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Introduction to the finite element method in 1D

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1 Abstract

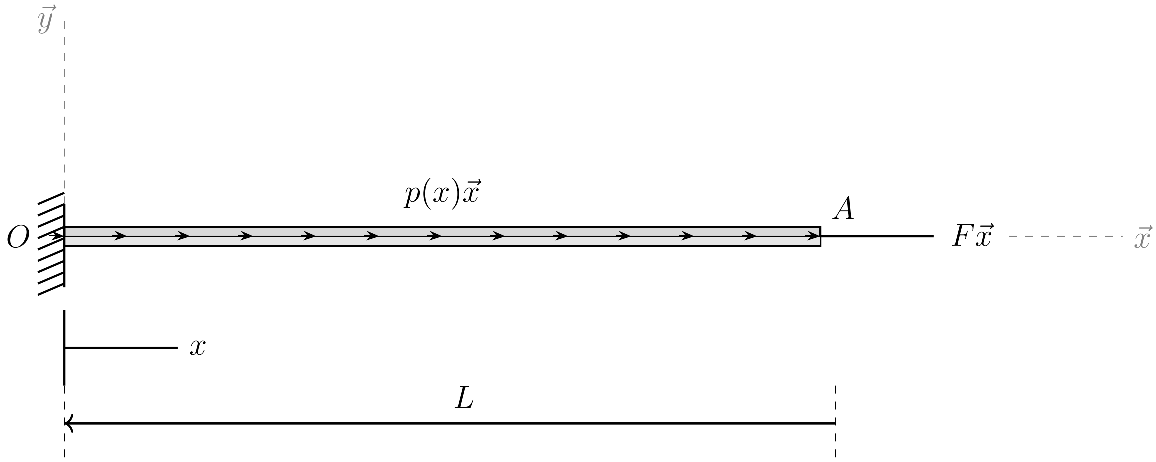
Finite element method (FEM) is one of the most commonly used numerical technique to find approximate solution for various problems in the field of mechanical engineering.

In this paper we are presenting the overview of solving mechanical 1D bar problem by Finite Element Method (FEM). First we will cover different methods of numerical analysis for structural mechanics. Doing so, we will introduce the governing differential equations of the problem with the strong formulation. Afterward, we will derive the other formulation from it, assuring that each formulation describes indeed the same problem.

We will then also present the equivalence between the different types of formulations. We will thus introduce the Ritz-Galërkin method which will allow us to find an approximation of a function by converting a continuous problem into a discrete one. Finally, we will solve the problem by the Finite element method which is the most widely used method in engineering to solve these type of problem by subdividing a large system into smaller, simpler parts that are called finite elements.

2 Model problem: the traction of a bar

We illustrate the basics of the Finite Element Method (FEM) by working out the example of the traction of a bar as a model problem. We consider a bar of length L occupying the domain $\Omega \equiv [0,L]$. Considering a one-dimensional bar model, let $ES(x)$ be its axial stiffness and $u(x)$ the axial displacement. The bar is clamped in $x = 0$ and loaded by an axial end-force F at $x = L$ as well as a distributed axial loading $p(x)$.



To briefly introduce the equations to solve, we analyse the problem as following:

The displacement $u(x)$ of the rod points produces the corresponding elongation $\epsilon(x)$:

$$\epsilon(x) = \frac{du}{dx}$$

and the normal stress σ in the rod is defined by Hooke law, with E is the Young modulus of the material:

$$\sigma = E\epsilon = E\frac{du}{dx}$$

The axial force $N(x)$ is defined as follows:

$$N(x) = S\sigma = ES\frac{du}{dx}$$

Using the forces equilibrium equation:

$$\frac{dN(x)}{dx} + F(x) = ES\frac{du}{dx} + p(x) = 0$$

With boundaries conditions :

$$u(0) = 0 \quad \text{and} \quad ESu'(L) = N(L) = F$$

This last equation is the one we need to solve using different problem formulations.

3 Problem formulation

In this section we will explain more precisely how to get different forms of equations.

3.1 Strong formulation (SF)

The strong formulation derivates in our case from the application of the resulting theorem applied to a plan curvilinear beam, under its derivied form :

$$\frac{d\vec{R}(s)}{ds} + f(\vec{s}) = \vec{0}$$

We won't go further on the provenance of this formula, but we will rather see how it applies and simplifies in our 1D case.

First, we consider an axial distributed load $p(x)$ along $\vec{ex}$ and an axial end force F at $x = L$. Therefore, the resulting forces as well as the displacement will only depend on x variable and be along $\vec{ex}$. Therefore, we have $s \equiv x$ and only component from projection on $\vec{ex}$.

For what is up to the resulting forces equilibrium, our equation rewrites as :

$$\frac{d\vec{N}_x(x)}{dx} + p(\vec{x}) = 0$$

Finally, considering an homogeneous material (static moments = 0), we get for N_x the following formula :

$$N_x(x) = ES(x)\epsilon_x + E(m_{gy}\gamma_y - m_{gz}\gamma_z) = ES(x)\epsilon_x = ES(x)\frac{du(x)}{dx} \equiv ES(x)u'(x)$$

with E the Young Modulus, S(x) the surface of a straight section. The product of these is called the axial rigidity.

