High-Fidelity Simulations for Turbulent Flows

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Part VIII

High-order schemes for compressible flows

1 Introduction

2 Methods for smooth flows
- High-order centred derivatives
- Stabilization for smooth flows
- Energy-consistent schemes

3 Methods for non smooth flows

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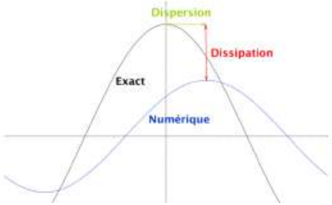
Motivation

- Numerical schemes introduce **dissipation** (**phase**) and **dispersion** (**amplitude**) errors altering the **representation** of a given solution mode
  - The smaller structures are the worst represented
  - Small structures may play a crucial role (turbulence, aeroacoustics, ..)
- Numerical errors interact with the model used to represent unresolved scales $\Longrightarrow$ Choice of the numerical method is **crucial**!
  - Remember: errors can be **one order of magnitude greater** than the subgrid modelling for low-order spatial discretizations
  - They produce a numerical cutoff well below the theoretical cutoff wavenumber associated with the grid



Two possibilities to palliate to the numerical errors:

1. **Increase grid resolution**: need for massively parallel solver, high memory load and storage, costly, ..
2. **Increase scheme resolution**: *i.e.* increase the scheme order p, and/or "optimize" the scheme so to lower the error constant C of the local truncation error $\varepsilon = C(\Delta x)^p$
   - More properly, one should look to the **convergence order** q, *i.e.*

$$E = ||w_{\text{num}} - w_{\text{ex}}|| = C_{\text{conv}} \Delta x^q$$

   - Typically $q < p$ because of boundary conditions, shocks, time integration, ..

Example: the Taylor-Green Vortex problem

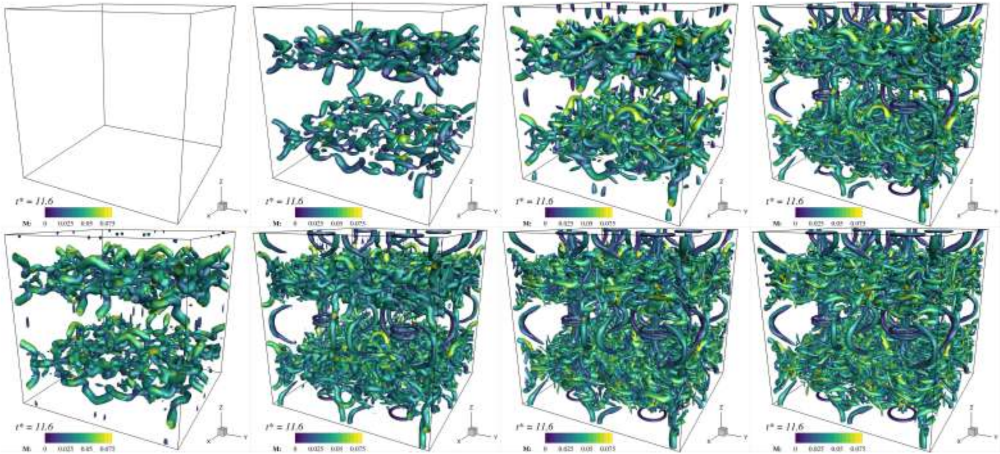
$$
\begin{cases}
u_0(x,y,z) = u_\infty \sin(x)\cos(y)\cos(z) \\
v_0(x,y,z) = -u_\infty \cos(x)\sin(y)\cos(z) \\
w_0(x,y,z) = 0 \\
p_0(x,y,z) = p_\infty + \dfrac{\rho_\infty u_\infty^2}{16}\left[\cos(2x)+\cos(2y)\right]\left[\cos(2z)+2\right]
\end{cases}
$$

- Classical test case for high-order methods
- Incompressible limit ($M < 0.3$)
- $\Omega \in [0, 2\pi]^3$
- Visualization of the $Q-$criterion:

64^3 | 128^3 | 256^3 | 512^3

3^{rd}-order scheme

9^{th}-order scheme



Problem statement

Hyperbolic system of conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \boldsymbol{w}\,\mathrm{d}\Omega + \sum_{i=1}^{3}\int_{\partial\Omega}\left(\boldsymbol{F}_i^{e} - \boldsymbol{F}_i^{v}\right) n_i\,\mathrm{d}S = 0 \quad \text{with} \quad \boldsymbol{w} = \begin{bmatrix}\rho\\ \rho u_j\\ \rho E\end{bmatrix} \quad \boldsymbol{F}^{e} = \begin{bmatrix}\rho u_i\\ \rho u_i u_j + p\delta_{ij}\\ \rho u_i H\end{bmatrix} \quad \boldsymbol{F}^{v} = \begin{bmatrix}0\\ \tau_{ij}\\ \tau_{ik}u_k - q_i\end{bmatrix}$$

