Lecture 5

Hyperbolic Equations of first order - Part 2

Finite Volume Method for Conservation Laws

General conservation law - Model problem

For

- a smooth flux $f : \mathbb{R}^m \to \mathbb{R}^m$
- and initial value $\mathbf{v} : [0, 1] \to \mathbb{R}^m$

we seek

$$\mathbf{u} = \mathbf{u}(x, t) : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^m$$

with

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \qquad \text{and} \qquad \mathbf{u}(x, 0) = \mathbf{v}(x).$$

for $0 < x < 1$ and $t \in (0, \infty)$.

General conservation law - Model problem

For $0 < x < 1$ and $t \in (0, \infty)$ find $\mathbf{u}(x, t)$ with

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \qquad \text{and} \qquad \mathbf{u}(x, 0) = \mathbf{v}(x).$$

Note: the problem can be

- Scalar; $m = 1$, e.g.
  - $f(\mathbf{u}) = \mathbf{a}\,\mathbf{u}$, linear wave propagation;
  - $f(\mathbf{u}) = \frac{1}{2}\mathbf{u}^2$, Burgers' equation;
- System; $m > 1$, e.g.
  - linear case: for $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$ we have $f(\mathbf{u}) = \mathbf{A} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$
  - nonlinear case: for $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$ we have $f(\mathbf{u}) = \begin{pmatrix} f_1(\mathbf{u}_1, \mathbf{u}_2) \\ f_2(\mathbf{u}_1, \mathbf{u}_2) \end{pmatrix}$

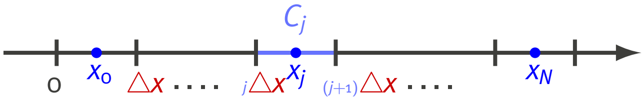
Finite Volume Discretization

For $0 < x < 1$ and $t \in (0, \infty)$ find $\mathbf{u}(x,t)$ with

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

Finite Volume Discretization: more or less as before.

1. Discretize in space into N cells of size $\triangle x = 1/N$.



Here $x_j = \frac{\triangle x}{2} + j\triangle x$ for $j = 0, 1, 2, \ldots, N-1$.

2. Discretize in time. For time step size $\triangle t$ we have $t^n = n\triangle t$ where $n = 0, 1, 2, \cdots$.

Finite Volume Discretization

3. Derive exact update formula as follows.

Conservation law on $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ scaled with $1/\triangle x$:

$$0 = \frac{1}{\triangle x}\int_{C_j} \partial_t \mathbf{u}(x,t)\,dx + \frac{1}{\triangle x}\int_{C_j} \partial_x f(\,\mathbf{u}(x,t)\,)\,dx$$

$$= \frac{1}{\triangle x}\int_{C_j} \partial_t \mathbf{u}(x,t)\,dx + \frac{f(\,\mathbf{u}(x_{j+\frac{1}{2}},t)\,) - f(\,\mathbf{u}(x_{j-\frac{1}{2}},t)\,)}{\triangle x}.$$

Integrating in time over $[t^n, t^{n+1}]$:

$$0 = \frac{1}{\triangle x}\int_{C_j}\int_{t^n}^{t^{n+1}} \partial_t \mathbf{u}(x,t)\,dt\,dx + \int_{t^n}^{t^{n+1}} \frac{f(\,\mathbf{u}(x_{j+\frac{1}{2}},t)\,) - f(\,\mathbf{u}(x_{j-\frac{1}{2}},t)\,)}{\triangle x}\,dt$$

$$= \frac{1}{\triangle x}\left(\int_{C_j} \mathbf{u}(x,t^{n+1})\,dx - \int_{C_j} \mathbf{u}(x,t^n)\,dx\right) + \int_{t^n}^{t^{n+1}} \frac{f(\,\mathbf{u}(x_{j+\frac{1}{2}},t)\,) - f(\,\mathbf{u}(x_{j-\frac{1}{2}},t)\,)}{\triangle x}\,dt$$

Finite Volume Discretization

4. Approximation of equation. We have

$$\underbrace{\frac{1}{\triangle x}\int_{C_j} \mathbf{u}(x,t^{n+1})\,dx}_{\text{cell average } \approx Q_j^{n+1}} = \underbrace{\frac{1}{\triangle x}\int_{C_j} \mathbf{u}(x,t^{n})\,dx}_{\text{cell average } \approx Q_j^{n}} - \frac{1}{\triangle x}\int_{t^n}^{t^{n+1}} f(\,\mathbf{u}(x_{j+\frac{1}{2}},t)\,) - f(\,\mathbf{u}(x_{j-\frac{1}{2}},t)\,)\,dt$$

Introduce suitable approximations:

- for the cell average of the wave

$$Q_j^n \approx \frac{1}{\triangle x}\int_{C_j} \mathbf{u}(x,t^n)\,dx.$$

- for the time average of the flux (depending on Q_j^n and Q_{j+1}^n)

$$F(Q_j^n, Q_{j+1}^n) \approx \frac{1}{\triangle t}\int_{t^n}^{t^{n+1}} f(\,\mathbf{u}(x_{j+\frac{1}{2}},t)\,)\,dt.$$

- Hence $$Q_j^{n+1} = Q_j^n - \frac{\triangle t}{\triangle x}\left(F(Q_j^n, Q_{j+1}^n) - F(Q_{j-1}^n, Q_j^n)\right)$$

Finite Volume Discretization

Finite Volume Scheme in conservation form:

$$Q_j^{n+1} = Q_j^n - \frac{\triangle t}{\triangle x}\left(F(Q_j^n, Q_{j+1}^n) - F(Q_{j-1}^n, Q_j^n)\right)$$

