Numerical solutions of differential equations

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General Finite Volumes Schemes of First Order

Monotone schemes

Monotone schemes

The Godunov scheme

Preliminary consideration

Consider the Riemann problem

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{for } t \geq 0$$

and initial value

$$u(x,0) := \begin{cases} u_l & \text{for } x \leq 0 \\ u_r & \text{for } x > 0 \end{cases}.$$

Then

- we (often) know the exact solution to this problem (for convex flux: shock or rarefaction wave)
- and we always know that it is of the form

$$u(x,t) = v\left(\frac{x}{t}\right)$$

for some function v.

Preliminary consideration

Consider the Riemann problem

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{for } t \geq t_0$$

and initial value

$$u(x,0) := \begin{cases} u_l & \text{for } x \leq x_0 \\ u_r & \text{for } x > x_0 \end{cases}.$$

Then

- $\hat{u}(x,t) := u(x - x_0, t - t_0)$ solves a Riemann problem as before.
- Hence, we know that u is of the form

$$u(x,t) = v\left(\frac{x - x_0}{t - t_0}\right)$$

for some function v and $t > t_0$.

The Godunov Scheme

In the following, we assume

- the initial value is given by $v_0 \in L^1(\mathbb{R})$,
- the numerical initial condition is selected as

$$Q_j^0 := \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} v_0 \qquad \text{for } j \in \mathbb{Z}.$$

Assume that $(Q_j^n)_{j\in\mathbb{Z}}$ is available, then the Godunov scheme is given by the following algorithm.

The Godunov Scheme - Algorithm Step 1

We consider an individual cell

$$(x_{j-1}, x_j) \times (t_n, t_{n+1})$$

and determine the exact solution of the Riemann problem

$$\partial_t u + \partial_x f(u) = 0$$

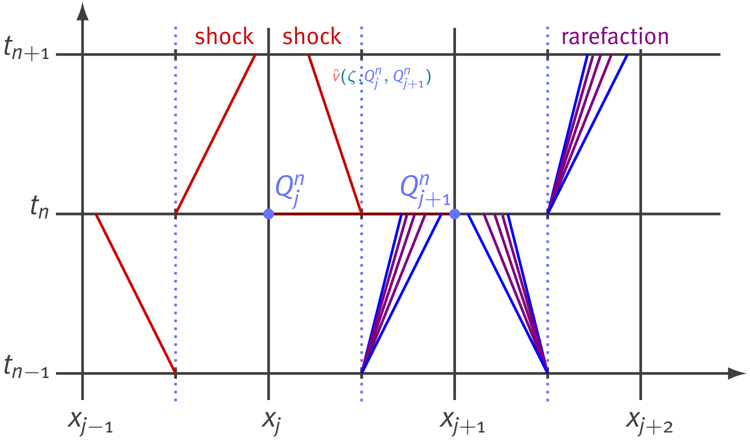
and initial value

$$u(x, t_n) := \begin{cases} Q_{j-1}^n & \text{for } x < x_{j-\frac{1}{2}} \\ Q_j^n & \text{for } x > x_{j-\frac{1}{2}} \end{cases}.$$

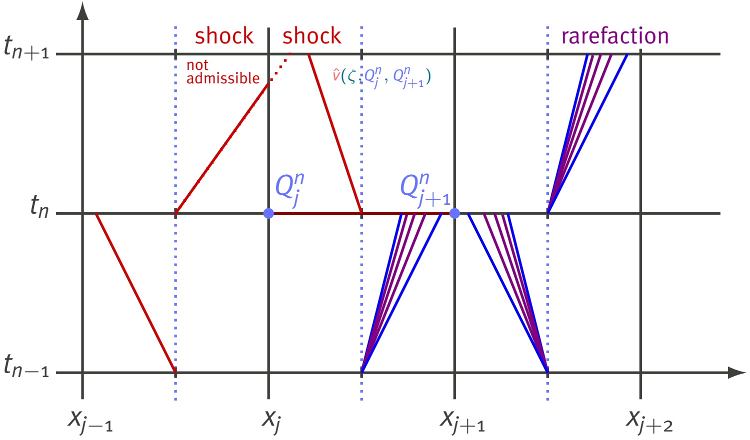
Recalling the preliminary considerations, we can write these local solutions as

$$\hat{v}(\zeta, Q_{j-1}^n, Q_j^n) := u(x, t) \qquad \text{where } \zeta = \frac{x - x_{j-\frac{1}{2}}}{t - t_n} \quad (t \neq t_n).$$

The Godunov Scheme - Algorithm Step 1



The Godunov Scheme - Algorithm Step 2



However: the local solutions must not interact.

