Class Note 5: Aeroelastic response to aerodynamic gusts

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PURPOSE: *This notebook presents widely used gust models for aeroelastic design of aircraft.*

1 Gust response

1.1 Definition

- **Continuous gust**: stochastic phenomena described by a large band spectral range, such as atmospheric turbulence and meteorological wind.
- **Discrete gust**: short-time elastic response of the aircraft with limited frequential contents. Example: Vortices due to relief or aircraft wake, ascendance due to to thermal gradient or clouds, wind shear at landing.

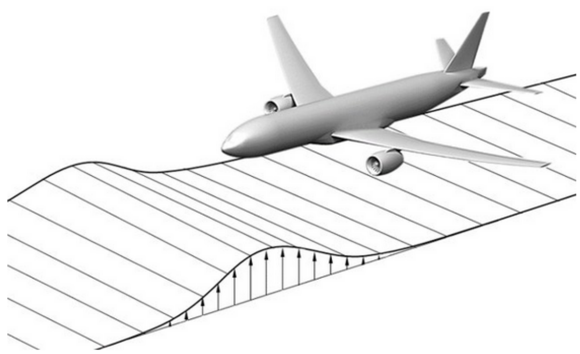

Fig. 1: Schematic of an aircraft impacting a 1D vertical gust. [1]

1.2 Governing equations

Compared to the equation of the dynamic investigated in Class Note 3, we now consider additionnal aerodynamic forces $\mathbf{F_{gust}}(w_G, t)$

$$\mathbf{M_S}\ddot{\mathbf{q}} + \mathbf{K_S}\mathbf{q} = \mathbf{F_{aero}}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, U_\infty) + \mathbf{F_{gust}}(w_G, t) \tag{1}$$

2 Extension of the Theodorsen solution to arbitrary airfoil motions

2.1 Downwash velocities

According to previous Class Notes, the PAPA system is described by

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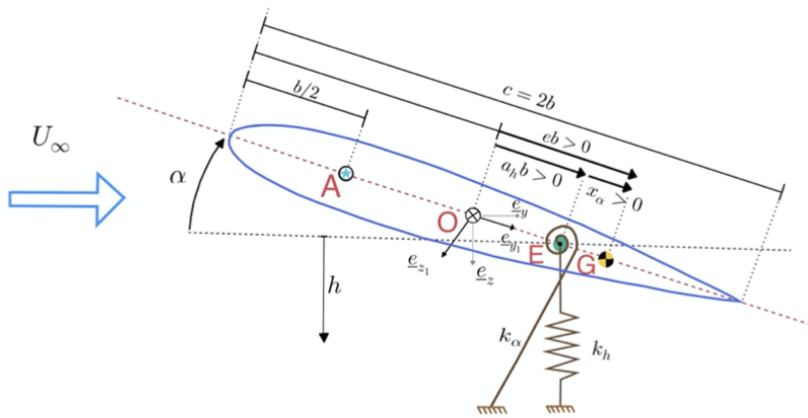

Figure 1: Schematic of the Pitch And Plunge Aeroelastic (PAPA) problem

the time-history of the PAPA flat plate is described by

$$z_a(x,t) = -h - \alpha(x - a_h b) \tag{2}$$

and the downwash velocity can be deduced from the

$$w_a(x, z = z_a, t) = \left.\frac{\partial \phi'}{\partial t}\right|_{z=z_a} = \frac{\partial z_a}{\partial t} + U_\infty \frac{\partial z_a}{\partial x}\ ; \quad -b \leq x \leq b \tag{3}$$

Thefore, we obtain

$$w_a(x) = -\dot{h} - \dot{\alpha}(x - a_h b) - U_\infty \alpha \tag{4}$$

Setting $x = x_{3/4c} = b/2$, gives the particular expression of the downwash at the 3/4 of chord

$$w_{3/4} = -\dot{h} - \dot{\alpha} b \left(\frac{1}{2} - a_h\right) - U_\infty \alpha \tag{5}$$

Now, we recall the Theodorsen solution for an harmonically oscillating flat plate in an incompressible flow (see Class Note 4)

$$L(t) = \underbrace{\pi \rho_\infty b^2 (\ddot{h} + U_\infty \dot{\alpha} - a_h b \ddot{\alpha})}_{L_{\Gamma=0}\text{:non-circulatory part}} + \underbrace{2\pi \rho_\infty U_\infty b C(k) \left[\overbrace{\dot{h} + U_\infty \alpha + b \left(\frac{1}{2} - a_h\right) \dot{\alpha}}^{w_{3/4}} \right]}_{L_\Gamma\text{:circulatory part}} \tag{6}$$

$$\begin{aligned} M_E(t) &= \underbrace{\pi \rho_\infty b^2 \left[a_h b \ddot{h} - U_\infty b \left(\frac{1}{2} - a_h\right) \dot{\alpha} - b^2 \left(\frac{1}{8} + a_h^2\right) \ddot{\alpha} \right]}_{M_{\Gamma=0}} \\ &\quad + \underbrace{2\pi \rho_\infty U_\infty b^2 \left(\frac{1}{2} + a_h\right) C(k) \left[\dot{h} + U_\infty \alpha + b \left(\frac{1}{2} - a_h\right) \dot{\alpha} \right]}_{M_\Gamma} \end{aligned} \tag{7}$$

According to Eq. (5), we can immediatly reformulate the aerodynamic forces as

$$L(t) \quad = \quad L_{\Gamma=0} \underbrace{-2\pi\rho_\infty U_\infty b C(k) w_{3/4}(t)}_{L_\Gamma} \tag{8}$$

$$M_E(t) \quad = \quad M_{\Gamma=0} \underbrace{-2\pi\rho_\infty U_\infty b^2 \left(\frac{1}{2} + a_h\right) C(k) w_{3/4}(t)}_{M_\Gamma} \tag{9}$$

It can then be seen from Eq. 6 that the Theodorsen solution can extented to compute the lift and the aerodynamic moment due to the circulatory flow for any prescribed time-history of the downwash at 3/4 chord point $w_{3/4}(t)$.

This statement will be used hereafter to adress two important issues in aeroelasticity modeling:

1. Flutter solution: The computation of $\mathbf{F_{aero}}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, \mathrm{U}_\infty)$ for an arbitrary motion of the airfoil
2. Gust response: The computation of $\mathbf{F_{gust}}(\mathrm{w_G}, \mathrm{t})$ for an arbitrary gust profile.

2.2 Extension to general motion

Let now consider the Fourier transform of the downwash at the $3/4c$ point[2]

$$w_{3/4}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \mathrm{e}^{i\omega t} d\omega \tag{10}$$

Reporting the expression of $w_{3/4}$ in Eq. (8), we obtain

$$L_\Gamma(t) = -2\pi\rho_\infty U_\infty b C(k) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \mathrm{e}^{i\omega t} d\omega\right) \tag{11}$$

Recall that the previous expression of circulatory lift is valid for harmonic flow **ONLY**, since the deficiency lift function $C(k)$ was derived for harmonic oscillations in pitch and plunge.

In order to take into account for a **general motion** of the airfoil, we have to compute the Theodorsen function according to the whole frequency range and not only for a single frequency (or harmonic), e.g.

$$L_\Gamma(t) \quad = \quad -2\pi\rho_\infty U_\infty b \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{C}(\frac{\omega b}{U_\infty}) f(\omega) \mathrm{e}^{i\omega t} d\omega\right) \tag{12}$$

$$\quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \underbrace{\left(-2\pi\rho_\infty U_\infty b C(\frac{\omega b}{U_\infty}) \mathrm{e}^{i\omega t}\right)}_{\Delta L_\Gamma(\omega, t)} d\omega \tag{13}$$

where $\Delta L_\Gamma(\omega, t)$ corresponds to the **indicial circulatory lift** due to a single frequency component with unit amplitude.

