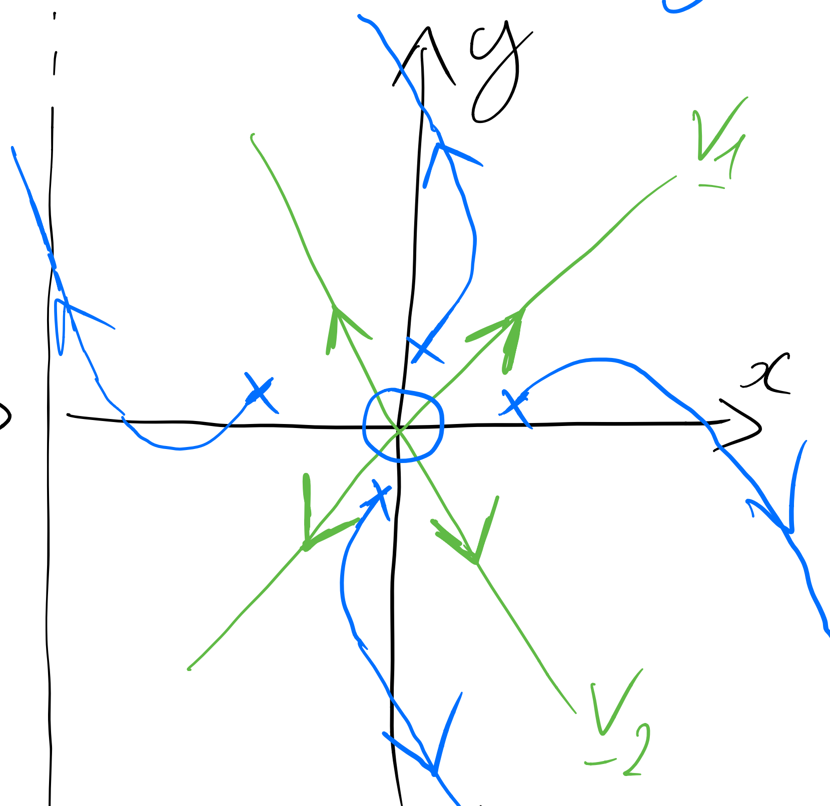
If you have one eigenvalue with positive real part, the fixed point $\underline{u}^* = \underline{0}$ is unstable.

* Phase portrait: $\lambda_i \in \mathbb{R}$ and both have same sign:



$\lambda_2 < \lambda_1 < 0$

Fixed point is stable node



$0 < \lambda_1 < \lambda_2$

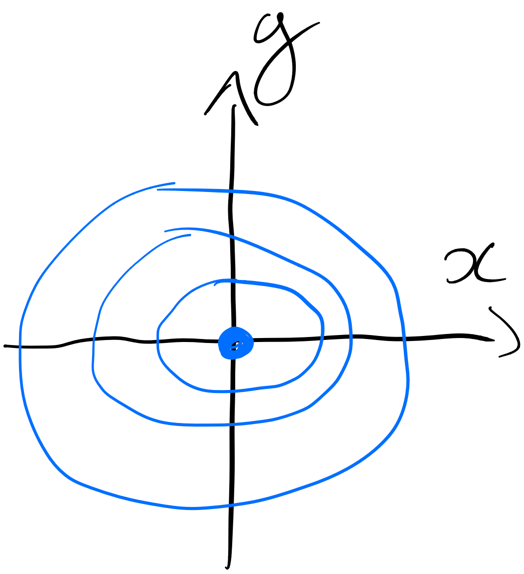
Fixed point is an unstable node

Trajectories approach the origin tangent to the slow direction $\underline{v}_1$.

As time $t \to +\infty$, trajectories become parallel to the fast direction $\underline{v}_2$.

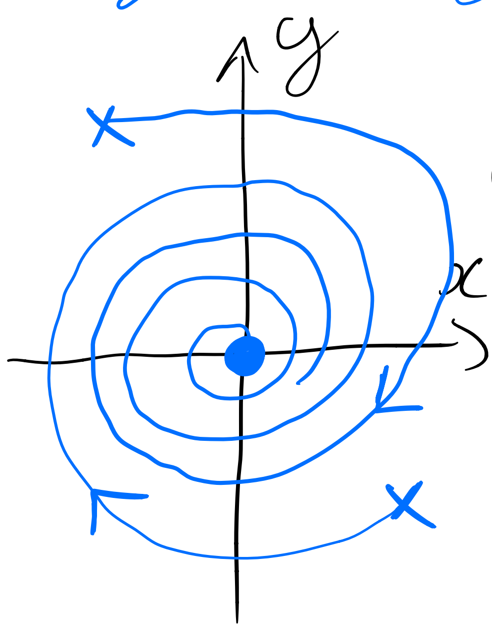
*If the eigenvalues are complex: $\lambda_i \in \mathbb{C}$: $\lambda = \alpha \pm i\omega$

≠ cases happen depending on the sign of $\alpha = Re(\lambda)$



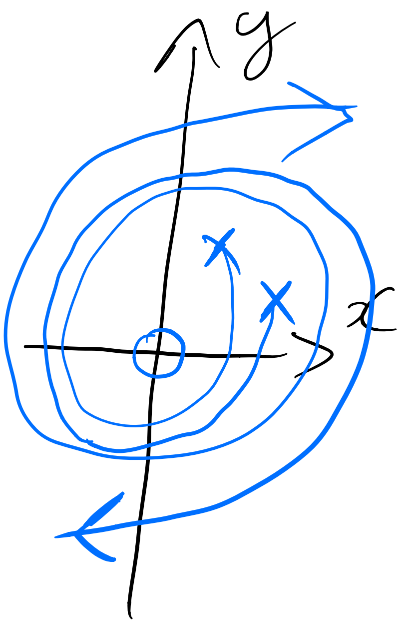
$\alpha = 0$

Fixed point is a center. It is neutrally or marginally stable



$\alpha < 0$

Fixed point is a stable spiral.



