High-Fidelity Simulations for Turbulent Flows

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Part VII

Numerical simulation of unsteady flows

Motivation

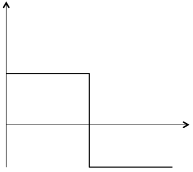
- Preliminary design of fluid systems mostly based on nominal conditions
- CFD of steady flow is now mature, but only **averages** are computed
  - These are not always enough (instabilities, growth rate, vortex shedding..)
  - They are even not always meaningful (steady solution may be different from time-averaged one)
  - Some flows are dominated by vortical structures (*e.g.*, separated flows)

Characteristics of unsteady flows

- Time is the fourth coordinate direction, which must be discretized
- Differences w.r.t. spatial discretization: **directionality**
  - A force at a given x may influence the flow anywhere else
  - A force at a given t will affect the flow only in the future
- Methods very similar to the ones applied to IVP for ODEs
- The basic problem is to find the solution w a short time Δt after the initial point
  - The solution at $t^1 = t^0 + \Delta t$ is used as new I.C.; the solution is then advanced to $t^2 = t^1 + \Delta t$, ..

Classification of unsteady flows

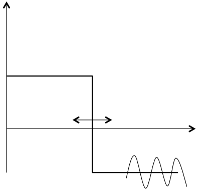
Unsteady flows classified as **slow** or **rapid** based on the ratio of a characteristic speed of flow unsteadiness to wave speed



Example 1: Inviscid Burgers equation

$$\frac{\partial w}{\partial t} + \frac{\partial (w^2/2)}{\partial x} = 0 \qquad x \in [-1,1], \quad \begin{cases} w(-1,t) = 1 \\ w(1,t) = -1 \end{cases}$$

Solution: steady shock located at $x = 0$



Example 2: Inviscid Burgers equation + periodic perturbation

$$\frac{\partial w}{\partial t} + \frac{\partial (w^2/2)}{\partial x} = 0 \qquad x \in [-1,1], \quad \begin{cases} w(-1,t) = 1 \\ w(1,t) = -1 + A\sin(\omega t) \quad \text{with} \quad A \ll 1 \end{cases}$$

Solution: shock oscillating around $x = 0$

- Characteristic speed of the unsteady perturbation:

$$u_p = \frac{\Delta x}{t_p} = \frac{\omega \Delta x}{2\pi}$$

- Wave speed (considering CFL$= \mathcal{O}(1)$):

$$u_0 = u, \quad \Delta t = \text{CFL}\frac{\Delta x}{|u|}$$

- Speed and time ratios:

$$\frac{u_p}{u_0} = \frac{\omega \Delta x}{2\pi u}, \qquad \frac{t_p}{\Delta t} = \frac{u}{\text{CFL}\Delta x}\frac{2\pi}{\omega} = \frac{1}{\text{CFL}}\frac{1}{u_p/u_0}$$

If $\dfrac{u_p}{u_0} \ll 1 \implies \dfrac{t_p}{\Delta t} \gg 1$

- **Lots of iterations** with an explicit method

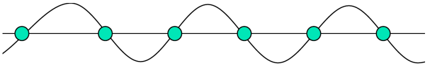
Remember: basics of Time-Marching Methods

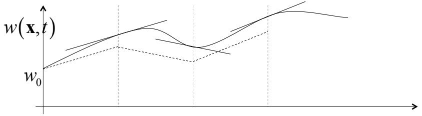
Unconditionally stable (implicit) schemes allow large time steps

- Too large time steps induce **aliasing**
  - **Nyquist barrier**: a signal of frequency f has to be sampled at a frequency $f_s > 2f$
- Large time steps induce **large truncation errors**
  - **Accuracy barrier**: depends on the numerical scheme
    Example for backward Euler scheme (1^{st}-order):
    - Geometric interpretation: solution is linearly extrapolated
    - Quick error accumulation
  - 1^{st}-order accuracy typically not sufficient
  - Achieving high-accurate **and** unconditionally stable schemes is a hard task

How to control the error? 2 possibilities:

1. **Decrease** $\Delta t \implies$ very costly if an implicit scheme is used
2. **Increase accuracy**: for a given level of accuracy, increasing the order of the time integration scheme is more efficient than decreasing Δt





Typical methods:

1. **One-step methods**
   - Forward Euler
   - Backward Euler
   - Trapezoidal (Crank-Nicolson)
   - Taylor-series Methods
   - Runge-Kutta Methods
2. **Multi-step methods**
   - Leapfrog (midpoint)
   - Linear Multistep Methods
     - Adams Methods
     - BDF Methods
3. ..

