Homework Assignment 3
SF2521, Spring 2020

(max. 4p)

Topics: Well-posedness and stability by Fourier analysis; Modified equations, dissipation and dispersion.

Purpose: To get acquainted with analysis techniques for solution schemes for initial-boundary value problems for hyperbolic systems

Instructions: Write a short report with the plots and answers to the questions posed. Make sure the plots are annotated and there is explanation for what they illustrate.

1 Well–posedness and von Neumann analysis

Consider 2π–periodic Cauchy problems for

$$\begin{aligned} u_t &= \alpha u_{xx} + \beta u_{xxxx} \\ u(x,0) &= \sin(x) \end{aligned}$$

1. **(0.5p)** Show that the problem is well-posed in L_2 for $\alpha > 0$ and $\beta = 0$. You may assume the existence of a solution and just do an energy estimate (using integration by parts or Fourier transform).

2. **(0.5p)** Show that the problem is ill-posed for $\beta > 0$, *no matter which sign α takes.*
   I.e. in general, the higher term determines the stability of the PDE.
   Note: To show **ill-posedness of a given problem** you must find a family of solutions with unity L^2 norm of the initial data whose growth rate is unbounded. *Check out the lecture slides for the example of ill-posedness of $\Delta u + \partial_t u = 0$.*

3. **(0.5p)** For $\alpha > 0$ and $\beta = 0$, use central difference in space for approximation of u_{xx} and forward Euler in time, derive a condition on Δt such that the numerical scheme is stable in the max norm; i.e.

$$\max_j |u_j^{n+1}| \leq \max |u_j^n|.$$

   Is it possible to have max norm stability property for the case $\beta \neq 0$ and a second order finite difference approximation of u_{xxxx}?

4. **(0.5p)** Use a finite difference approximation with central differences on a uniform grid in space and forward difference in time. Apply von Neumann analysis to determine how Δt and Δx should be related for the method to be stable in the well–posed case(s). Is there a stable discretization of the ill-posed case(s)?

5. **(0.5p)** Implement the scheme in a Matlab code and illustrate the conclusions by numerical experimentation. Compare the theoretical conditions with the numerical results. Give examples of α, $\beta \neq 0$ from parts 3 and 4.

2 Shallow water equations, dissipation and dispersion

- **(0.5p)** Run your Lax-Friedrichs program for Homework 2 with different values of $\Delta t/\Delta x$. How large can Δt be without violating the CFL condition? Plot solutions for different $\Delta t/\Delta x$. When is the damping of waves largest/smallest? Compare with your prediction in the previous question.

- **(1.0p)** Write a similar program that solves the problem by the two-step McCormack's scheme (a variant of the Lax-Wendroff method that avoid approximation of the Jacobian matrix):

$$
\begin{aligned}
u_j^* &= u_j^n - \frac{\Delta t}{\Delta x}\left[f(u_{j+1}^n) - f(u_j^n)\right] \\
u_j^{n+1} &= \frac{1}{2}(u_j^n + u_j^*) - \frac{\Delta t}{2\Delta x}\left[f(u_j^*) - f(u_{j-1}^*)\right].
\end{aligned}
$$

  According to LeVeque [**?**] §8.6.2, the method is dispersive.

  - Show that this method is formally second order accurate in Δx and Δt, i.e. it is consistent.
  - Hand in plots of the solution where effects of dispersion errors can be seen. Don't use the "*magic timestep*".
  - How is this effect changed by higher spatial resolution? Do you see any damping?
  - Compare (disspation/dispersion ...) with the solution obtained with the Lax-Friedrichs method.

References

[1] Leveque R.J. - *Finite-Volume Methods for Hyperbolic Problems.* Cambridge University Press (2002).

[2] Gustafsson B., Kreiss H.-O. - *Time-Dependent Problems and Difference Methods (2nd ed.).* John Wiley & Sons (2013).

[3] Kreiss H.-O., Lorenz J. - *Initial-Boundary Value Problems and the Navier-Stokes Equations.* SIAM Society for Industrial and Applied Mathematics.