Note : The component of displacement along $\vec{ex}$ is noted u.

By replacing $N(x)$ in the equation 3.1 and by projection on $\vec{e_x}$, we finally get :

$$(ES(x)u'(x))' + p(x) = 0$$

For the strong formulation to be well defined, this equation needs additional conditions. In fact, we will have BC in displacement and in stress. We therefore define the kinematically and statically admissible fields :

- Statically admissible field :
  $\Sigma_{ad} = (\bar{\bar{\tau}}(x)$ defined $\forall x \in \Omega$, symmetrical, smooth such that $[\tau(\bar{\bar{x}})]\vec{n} = \vec{0}\ \forall x \in \Omega$ || $\vec{div}(\bar{\bar{\tau}}) + \rho\vec{f} = \vec{0}\ \forall x \in \Omega$ and the BC $\quad \bar{\bar{\tau}}(x)\vec{n}(x) = F\vec{ex} \quad \forall x \in \partial\Omega_F \equiv x = L)$

- Kinematically admissible field :
  $U_{ad} = (\bar{v}(x)$ defined $\forall x \in \Omega$, smooth such that $[v(\vec{x})] = \vec{0}\ \forall x \in \Omega$ || the BC for the beam being clamped : $v(\vec{x}) = \vec{0} \quad \forall x \in \partial\Omega_U \equiv x = 0)$

The continuity conditions will be fundamental here as we will in a first time consider an x depending surface. We study the general case to stick with the most general 1D case possible.

We will in the following consider a surface of the form :

$$S(x) = \begin{cases} S1, & \forall x \in [0,K] \\ S2, & \forall x \in [K,L] \end{cases}$$

with K, S1, S2 $\in R$ and $K \in [0,L]$.

This concludes the strong formulation development. We see that we need to solve a differential equation to end up with the solution in displacement and in stress, applying the BC. We here below synthesise this strong formulation in our 1D case :

SF => To find $(\vec{u}, \bar{\bar{\sigma}}) \in U_{ad} \times \sum_{ad}$ with:

$$\begin{cases} (ES(x)u'(x))' + p(x) = 0 \\ ES(l)u'(l) = N(L) = F \\ u(0) = 0 \end{cases}$$

3.2 Weak formulation (WF)

Even though the strong formulation (SF) looks solvable analytically, it demands a few BC and requires smoothness. Solving the strong form (governing differential equations) is thus not always efficient and there may not be smooth (classical) solutions to a particular problem. This is true especially in the case of complex domains and/or different material interfaces. Moreover, incorporating boundary conditions is always a daunting task with solving strong forms directly. The requirement on continuity of field variables is much stronger.

In order to overcome the above difficulties, weak formulations are preferred. They reduce the continuity requirements on the approximation functions (or basis functions u(x)), thereby allowing the use of easy-to-construct and implement polynomials.

3.2.1 $SF \Rightarrow WF$

To get this weak formulation and justify that it solves the same problem, we will go from the strong one. Thereby, we consider the SF and we apply a test function to it. We thus multiply the SF by a test function v(x) and then integrate over the bar (i.e over the domain $\Omega \equiv [0,L]$).

We get the following formulation :

$$\begin{aligned} &\int_0^L ((ES(x)u'(x))' + p(x))v(x)dx &&= 0 \\ \Rightarrow &\int_0^L (ES(x)u'(x))'v(x)dx &&= -\int_0^L p(x)v(x)dx \end{aligned}$$

We now do an integration by part (IPP) on u(x) on the left hand term, developing as well the integrals for our two surfaces :

$$\int_0^L (ES(x)u'(x))'v(x)dx = -\int_0^L p(x)v(x)dx$$
$$\int_0^{K^-} (ES_1u'(x))'v(x)dx + \int_{K^+}^L (ES_2u'(x))'v(x)dx = -\int_0^L p(x)v(x)dx$$

$$\Rightarrow [(ES_1u'(x))v(x)]_0^{K^-} + [(ES_2u'(x))v(x)]_{K^+}^L$$
$$-\int_0^{K^-} (ES_1u'(x))v'(x)dx - \int_{K^+}^L (ES_2u'(x))v'(x)dx = -\int_0^L p(x)v(x)dx$$

$$\Rightarrow ES_1(u'(K^-)v(K^-) - u'(0)v(0)) + ES_2(u'(L)v(L) - u'(K^+)v(K^+))$$
$$-\int_0^{K^-} ES_1u'(x)v'(x)dx - \int_{K^+}^L ES_2u'(x)v'(x)dx = -\int_0^L p(x)v(x)dx$$

Before going any further, one must define the fields in which our functions u(x) and v(x) are defined. We in fact search for our solution u(x) such that it is kinematically admissible and for our overall field set to be a vectorial space.