$\alpha > 0$

Fixed point is an unstable spiral.

Remark:

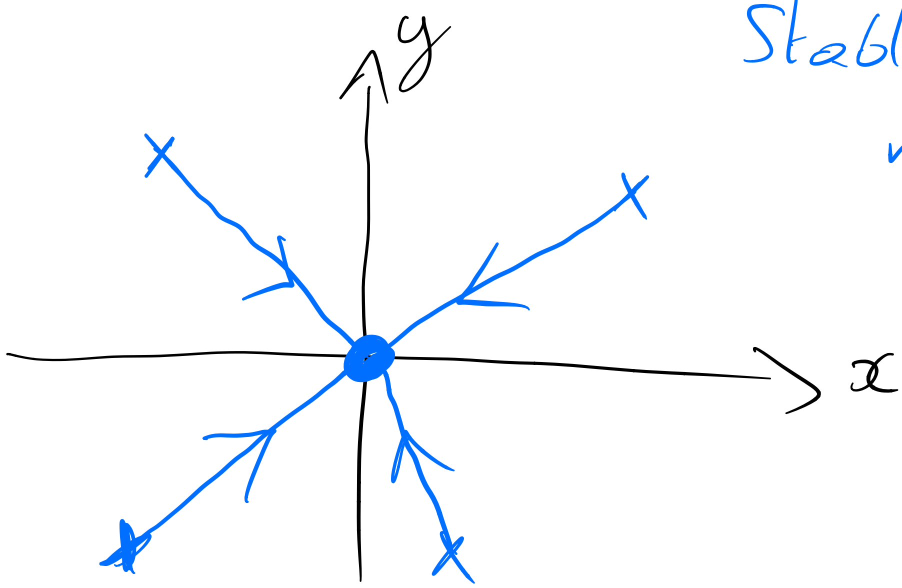
Two different trajectories never cross!

* If the eigenvalues are equal: $\lambda_1 = \lambda_2 = \lambda$

→ Two independent eigenvectors and $\lambda \neq 0$.

Fixed point is a star node: trajectories are straight lines through the origin:

Example: $\lambda_1 = \lambda_2 = \lambda < 0$



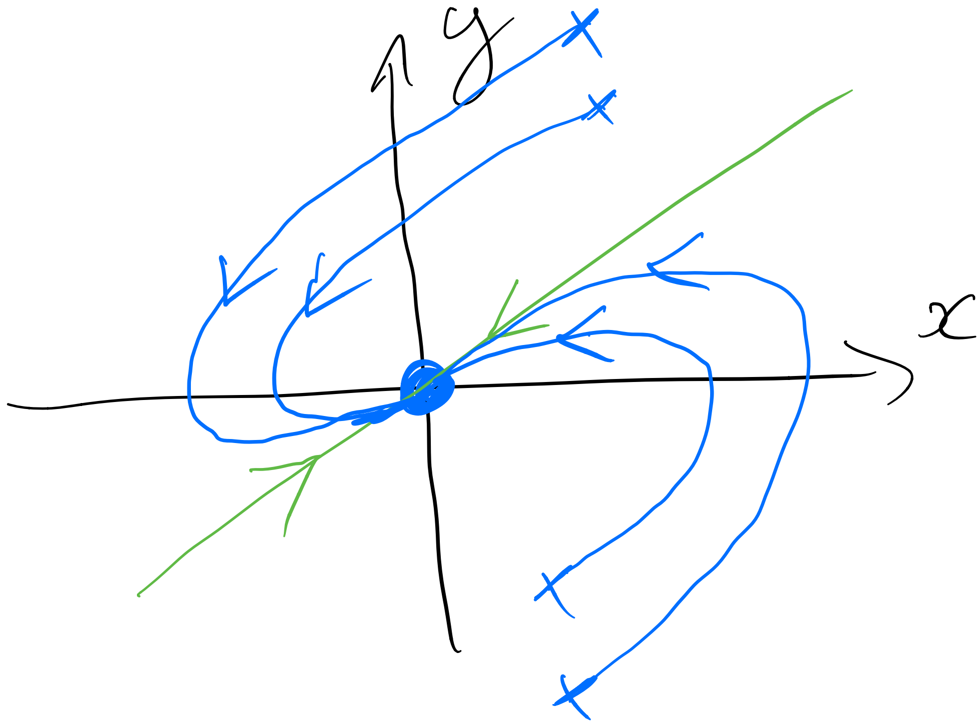
Stable star node

→ Two independent eigenvectors and $\lambda = 0$.

The whole phase plane is filled with fixed points.

→ Only one eigendirection:

The fixed point is a degenerate node.



Classification of stability of fixed points: