Numerical solutions of differential equations

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Lecture 10

Entropy solutions

Applications of the Entropy Condition

Entropy - Consequence for numerical schemes

Recall: von Neumann analysis for linear problem with periodic BC revealed:

Example ($\partial_t u + \partial_x u = 0$ - Advection equation)

Consider forward Euler and central differences for the advection equation:

$$Q_j^{n+1} = Q_j^n + \frac{1}{2}\lambda_{\mathrm{CFL}}(Q_{j+1}^n - Q_{j-1}^n), \qquad \lambda_{\mathrm{CFL}} = \frac{\Delta t}{\Delta x}.$$

Again, here $m = M = 1$, but $b_{-1} = -\frac{\lambda_{\mathrm{CFL}}}{2}$, $b_0 = 1$, $b_1 = \frac{\lambda_{\mathrm{CFL}}}{2}$. Then

$$g_k(\Delta t, \Delta x) = -\frac{\lambda_{\mathrm{CFL}}}{2} e^{-ik\Delta x} + 1 + \frac{\lambda_{\mathrm{CFL}}}{2} e^{ik\Delta x} = 1 + \lambda_{\mathrm{CFL}} \mathrm{i} \sin(k\Delta x).$$

Hence,

$$\max_k |g_k(\Delta t, \Delta x)| = \max_k \sqrt{1 + \lambda_{\mathrm{CFL}}^2 \sin(k\Delta x)^2} > 1,$$

and the method is unstable for all fixed λ_{CFL}.

Entropy - Consequence for numerical schemes

General (nonlinear) case with convex flux f:

central differences are inappropriate to discretize the problem.

Example

Let $f'' > 0$ with $f'(-1) < 0 < f'(1)$ and $f(1) = f(-1)$, e.g. $f(u) = u^2$.

Consider initial value $v_0(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$.

Central difference discretization:

$$Q_j^{n+1} := Q_j^n - \frac{\Delta t}{2\Delta x}\left(f(Q_{j+1}^n) - f(Q_{j-1}^n)\right),$$

where $Q_j^n \approx u(j\Delta x, n\Delta t)$ on an equidistant mesh.

Entropy - Consequence for numerical schemes

Example (- Part 2)

Let $f'' > 0$ with $f'(-1) < 0 < f'(1)$ and $f(1) = f(-1)$.

Consider initial value $v_0(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$.

Hence: $f(Q_j^0) = f(\pm 1) = f(1) \quad \Rightarrow \quad f(Q_{j+1}^0) - f(Q_{j-1}^0) = 0$ for all j.
Since

$$Q_j^{n+1} := Q_j^n - \frac{\Delta t}{2\Delta x}(f(Q_{j+1}^n) - f(Q_{j-1}^n)),$$

we conclude

$$Q_j^n = \begin{cases} 1, & j > 0 \\ -1, & j < 0 \end{cases} \qquad \text{for all } n \in \mathbb{N}.$$

Entropy - Consequence for numerical schemes

Example (- Part 3)

Let $f'' > 0$ with $f'(-1) < 0 < f'(1)$ and $f(1) = f(-1)$ and

initial value $v_0(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$.

Central difference approximation:

$$Q_j^n = \begin{cases} 1, & j > 0 \\ -1, & j < 0 \end{cases} \qquad \text{for all } n \in \mathbb{N}.$$

Question:

Is $u(x,t) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$ also Lax-Entropy solution?

Entropy - Consequence for numerical schemes

Example (- Part 4)

Is $u(x,t) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$ also Lax-Entropy solution?

1. Verify Rankine-Hugoniot jump condition:

$$s = 0, \quad f(u_l) - f(u_r) = 0 \quad \Rightarrow \quad (u_l - u_r)s = f(u_l) - f(u_r)$$

$\Rightarrow$ u is weak solution.

2. Verify Lax entropy condition:

We have $f'(u_l) = f'(-1)$ and $f'(u_r) = f'(1)$. Hence:

$$f'(-1) = f'(u_l) < \underbrace{s}_{=0} < f'(u_r) = f'(1) \quad \text{“Contradiction”.}$$

$\Rightarrow$ u does not fulfill Lax entropy condition.

The central difference scheme does not produce an entropy solution!

Entropy - Consequence for numerical schemes

Observe:

If $f'' > 0$ with $f'(-1) < 0 < f'(1)$ and $f(1) = f(-1)$, and for initial value $v_0(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$,

the upwind scheme

$$Q_j^{n+1} := Q_j^n - \frac{\Delta t}{\Delta x}(f(Q_j^n) - f(Q_{j-1}^n)),$$

suffers from the same problem.

Entropy solutions to the Riemann-problem

Riemann-problem.

Let $f \in C^2(\mathbb{R})$ be a convex flux, i.e. $f'' > 0$, and let

$$v_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

We seek the entropy solution $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ to

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{and} \qquad u(\cdot, 0) = v_0.$$

The equation is understood in the weak sense!

Can we explicitly state the entropy solution to this problem?