We then define 2 admissible fields, one for u(x) and the other one for v(x)

$$u \in C = (u : [0,L] \to R, \quad \text{smooth} \quad || \quad u(0) = 0)$$
$$v \in C^0 = (v : [0,L] \to R, \quad \text{smooth} \quad || \quad v(0) = 0)$$

You can already note that these two domains C and C^0 are equivalent such that they are both vectorial spaces (this stands as we have homogeneous conditions).

From our integral developed above, and using these spaces C and C^0 we finally get :

$$N(K^-)v(K^-) - N(0)v(0) + N(L)v(L) - N(K^+)v(K^+)$$
$$-\int_0^{K^-} ES_1u'(x)v'(x)dx - \int_{K^+}^L ES_2u'(x)v'(x)dx = -\int_0^L p(x)v(x)dx$$
$$\Rightarrow N(K^-)v(K^-) - N(K^+)v(K^+) + N(L)v(L)$$
$$-\int_0^{K^-} ES_1u'(x)v'(x)dx - \int_{K^+}^L ES_2u'(x)v'(x)dx = -\int_0^L p(x)v(x)dx$$

We recall that the SF defines the continuity inside the bar, and so in K, and that we have the BC $N(L) = F$. Replacing and changing sides some elements :

N(K$^-$)$v(K^-) - N(K^+)v(K^+) = [N]_{(K)} = 0$ and $\quad N(L)v(L) = Fv(L)$

$$\Rightarrow \underbrace{\int_0^{K^-} ES_1u'(x)v'(x)dx + \int_{K^+}^L ES_2u'(x)v'(x)dx}_{a(u,v)} = + \underbrace{\int_0^L p(x)v(x)dx + Fv(L)}_{l(v)}$$

This last formulation stands as the WF, and we deduced it from the SF, by redefining as mentioned the admissible spaces, while sticking to the same initial problem.

To finish with the WF, we synthesize the purpose of it here below :

WF => To find $\vec{u} \in C$ such that :

$$\begin{aligned} a(\vec{u},\vec{v}) &= l(\vec{v}), \qquad && \forall \vec{v} \in \ \mathrm{C}^0 \\ \iff a(\vec{u},\vec{v}-\vec{u}) &= l(\vec{v}-\vec{u}), && \forall \vec{v} \in \ \mathrm{C} \end{aligned}$$

with $a(\vec{u},\vec{v}) = \int_0^{K^-} ES_1 u'(x)v'(x)dx + \int_{K^+}^{L} ES_2 u'(x)v'(x)dx$

and $l(\vec{v}) = \int_0^L p(x)v(x)dx + Fv(L)$

The advantages of the weak formulation is that we get an integration formulation, and we pass from a second order equation to a first order equation.

3.2.2 $WF \Rightarrow SF$

Before we discuss the variational formulation, we will show the implication $WF \Rightarrow SF$. To do so we have at use the WF as well as the fields C and C^0 defining it. The process we will go through is :

- i. to do an integration by part on v(x)
- ii. to apply the **F**undamental **L**emma of the **C**alculus of **V**ariations (LFCV)
- iii. to conclude on the BC and equivalence of kinematically and statically admissible fields.

i.

$$\int_0^{K^-} ES_1 u'(x)v'(x)dx + \int_{K^+}^{L} ES_2 u'(x)v'(x)dx \qquad = \int_0^L p(x)v(x)dx + Fv(L)$$

$$\begin{aligned} \Rightarrow & [(ES_1u'(x))v(x)]_0^{K^-} + [(ES_2u'(x))v(x)]_{K^+}^{L} \\ & - \int_0^{K^-} (ES_1u'(x))'v(x)dx - \int_{K^+}^{L} (ES_2u'(x))'v(x)dx \qquad = \int_0^L p(x)v(x)dx + Fv(L) \end{aligned}$$

$$\begin{aligned} \Rightarrow & N(K^-)v(K^-) - N(0)v(0) + N(L)v(L) - N(K^+)v(K^+) \\ & - \int_0^{K^-} (ES_1u(x))'v(x)dx - \int_{K^+}^{L} (ES_2u'(x))'v(x)dx \qquad = \int_0^L p(x)v(x)dx + Fv(L) \end{aligned}$$

Yet, from C^0 we have that the function v(x) is smooth throughout the bar, and so that $v(K^-) = v(K^+) = v(K)$. Recalling that $v(0) = 0$ we get :

$$
\begin{aligned}
&(N(K^-) - N(K^+))v(K) + N(L)v(L)\\
&-\int_0^{K^-} (ES_1u'(x))'v(x)dx - \int_{K^+}^{L} (ES_2u'(x))'v(x)dx \quad = \int_0^L p(x)v(x)dx + Fv(L)
\end{aligned}
$$

$$
\begin{aligned}
&[N]_{(K)}v(K) + (N(L) - F)v(L) -\\
&\int_0^{K^-} (ES_1u'(x))'v(x)dx - \int_{K^+}^{L} (ES_2u'(x))'v(x)dx \qquad = \int_0^L p(x)v(x)dx
\end{aligned}
$$

ii.

As v(x) is any test function, we will first consider v(x) = 0 at the right bar side, and then do the same consideration to the left.