According to Eqs. (8,9), the final expression of the Lift and the aerodynamic moment due to a general motion reads

$$L(t) \quad = \quad L_{\Gamma=0}(t) + L_\Gamma(t), \qquad M_E(t) = M_{\Gamma=0} + b\left(\frac{1}{2} + a_h\right) L_\Gamma(t) \tag{14}$$

[2]with the corresponding inverse Fourier transform:

$$f(\omega, t) = \int_{-\infty}^{\infty} w_{3/4}(t) \mathrm{e}^{-i\omega t} dt$$

2.3 Lift due to indicial step change in the angle of attack

Assume that the airfoil is subject to a sudden change in the angle of attack. This leads to the corresponding downwash at the $3/4c$ point

$$w_{3/4}(t) = \begin{cases} 0 & t < 0 \\ -U\alpha_0 & t \geq 0 \end{cases} \tag{15}$$

whose Fourier transfrom gives

$$w_{3/4}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\underbrace{-\frac{U_\infty\alpha_0}{i\omega}}_{f(\omega)} \mathrm{e}^{i\omega t} d\omega\right) \tag{16}$$

Starting from Eq. (13)

$$L_\Gamma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\Delta L_\Gamma(\omega, t) d\omega \tag{17}$$

and introducing $s = U_\infty t/b$ the distance in semi-chord traveled by the airfoil and $k = \omega b/U_\infty$ the reduced frequency, we obtain[3]

$$L_\Gamma(s) = 2\pi\rho_\infty U_\infty^2 b\alpha_0 \underbrace{\int_{-\infty}^{\infty}\frac{C(k)}{ik}e^{iks}dk}_{=\Psi^{WAG}(s):\text{Wagner function}} \tag{18}$$

Another expression of the Wagner function can be obtained by taking $C(k) = F(k) + iG(k)$ in Eq. (18)

$$\Psi^{WAG}(s) = \mathrm{ı}(s) + \frac{2}{\pi}\int_0^{\infty}\frac{G(k)}{k}\cos(ks)dk \tag{19}$$

where the step function is defined by : $\mathrm{ı}(s < 0) = 0$ $\mathrm{ı}(s \geq 0) = 1$

Figure (1) shows the time-history for two following widely used approximations of the Wagner function

$$\Psi^{WAG}(s) \cong \frac{s+2}{s+4}; \quad \Psi^{WAG}(s) \cong 1 - 0.165\mathrm{e}^{-0.0455s} - 0.335\mathrm{e}^{-0.3s} \tag{20}$$

Remark: $\Psi^{WAG}(0) = 0.5$ and $\lim_{s\to\infty}\Psi^{WAG}(s) = 1$

2.4 Lift due to arbitrary motion

In order to evaluate the response to an airfoil motion **varying arbitrarily in time**, we can make use of **Duhamel's integral** which consists in *superimposing a number of step functions* in order to recover the time history of the motion.

Considering that the airfoil starts from rest at $t = 0$, the lift obtained for an arbitrary motion can be computed by (ref. 2, p.285)

$$L(s) = L_{\Gamma=0} - 2\pi\rho_\infty U_\infty b\left[w_{3/4}(t=0)\Psi^{WAG}(s) + \int_0^s \frac{dw_{3/4}(\sigma)}{d\sigma}\Psi^{WAG}(s-\sigma)d\sigma\right] \tag{21}$$

[3] after elementary calculations to be done during the course

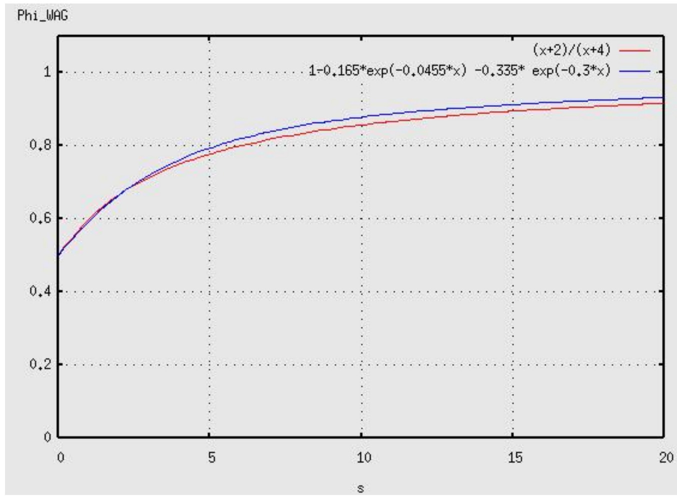

Fig. 2: Plot of approximated Wagner functions for indicial step change in the angle of attack