Possible choices for $F(Q_j^n, Q_{j+1}^n)$? Examples:

- Lax-Friedrich scheme

$$F(U,V) = \frac{1}{2}\left(f(U) + f(V) - \frac{\triangle x}{\triangle t}(U - V)\right).$$

- Lax-Wendroff scheme

$$F(U,V) = \frac{1}{2}\left(f(U) + f(V) - \frac{\triangle t}{\triangle x}\frac{(f(V) - f(U))^2}{V - U}\right).$$

Finite Volume Discretization - Lax-Friedrich

Lax-Friedrich scheme Motivation:

in $\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$ we approximate

$$\partial_t \mathbf{u}(x_j, t^n) \approx \frac{\mathbf{u}(x_j, t^{n+1}) - \frac{\mathbf{u}(x_{j+1}, t^n) + \mathbf{u}(x_{j-1}, t^n)}{2}}{\triangle t} \approx \frac{Q_j^{n+1} - \frac{Q_{j+1}^n + Q_{j-1}^n}{2}}{\triangle t}$$

and

$$\partial_x f(\,\mathbf{u}(x_j, t^n)\,) \approx \frac{f(\,\mathbf{u}(x_{j+1}, t^n)\,) - f(\,\mathbf{u}(x_j, t^n)\,)}{\triangle x} \approx \frac{f(\,Q_{j+1}^n\,) - f(\,Q_{j-1}^n\,)}{2\triangle x}$$

Hence

$$Q_j^{n+1} = \frac{1}{2}\left(Q_{j+1}^n + Q_{j-1}^n\right) - \frac{\triangle t}{2\triangle x}\left(f(\,Q_{j+1}^n\,) - f(\,Q_{j-1}^n\,)\right)$$

Finite Volume Discretization - Lax-Friedrich

This coincides with the Lax-Friedrichs scheme, because

$$
\begin{aligned}
Q_j^{n+1} &= \frac{1}{2}\left(Q_{j+1}^n + Q_{j-1}^n\right) - \frac{\triangle t}{2\triangle x}\left(f(\,Q_{j+1}^n\,) - f(\,Q_{j-1}^n\,)\right) \\
&= Q_j^n - \frac{\triangle t}{2\triangle x}\left(f(\,Q_{j+1}^n\,) - f(\,Q_{j-1}^n\,)\right) + \frac{1}{2}\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right) \\
&= Q_j^n - \frac{\triangle t}{\triangle x}\left\{\frac{1}{2}\left(f(\,Q_{j+1}^n\,) + f(\,Q_j^n\,) - f(\,Q_j^n\,) - f(\,Q_{j-1}^n\,)\right)\right. \\
&\qquad\qquad \left. + \frac{\triangle x}{2\triangle t}\left((Q_{j+1}^n - Q_j^n) - (Q_j^n - Q_{j-1}^n)\right)\right\} \\
&= Q_j^n - \frac{\triangle t}{\triangle x}\left(F(Q_{j+1}^n, Q_j^n) - F(Q_j^n, Q_{j-1}^n)\right)
\end{aligned}
$$

with the Lax-Friedrich flux

$$
F(U,V) := \frac{1}{2}\left(f(U) + f(V) - \frac{\triangle x}{\triangle t}(U - V)\right).
$$

Finite Volume Discretizations - Short comparison

Example: linear, scalar Riemann problem

$$\partial_t \mathbf{u} + \mathbf{a}\partial_x \mathbf{u} = 0 \qquad \text{and} \qquad \mathbf{u}(x,0) = \mathbf{v}(x).$$

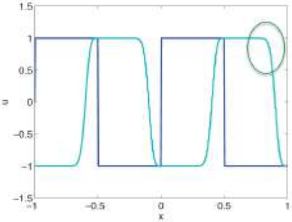
Lax-Friedrichs scheme

$$Q_j^{n+1} = \frac{1}{2}\left(Q_{j+1}^n + Q_{j-1}^n\right) - \frac{\triangle t a}{2\triangle x}\left(Q_{j+1}^n - Q_{j-1}^n\right).$$

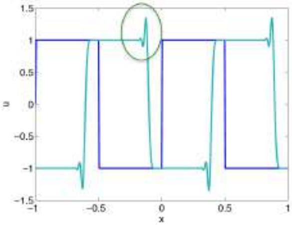
Lax-Wendroff scheme

$$Q_j^{n+1} = Q_j^n - \frac{\triangle t a}{2\triangle x}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{1}{2}\left(\frac{\triangle t a}{\triangle x}\right)^2\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right).$$

First order methods - diffusion

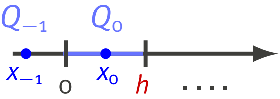


Second order methods - dispersion



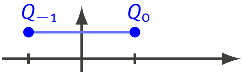
Finite Volume Discretizations - Boundary conditions

Note on boundary conditions (cf. HW2):



If boundary does not coincide with nodes $\Rightarrow$ use ghost points.

- Reflecting boundary condition at $x = 0$, e.g. $\mathbf{u}(0, t) = 0$.
  Approximated by average $\frac{Q_{-1}+Q_0}{2} = 0 \Rightarrow Q_0 = -Q_{-1}$.
- Non-reflecting boundary condition at $x = 0$: wave behaves as if there is no boundary $\Rightarrow$ Approximated by extrapolation.
  - 0'th order: $Q_{-1} = Q_0$.



  - 1'th order: $Q_{-1} = 2Q_0 - Q_1$.