Categorizing a Linear ODE: absolute stability

A linear semi-discrete problem yields the ODE

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}t} = R(w)}$$

with $R(w) = Aw$ and A a matrix.

Definitions:

- **Linear system**: $F(w, t) = A(t)w + g(t)$ where $A(t) \in \mathbb{R}^{s\times s}$ and $g(t) \in \mathbb{R}^s$
  - *Constant coefficient* if A is a constant matrix
  - *Homogeneous* if $g(t) \equiv 0$
- **System of ODEs**: $\dfrac{\mathrm{d}w}{\mathrm{d}t} = F(w(t), t)$ for $t > t^0$ and $w(t^0) = w^0$
- **Solution** of the IVP: Assuming periodic BCs, solutions under the form of Fourier modes:

$$\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = \widehat{A}\widehat{w} \implies \widehat{w}(t) = \widehat{w}^0 e^{A(t-t^0)} = \widehat{w}^0 e^{\lambda_i t}$$

  - λ_i eigs of $\widehat{A}$ (possibly complex)

Consider the scalar model problem (=homogen. modal eq.)

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}t} = \lambda w(t)}$$

Exact (w^0) and **numerical** ($\widetilde{w}^0$) solutions:

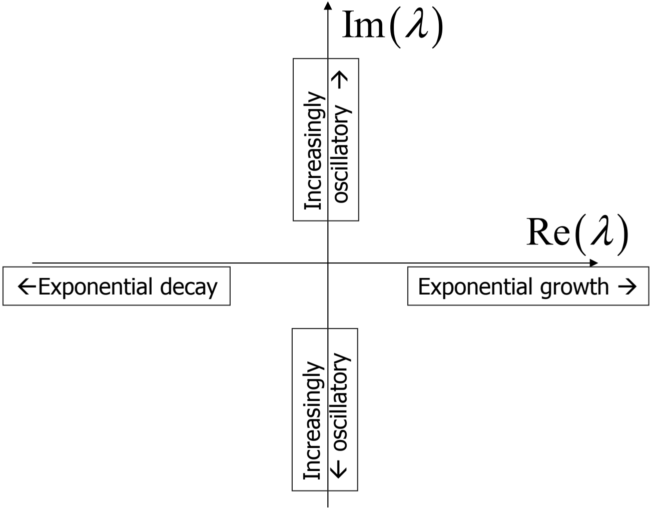
$$w(t) = w^0 e^{\lambda t}$$
$$\widetilde{w}(t) = \widetilde{w}^0 e^{\lambda t}$$

- **Difference**:

$$|w - \widetilde{w}| = |(w^0 - \widetilde{w}^0)e^{\lambda t}| = |(w^0 - \widetilde{w}^0)|e^{\mathrm{Re}(\lambda)t}$$

  - $\mathrm{Re}(\lambda) \le 0$: difference bounded, **stable**
  - $\mathrm{Re}(\lambda) < 0$: difference decays, **asymptotically stable**
  - $\mathrm{Re}(\lambda) > 0$: difference unbounded, **unstable**
  - For a system, each λ_i is considered
- **Stability** completely determined by the eigs of the space discretization matrix A:
  - In **linear cases**: eigenvalues of the coefficient matrix
  - In **nonlinear cases**: eigenvalues of the Jacobian matrix (after linearization)

Stability of the spatial discretization



$$\boxed{\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = \lambda \widehat{w} \qquad \Longrightarrow \qquad \widehat{w} = \widehat{w}^0 e^{\lambda t}}$$

