High-Fidelity Simulation for Turbulent Flows

Contents

1 Estimation of HPC resources

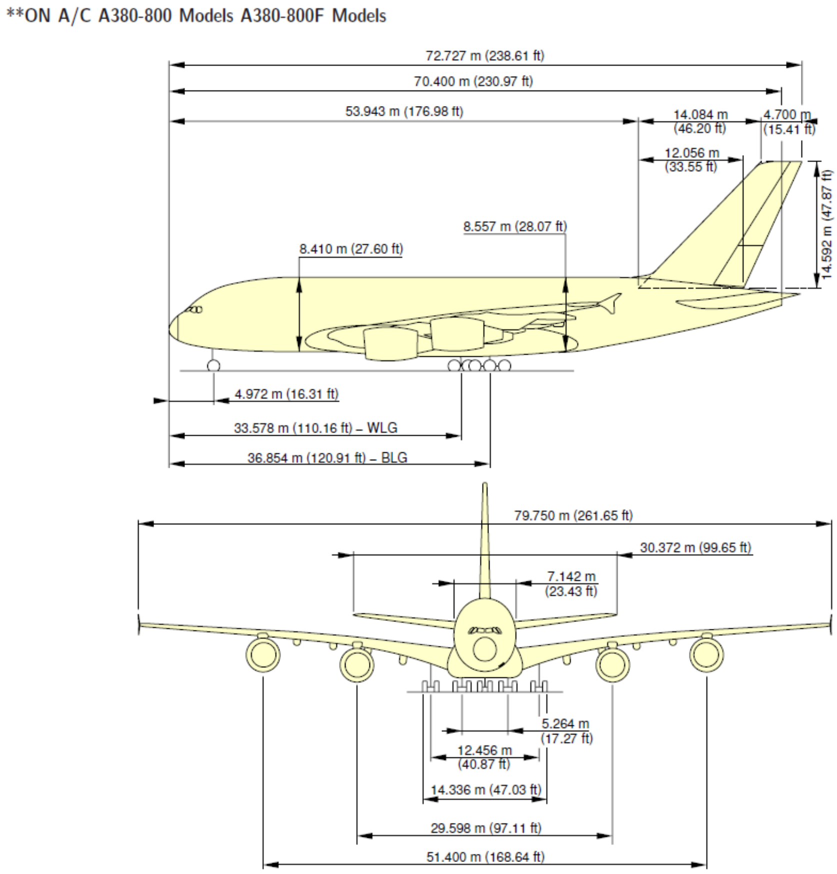
A new turbulence model has been proposed, whose coefficients need to be calibrated by performing a DNS of homogeneous isotropic turbulence at a Reynolds number of $\mathrm{Re} = 10^3$. In order to have access to the local High Performance Computing (HPC) resources, a proposal must be submitted with details concerning the overall cost of the simulation.
The code to be used has the following characteristics:

- The sequential fraction of the code (that is, the fraction of the code that cannot be parallelized) is $S = 0.002$;
- The code has been previously used for the same configuration, but at $\mathrm{Re} = 200$. For this Reynolds number, the total number of grid points needed to ensure DNS resolution is $N = 12 \cdot 10^6$ points and the total computational cost of the simulation is $C = 1500$ CPU hours.

1. Estimate the number of grid points for the new configuration and the total amount of CPU hours that should be demanded to carry out the simulation.
2. A condition for obtaining the computational hours required is that the run should be completed in 7 days. Assuming that there is no limit in the number of cores N_{cores} that can be used and the scalability of the code is ideal, can the simulation run be completed on time?
3. Someone finally finds the solution to remove completely the sequential fraction of the code (thus $S = 0$), but now unfortunately the code runs 10% slower than before. Estimate again the number of CPU hours that are needed in these new conditions and the number of cores N_{cores} that should be used to end the run in 7 days.
4. Determine how much time you will need to perform the simulation on a single core, assuming that the efficiency of the multi-core run is 90%

2 Propulsion of an Airbus A380

The propulsion of an Airbus A380 is ensured by 4 engines, each one providing a thrust of $F = 300\,\mathrm{kN}$ allowing to fly at a cruise velocity of $U = 910\,\mathrm{km/h}$ at an altitude of $h = 10\,700\,\mathrm{m}$, where the air properties are as follows: temperature $T = -55\,^\circ\mathrm{C}$, density $\rho = 0.4\,\mathrm{kg/m^3}$, kinematic viscosity $\nu = 4 \cdot 10^{-5}\,\mathrm{m^2 s^{-1}}$. The aim of this problem is to explore the feasibility of a Direct Numerical Simulation (DNS) of the flow around the airplane.