3 Lift due to discrete gusts

3.1 The sinusoidal gust model

Let the coordinates axes be fixed on the airfoil, with the origin $x = 0$ located at mid-chord point. We consider that the gust is convected by the uniform flow $U_\infty \vec{e}_x$. The vertical gust velocity is given by

$$w_G(x,t) = \bar{w}_G \mathrm{e}^{i\omega\left(t - \frac{x}{U_\infty}\right)} \tag{22}$$

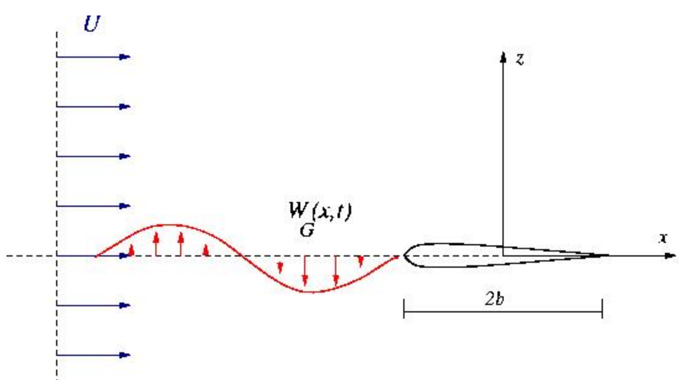

Fig. 3: The harmonic gust model

Using $x^* = x/b$, $s = U_\infty t/b$ and $k = \omega b/U_\infty$, the non-dimensional form induced velocity by the gust reads

$$w_G(x^*, s) = \bar{w}_G \mathrm{e}^{ik(s - x^*)} \tag{23}$$

Now, the aim is to relate the gust velocity to the downwash at the $3/4c$ point in order to be able to compute the circulatory lift through Eq. (7).

To this end, we make use of the boundary condition on the airfoil which stipulates that the global vertical velocity due to the airfoil motion w_a and due to the gust w_G must be equal to zero, therefore

$$w_a(x^*, s) = -\bar{w}_G \, e^{iks} \, e^{-ikx^*} \tag{24}$$

Let $\bar{w}_a = -\bar{w}_G e^{-ikx^*}$, it is possible to show[4] that the lift due to the sinusoidal gust may be written under the form (ref. 2, p.287)

$$L(t) \quad = \quad 2\pi\rho_\infty U_\infty b \bar{w}_G \underbrace{\left([J_0(k) - iJ_1(k)]\, C(k) + J_1(k)\right)}_{S(k)} e^{i\omega t} \tag{25}$$

The Sears function $S(k)$ is plotted in Fig. (3) and it can be admitted than $|S(k)|^2 \cong (1 + 2\pi k)^{-1}$.

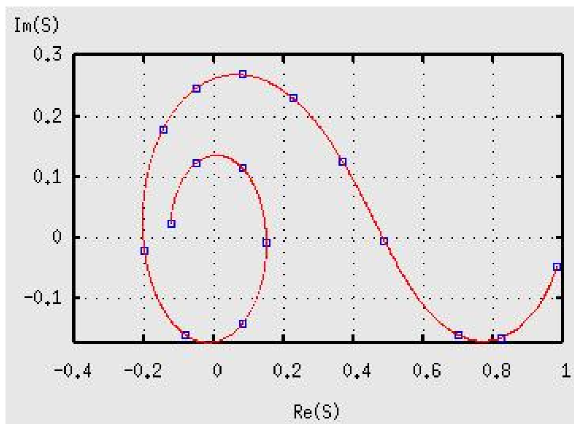

Fig. 4: Plot of the Sears function

3.2 The "sharp edge" gust problem

The gust profile shown in Fig. (4) can be modeled by

$$w_G(x, t) = 1(U_\infty t - b)\, w_0 \tag{26}$$

Making use of the Fourier transform of w_G, we obtain

$$w_G(x, t) = \frac{w_0}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\left(t - \frac{b}{U_\infty} - \frac{x}{U_\infty}\right)}}{i\omega} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega \tag{27}$$

where

$$f(\omega) = \frac{w_0}{i\omega} e^{-i\omega\left(\frac{b}{U_\infty} + \frac{x}{U_\infty}\right)} \tag{28}$$

Whithout loss of generality, we consider that the gust velocity acts on the mid-chord point $(x = 0)$, giving

$$f(\omega) = \frac{w_0}{i\omega} e^{-i\omega\left(\frac{b}{U_\infty}\right)} \tag{29}$$

[4]The demonstration is beyond the scope of the present Class Note.