Real part: **dissipation** (damping)
Imaginary part: **dispersion**

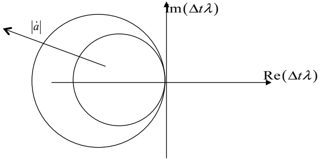
Examples: FOU and CS

- FOU scheme for scalar linear advection:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -a\frac{w_{j+1}-w_{j-1}}{2\Delta x} + \frac{1}{2}|a|\frac{w_{j+1}-2w_j+w_{j-1}}{\Delta x}$$

$$\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -\frac{1}{\Delta t}[i\dot{a}\sin\beta + |\dot{a}|(1-\cos\beta)]\widehat{w}$$

$$\lambda\Delta t = -[i\dot{a}\sin\beta + |\dot{a}|(1-\cos\beta)], \quad Re(\lambda\Delta t) = -|\dot{a}|(1-\cos\beta) \in [-2|\dot{a}|, 0]$$

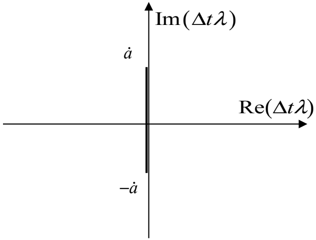


  - Locus of FOU in the complex plane: circle of radius $|\dot{a}|$

- Second-order centred scheme for scalar linear advection:

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = -a\frac{w_{j+1}-w_{j-1}}{\Delta x} \implies \frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -\frac{1}{\Delta t}(i\dot{a}\sin\beta)\widehat{w}$$

$$\implies \lambda\Delta t = -i\dot{a}\sin\beta, \qquad Re(\lambda\Delta t) = 0$$



- Zero dissipation!
- Locus of CS in the complex plane: segment of imaginary axis
  - The ODE is marginally stable

Stability of the time discretization

Applying one-step methods to test problem $w' = \lambda w$, one typically obtains an expression of the form:

$$w^{n+1} = P(\lambda \Delta t) w^n$$

- Iterations diverge for $|P(\lambda\Delta t)| > 1$!
- Only the product $\lambda \Delta t$ matters
- For the solution not to grow without bound, the eigenvalues z of $P(\lambda\Delta t)$ should have $|z| \leq 1 \quad \forall \lambda$

Condition for the stability of **fully discrete scheme**: The spectrum of the **spatial operator** $A(w)$ must lie in the stability region of the **time integration method**

- When more than one value is present, consistency requires that one of the eigs should represent an approximation to the physical behaviour
  - This solution of the eigenvalue eq., called **principal solution**, is recognized by the fact that

$$\lim_{\lambda\Delta t \to 0} z(\lambda) = 1$$

Definition of stability regions:

1. **Absolute stability region** is the set

$$\{z \in \mathbb{C} \,|\, |P(z)| \leq 1\}$$

2. **Relative stab. region** (or **Order star**) is the set

$$\{z \in \mathbb{C} \,|\, |P(z)| \leq |e^z|\}$$

   It compares the growth of the iteration to the growth of the exact solution $\widehat{w}^0 e^{\lambda t}$

A **time discretization** is:

- **Linearly stable** if it admits an absolute stab. region
- **A-stable** if its absolute stab. region contains the entire left half-plane ($|P(z)| < 1$ for $\mathrm{Re}(z) < 0$)
- **A(α)-stable**: $|P(z)| < 1$ for $\mathrm{Re}(z) < -\tan\alpha |\mathrm{Im}(z)|$
  - *i.e.*, the wedge $\pi - \alpha \leq \arg(z) \leq \pi + \alpha$ is entirely contained in the stability region
  - Weaker than A-stable
  - A-stable $\equiv$ A($\pi/2$)-stable method

One-step methods (I)

Forward Euler

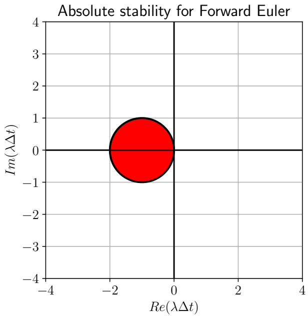
$$\frac{w^{n+1} - w^n}{\Delta t} = R(w^n) = \lambda w^n$$
$$w^{n+1} = w^n + \lambda \Delta t w^n = (1 + \lambda \Delta t) w^n$$
$$w^{n+1} = P(z) w^n = (1 + z) w^n$$

