Lecture 4

Hyperbolic Equations of first order - Part 1

Characteristics

Characteristics

Important tool to understand Conservation Laws: Characteristics.

Consider a solution $u \in C^1(\mathbb{R} \times [0, \infty))$ to

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{and} \qquad u(x, 0) = v(x)$$

where $v \in C^1(\mathbb{R})$ and $f \in C^2(\mathbb{R})$.

Definition

For every $x_0 \in \mathbb{R}$ there is a maximum time $T > 0$ such that (by Picard-Lindelöf theorem) there is a unique solution γ to

$$\begin{aligned} \gamma'(t) &= f'\big(u(\gamma(t), t)\big) \quad \text{for } t \in (0, T), \\ \gamma(0) &= x_0 . \end{aligned}$$

The curve $\{(\gamma(t), t) \mid t \in [0, T]\}$ is called *Characteristic* of the Conservation law.

Characteristics - Example: Linear advection

Definition

For $x_0 \in \mathbb{R}$ and maximum time $T > 0$, let γ solves

$$\begin{aligned} \gamma'(t) &= f'(u(\gamma(t), t)) \quad \text{for } t \in (0, T), \\ \gamma(0) &= x_0. \end{aligned}$$

The curve $\{(\gamma(t), t) \mid t \in [0, T]\}$ is called *Characteristic*.

Example: $f(u) = u$. Let $u \in C^1(\mathbb{R} \times [0, \infty))$ solve

$$\partial_t u + \partial_x u = 0 \qquad \text{and} \qquad u(x, 0) = v(x)$$

Solution: $u(x, t) = v(x - t)$. Characteristic (since $f'(u) = 1$):

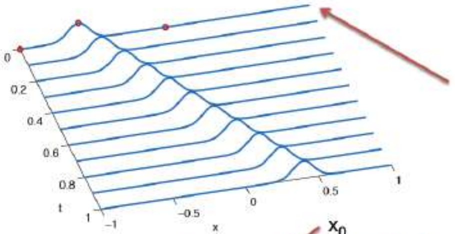
$$\gamma'(t) = 1 \qquad \Rightarrow \qquad \gamma(t) = t + x_0.$$

Hence

$$u(\gamma(t), t) = v(\gamma(t) - t) = v(x_0),$$

i.e. the characteristic is a linear curve on which $u(x, t)$ is constantly $v(x_0)$.

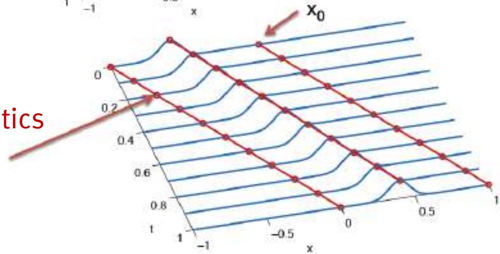
Characteristics - Example: Linear advection



$$\partial_t u + \mathbf{a}\partial_x u = 0,$$

$$u(x,0) = v(x),$$

$$u(x,t) = v(x - \mathbf{a}t)$$



Characteristics

$\gamma'(t) = \mathbf{a}.$

Sol. const. along

$\gamma(t) = x_0 + \mathbf{a}t$

with value $v(x_0)$.

Characteristics

Definition

For $x_0 \in \mathbb{R}$ and maximum time $T > 0$, let γ solves

$$\begin{aligned} \gamma'(t) &= f'\big(u(\gamma(t), t)\big) \quad \text{for } t \in (0, T), \\ \gamma(0) &= x_0. \end{aligned}$$

The curve $\{(\gamma(t), t) \mid t \in [0, T]\}$ is called *Characteristic*.

General properties:

- The map $t \mapsto u(\gamma(t), t)$ is always constant on $[0, T]$.
- The function γ has the form

$$\gamma(t) = f'\big(v(x_0)\big)\, t + x_0.$$

Next, we prove these properties.

Characteristics - Proof: $t \mapsto u(\gamma(t), t)$ is constant.

Differentiating yields

$$
\begin{aligned}
\partial_t\left[u(\gamma(t),t)\right] &= \partial_x u(\gamma(t),t)\,\gamma'(t) + \partial_t u(\gamma(t),t) \\
&\overset{\text{Def. }\gamma}{=} \partial_x u(\gamma(t),t)\,f'(u(\gamma(t),t)) + \partial_t u(\gamma(t),t) \\
&\overset{\text{Product rule}}{=} \partial_x f(u(\gamma(t),t)) + \partial_t u(\gamma(t),t) \\
&\overset{u\text{ solves PDE}}{=} 0.
\end{aligned}
$$

Hence, $t \mapsto u(\gamma(t), t)$ is constant, i.e.

$$u(\gamma(t), t) = v(x_0) \qquad \text{for all } t \in [0, T].$$

Characteristics - Proof: $\gamma(t) = f'\big(v(x_0)\big)\, t + x_0$.

Since $t \mapsto u(\gamma(t), t)$ is constant

$$\gamma'(t) \;=\; f'\big(u(\gamma(t), t)\big) \;=\; f'\big(u(\gamma(0), 0)\big) \;=\; f'\big(v(x_0)\big).$$

Hence with $\gamma(0) = x_0$ we have

$$\gamma(t) = f'\big(u_0(\gamma(0))\big)t + x_0.$$

Characteristics - Summary

Definition

For $x_0 \in \mathbb{R}$ and maximum time $T > 0$, let γ solves

$$\begin{aligned} \gamma'(t) &= f'\big(u(\gamma(t), t)\big) \quad \text{for } t \in (0, T), \\ \gamma(0) &= x_0 . \end{aligned}$$

The curve $\{(\gamma(t), t) \mid t \in [0, T]\}$ is called *Characteristic*.

General properties:

- The map $t \mapsto u(\gamma(t), t)$ is <u>always</u> constant on $[0, T]$.
- The function γ has the form $\gamma(t) = f'\big(v(x_0)\big)\, t + x_0$.

Why characteristics?

1. Understanding solutions.
2. Constructing solutions.

Characteristics - Understanding solutions

Some observations:

- the characteristic is explicitly known and solving is not necessary

$$\gamma(t) = f'(v(x_0))\, t + x_0;$$

- we know u is constant on $(\gamma(t), t)$.
- we only know that a characteristic exists (uniquely) for some maximum time T;
- in general, we do not know the value of T;
- what does that mean for $u(t)$ for $t \geq T$?