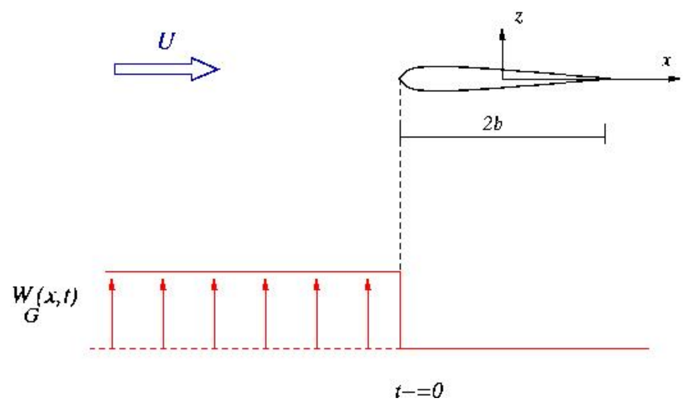

Fig. 5: Schematic of the sharp-egde gust model

By analogy with the expression of the circulatory lift ΔL for one frequency component in a step change of angle of attack in Eq. 13, we can deduce from Eq. 25 the indicial lift ΔL_G due to a **harmonic gust with unit amplitude**

$$\Delta L_G = 2\pi\rho_\infty U_\infty b S(k) \mathrm{e}^{i\omega t} \tag{30}$$

Using the superposition principle, the resulting lift is given by the compatation of the following Duhamel integral

$$\begin{aligned} L_G(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\Delta L_G(\omega,t)d\omega & (31)\\ &= \rho_\infty U_\infty b \int_{-\infty}^{\infty} \frac{S(k)}{i\omega} e^{i\omega\left(t-\frac{b}{U_\infty}\right)} d\omega & (32)\end{aligned}$$

Introducing $s = U_\infty t/b$ the distance in semi-chord traveled by the airfoil and $k = \omega b/U_\infty$ the reduced frequency, we obtain

$$L_G(t) = 2\pi\rho_\infty U_\infty b w_0 \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S(k)}{ik} e^{ik(s-1)dk}}_{=\Psi^{KUS}(s)} \tag{33}$$

where $\Psi^{KUS}(s)$ is the Küssner function which can be approximated by

$$\Psi_1^{KUS}(s) = 1 - 0.5\mathrm{e}^{-0.13s} - 0.5\mathrm{e}^{-s}; \quad \Psi_2^{KUS}(s) = \frac{s(s+1)}{s^2+2.82s+0.8} \tag{34}$$

3.3 Application to arbitrary gusts

Similary to the computation of the circulatory lift due to an arbitrary motion, the lift due to an arbitrary gust profile w_G can be derived using the **Duhamel integral** as (ref. 2, p.288)

$$L_G(s) = 2\pi\rho_\infty U_\infty b \left[w_G(0)\Psi^{KUS}(s) + \int_0^s \frac{dw_G(\sigma)}{d\sigma}\Psi^{KUS}(s-\sigma)d\sigma \right] \tag{35}$$

Fig. 6: Plot of widely used approximations of the Küssner function $\Psi^{KUS}(s)$

We can remark the similar form of this expression of the fift for arbitrary gust with those obtained in Eq (21) for an arbitrary motion :

$$L_M(s) \quad = \quad L_{\Gamma=0} - 2\pi\rho_\infty U_\infty b \left[w_{3/4}(t=0)\Psi^{WAG}(s) + \int_0^s \frac{dw_{3/4}(\sigma)}{d\sigma}\Psi^{WAG}(s-\sigma)d\sigma \right] \tag{36}$$

References

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