Absolutely stable $\iff$ $|1 + z| \leq 1$

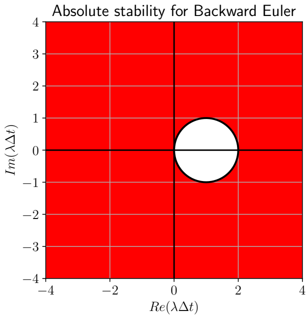
Backward Euler

$$\frac{w^{n+1} - w^n}{\Delta t} = R(w^{n+1}) = \lambda w^{n+1}$$
$$(1 - \lambda \Delta t) w^{n+1} = w^n$$
$$w^{n+1} = P(z) w^n = (1 - z)^{-1} w^n$$

Absolutely stable $\iff$ $|1 - z|^{-1} \leq 1$

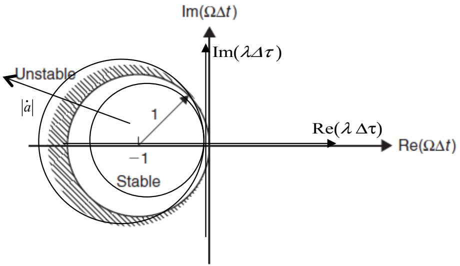


- Solve for $z = x + iy$
- Absolute stability interval for Forward Euler: $[-2, 0]$
- Backward Euler: A-stable!
- Both are first-order..

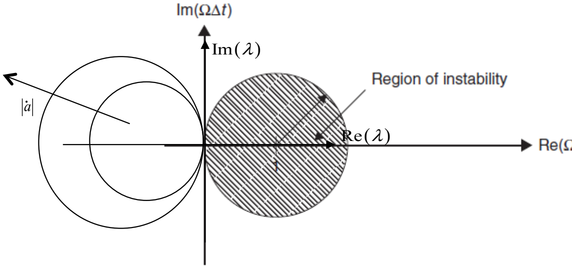


Example: FOU in space + Euler schemes

FOU + Forward Euler: stable for $|\dot{a}| \leq 1$



FOU + Backward Euler: unconditionally stable



- **centred 2nd order + FE**: unconditionally unstable
- **centred 2nd order + BE**: unconditionally stable

An A-stable time integration method leads to a **unconditionally stable** space discretization if combined with a dissipative (stable) operator

One-step methods (II)

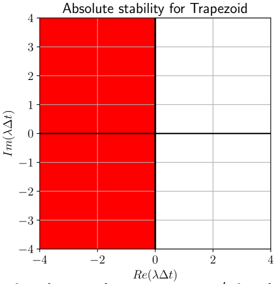
Trapezoidal (CN) Method

$$\frac{w^{n+1}-w^n}{\Delta t} = \frac{1}{2}\left[R(w^n)+R(w^{n+1})\right] = \frac{1}{2}\left[\lambda w^n + \lambda w^{n+1}\right]$$

$$\left[1-\frac{1}{2}\lambda\Delta t\right] w^{n+1} = \left[1+\frac{1}{2}\lambda\Delta t\right] w^n$$

$$w^{n+1} = P(z) = \frac{2+z}{2-z} w^n$$

$$\text{Absolutely stable} \iff \left|\frac{2+z}{2-z}\right| \leq 1$$



$$\text{Re}(z) > 0: \left|\frac{2+z}{2-z}\right| > 1$$

$$\text{Re}(z) \leq 0: \left|\frac{2+z}{2-z}\right| \leq 1$$

- It is the only one-step/single-stage/A-stable 2^{nd} order scheme!