Entropy solutions to the Riemann-problem

Observation:

Let u be a weak solution to $\partial_t u + \partial_x f(u) = 0$.
Then, $u_\lambda(x,t) := u(\lambda x, \lambda t)$ is also a weak solution for all $\lambda > 0$. In particular:

$$u_\lambda(x,0) = u(\lambda x, 0) = \begin{cases} u_l, & \lambda x < 0 \\ u_r, & \lambda x > 0 \end{cases} = v_0(x).$$

Since the entropy solution is unique, it must hold $u(\lambda x, \lambda t) = u(x,t)$ for all $\lambda > 0$.

We therefore consider solutions of the form

$$u(x,t) = v\left(\frac{x}{t}\right).$$

Entropy solutions to the Riemann-problem

Let $u(x,t) = v\left(\frac{x}{t}\right)$.

In regions in which v is smooth we have:

$$\begin{aligned} 0 &= \partial_t u(x,t) + \partial_x f(u(x,t)) \\ &= -\frac{x}{t^2} v'\left(\frac{x}{t}\right) + f'\left(v\left(\frac{x}{t}\right)\right) v'\left(\frac{x}{t}\right)\frac{1}{t} \\ &= v'\left(\frac{x}{t}\right)\frac{1}{t}\left(f'\left(v\left(\frac{x}{t}\right)\right) - \frac{x}{t}\right). \end{aligned}$$

Hence for all $\xi = \frac{x}{t} \in \mathbb{R}$ we have either

$$f'(v(\xi)) - \xi = 0$$

or

$$v'(\xi) = 0.$$

Entropy solutions to the Riemann-problem

From the conservation we have

$$f'(v(\xi)) - \xi = 0 \qquad \text{or} \qquad \underline{v'(\xi) = 0} \qquad \text{for all } \xi = \frac{x}{t} \in \mathbb{R}.$$

We can distinguish 3 cases.

Case 1: $u_l = u_r$.

We have the classical solution $u(x,t) \equiv u_l$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Since it is a classical solution, it must be the unique entropy solution.

Entropy solutions to the Riemann-problem

From the conservation we have

$$f'(v(\xi)) - \xi = 0 \qquad \text{or} \qquad \underline{v'(\xi) = 0} \qquad \text{for all } \xi = \frac{x}{t} \in \mathbb{R}.$$

Case 2: $u_l > u_r$. Then

$$u(x,t) = \begin{cases} u_l & \text{for} \quad x < st \\ u_r & \text{for} \quad x > st \end{cases}$$

with $s = \frac{f(u_l) - f(u_r)}{u_l - u_r}$ is the unique entropy solution, because

$$f'(u_r) < \frac{f(u_l) - f(u_r)}{u_l - u_r} < f'(u_l).$$

This is the *Lax shock*.

Entropy solutions to the Riemann-problem

Ansatz: $u(x,t) = v(\frac{x}{t})$. From the conservation we have

$$\underline{f'(v(\xi)) - \xi = 0} \qquad \text{or} \qquad v'(\xi) = 0 \qquad \text{for all } \xi = \frac{x}{t} \in \mathbb{R}.$$

Case 3: $u_l < u_r$. Recall: discontinuous solutions are only possible if the Lax entropy condition is violated. Since $v'(\xi) = 0$ is therefore not everywhere possible, it must hold $f'(v(\xi)) - \xi = 0$. Hence:

$$f'(v(\xi)) = \xi \quad \Rightarrow \quad (f')^{-1}(f'(v(\xi))) = (f')^{-1}(\xi)$$

$$\Rightarrow \qquad v(\xi) := (f')^{-1}(\xi).$$

We obtain

$$u(x,t) := \begin{cases} u_l & \text{for} \quad \frac{x}{t} \leq f'(u_l) \\ v(\frac{x}{t}) & \text{for} \quad f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & \text{for} \quad \frac{x}{t} > f'(u_r) \end{cases}$$

This entropy solution is called rarefaction-wave.

Entropy solutions to the Riemann-problem

Summary. Riemann-problem: $\partial_t u + \partial_x f(u) = 0$.

Let $f \in C^2(\mathbb{R})$ be a convex flux, i.e. $f'' > 0$, and let

$$u(x,0) = v_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

The entropy solution $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ is given by:

If $u_l = u_r$:

$$u(x,t) \equiv u_l.$$

If $u_l > u_r$:

$$u(x,t) = \begin{cases} u_l & \text{for} \quad x < st \\ u_r & \text{for} \quad x > st \end{cases} \qquad \text{where} \quad s = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

If $u_l < u_r$:

$$u(x,t) := \begin{cases} u_l & \text{for} \quad \frac{x}{t} \leq f'(u_l) \\ (f')^{-1}(\frac{x}{t}) & \text{for} \quad f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & \text{for} \quad \frac{x}{t} > f'(u_r) \end{cases}$$

Entropy solutions to the Riemann-problem

Remarks:

- It is possible to show that these unique entropy solutions are <u>obtained by the viscosity limit</u>.
- For the Riemann-problem we have now an explicit formula to state the solutions for quite <u>general nonlinearities</u>.
- Unfortunately, it is not always that easy and we mostly need numerical methods.
- Can we use the Lax entropy condition for numerical solutions?