The Godunov Scheme - Algorithm Step 2

Necessary CFL condition:

- Require Δt is chosen small enough, so that the local solutions do not interact.
- The shock speed is

$$s = \frac{f(Q_j) - f(Q_{j-1})}{Q_j - Q_{j-1}}.$$

- Hence: max-distance that information can travel in time Δt with speed s in x-direction is $d_{\max} := s\Delta t$.
- Origin of discontinuity is at $x_{j-\frac{1}{2}}$. Distance to left and right cell boundary is $|x_j - x_{j-\frac{1}{2}}| = |x_{j-\frac{1}{2}} - x_{j-1}| = \Delta x/2$.

  Hence, we demand

$$d_{\max} \leq \frac{\Delta x}{2} \quad \Leftrightarrow \quad \frac{s\Delta t}{\Delta x} \leq \frac{1}{2}.$$

The Godunov Scheme - Algorithm Step 2

Necessary CFL condition:

- For shock speed

$$s = \frac{f(Q_j) - f(Q_{j-1})}{Q_j - Q_{j-1}}.$$

  we demand

$$\frac{s\Delta t}{\Delta x} \leq \frac{1}{2}.$$

- Condition is fulfilled if Δt is such that

$$\frac{|f'(\eta)|\,\Delta t}{\Delta x} \leq \frac{1}{2} \qquad \text{for all } \eta \in \mathbb{R}.$$

- This condition is the general CFL condition with CFL number $\frac{1}{2}$. (after Courant-Friedrichs-Levy)

The Godunov Scheme - Algorithm Step 3

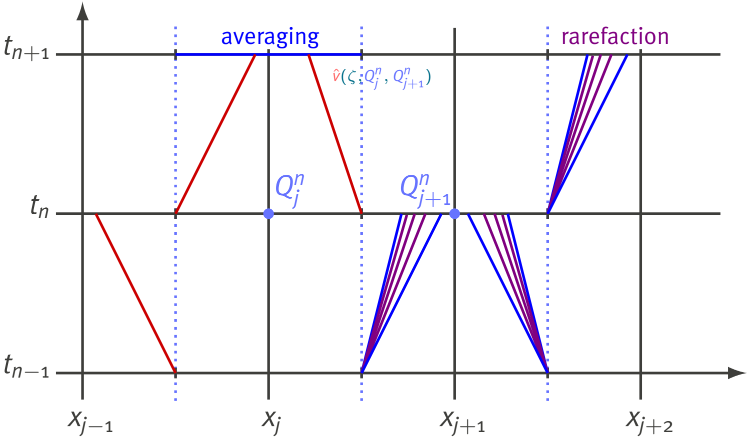
We define

$$Q(x,t) := \begin{cases} \hat{v}\left(\frac{x-x_{j-\frac{1}{2}}}{t-t_n}, Q_{j-1}^n, Q_j^n\right) & \text{for } x_{j-\frac{1}{2}} \le x < x_j \\ \hat{v}\left(\frac{x-x_{j+\frac{1}{2}}}{t-t_n}, Q_j^n, Q_{j+1}^n\right) & \text{for } x_j \le x < x_{j+\frac{1}{2}}. \end{cases}$$

Then Q_j^{n+1} is defined as the average:

$$Q_j^{n+1} := \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} Q(x, t_{n+1})\, dx.$$

The Godunov Scheme - Algorithm Step 3



The Godunov Scheme - Algorithm Step 4

Goal: Simplify Godunov scheme to an acceptable numerical scheme in conservation form.

From the conservation for the exact solution we have:

$$\int_{x_{j-\frac{1}{2}}}^{x_j} u(x,t_{n+1},Q_{j-1}^n,Q_j^n) - u(x,t_n,Q_{j-1}^n,Q_j^n)\,dx$$

$$= \int_{t_n}^{t_{n+1}} f(u(x_{j-\frac{1}{2}},t,Q_{j-1}^n,Q_j^n)) - f(u(x_j,t,Q_{j-1}^n,Q_j^n))\,dt$$

and

$$\int_{x_j}^{x_{j+\frac{1}{2}}} u(x,t_{n+1},Q_j^n,Q_{j+1}^n) - u(x,t_n,Q_j^n,Q_{j+1}^n)\,dx$$

$$= \int_{t_n}^{t_{n+1}} f(u(x_j,t,Q_j^n,Q_{j+1}^n)) - f(u(x_{j+\frac{1}{2}},t,Q_j^n,Q_{j+1}^n))\,dt$$

The Godunov Scheme - Algorithm Step 4

Using the definition of $Q(x,t)$ yields

$$\int_{x_{j-\frac{1}{2}}}^{x_j} Q(x,t_{n+1}) - Q(x,t_n)\,dx \quad = \quad \int_{t_n}^{t_{n+1}} f(Q(x_{j-\frac{1}{2}},t)) - f(Q(x_j,t))\,dt$$

and

$$\int_{x_j}^{x_{j+\frac{1}{2}}} Q(x,t_{n+1}) - Q(x,t_n)\,dx = \int_{t_n}^{t_{n+1}} f(Q(x_j,t)) - f(Q(x_{j+\frac{1}{2}},t))\,dt$$