- **Smooth flow assumption**: NSE can be recast as $\dfrac{\partial \boldsymbol{w}}{\partial t} + \sum_{i=1}^{3}\dfrac{\partial \boldsymbol{F}_i^{e}}{\partial x_i} - \sum_{i=1}^{3}\dfrac{\partial \boldsymbol{F}_i^{v}}{\partial x_i} = 0$
- **Euler equations** ($\boldsymbol{F}_i^{v} = 0$) have **two important properties** for the development of numerical methods:
  1. **Hyperbolicity**: Recast in characteristic form, the projection in any direction gives rise to a system of coupled wave-like eqs, motivating the study of the model 1D scalar conservation law

$$\frac{\partial w}{\partial t} + \frac{\partial F(w)}{\partial x} = \frac{\partial w}{\partial t} + a(w)\frac{\partial w}{\partial x} = \nu\frac{\partial^2 w}{\partial x^2}$$

  2. **Conservation properties**: Apart from conservation of integrals of $\boldsymbol{w}$ components, kinetic energy (KE) is also conserved as shown from balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\frac{u_k u_k}{2}\,\mathrm{d}\Omega = -\int_{\partial\Omega}\left(\rho\frac{u_k u_k}{2} + p\right) u_i n_i\,\mathrm{d}S + \int_\Omega p\frac{\partial u_i}{\partial x_i}\,\mathrm{d}\Omega$$

     - Varies only for boundary flux or volumetric pressure work: **convective terms do not cause variations**
     - Lead to attempt of building schemes that enforce "KE preservation" **in the discrete sense**
     - Similar considerations for thermodynamic **entropy**-related functions (Harten, 1983)

Common choices for high-resolution numerical methods

- **Spectral methods**: Fourier for homogeneous dirs, Chebychev/Legendre polyn. for inhomogeneous dirs
  - ✔ Spectral resolution
  - ✘ Simple geometries, incompressible flows
- **High-order centred schemes**: at least 4th-order; spectral resolution enhanced by minimizing the dispersion error (reducing the formal order), or by using Padé-like fractions (compact schemes)
  - ✔ High-order straightforward to achieve
  - ✘ Non-dissipative, need stabilization
- **High-order non-centred schemes**: flux decomposition schemes (Roe/AUSM with MUSCL reconstr.), ENO/WENO schemes, OSMP, ...
  - ✔ Stabilization embedded in the scheme
  - ✘ The intrinsic dissipation can be harmful for LES
- **Energy/Entropy-consistent schemes**: skew-symmetric splitting for the convective term ensuring semidiscrete KE/entropy preservation
  - ✔ Stabilization embedded in the scheme
  - ✘ Not adapted to shocked flows

Compressible vs Incompressible

Methods for incompressible flows not suited for compressible (especially low-M) ones

$$\frac{\Delta t_c}{\Delta t_i} = \frac{u}{u+c} = \frac{M}{1+M}$$

$\Delta t_c \ll \Delta t_i$ for $M \to 0$

Smooth vs shocked compressible flows

- Standard smooth-flow discretizations can cause strong Gibbs oscillations in presence of shocks
- Shocked-flow methods exhibit excessive numerical dissipation for smooth flows

$\Longrightarrow$ **two distinct classes exist!**

The perfect scheme for any flow problem does not exist, it depends on what you are looking for!

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High-order standard centred differences

$$\frac{\partial f}{\partial x}(x_0) = \frac{1}{\Delta x}\sum_{j=-N}^{N} a_j[f(x_0+j\Delta x))] = \frac{1}{\Delta x}\sum_{j=1}^{N} a_j[f(x_0+j\Delta x)-f(x_0-j\Delta x)] \quad \text{where} \quad a_j = -a_{-j}$$