These manipulations together with the application of **LFCV** lead to :

$$
\begin{cases}
\text{E S}_1u''(x) + p(x) = 0,\ \forall x \in [0,K^-]\\
\text{E S}_2u''(x) + p(x) = 0,\ \forall x \in [K^+,L]\\
[\text{N}]_{(K)} = 0\\
N(L) = F\\
u(0) = 0
\end{cases}
$$

iii.

In the end, we have for the SF formulation the BC in space $u(0) = 0$ from the kinematic field C, as well as the stress BC $N(L) = F$ in $x = L$ and the continuity over the bar. We as well get back the local form of the displacement over the beam, i.e, the strong formulation.

To have shown the double implication makes reliable and consistent the use of the WF. We will now now see that this expression is equivalent to that of the variational formulation (VF), giving as well a physical comprehension of our terms.

3.3 Variational formulation (VF)

To illustrate the variational formulation, the finite element equations of the bar will be derived from the Minimum Potential Energy principle. On wards we will consider only one surface S in the calculation.

$$
J(u) = \int_0^L \frac{ES}{2}u'(x)^2 - \int_0^L p(x)u(x)dx + Fv(L) \tag{1}
$$

In order to get an expression of the variational formulation, we use the potential energy as in equation 1 and we try to find the value of u that minimizes this energy.

$$
J(u(x)+hv(x)) = \int_0^L \frac{ES}{2}(u'(x)+hv'(x))^2dx - \int_0^L p(x)(u(x)+hv(x))dx - F(x)(u+hv)(L)
$$

$$\frac{dJ(u(x)+hv(x))}{dh} = \int_0^L ES(u'(x)+hv'(x))v'(x)dx - \int_0^L p(x)v(x)dx - Fv(L)$$

We evaluate the derivative when h is close to 0 :

$$\left.\frac{dJ(u(x)+hv(x))}{dh}\right|_{\mathrm{h=0}} = \int_0^L ESu'v'dx - \int_0^L p(x)vdx - Fv(L)$$

The function J(u) is minimized when its derivative is equal to 0 so we can end up with variational formulation :

$$\int_0^L ESu'v'dx - \int_0^L p(x)vdx - Fv(L) = 0$$

Note that to get $\frac{dJ(u(x)+hv(x))}{dh}$ is a necessary condition to get a minimum, but not a sufficient one, as it could be as well a maximum.

To in fact conclude on a minimum one would need to need to show that the function is only convex.

3.3.1 $VF \Rightarrow WF$

In this setion we will be exploiting the variational formulation to show how it is equivalent to the weak formulation .

We define the space of cinematically admissible displacements : $\mathcal{C}$ = {u(x), smooth | u(0)=0 } and $\mathcal{C}_0$ = {v(x), smooth | v(0)=0 } = $\mathcal{C}$ in this problem.

Let's find u$\in \mathcal{C}$ such that $J(V) - J(u) \geq 0$, with $V = u + hv$ ($V \in \mathcal{C}$,$v(x) \in \mathcal{C}_0$ and $h \in R$): By Taylor expansion we have :

$$J(u(x)+hv(x)) - J(u(x)) = \underbrace{\frac{dJ(u(x)+hv(x))}{dh}|_{\mathrm{h=0}}}_{J'(u(x))(v(x))} *h + o(h)$$

If we do the directional derivative of J in u in the direction of v and from the variational form we have :

$$(J'(u(x))(v(x)) * h \geq 0 \ \forall h \in R \implies J'(u(x))(v(x)) = 0 \ \forall v(x) \in \mathcal{C}_0$$

Now let's calculate the first derivative.

we have:

$$J(u) = \int_0^L \frac{ES}{2} u'(x)^2 - \int_0^L p(x)u(x)dx$$

and

$$J'(u(x)+hv(x)) = \frac{dJ(u(x)+hv(x))}{dh}|_{\mathrm{h=0}}$$

Thus:

$$(i) \ \ J(u(x)+hv(x)) = \int_0^L \frac{ES}{2}(u'(x)+hv'(x))^2 dx - \int_0^L p(x)(u(x)+hv(x))dx$$

$$(ii)\quad \frac{dJ(u(x)+hv(x))}{dh} = \int_0^L ES(u'(x)+hv'(x))v'(x)dx - \int_0^L p(x)dx$$

$$(iii)\quad \frac{dJ(u(x)+hv(x))}{dh}|_{\mathrm{h=0}} = \underbrace{\int_0^L u'(x)v'(x)ESdx}_{a(u,v)} = \underbrace{\int_0^L p(x)v(x)dx}_{l(v)}\,,\ \forall v(x) \in \mathcal{C}_0$$

We can see that in the end as $J'(u(x))(v(x)) = 0\ \forall v(x) \in \mathcal{C}_0 \implies a(u,v) = l(v)$, hence the equivalence to the weak formulation.