Taylor-series methods

TS method of order p can be derived by keeping the first $p+1$ terms of the TS expansion and dropping the higher ones:

$$w^{n+1} \approx w^n + \Delta t w'^n + \frac{(\Delta t)^2}{2} w''^n + \frac{(\Delta t)^3}{6} w'''^n + \ldots + \frac{(\Delta t)^p}{p!} w^{p,n}$$

with

$$w'^n = R(w^n)$$

$$w''^n = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial t} = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial w} R$$

$$w'''^n = \frac{\partial^2 R}{\partial t^2} + 2\frac{\partial^2 R}{\partial t \partial w} R + 2\frac{\partial R}{\partial w}\frac{\partial R}{\partial t} + \frac{\partial^2 R}{\partial w^2} R^2 + 2\left[\frac{\partial R}{\partial w}\right]^2 R$$

- Dropping terms in $(\Delta t)^2$: Forward Euler
- Becomes messy and cumbersome for higher-derivatives!

One-step methods (III)

Runge–Kutta methods

- **Multistage** method: intermediate values of the solution and its derivatives are generated and used within a single timestep

$$\begin{cases} w^{n+1} = w^n + \Delta t \sum_{i=1}^{s} b_i R(w^{(i)}) \\ w^{(i)} = w^n + \Delta t \sum_{j=1}^{s} a_{ij} R(w^{(j)}) \end{cases}$$

- Both explicit and implicit methods exist
  - For **explicit**: $a_{ij} = 0$ for $j \geq i$
  - Only **implicit** methods may give A-stability
- Increasing the **number of stages** one can:
  - Increase the **order** of the method
    An $r-$stage ERK can have order at most r
    (albeit for $r \geq 5$ the order is strictly lower than r)
  - Increase the **stability** of the method
- Explicit schemes are the most used, but lots of combinations possibles

Examples

2nd-order 2-stage RK

$$\begin{cases} w^* = w^n + \frac{1}{2}\Delta t R(w^n) \\ w^{n+1} = w^n + \Delta t R(w^*) \end{cases}$$

It is a 1-step method, can be combined in

$$w^{n+1} = w^n + \Delta t R\left(w^n + \frac{\Delta t}{2} R(w^n)\right)$$

3rd-order 3-stage RK

$$w^{(1)} = w^n + \Delta t R(w^n)$$

$$w^{(2)} = \frac{3}{4} w^n + \frac{1}{4} w^{(1)} + \frac{1}{4} \Delta t R(w^{(1)})$$

$$w^{n+1} = \frac{1}{3} w^n + \frac{2}{3} w^{(2)} + \frac{2}{3} \Delta t R(w^{(2)})$$

One-step methods (IV)

Examples

2nd-order 4-stage RK (Jameson)

$$\begin{cases} w^{(0)} = w^n \\ w^{(i+1)} = w^{(0)} - \Delta t \alpha_i R(w^{(i)}) \\ w^{n+1} = w^{(4)} \end{cases}$$

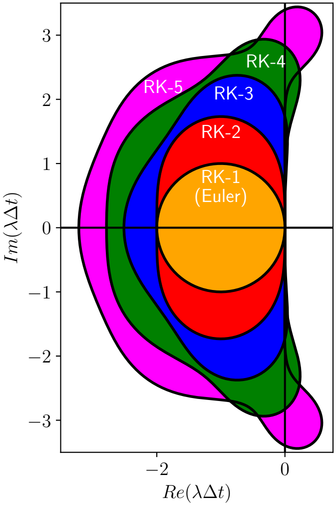
with $i = 1, 2, 3, 4$ and $\alpha = \dfrac{1}{5 - i}$

- 4-stage RK + centred scheme: stable for $CFL \leq 2\sqrt{2}$

For order r, $P(z)$ given by first $r+1$ terms of Taylor expansion of e^z:

$$P(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + .. + \frac{z^r}{r!} + ..$$

- if $s = r$, $P(z)$ does not depend on a_{ij}, b_i
  - all ERK with $s{=}r$ have the same absolute stability region
- $|P(z)| \to \infty$ as $|z| \to \infty$
  - Bounded stability region, not good stiff solver
    (For stiff problems, Δt restricted by stability, not accuracy)



Linear Multistep Methods

LMM: values of $R(w)$ computed in previous times are **reused** to obtain higher-order accuracy

$$\boxed{\sum_{j=0}^{r} \alpha_j w^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j R(w^{n+j})}$$