1. Compute the power developed by the engines.

2. In a steady flight regime, the power developed by the engines is equal to the power dissipated by the turbulent motions of the air flow. Assuming that the air volume moved by the plane is comparable to the volume of the plane itself, compute the power per unit mass dissipated in the air flow, ε.

3. Derive the expression of the Kolmogorov scale η as a function of ε and ν, and compute it for the present case.

4. We want to perform the DNS in a computational domain of size $6\times$ the length, $3\times$ the width and $3\times$ the height of the plane. We consider a regular, cubic Cartesian mesh. Estimate the number of grid points N and the number of time steps N_t needed to run the simulation for a total time of L/U. Are these orders of magnitude achievable with modern computational resources?

5. Estimate the plane surface (consider only the fuselage and the wings) and derive the stress τ_w exerted on the plane from the flow.

6. Derive the expression of the friction velocity u_τ as a function of τ_w and ρ and of the viscous length δ_v, and compute u_τ and δ_v.

7. Is the previously estimated grid sufficiently refined for resolving the viscous sublayer? What should be done if it is not the case?

3 LES of Homogeneous Isotropic Turbulence

A problem of homogeneous isotropic turbulence at high-Reynolds numbers is to be solved by means of Large-Eddy simulations. To this aim, we need to estimate the width of the LES filter needed in order to ensure that a given portion of the turbulent spectrum is resolved. It is assumed that, in the inertial range, the turbulent kinetic energy spectrum follows Kolmogorov's law $E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3}$, with $C = 1.5$ and where κ and ε denote the wavenumber and the dissipation rate of the turbulet kinetic energy k_t, respectively.

1. Define a lengthscale L as a function of k_t and ε.

2. For homogeneous turbulence, the filtered kinetic energy spectrum function reads

$$\overline{E}(\kappa) = \widehat{G}^2(\kappa)E(\kappa) \tag{1}$$

   where $\overline{(\bullet)}$ denotes the filtering operation and $\widehat{G}(\kappa)$ the transfer function of the filter. Use equation (1) to derive a relation for computing $\langle k_{\text{sgs}}\rangle$, where k_{sgs} denotes the subgrid-scale kinetic energy and $\langle\bullet\rangle$ a volume integration operation.

3. Let's consider now an isotropic sharp spectral filter of width Δ and the related cutoff wavenumber $\kappa_c = \dfrac{\pi}{\Delta}$. The transfer function of such a filter reads

$$\widehat{G}(\kappa) = \begin{cases} 0 & \text{if } |\kappa| \geq \kappa_c \\ 1 & \text{if } |\kappa| < \kappa_c \end{cases} \tag{2}$$

   Using the equation derived in step 2, derive a relation of the type

$$\frac{\langle k_{\text{sgs}}\rangle}{k_t} = A \cdot L^{\alpha} \cdot \kappa_c^{\beta} \tag{3}$$

   where $\dfrac{\langle k_{\text{sgs}}\rangle}{k_t}$ denotes the ratio of the subgrid-scale kinetic energy to the total turbulent kinetic energy. Find the numerical values of the constants A, α and β.

4. What assumptions must hold for the previous result to be valid?

5. Find the value of $\dfrac{\Delta}{L}$ for which 75% of the turbulent kinetic energy spectrum is resolved.

6. Repeat the same analysis of step 3 and 5 for a Gaussian filter, whose transfer function reads

$$\widehat{G}(\kappa) = \exp\left(-\frac{\kappa^2\Delta^2}{24}\right) \tag{4}$$

   In the calculations, the following result should be used:

$$I_0 \equiv \int_0^{\infty} (1 - e^{-x})x^{-4/3}\,\mathrm{d}x \approx 4.062 \tag{5}$$

7. For a given numerical cutoff κ_c, compute the ratio of the subgrid-scale kinetic energies obtained with the two filters, $\dfrac{\langle k_{\text{sgs}}\rangle|_{\Delta=\Delta_{\text{gauss}}}}{\langle k_{\text{sgs}}\rangle|_{\Delta=\Delta_{\text{spect}}}}$. Which filter drains more energy? Why?

8. Consider again the sharp spectral filter. In 1980, Deardoff proposed an eddy viscosity model given by
$$\nu_{\mathrm{sgs}} = C_v \sqrt{k_{\mathrm{sgs}}}\,\Delta \tag{6}$$
Assuming equilibrium conditions and knowing that the production reads $\mathcal{P} = \nu_{\mathrm{sgs}} |\overline{S}|^2$ with $|\overline{S}|^2 \approx 2\int_0^{\kappa_c} \kappa^2 E(\kappa)\,\mathrm{d}\kappa$, estimate the value of the coefficient C_v.