4 Variational approximations: the Galërkin method

Steps of the Galërkin's method :

1. Find the analytical solution
2. Make an approximate resolution of the Weak Formulation using the Galerkin approach
3. Define a basis function Φ

The main idea of the Ritz-Galerkin method is to build an approximation of $u(x)$ by projection in a space of functions of finite dimension.

$$u(x) \simeq u_n(x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

Where $\phi_i(x)$ is a basis function. We admit the same rule for the approximation of $v(x)$

$$v(x) \simeq v_n(x) = \sum_{i=1}^{n} \hat{a}_i \phi_i(x)$$

Now we can project the weak formulation, in order to find $u_n(x)$ such as:

$$a(u_n,v_n) = l(v_n) \quad \forall \quad v_n \in C_0$$

we denote in matrix form the projection :

$$u(x) \simeq u_n(x) = \sum_{i=1}^{n} a_i \phi_i(x) = \{\hat{a}\}\{\phi\} = \{a\}^t\{\phi\} = \{\phi\}^t\{a\}$$

with :

$$\{a\} = [a_1,a_2,...,a_n]^t \quad \text{and} \quad \{\phi\} = [\phi_1,\phi_2,...,\phi_n]^t$$

And then using the variational formulation we get:

$$a(u_n,v_n) = \{a\}^t\{K\}\{\phi\} \quad \text{and} \quad l(v_n) = \{\hat{a}\}^t\{F\}$$

With K the stiffness matrix and F a vector of generalised form.

$$K_{ij} = a(\phi_i,\phi_j) \quad \text{and} \quad F_i = l(\phi_i)$$

And now to find the solution, we have to solve the matrix solution such as:

$$\{a\}^t\{K\}\{a\} = \{\hat{a}\}^t\{F\} \quad \Leftrightarrow \quad \{K\}\{a\} = \{F\}$$

so we will finally find :

$$\{a_{sol}\} = \{K\}^{-1}\{F\} \quad \text{then} \quad u_n(x) = \{a_{sol}\}\{\phi\}$$

by choosing the basic conditions :

$$\phi_0 = 1 \quad \phi_1 = \xi \quad \phi_1 = \xi^2 \quad \xi = \frac{x}{L}$$

$$u_n(x) = \sum_{i=1}^{n} a_i \phi_i(x) = a_0\phi_0 + a_1\phi_1 + a_2\phi_2 = a_0 + a_1\xi + a_2\xi^2$$

$$u(x=0) = 0 \quad \Rightarrow \quad a_0 = 0$$

$$u(x=L) = 0 \quad \Rightarrow \quad a_1 = -a_2$$

so finally we get the solution:

$$v_n(x) = \hat{a}_1(\xi - \xi^2) \quad u_n(x) = a_1(\xi - \xi^2) + \bar{u}\xi \quad \in C$$

5 The finite element method

In the previous part we describes the the Galërkin method, in which we search for the approximated solution in form of a combination of basis functions. However, this method has some limits, for example, we should calculate integration, which can be complex for computer implementation, and the complexity of defying basis functions to verify of boundary conditions in 2D and 3D cases.

In this part, we introduce the finite element method. The general idea is to separate the studied domain into small 'elements', and each element has several points that we called 'nodes'. We calculate the interested properties on this nodes. The solution then can be approximated by property values on this nodes. This kind of separation can turn integration into summation, which computers are more capable to calculate.

The description above may be not clear enough, we will then illustrate the concept by applying the method on a 1D problem, the traction of a bar.

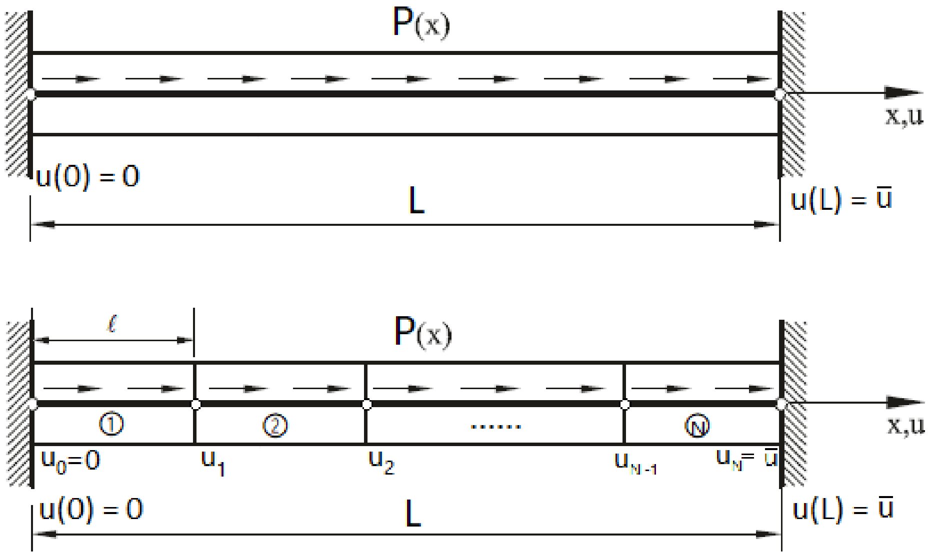


FIG. 1 – *Traction of a bar and elements*

5.1 Discretization of a bar and approximation of the displacement field

The figure1 shows how to separate a bar into N parts, called 'element', each element has 2 nodes. We note the x position as x_0, x_1, ..., x_n, ...,x_N. For the n^{th} element, the 2 nodes are situated in x_{n-1} and x_n. The displacements on these 2 points are noted $u_{n-1} = u(x_{n-1})$ and $u_n = u(x_n)$.