- w^{n+r} computed from previous values w^{n+r-1}, w^{n+r-2}..
- Method **explicit** for $\beta_r = 0$, **implicit** for $\beta_r \neq 0$

Examples: Adams methods $w^{n+r}=w^{n+r-1}+\Delta t \sum_{j=0}^{r} \beta_j R(w^{n+j})$, thus $\alpha_r=1$, $\alpha_{r-1}=-1$ and $\alpha_j=0$ for $j<r-1$

Explicit Adams-Bashforth methods (order r)

$$\begin{aligned}
\text{1-step:} \quad & w^{n+1} = w^n + \Delta t R(w^n) \quad \text{(forward Euler)} \\
\text{2-step:} \quad & w^{n+2} = w^{n+1} + \frac{\Delta t}{2} \left[-R(w^n) + 3R(w^{n+1})\right] \\
\text{3-step:} \quad & w^{n+3} = w^{n+2} + \frac{\Delta t}{12} \left[5R(w^n) - 16R(w^{n+1}) + 23R(w^{n+2})\right] \\
\text{4-step:} \quad & w^{n+4} = w^{n+3} + \frac{\Delta t}{24} \left[-9R(w^n) + 37R(w^{n+1}) - 59R(w^{n+2}) + 55R(w^{n+3})\right]
\end{aligned}$$

Implicit Adams-Moulton methods (order $r+1$)

$$\begin{aligned}
\text{1-step:} \quad & w^{n+1} = w^n + \frac{\Delta t}{2} \left[R(w^n) + R(w^{n+1})\right] \quad \text{(trapezoidal method)} \\
\text{2-step:} \quad & w^{n+2} = w^{n+1} + \frac{\Delta t}{12} \left[-R(w^n) + 8R(w^{n+1}) + 5R(w^{n+2})\right] \\
\text{3-step:} \quad & w^{n+3} = w^{n+2} + \frac{\Delta t}{24} \left[R(w^n) - 5R(w^{n+1}) + 19R(w^{n+2}) + 9R(w^{n+3})\right] \\
\text{4-step:} \quad & w^{n+4} = w^{n+3} + \frac{\Delta t}{720} \left[-19R(w^n) + 106R(w^{n+1}) - 264R(w^{n+2}) + 646R(w^{n+3}) + 251R(w^{n+4})\right]
\end{aligned}$$

Stability for LMM

Applying LMM to $w' = \lambda w$, one has

$$\sum_{j=0}^{r} \alpha_j w^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j \lambda w^{n+j}$$

$$\sum_{j=0}^{r} (\alpha_j - z\beta_j) w^{n+j} = 0$$

General form of the **solution**:

$$w^n = c_1 \xi_1^n + c_2 \xi_2^n + \ldots + c_r \xi_r^n$$

with ξ_j roots of the **stability polynomial**:

$$\pi(\xi; z) = \sum_{j=0}^{r} (\alpha_j - z\beta_j)\xi^j = \rho(\xi) - z\sigma(\xi)$$

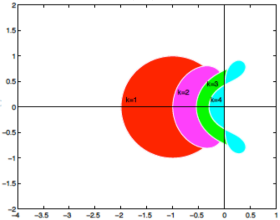
with the characteristics polynomial

$$\rho(\xi) = \sum_{j=0}^{r} \alpha_j \xi^j \quad \text{and} \quad \sigma(\xi) = \sum_{j=0}^{r} \beta_j \xi^j$$

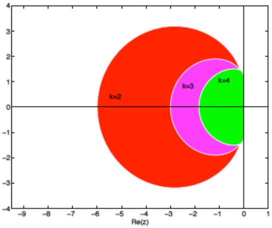
- Polynomial in ξ, with coefficients depending on z
- The region of absolute stability is the set of z for which $\pi(\xi; z)$ satisfies the root condition