9. We are now interested in applying a dynamic model, for which two filters must be defined: the "grid" filter, of width $\overline{\Delta}$, and the "test" filter, of width $\widetilde{\Delta}$, with $\widetilde{\Delta} = 2\overline{\Delta}$. Using the Kolmogorov spectrum, estimate the ratio of the energy contained in the smallest resolved scales (*i.e.*, the scales between $\overline{\Delta}$ and $\widetilde{\Delta}$) to the total subgrid-scale energy.

4 Hybrid Methods

4.1 DES

The Detached Eddy Simulation (DES) model initially proposed by Spalart in 1997 consists in a modification of the one-equation Spalart-Allmaras model. Specifically, the equation for the modified turbulent viscosity reads

$$\frac{\partial \nu_{\mathrm{sa}}}{\partial t} + \widetilde{u}_j \frac{\partial \nu_{\mathrm{sa}}}{\partial x_j} = c_{b1} S_{\mathrm{sa}} \nu_{\mathrm{sa}} - c_{w1} f_w^{\mathrm{DES}} \left(\frac{\nu_{\mathrm{sa}}}{d}\right)^2 + \frac{1}{\sigma}\left[\frac{\partial}{\partial x_k}\left((\widetilde{\nu} + \nu_{\mathrm{sa}})\frac{\partial \nu_{\mathrm{sa}}}{\partial x_k}\right) + c_{b2}\frac{\partial \nu_{\mathrm{sa}}}{\partial x_k}\frac{\partial \nu_{\mathrm{sa}}}{\partial x_k}\right] \tag{1}$$

where the following relations hold:

$$\mu_t = \rho \nu_{\mathrm{sa}} f_{v1} \qquad S_{\mathrm{sa}} = \sqrt{2\widetilde{\Omega}_{ij}\widetilde{\Omega}_{ij}} + \frac{\nu_{\mathrm{sa}}}{\kappa^2 d^2} f_{v2} \qquad \widetilde{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \widetilde{u}_i}{\partial x_j} - \frac{\partial \widetilde{u}_j}{\partial x_i}\right) \tag{2}$$

$$\chi = \frac{\nu_{\mathrm{sa}}}{\widetilde{\nu}} \qquad f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \qquad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \tag{3}$$

$$f_w^{\mathrm{DES}} = g\left(\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right)^{1/6} \qquad g = r + c_{w2}(r^6 - r) \qquad r^{\mathrm{DES}} = \frac{\nu_{\mathrm{sa}}}{S_{\mathrm{sa}}\kappa^2 d^2} \tag{4}$$

Spalart proposed to replace the d_w in r^{DES} with $d = \min(d_w, C_{\mathrm{DES}}\Delta)$.

1. Prove that in the asymptotic limit (i.e., under the hypotheses of equilibrium and high-Re flow), ν_{sa} behaves as a subgrid viscosity.

4.2 Blending methods

Fan *et al.* in 2002 proposed a hybrid turbulence model built on a blending between a RANS and a LES model. Specifically, they considered the k_t-ω model and considered a "hybrid viscosity" and a "hybrid" turbulent kinetic energy equation defined as a weighted linear combination of the two models; that is:

$$\nu_t = \Gamma \frac{k_t}{\omega} + (1 - \Gamma) C_s \sqrt{k_t}\Delta \tag{1}$$

$$\frac{\mathrm{d}k_t}{\mathrm{d}t} = \nu_t \Omega^2 - \left[\Gamma(C_\mu k_t \omega) + (1 - \Gamma) C_d \frac{k_t^{3/2}}{\Delta}\right] + \text{diffusion} \tag{2}$$

with $C_\mu = 0.09$, $C_s = 0.1$ and $C_d = 0.05$. The turbulence frequency ω is obtained from its own transport equation. The weighting function Γ is defined as

$$\Gamma = \tanh(\eta^4) \qquad \text{with} \qquad \eta = \frac{1}{\omega}\max\left(\frac{\sqrt{k_t}}{C_\mu d_w}, \frac{500\nu}{d_w^2}\right) \tag{3}$$

1. Describe the behavior of the model with respect to the weighting function Γ.
2. Show that under the equilibrium hypothesis and in the limit $\Gamma = 0$, one obtains a Smagorinsky-type subgrid viscosity:
$$\nu_t = C_s\sqrt{\frac{C_s}{C_d}}\Delta^2\Omega = C_F\Delta^2\Omega \tag{4}$$
3. Under the same hypotheses, derive a one-equation model for the transport of the subgrid viscosity. Does an increase of k_t directly translate in an increasing ν_t? Why?