Each element can have different length, but to simplify the problem, we consider they are of the same length l ($x_{i+1} - x_i = l$).

Now we focus on one element:

Not to loss generality, we consider a ***reference element***, the nodes' positions of which is described in another coordinate, situated respectively in 0 and 1 (figure2). To change from the original coordinate, we simply need to do a coordinate transformation:

$$\xi = \frac{x - x_{n-1}}{x_n - x_{n-1}} = \frac{x - x_{n-1}}{l} \qquad x \in [x_n, x_{n-1}]$$

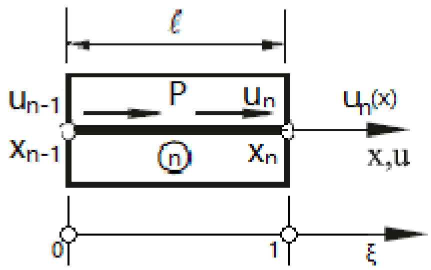


FIG. 2 – *Illustration of one element*

Reciprocally, we can change from position description by ξ back to the original coordinate.

$$x = N_1(\xi)x_{n-1} + N_2(\xi)x_n \qquad N_1(\xi) = 1 - \xi \qquad N_2(\xi) = \xi$$

The functions $N_1(\xi)$ and $N_2(\xi)$ are called ***shape functions***.

Remain to the analysis on the reference element, suppose we know the results of the displacement at points $\xi = 0$ and $\xi = 1$, noted as $\bar{u}_0$ and $\bar{u}_1$ respectively. How to represent the displacement in between those 2 points ? We can approximate it by a linear form: $\bar{u}(\xi) = a + b\xi$, $\xi \in [0,1]$. As we know the values of $\bar{u}(\xi)$:

$$\left\{ \begin{array}{l} a + b = \bar{u}_1 \\ a = \bar{u}_0 \end{array} \right. \longrightarrow \left\{ \begin{array}{l} a = \bar{u}_0 \\ b = \bar{u}_1 - \bar{u}_0 \end{array} \right. \longrightarrow \begin{array}{ll} \bar{u}(\xi) & = \bar{u}_0 + (\bar{u}_1 - \bar{u}_0)\xi \\ & = (1-\xi)\bar{u}_0 + \xi\bar{u}_1 \end{array}$$

We can note $\bar{u}(\xi)$ as $\bar{u}(\xi) = \Phi_0(\xi)\bar{u}_0 + \Phi_1(\xi)\bar{u}_1$ with $\Phi_0(\xi) = 1 - \xi$ and $\Phi_1(\xi) = \xi$. The functions $\Phi_0(\xi)$ and $\Phi_1(\xi)$ are the ***basis functions***. Similar to the Galërkin method, we represent the displacement as a combination of the basis functions, but just for one element.

Curiously, the shape functions and the basis functions have the same form, this can be partly explained by the fact that they all verify the condition that at one extremity ($\xi = 0$ or 1), the function value should be 0 and at the other the value should be 1.

5.2 Matrix form equation for one element

We still consider the element in the previous part (figure2), but this time focus on the resolution of its deformation.

The weak formulation that we discussed in the second part is used:

Find $u \in U_{ad}$ that verify the following relation:

$$\underbrace{\int_{x_{n-1}}^{x_n} ESu'(x)v'(x)dx}_{a(u,v)} = \underbrace{\int_{x_{n-1}}^{x_n} p(x)v(x)dx}_{l(v)} \qquad \forall v \in U_{ad}^0 \tag{2}$$

To calculate the values of u_{n-1} and u_n, extremity displacements of the n^{th} element, we inject the approximated displacements $\bar{u_n}(\xi)$ in the formulation(2):

$$u'(x) \approx \frac{\partial \bar{u}_n(\xi(x))}{\partial x} = u_{n-1}\frac{d\Phi_0(\xi)}{d\xi}\frac{d\xi}{dx} + u_n\frac{d\Phi_1(\xi)}{d\xi}\frac{d\xi}{dx}$$

As $\xi = \frac{x - x_{n-1}}{l}$, $\frac{d\xi}{dx} = \frac{1}{l}$. Finally we have:

$$u'(x) \approx \frac{1}{l}(-u_{n-1} + u_n)$$

If we present the displacements u_{n-1} and u_n in form of a vector $\{u_e\} = {}^t[u_{n-1}\ u_n]$:

$$u(x) \approx [\Phi_0(\xi)\ \Phi_1(\xi)] \begin{bmatrix} u_{n-1} \\ u_n \end{bmatrix} = [\Phi_0(\xi)\ \Phi_1(\xi)]\{u_e\} = {}^t\{u_e\} \begin{bmatrix} \Phi_0(\xi) \\ \Phi_1(\xi) \end{bmatrix}$$
$$u'(x) \approx [Be]\{u_e\},\ \ in\ which:\ [Be] = \frac{1}{l}[\frac{d\Phi_0(\xi)}{d\xi}\ \frac{d\Phi_1(\xi)}{d\xi}]$$