- $\pi(\xi; z)$ satisfies the **root condition** $\iff$
  - $|\xi_j| \leq 1$ for $j = 1, 2, .., r$
  - $|\xi_j| < 1$ if ξ_j is a repeated root
- For $\Delta t \to 0$: $\sum_{j=0}^{r} \alpha_j w^{n+j} = 0$
  - The formula reduces to the linear recursion with characteristic polynomial $\rho(\xi)$

Adams-Bashforth for r=1,2,3,4



Adams-Moulton for r=2,3,4



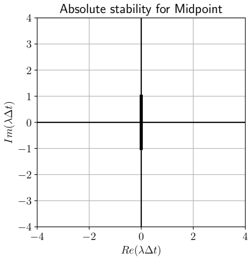
Stability polynomials for different methods

Forward Euler: $\pi(\xi;z) = \xi - (1+z)$ $\quad \xi_1 = 1+z$

Backward Euler: $\pi(\xi;z) = (1-z)\xi - 1$ $\quad \xi_1 = (1-z)^{-1}$

Trapezoidal: $\pi(\xi;z) = \left[1 - \frac{z}{2}\right]\xi - \left[1 + \frac{z}{2}\right]$ $\quad \xi_1 = \frac{2+z}{2-z}$

Midpoint: $\pi(\xi;z) = \xi^2 - 2z\xi - 1$ $\quad \xi_{1,2} = z \pm \sqrt{z^2+1}$



For midpoint, root condition never satisfied!

- if $z = \pm i$: $\xi_1 = \xi_2$ repeated root of modulus 1 (Stability region: interval $]-i, i[$)
- A 1-step method is **L-stable** iff it is A-stable and $\lim_{z\to\infty} |P(z)| = 0$
  - Difference in the Riemann sphere, on the right half-plane
  - BE is L-stable, Trapezoidal is not! Important in case of rapid transients, that we are not interested in

***r*-step AB and AM**: $\pi(\xi;z)$ of degree $r \implies r$ roots, more complex to find stability region!

- **Dahlquist's second barrier theorem**: any A-stable LMM is at most second-order accurate
- For many stiff problems, eigenvalues are far out in the left half-plane but near the real axis
  - no reason to require A-stability, but $A(\alpha)$ stability is sufficient
  - $\implies$ BDF Methods!

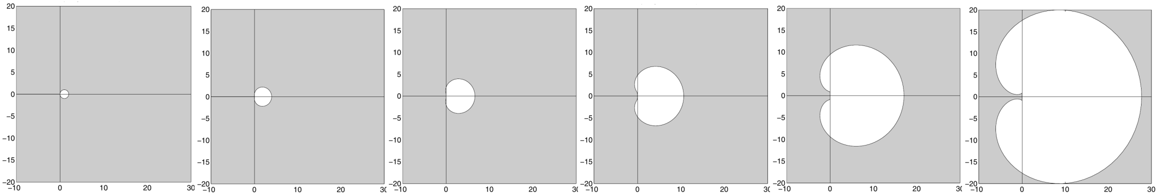
BDF Methods

Backward Differentiation Formula methods: Methods having $\sigma(\xi) = \beta_r \xi^r$:

$$\alpha_0 w^n + \alpha_1 w^{n+1} + ... + \alpha_r w^{n+r} = \Delta t \beta_r R(w^{n+r}) \qquad \text{with} \qquad \beta_0 = \beta_1 = ..\beta_{r-1} = 0$$

$$\begin{aligned}
&\text{1-step :} & w^{n+1} - w^n &= \Delta t R(w^{n+1}) \\
&\text{2-step :} & 3w^{n+2} - 4w^{n+1} + w^n &= 2\ \Delta t R(w^{n+2}) \\
&\text{3-step :} & 11w^{n+3} - 18w^{n+2} + 9w^{n+1} - 2w^n &= 6\ \Delta t R(w^{n+3}) \\
&\text{4-step :} & 25w^{n+4} - 48w^{n+3} + 36w^{n+2} - 16w^{n+1} + 3w^n &= 12 \Delta t R(w^{n+4}) \\
&\text{5-step :} & 137w^{n+5} - 300w^{n+4} + 300w^{n+3} - 200w^{n+2} + 75w^{n+1} - 12w^n &= 60 \Delta t R(w^{n+5})
\end{aligned}$$