All the terms of the Taylor expansion of f are cancelled until the order Δx^{2N-1} included:

$$f(x_0+j\Delta x) = \cancel{f(x_0)} + j\Delta x f'(x_0) + \cancel{\frac{(j\Delta x)^2}{2!}} f''(x_0) + \frac{(j\Delta x)^3}{3!} f'''(x_0) + \cancel{\frac{(j\Delta x)^4}{4!} f^{(iv)}}(x_0) + \ldots$$

$$f(x_0-j\Delta x) = \cancel{f(x_0)} - j\Delta x f'(x_0) + \cancel{\frac{(j\Delta x)^2}{2!}} f''(x_0) - \frac{(j\Delta x)^3}{3!} f'''(x_0) + \cancel{\frac{(j\Delta x)^4}{4!} f^{(iv)}}(x_0) + \ldots$$

thus

$$\frac{\partial f}{\partial x}(x_0) = \frac{1}{\Delta x}\sum_{j=1}^{N} a_j \left[2j\Delta x f'(x_0) + \frac{2j^3\Delta x^3}{3!} f'''(x_0) + \frac{2j^5\Delta x^5}{5!} f^{(v)}(x_0) + \frac{2j^7\Delta x^7}{7!} f^{(vii)}(x_0) + \ldots\right]$$

$$\begin{cases} \sum_{j=1}^{N} 2ja_j &= 1 \\ \sum_{j=1}^{N} j^3 a_j &= 0 \\ \quad\vdots & \\ \sum_{j=1}^{N} j^{2N-1} a_j &= 0 \end{cases} \quad \text{and} \quad a_j = -a_{-j}$$

$2N+1$ **points,** N **relations, order** $2N$

- Use additional points to increase the formal order of accuracy of the approximation
- Explicit recursive correction of the dispersive error
- Non-dissipative, dispersive at order 2N+1
  - **Need for numerical stabilization**

High-order standard centred differences - delta notation

Recursive correction of the dispersive error: derivation in delta form

3-points, 2nd-order centred scheme: $\dfrac{\partial f}{\partial x} = f_j' \approx \dfrac{f_{j+1} - f_{j-1}}{2\Delta x} = \dfrac{\delta \mu f_j}{\Delta x}$

$$\begin{cases} f_{j+1} = \cancel{f_j} + \Delta x f_j' + \cancel{\dfrac{\Delta x^2}{2}} f_j'' + \dfrac{\Delta x^3}{6} f_j''' + \dots \\ f_{j-1} = \cancel{f_j} - \Delta x f_j' + \cancel{\dfrac{\Delta x^2}{2}} f_j'' - \dfrac{\Delta x^3}{6} f_j''' + \dots \end{cases} \implies (\star) \quad f_j' = \frac{\delta \mu f_j}{\Delta x} - \frac{\Delta x^2}{6} f_j''' + \dots$$

Discretization of n-th deriv. in delta form: $f_j^n = \dfrac{\delta^n \mu f_j}{\Delta x^n} \implies f_j''' \approx \dfrac{\delta^3 \mu f_j}{\Delta x^3} = \dfrac{f_{j+2} - 2f_{j+1} + 2f_{j-1} - f_{j-2}}{2\Delta x^3} \quad (\bullet)$

$$\begin{cases} f_{j+2} = f_j + 2\Delta x f_j' + 4\dfrac{\Delta x^2}{2} f_j'' + 8\dfrac{\Delta x^3}{6} f_j''' + 16\dfrac{\Delta x^4}{24} f^{iv} + 32\dfrac{\Delta x^5}{120} f^{v} + \dots \\ -2f_{j+1} = -2f_j - 2\Delta x f_j' - 2\dfrac{\Delta x^2}{2} f_j'' - 2\dfrac{\Delta x^3}{6} f_j''' - 2\dfrac{\Delta x^4}{24} f^{iv} - 2\dfrac{\Delta x^5}{120} f^{v} + \dots \\ 2f_{j-1} = 2f_j - 2\Delta x f_j' + 2\dfrac{\Delta x^2}{2} f_j'' - 2\dfrac{\Delta x^3}{6} f_j''' + 2\dfrac{\Delta x^4}{24} f^{iv} - 2\dfrac{\Delta x^5}{120} f^{v} + \dots \\ -f_{j-2} = -f_j + 2\Delta x f_j' - 4\dfrac{\Delta x^2}{2} f_j'' + 8\dfrac{\Delta x^3}{6} f_j''' - 16\dfrac{\Delta x^4}{24} f^{iv} + 32\dfrac{\Delta x^5}{120} f^{v} + \dots \end{cases}$$

$$\implies f_j''' = \frac{f_{j+2} - 2f_{j+1} + 2f_{j-1} - f_{j-2}}{2\Delta x^3} - \frac{\Delta x^2}{4} f^{v} + \dots$$

High-order standard centred differences - delta notation (2)