Idem for the test function $v(x)$, $v'(x) \approx [Be]\{v_e\}$, with $\{v_e\} = {}^t[v_{n-1}\ v_n]$.
Using matrix representation in the equation(2), we have:

$$\begin{aligned} a(u_e,v_e) = \int_{x_{n-1}}^{x_n} ESu'(x)v'(x)dx &\approx \int_{x_{n-1}}^{x_n} \frac{ES}{l^2}{}^t([Be]\{u_e\})[Be]\{v_e\}dx \\ &= {}^t\{u_e\}\left(\int_0^1 \frac{ES}{l^2}{}^t[Be][Be]d(x_{n-1} + l\xi)\right)\{v_e\} \\ &= {}^t\{u_e\}\left(\int_0^1 \frac{ES}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi\right)\{v_e\} \\ &= {}^t\{u_e\}[K_e]\{v_e\} = {}^t\{v_e\}[K_e]\{u_e\} \end{aligned}$$

$[K_e]$ is the ***local stiffness matrix***:

$$[K_e] = \int_0^1 \frac{ES}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi \tag{3}$$

We do the same reasoning for $l(v)$:

$$\begin{aligned} l(v_e) = \int_{x_{n-1}}^{x_n} P(x)v(x)dx &\approx \int_0^1 p(x(\xi))^t\{v_e\}\begin{bmatrix} \Phi_0(\xi) \\ \Phi_1(\xi) \end{bmatrix} ld\xi \\ &= {}^t\{v_e\}\int_0^1 lP(x(\xi))\begin{bmatrix} \Phi_0(\xi) \\ \Phi_1(\xi) \end{bmatrix} d\xi \\ &= {}^t\{v_e\}\{F_e\} \end{aligned}$$

$\{F_e\}$ is the ***local force vector*** (or ***local load vector***):

$$\{F_e\} = \int_0^1 lP(x(\xi))\begin{bmatrix} \Phi_0(\xi) \\ \Phi_1(\xi) \end{bmatrix} d\xi \tag{4}$$

The the expression $a(u_e,v_e) = l(v_e)$ becomes:

$$\begin{aligned} &{}^t\{v_e\}[K_e]\{u_e\} = {}^t\{v_e\}\{F_e\} \qquad \forall\{v_e\} \\ &\Rightarrow [K_e]\{u_e\} = \{F_e\} \end{aligned} \tag{5}$$

By resolving the matrix form equation(5), we get the values of the displacements on 2 nodes of the considered element, then we can use the method described in part 5.1 to approximate displacement on very point of the element.

5.3 Assembly of matrices and vectors

Obviously it's not enough to solve the problem locally, as the equilibrium equation relates one element to the others and we also need to consider the boundary conditions.

To further explain how to use the finite element method, we try to resolve the problem presents in figure 1 with 2 elements:

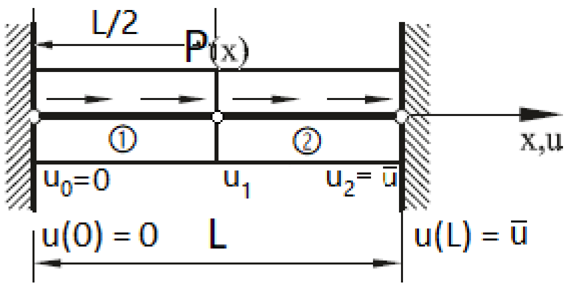


FIG. 3 – *Axially loaded bar with fixed extremity displacements*

Here we have an axially loaded bar of length L. To simplify the calculation, we consider the load is constant ($P(x) = P$). One extremity is fixed on the axis origin and the other is known to have a displacement of $\bar{u}$. We divide the bar into 2 elements of the same length $L/2$.

From the part 5.2, we observe that the expression of stiffness matrix $[K_e]$ (equation 3) does not depend on the position x, so $[K_e]$ have the same form for the 2 elements. Because they are defined for different element so as different nodes, we note them separately as $[K_e]^{(1)}$ and $[K_e]^{(2)}$:

$$[K_e]^{(1)} = \frac{2ES}{L} \begin{array}{cc} x_0 & x_1 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} x_0 \\ x_1 \end{array} \qquad [K_e]^{(2)} = \frac{2ES}{L} \begin{array}{cc} x_1 & x_2 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \end{array}$$

Usually the calculation of the load vector is more complicated but in this case, as we have constant axial load, we have:

$$\{F_e\} = \int_0^1 \frac{PL}{2} \begin{bmatrix} \Phi_0(\xi) = 1 - \xi \\ \Phi_1(\xi) = \xi \end{bmatrix} d\xi = \frac{PL}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In this example the load vector doesn't depend on position, but for the same reason as the stiffness matrix, we note separately for the 2 elements, $\{F_e\}^1$, $\{F_e\}^2$:

$$\{F_e\}^1 = \frac{PL}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{array}{c} x_0 \\ x_1 \end{array} \qquad \{F_e\}^2 = \frac{PL}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{array}{c} x_1 \\ x_2 \end{array}$$

Assembly of global matrix and vector The global equilibrium equation can be written into a matrix form like equation(5), the only difference is that the stiffness matrix and the load vector is written for the whole structure. In our example, the local matrix and vector only consider 2 nodes of the element, but the global matrix and global vector consider all 3 nodes of the structure.

