# Numerical solutions of differential equations

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**Course SF2521**, 7.5 ECTS, VT18

# General Finite Volumes Schemes of First Order

## Monotone schemes

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## Motivation

# Monotone schemes - motivation

We saw: a consistent scheme in conservation form is a order 2 approximation to

$$\partial_t u + \partial_x f(u) = \frac{\Delta x}{2}\,\partial_x(b(u)\partial_x u) \qquad (*)$$

with $$b(u) = \partial_1 g(u,u) - \partial_2 g(u,u) - \lambda(f'(u))^2, \quad \lambda = \frac{\Delta t}{\Delta x}.$$

Observations:

1. With $\Delta x \to 0$, the right hand side of $(*)$ is vanishing.
2. If $b(u) > 0$, the right hand side of $(*)$ can be interpreted as viscosity term. In this case, the scheme tries to mimic the viscosity limit and we can expect convergence to the unique entropy solution.

   The term $\frac{\Delta x}{2}\partial_x(b(u)\partial_x u)$ is called numerical viscosity.
3. Motivated by 2) we shall in the following consider schemes for which $g$ and $\lambda$ are chosen such that:

$$\partial_1 g(u,u) - \partial_2 g(u,u) - \lambda(f'(u))^2 > 0.$$

   For example fulfilled for: $\partial_1 g > 0, \quad \partial_2 g < 0$ and $\lambda$ sufficiently small.

# Monotone schemes - motivation

We want to chose $g$ and $\lambda$ such that:

$$\partial_1 g(u,u) - \partial_2 g(u,u) - \lambda (f'(u))^2 > 0.$$

What is a simple condition that guarantees this property?

Assume that $\partial_1 g > 0$ and $\partial_2 g < 0$. Then for any $u \in \mathbb{R}$:

$$\begin{aligned}
|f'(u)| &= \lim_{h\to 0} \frac{|f(u+h) - f(u)|}{h} = \lim_{h\to 0} \frac{|g(u+h,u+h) - g(u,u)|}{h} \\
&\le \lim_{h\to 0} \frac{|g(u+h,u+h) - g(u,u+h)| + |g(u,u+h) - g(u,u)|}{h} \\
&= |\partial_1 g(u,u)| + |\partial_2 g(u,u)| = \partial_1 g(u,u) - \partial_2 g(u,u).
\end{aligned}$$

Hence

$$(f'(u))^2 \le (\partial_1 g(u,u) - \partial_2 g(u,u))^2.$$

# Monotone schemes - motivation

We want to chose $g$ and $\lambda$ such that:

$$\partial_1 g - \partial_2 g - \lambda (f')^2 > 0.$$

What is a simple condition that guarantees this property?

Assume that $\partial_1 g > 0$ and $\partial_2 g < 0$, then we saw that

$$(f')^2 \leq (\partial_1 g - \partial_2 g)^2. \qquad (*)$$

If we pick the condition that

$$1 - \lambda (\partial_1 g - \partial_2 g) > 0$$

then we have

$$\begin{aligned} \partial_1 g - \partial_2 g - \lambda (f')^2 &\overset{(*)}{\geq} \partial_1 g - \partial_2 g - \lambda (\partial_1 g - \partial_2 g)^2 \\ &\geq \underbrace{(\partial_1 g - \partial_2 g)}_{>0} \underbrace{(1 - \lambda (\partial_1 g - \partial_2 g))}_{>0} > 0. \end{aligned}$$

# Monotone schemes - motivation

**Summary:**

We saw that if we pick $g$ and $\lambda$ such that:

1. $\partial_1 g > 0$,
2. $\partial_2 g < 0$,
3. $1 - \lambda \left(\partial_1 g - \partial_2 g\right) > 0$,

then it holds

$$\partial_1 g(u,u) - \partial_2 g(u,u) - \lambda (f'(u))^2 > 0,$$

which was the
condition so that the consistent scheme mimics the viscosity limit.

# Monotone schemes - motivation

Recall: For a numerical flux $g \in C^1(\mathbb{R} \times \mathbb{R})$ and with

$$\Phi(v, w, z) := w - \lambda[g(w, z) - g(v, w)],$$

the scheme in conservation form (with $\lambda = \frac{\Delta t}{\Delta x}$) reads

$$Q_j^{n+1} = \Phi(Q_{j-1}^n, Q_j^n, Q_{j+1}^n).$$

We observe

1. $\partial_v \Phi(v, w, z) = \lambda \partial_1 g(v, w) > 0 \quad \Leftrightarrow \quad \partial_1 g > 0,$
2. $\partial_z \Phi(v, w, z) = -\lambda \partial_2 g(w, z) > 0 \quad \Leftrightarrow \quad \partial_2 g < 0,$
3. $\partial_w \Phi(v, w, z) = 1 - \lambda(\partial_1 g(w, z) - \partial_2 g(v, w)) > 0$
   $\Leftrightarrow \quad 1 - \lambda\left(\partial_1 g - \partial_2 g\right) > 0.$

Hence: the desired properties are equivalent to
a scheme $\Phi$ that is monotone increasing in each argument.

# Monotone schemes

The previous considerations lead us to the following definition.

## Definition (Monotone scheme)

A scheme in conservation form written as

$$Q_j^{n+1} = \Phi(Q_{j-1}^n, Q_j^n, Q_{j+1}^n),$$

is called a monoton scheme if $\Phi$ is monotonically increasing in every argument, i.e.

- $\Phi(v_1, w, z) < \Phi(v_2, w, z)$ if $v_1 < v_2$ and all $w, z \in \mathbb{R}$.
- $\Phi(v, w_1, z) < \Phi(v, w_2, z)$ if $w_1 < w_2$ and all $v, z \in \mathbb{R}$.
- $\Phi(v, w, z_1) < \Phi(v, w, z_2)$ if $z_1 < z_2$ and all $v, w \in \mathbb{R}$.

If $\Phi \in C^1(\mathbb{R}^3)$, then monotone schemes fulfill $\partial_j \Phi > 0$ for $j = 1, 2, 3$.