Plugging (•) in ($\star$), one has a 5-point, 4$^{\text{th}}$-order centred scheme:

$$f_j' \approx \frac{\delta\mu f_j}{\Delta x} - \frac{\Delta x^2}{6}\frac{\delta^3\mu f_j}{\Delta x^3} = \left(\mathcal{I} - \frac{1}{6}\delta^2\right)\frac{\delta\mu f_j}{\Delta x} = \frac{-f_{j+2} + 8f_{j+1} - 8f_{j-1} + f_{j-2}}{12\Delta x}$$

$$\begin{cases} -\ f_{j+2} = -\ f_j - 2\Delta x f_j' - 4\frac{\Delta x^2}{2}f_j'' - 8\frac{\Delta x^3}{6}f_j''' - 16\frac{\Delta x^4}{24}f^{iv} - 32\frac{\Delta x^5}{120}f^{v} - 64\frac{\Delta x^6}{720}f^{vi} - 128\frac{\Delta x^7}{5040}f^{vii} + \dots \\ 8f_{j+1} = 8f_j + 8\Delta x f_j' + 8\frac{\Delta x^2}{2}f_j'' + 8\frac{\Delta x^3}{6}f_j''' + 8\frac{\Delta x^4}{24}f^{iv} + 8\frac{\Delta x^5}{120}f^{v} + 8\frac{\Delta x^6}{720}f^{vi} + 8\frac{\Delta x^7}{5040}f^{vii} + \dots \\ -8f_{j-1} = -8f_j + 8\Delta x f_j' - 8\frac{\Delta x^2}{2}f_j'' + 8\frac{\Delta x^3}{6}f_j''' - 8\frac{\Delta x^4}{24}f^{iv} + 8\frac{\Delta x^5}{120}f^{v} - 8\frac{\Delta x^6}{720}f^{vi} + 8\frac{\Delta x^7}{5040}f^{vii} + \dots \\ f_{j-2} = f_j - 2\Delta x f_j' + 4\frac{\Delta x^2}{2}f_j'' - 8\frac{\Delta x^3}{6}f_j''' + 16\frac{\Delta x^4}{24}f^{iv} - 32\frac{\Delta x^5}{120}f^{v} + 64\frac{\Delta x^6}{720}f^{vi} - 128\frac{\Delta x^7}{5040}f^{vii} + \dots \end{cases}$$

$$f_j' \approx \frac{\delta\mu f_j}{\Delta x} - \frac{\Delta x^2}{6}\frac{\delta^3\mu f_j}{\Delta x^3} + \frac{\Delta x^4}{30}f_j^{v} + \dots$$

Recursive correction to obtain higher orders:

$$f_j^{v} = \frac{\partial^5 f}{\partial x^5} = \frac{\delta^5\mu f}{\Delta x^5} + \mathcal{O}(\Delta x^2) \quad \Longrightarrow \quad f_j' \approx \left(\mathcal{I} - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4\right)\frac{\delta\mu f_j}{\Delta x} = 0 \quad \Longrightarrow \quad \varepsilon = \frac{\Delta x^6}{140}f_j^{vii} + \mathcal{O}(\Delta x^8)$$

High-order standard centred differences - delta notation (3)

General formula for Directional Non Compact (DNC) centred schemes of any order:

$$\frac{\partial f}{\partial x} \approx \left(\mathcal{I} - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4 - \frac{1}{140}\delta^6 + \frac{1}{630}\delta^8 + ..\right)\frac{\delta\mu f_j}{\Delta x} = \left(\mathcal{I} - \sum_{p=0}^{P}(-1)^p a_p \delta^{2p+2}\right)\frac{\delta\mu f_j}{\Delta x}$$

- Approximation of order $2(P+2)$, dispersive at order $2(P+2)+1$, using $2(P+2)+1$ points in each dir.
- Non-dissipative schemes $\Longrightarrow$ cannot damp spurious oscillations $\Longrightarrow$ need for numerical dissipation

Fourier analysis

- On a grid with spacing Δx we can resolve wavenumbers k for which $|k\Delta x| \leq \pi$
- Order of accuracy tell us what happens when $k\Delta x \to 0$, but how the methods handle wavenumbers that are **not so well resolved**?