4.3 VLES

In 1998, Speziale proposed an hybrid RANS/LES method called Very Large Eddy Simulation (VLES). The main idea is to damp the Reynolds stresses in regions where the grid spacing Δ approaches the Kolmogorov scale η:

$$\tau_{ij} = \alpha\tau_{ij}^{\mathrm{RANS}} \qquad \text{with} \qquad \alpha = \left[1 - \exp\left(-\frac{\beta\Delta}{\eta}\right)\right] \tag{1}$$

Here, β and n are some modelling parameters and $\eta = (\nu^3/\varepsilon)^{1/4}$.

1. Describe how this VLES switch from RANS to LES mode.
2. In your opinion, what are the strength and the weaknesses of such a model? How does it perform when turbulence is not fully developed?
3. In 2001, Magnient proposed to modify the hybrid eddy viscosity using the following form
$$\nu_t = \nu_t^{\mathrm{RANS}} f\left(\frac{\Delta}{L_{\mathrm{RANS}}}\right) g\left(\frac{\Delta}{\eta}\right) \tag{2}$$
and used a k_t-ε model to determine the RANS quantities. In the VLES framework, one can consider $g(\Delta/\eta) \approx 1$. Provide expressions for ν_t in the RANS and LES modes as a function of ε. Then, using the equilibrium hypothesis, provide a formulation for $f(\Delta/L_{\mathrm{RANS}})$.

5 PANS Modelling

The hybrid modelling RANS/LES named PANS (Partially-Averaged Navier–Stokes) has been developed by Girimaji *et al.* in 2005. It is based on a modified formulation of the k_t-ε turbulence model, by making it sensible to local grid refinements. In such a way, it can locally behave as a subgrid model in flow regions where the mesh has a LES-like resolution. Specifically, the PANS approach consists in introducing the unresolved-to-total ratio of kinetic energy and dissipation:

$$f_k = \frac{k_u}{k} \qquad f_\varepsilon = \frac{\varepsilon_u}{\varepsilon} \tag{1}$$

entering in the two-equation PANS model as

$$\frac{\mathrm{d}k_u}{\mathrm{d}t} = P_u - \varepsilon_u + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_u}{\sigma_{ku}}\right)\frac{\partial k_u}{\partial x_j}\right] \tag{2}$$

$$\frac{\mathrm{d}\varepsilon_u}{\mathrm{d}t} = f_k\left(C_{\varepsilon_1}\frac{P_u\varepsilon_u}{k_u} - C_{\varepsilon_2}^*\frac{\varepsilon_u^2}{k_u}\right) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_u}{\sigma_{\varepsilon u}}\right)\frac{\partial \varepsilon_u}{\partial x_j}\right] \tag{3}$$

where P_u and ε_u are the underresolved kinetic energy production and dissipation, and $\nu_u = C_\mu\dfrac{k_u^2}{\varepsilon_u}$. In these equations, the coefficients $C_{\varepsilon_2}^*$, σ_{ku} and $\sigma_{\varepsilon u}$ are modified with respect the classical k_t-ε RANS model coefficients, being

$$C_{\varepsilon_2}^* \equiv C_{\varepsilon_1} + \frac{f_k}{f_\varepsilon}(C_{\varepsilon_2} - C_{\varepsilon_1}) \qquad \sigma_{ku} \equiv \frac{f_k^2}{f_\varepsilon}\sigma_k \qquad \sigma_{\varepsilon u} \equiv \frac{f_k^2}{f_\varepsilon}\sigma_\varepsilon \tag{4}$$

1. Specify the interval limits of variation for the coefficients f_k and f_ε. For a given cutoff scale in the inertial range, which of them is bigger?

2. Discuss the limiting behavior of the PANS model as a function of f_k; that is, when PANS recovers a pure RANS or a pure DNS behavior.

3. What would be the value of f_ε for a high-Reynolds flow? And for low-Reynolds one? (Assume always a cutoff in the inertial region).

4. In this model, it would be convenient to find an estimation for the lowest value of f_k that a given grid of grid size Δ could support at a given location. Show that one has:

$$f_k = \min\left[1, \frac{1}{\sqrt{C_\mu}}\left(\frac{\Delta}{L_T}\right)^{2/3}\right] \tag{5}$$

where $L_T = \frac{k_t^{3/2}}{\varepsilon}$ is the local Taylor turbulent scale. To do this, assume that $f_\varepsilon = 1$, and that the smallest resolved length scale η_r can be computed as $\eta_r = \left[\frac{\nu_u^3}{\varepsilon}\right]^{\frac{1}{4}}$.