Numerical solutions of differential equations

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Lecture 5

Hyperbolic Equations of first order - Part 2

Characteristics: Example 2, Linear advective systems

Preliminary

For $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ we denote

$$\partial_t \mathbf{u}(x,t) := \begin{pmatrix} \partial_t \mathbf{u}_1(x,t) \\ \vdots \\ \partial_t \mathbf{u}_m(x,t) \end{pmatrix} \quad \text{and} \quad \partial_x \mathbf{u}(x,t) := \begin{pmatrix} \partial_x \mathbf{u}_1(x,t) \\ \vdots \\ \partial_x \mathbf{u}_m(x,t) \end{pmatrix}.$$

Characteristics: Linear advective systems

We consider a linear system of hyperbolic equations.

Find $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ with

$$\partial_t \mathbf{u} + \mathbf{A}\,\partial_x \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

If the system is hyperbolic we have

- $\mathbf{v} = \mathbf{v}(x) : \mathbb{R} \to \mathbb{R}^m$;
- $\mathbf{A} \in \mathbb{R}^{m\times m}$ is diagonalizable with real non-zero eigenvalues $\lambda_1 \leq \ldots \leq \lambda_m$;
- corresponding diagonal matrix:

$$\boldsymbol{\Lambda} := \mathrm{diag}(\lambda_1, \cdots, \lambda_m);$$

- eigenvectors: $\mathbf{r}_1, \cdots, \mathbf{r}_m \in \mathbb{R}^m$; i.e. $\mathbf{A}\mathbf{r}_p = \lambda_p \mathbf{r}_p$ for $1 \leq p \leq m$.
- matrix with eigenvectors as columns $\mathbf{R} := [\mathbf{r}_1, \ldots, \mathbf{r}_m]$.
- $\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$.

Characteristics: Linear advective systems

Multiplying the equation with $\mathbf{R}^{-1}$ we obtain

$$
\begin{aligned}
0 &= \mathbf{R}^{-1}\partial_t \mathbf{u} + \mathbf{R}^{-1}\mathbf{A}\,\partial_x \mathbf{u} \\
&\overset{\mathbf{A}=\mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}}{=} \mathbf{R}^{-1}\partial_t \mathbf{u} + \mathbf{R}^{-1}\mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}\,\partial_x \mathbf{u} \\
&= \mathbf{R}^{-1}\partial_t \mathbf{u} + \mathbf{\Lambda}\mathbf{R}^{-1}\,\partial_x \mathbf{u} \\
&\overset{\mathbf{z}=\mathbf{R}^{-1}\mathbf{u}}{=} \partial_t \mathbf{z} + \mathbf{\Lambda}\,\partial_x \mathbf{z}.
\end{aligned}
$$

Hence for $p = 1, \dots, m$

$$\partial_t \mathbf{z}_p + \lambda_p\,\partial_x \mathbf{z}_p = 0 \qquad \text{and} \qquad \mathbf{z}_p(0) = (\mathbf{R}^{-1}\mathbf{v})_p.$$

The corresponding characteristic for $x_0 \in \mathbb{R}$ is given by

$$\gamma_p(t) = \lambda_p\, t + x_0.$$

We call γ_p the <u>p-characteristic of the system</u>.

Characteristics: Linear advective systems

Since $\partial_t \mathbf{z}_p + \lambda_p \, \partial_x \mathbf{z}_p = 0$ and $\mathbf{z}_p(0) = (\mathbf{R}^{-1}\mathbf{v})_p$
and since γ_p is the characteristic for $x_0 \in \mathbb{R}$ with

$$\gamma_p(t) = \lambda_p \, t + x_0,$$

$\mathbf{z}_p$ is constant along $(\gamma_p(t), t)$ and hence with $x = \lambda_p \, t + x_0$

$$\mathbf{z}_p(x,t) = \mathbf{z}_p(x_0, 0) = (\mathbf{R}^{-1}\mathbf{v}(x_0))_p = (\mathbf{R}^{-1}\mathbf{v}(x - \lambda_p \, t))_p =: \mathbf{z}_p^0(x - \lambda_p \, t).$$

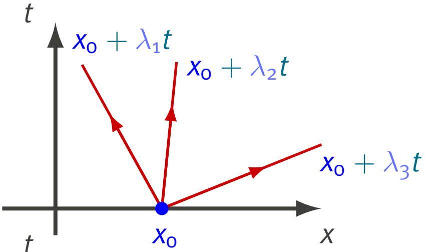
and

$$\mathbf{u} = \mathbf{R}\mathbf{z} = \sum_{p=1}^{m} \mathbf{z}_p(x,t)\mathbf{r}_p = \sum_{p=1}^{m} \mathbf{z}_p^0(x - \lambda_p \, t) \, \mathbf{r}_p$$

Hence

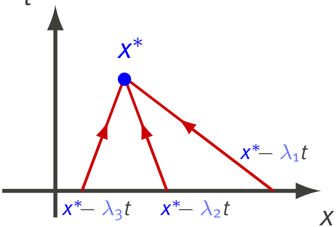
- solution $\mathbf{u}$ is given as linear combination (superposition) of $\mathbf{z}_p$;
- superposition of waves propagating with speed λ_p.