To better explain the assembly of the global matrix and vector, we can firstly enlarge a local matrix/vector by leaving places for others nodes, but the local equation remain valid only for the element.

For the first element:

$$[K_e]^1\{u\} = \{F_e\}^1$$

$$\frac{2ES}{L}\begin{array}{cc} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \end{array} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \frac{PL}{4}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array}$$

For the second element:

$$[K_e]^2\{u\} = \{F_e\}^2$$

$$\frac{2ES}{L}\begin{array}{cc} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \end{array} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \frac{PL}{4}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array}$$

As these 2 are linear systems, we can add them together to form the global equation in matrix form:

$$\frac{2ES}{L}\begin{array}{cc} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \end{array} \cdot \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \frac{PL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array}$$

$$[K] \cdot \{u\} = \{F\} \tag{6}$$

$[K]$ is the ***global stiffness matrix*** and $\{F\}$ is the ***global load vector***:

$$[K] = \frac{2ES}{L}\begin{array}{cc} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \end{array} \qquad \{F\} = \frac{PL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array}$$

We observe that $det([K]) = 0$, the equation(6) itself doesn't have a unique solution for $\{u\}$, but if we add boundary conditions, we can find the unique solution.

Let's consider the boundary condition of our example: $u(x = 0) = 0$ and $u(x = L) = \bar{u}$, so the displacement vector can be written as $\{u\} = {}^t[u_0 = 0 \quad u_1 \quad u_2 = \bar{u}] = {}^t[0 \quad u_1 \quad \bar{u}]$. The equation(6) becomes:

$$\frac{2ES}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_1 \\ \bar{u} \end{bmatrix} = \frac{PL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

We cross off the lines of nodes x_0 and x_1 because they are already defined. Finally we have only $2ES \cdot u_1/L = PL/2$, which gives us the value of u_1. Imagining we divide the bar into more elements, we will have a matrix form equation for solving more nodes' values.

Different boundary conditions In the previous example, we have Dirichlet condition on 2 extremities, but if we have Neumann condition (defined force), how would it change ?

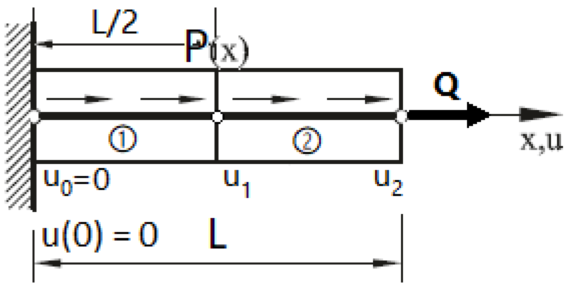


FIG. 4 – *Axially loaded bar with a force at the end point*

We represent this situation in figure(4)

The external forces don't change the properties material, so the stiffness matrix remains the same. However, the load vector, representing the external loads, has to change. In this example, compered to the previous one, we added a force to the end point (position $x = L$, noted as x_2), so it's enough to add this contribution to the load vector in the previous example. As for the displacement vector, we only know the value of $x_0 = 0$, so it's a vector of 2 unknown variables (u_1, u_2).

Finally, the matrix form equation becomes:

$$\frac{2ES}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} = \frac{PL}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix}$$

We get the values of u_1 and u_2 by solving this equation.

6 Conclusions

In this report, by solving the problem on the deformation of a bar under traction, we introduced the basic theory and concepts of 1D finite element method. We firstly introduced the the equations: strong formulation and weak formulation and variational equation, as well as a brief explanation on how to find them. We also proved the equivalence between these equations. To calculate the deformation, 2 methods are introduced in this report: the Galerkin method and the finite element method, both of which use the weak formulation to find an approximate solution. The difference is that the first one use a finite space of admissible function to approximate the solution, and the other separate the structure on elements, calculating values on nodes to achieve the approximation.

By doing this report, we revise the basic ideas of resolution of material deformation and concepts of 2 approximation methods. For further exploration, we continue our study on finite element methods for more complex structure in 2D and 3D .

Authors contributions :

In this report, we divided the work between the four of us but we also worked together as a team and helped one another to understand the problem. Ackbarally wrote the abstract(part 1) and worked on part 3(different types of formulations part) together with Duvivier Valentin who made the weak and strong formulation as well as the design on part 2. Richard wrote the part 2 (model problem) and 4 (Galerkin method) and Jiayu Wang for the part 5 (finite element method) and the conclusion.

All four of us revised the manuscript. All authors gave final approval for publication and agree to be held accountable for the work performed therein.