- r-step BDF method is r-th order accurate
- They are stable only for $r \leq 6$ (higher orders cannot be used!)
- A-stable for $r = 1, 2$, A(α) stable for $r > 2$: $\alpha = 90°, 90°, 88°, 73°, 51°, 18°$ for $r = 1, 2, 3, 4, 5, 6$
- BDF methods sacrifice A-stability for stiff decay (*i.e.*, L-stability)



Time integration schemes for unsteady flows

One-step methods:

- ✔ Self-starting methods
- ✔ Δt can be changed at any point (more care is needed for LMM)

LMM Methods:

- ✔ For explicit methods, only one R-evaluation is needed (s for RK)
- ✔ For implicit methods, only one m dimension system of nonlinear eqs has to be solved (ms for RK)

Practical choice of step size: Δt must be small enough

1. for the LTE to be acceptably small: $\Delta t \leq \Delta t_{\text{acc}}$, where Δt_{acc} depends on several things:
   - What method is being used
   - How smooth the solution is
   - What accuracy is required
2. for the method to be absolutely stable on this particular problem: $\Delta t \leq (\Delta t)_{\text{stab}}$, depending on λ_i's

Explicit or implicit schemes?

- For problems dominated by **small time scales** or **propagation of discontinuities**, **explicit TVD**
- For problems where dominant time scales are **much larger than the acoustic scale**, **implicit schemes**
- A way for removing stability constraint on the time step is using implicit schemes
  - Generally lead to the solution, at each physical time step, of nonlinear systems of the form:

$$R^*(w) = 0, \qquad \text{with} \qquad R^*(w) = T(w) + R(w)$$

  with $T(w)$ the time discretization operator
  - Direct methods cannot be applied without **linearization**, **Jacobian approximation**, **factorization**, ...
    $\Longrightarrow$ **significant errors**! Alternative: use **iterative methods**, such as **Dual Time Stepping** and **Newton**

Implicit schemes for unsteady flows

Dual Time Stepping

Add to the unsteady residual R^* a derivative with respect to a fictitious or **pseudo time** τ:

$$w_\tau + R^*(w) = 0$$

- Unsteady problem transformed in **steady** w.r.t. τ
  - Solve the false transient by applying any time-stepping technique

✔ Any error allowed in τ, **efficiency** only matters!

✔ Can use linearization, factorization, local time stepping, multigrid.. to speed-up convergence in pseudo time

✘ Important numerical errors or even instabilities if insufficient convergence

✘ Even if large (physical) Δt are allowed, efficiency depends on the number of sub-iterations required

Newton Methods

Solve directly $R^*(w) = 0$ with

$$\frac{\partial R^*}{\partial w}(w^{n+1,m+1} - w^{n+1,m}) = -R^*(w^{n+1,m})$$

- An approximate (or frozen) Jacobian can be used
- Factorization and multigrid also allow to speed-up convergence

✔ Typically much faster than dual time stepping

✘ Similar problems if insufficient convergence

Time advancement

It is chosen following stability and accuracy considerations (unsteady flows)

$$\textit{viscous criterion:} \quad \Delta t_v = \sigma \frac{\Delta x^2}{\nu}$$

$$\textit{convective criterion:} \quad \Delta t_c = \mathrm{CFL} \frac{\Delta x}{u}$$

- **Incompressible regime:** viscous criterion is often very restrictive (except without solid boundaries)
  We can use:
  - an implicit integration for viscous terms (second-order Crank-Nicolson scheme for ex.)
  - an explicit integration for convective terms (3^{rd}- or 4^{th}-order Runge Kutta algorithms or linear multistep methods such as 2^{nd}- or 3^{rd}-order Adams-Bashforth).
- **Compressible regime:** the convective CFL criterion becomes limiting since it implies sound speed $\Delta t_c = CFL \frac{\Delta x}{u+c}$. An explicit method is generally preferred when strong unsteadiness is present.
  - Not always the case, depends on the physics (e.g., chemistry, low-frequency phenomena, ..)