Characteristics: Linear systems. Example: $m = 3$



$\gamma_p(t) = \lambda_p t + x_0$.
Range of influence of x_0:
All points along $\gamma_p(t) = \lambda_p t + x_0$ for $p = 1, 2, 3$.

"Information" in x_0 spreads along the red lines.



Domain of dependence of x^* at time T:
Set of points
$D(x^*, T) := \{x^* - \lambda_p T | \; p = 1, 2, 3\}$.

Solution in (x^*, T) is given as linear combination of $\mathbf{z}_p^0$ in $x^* - \lambda_p T$.

Characteristics: Linear systems

Explicit example:

Find $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}^2$ with

$$\partial_t \mathbf{u} + \mathbf{A}\, \partial_x \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

Here, for $c > 0$

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}(x) = \begin{pmatrix} 0 \\ v_0(x) \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$$

For the eigenvalues we have $\lambda_1 = -c$ and $\lambda_2 = c$

$$\begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ c^{-1} \end{pmatrix}}_{=\mathbf{r}_1} = \lambda_1 \begin{pmatrix} 1 \\ c^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ -c^{-1} \end{pmatrix}}_{=\mathbf{r}_2} = \lambda_2 \begin{pmatrix} 1 \\ -c^{-1} \end{pmatrix}$$

Characteristics: Linear systems

We have

$$\mathbf{\Lambda} := \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}, \qquad \mathbf{R} := \begin{pmatrix} 1 & 1 \\ \frac{1}{c} & -\frac{1}{c} \end{pmatrix}, \qquad \mathbf{R}^{-1} := \frac{1}{2}\begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}.$$

From $\partial_t \mathbf{u} + \mathbf{A}\,\partial_x \mathbf{u} = 0$ and using the transformation $\mathbf{u} = \mathbf{R}\mathbf{z}$ we have

$$\partial_t \mathbf{z} + \mathbf{\Lambda}\,\partial_x \mathbf{z} = 0$$

and

$$\mathbf{z}(0,x) = \mathbf{R}^{-1}\mathbf{v}(x) = \frac{1}{2}\begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}\begin{pmatrix} 0 \\ v_0(x) \end{pmatrix} = \frac{c}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} v_0(x)$$

The p-characteristics are given by

$$\gamma_1(t) = -c\,t + x_0 \qquad \text{and} \qquad \gamma_2(t) = c\,t + x_0.$$

Characteristics: Linear systems

From

$$\partial_t \mathbf{z} + \Lambda\, \partial_x \mathbf{z} = 0; \qquad \mathbf{z}(x,0) = \frac{c}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} v_0(x)$$

and the p-characteristics

$$\gamma_1(t) = -c\,t + x_0 \qquad \text{and} \qquad \gamma_2(t) = c\,t + x_0,$$

we conclude that

$$\mathbf{z}(x,t) = \begin{pmatrix} \mathbf{z}(x+ct,0) \\ \mathbf{z}(x-ct,0) \end{pmatrix} = \frac{c}{2}\begin{pmatrix} v_0(x+ct) \\ -v_0(x-ct) \end{pmatrix}$$

and hence

$$\mathbf{u}(x,t) = \mathbf{R}\mathbf{z}(x,t) = \frac{c}{2}\begin{pmatrix} 1 & 1 \\ \frac{1}{c} & -\frac{1}{c} \end{pmatrix}\begin{pmatrix} v_0(x+ct) \\ -v_0(x-ct) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} cv_0(x+ct) - cv_0(x-ct) \\ v_0(x+ct) + v_0(x-ct) \end{pmatrix}.$$

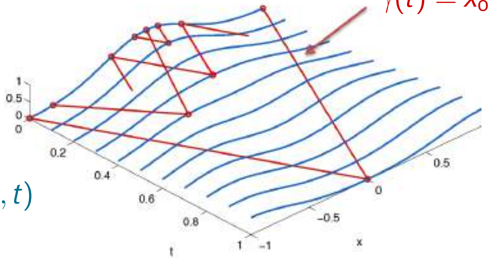
Characteristics: Linear systems

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = 0 \quad \text{and} \quad \begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} 0 \\ v_0(x) \end{pmatrix}.$$

$$u_1(x,t) = \frac{c}{2}\left(v_0(x+ct) - v_0(x-ct)\right) \quad \text{and} \quad u_2(x,t) = \frac{1}{2}\left(v_0(x+ct) + v_0(x-ct)\right).$$

$$\gamma(t) = x_0 \pm ct.$$

$u_2(x,t)$



If $v_0(x) = 2\sin(2\pi x)$ we have $u_2(x,t) = \sin(2\pi(x+ct)) + \sin(2\pi(x-ct))$.
At $T = 1$ and for $c = 1$ we have $u_2(0,T) = \sin(2\pi) + \sin(-2\pi) = 0$.

Characteristics: Linear systems