Hence

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} Q(x,t_{n+1}) - Q(x,t_n)\,dx \quad = \quad \int_{t_n}^{t_{n+1}} f(Q(x_{j-\frac{1}{2}},t)) - f(Q(x_{j+\frac{1}{2}},t))\,dt$$

The Godunov Scheme - Algorithm Step 4

We obtained

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} Q(x,t_{n+1}) - Q(x,t_n)\,dx \quad = \quad \int_{t_n}^{t_{n+1}} f(Q(x_{j-\frac{1}{2}},t)) - f(Q(x_{j+\frac{1}{2}},t))\,dt$$

Recalling that $Q_j^n := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} Q(x,t_n)\,dx$ we obtain

$$\begin{aligned}
Q_j^{n+1} &= Q_j^n + \frac{1}{\triangle x}\int_{t_n}^{t_{n+1}} f(Q(x_{j-\frac{1}{2}},t)) - f(Q(x_{j+\frac{1}{2}},t))\,dt \\
&= \frac{1}{\triangle x}\int_{t_n}^{t_{n+1}} f(\hat{v}(\frac{x_{j-\frac{1}{2}} - x_{j-\frac{1}{2}}}{t-t_n}, Q_{j-1}^n, Q_j^n)) - f(\hat{v}(\frac{x_{j+\frac{1}{2}} - x_{j+\frac{1}{2}}}{t-t_n}, Q_j^n, Q_{j+1}^n)\,dt \\
&= \frac{1}{\triangle x}\int_{t_n}^{t_{n+1}} f(\hat{v}(0, Q_{j-1}^n, Q_j^n)) - f(\hat{v}(0, Q_j^n, Q_{j+1}^n))\,dt \\
&= \frac{\Delta t}{\triangle x}\left(f(\hat{v}(0, Q_{j-1}^n, Q_j^n)) - f(\hat{v}(0, Q_j^n, Q_{j+1}^n))\right).
\end{aligned}$$

The Godunov Scheme - Algorithm Step 4

We obtained

$$Q_j^{n+1} = \frac{\Delta t}{\Delta x}\left(f(\hat{v}(0, Q_{j-1}^n, Q_j^n)) - f(\hat{v}(0, Q_j^n, Q_{j+1}^n))\right).$$

With

$$g(v, w) := f(\hat{v}(0, v, w))$$

we obtain

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(g(Q_j^n, Q_{j+1}^n) - g(Q_{j-1}^n, Q_j^n)\right).$$

Hence, the scheme is in conservation form and with a consistent numerical flux.

The Godunov Scheme - Algorithm Step 5

Additional simplifications for convex flux f. For $f'' > 0$ it holds:

- Case 1. $w > v$ "Shock"

$$\Rightarrow \qquad \hat{v}(0, w, v) = \begin{cases} w & \text{if } \ 0 \leq s \ \Leftrightarrow \ f(w) \geq f(v) \\ v & \text{if } \ s < 0 \ \Leftrightarrow \ f(w) < f(v) \end{cases}.$$

- Case 2: $w < v$ "Rarefaction wave"

$$\Rightarrow \qquad \hat{v}(0, w, v) = \begin{cases} w & \text{if } \ 0 < f'(w) \\ (f')^{-1}(0) & \text{else} \\ v & \text{if } \ f'(v) < 0 \end{cases}.$$

Combining this, we have with $g(v, w) := f(\hat{v}(0, v, w))$ that

$$g(w, v) = \begin{cases} f(w) & \text{if } \ w \geq v \ \text{and } \ f(w) \geq f(v) \\ f(v) & \text{if } \ w \geq v \ \text{and } \ f(w) < f(v) \\ f((f')^{-1}(0)) & \text{else} \\ f(w) & \text{if } \ w < v \ \text{and } \ f'(w) > 0 \\ f(v) & \text{if } \ w < v \ \text{and } \ f'(v) < 0. \end{cases}$$

The Godunov Scheme

Summary of properties of the Godunov scheme:

- it is a scheme in conservation form,
- $g(u, u) = f(u)$ "consistency",
- g is Lipschitz-continuous, if f is Lipschitz-continuous,
- g is monotone, i.e. $\partial_1 g \geq 0, \quad \partial_2 g \leq 0$.

Montone schemes

General remark on montone schemes:

- monotone schemes converge to the entropy solution,
- the consistency order of monotone schemes is at most 1
- the convergence of monotone schemes can be generalized to non-uniform meshes (in space and time),
- On uniform meshes, it is possible to prove a priori error estimates of the form

$$\|u(\cdot,t) - Q_{\Delta x,\Delta t}(\cdot,t)\| \leq C\Delta x^{\frac{1}{2}}.$$

- **Question:** What can we do to construct higher order methods? (as monotone schemes can only be first order schemes)