- One should check the **dispersion relation** in $0 \leq k\Delta x \leq \pi$. How? **Fourier analysis**! It allows to:
  - Control the **resolvability limits** of the scheme on a mesh
  - Know the cutoff between well- and badly-resolved k

Consider the semi-discrete linear advection $\dfrac{\mathrm{d}w_j}{\mathrm{d}t} = -aDw_j$ and a wave $w_j(t) = \widehat{w}_j(t)e^{ikx_j} = \widehat{w}_j(t)e^{ikj\Delta x}$

- **Exact representation** of first derivative:

$$\widehat{\frac{\partial w}{\partial x}} = ik\widehat{w}$$

- **Numerical FD approximation** of first derivative:

$$\frac{\partial w}{\partial x} = \frac{1}{\Delta x}\sum_{j=-N}^{N} a_j[w(x_0+j\Delta x))]$$

$$\implies \widehat{\frac{\partial w}{\partial x}} = \left(\frac{1}{\Delta x}\sum_{j=-N}^{N} a_j e^{ikj\Delta x}\right)\widehat{w} = ik^\star \widehat{w}$$

- Thus one has

$$k^\star = \frac{1}{i\Delta x}\sum_{j=-N}^{N} a_j e^{ikj\Delta x} = \frac{2}{\Delta x}\sum_{j=1}^{N} a_j \sin(jk\Delta x)$$

$k^\star$ is the **modified wavenumber**

- It measures the accuracy with which derivatives are represented in wavenumber space
- $k^\star\Delta x$ is the **reduced wavenumber**

High-order finite-difference schemes (2): Fourier analysis

$$(\bullet) \quad k^\star = \frac{2}{\Delta x}\sum_{j=1}^{N} a_j \sin(jk\Delta x)$$

- **Analytical solution** of semi-discrete equation:

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = -aDw_j \quad \Longrightarrow \quad \frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -iak^\star\widehat{w}$$

$$(\dagger) \quad \widehat{w}(t) = \widehat{w}^0 e^{-iak^\star t} = \widehat{w}^0 e^{-ia\mathrm{Re}(k^\star)t} e^{a\mathrm{Im}(k^\star)t}$$

- $\mathrm{Im}(k^\star) = 0$: constant amplitude, damping for <0
- $\mathrm{Im}(k^\star) = 0$ for centred schemes (*e.g.* $\bullet$)
- $\mathrm{Im}(k^\star) \neq 0$ for upwind-biased schemes

$\Longrightarrow$ **not ideal candidates** for DNS!

From ($\dagger$) with $a = 1$, one has

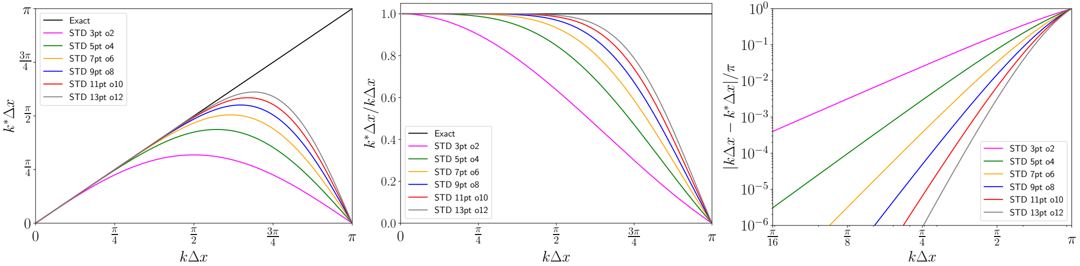
$$w(t) = \widehat{w}(t)e^{ikx_j} = \widehat{w}^0 e^{ikx_j - ik^\star t} = \widehat{w}^0 e^{i(kx_j - k^\star t)}$$

Phase velocity for the k mode:

$$c_p(k) = \frac{\omega(k)}{k} = \frac{k^\star}{k} = \frac{k^\star\Delta x}{k\Delta x}$$

Comparisons usually shown in terms of

1. $k^\star\Delta x$ or $\dfrac{k^\star\Delta x}{k\Delta x}$ vs $k\Delta x$ (**phase error**)
2. $\dfrac{|k^\star\Delta x - k\Delta x|}{\pi}$ vs $k\Delta x$ (**dispersion error**)



Optimized schemes: Dispersion Relation Preserving (DRP)

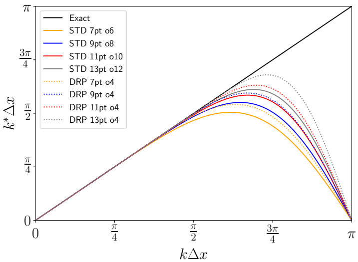
Idea (Tam and Webb, 1993): instead of increasing the formal order of accuracy, **minimize the dispersion error**

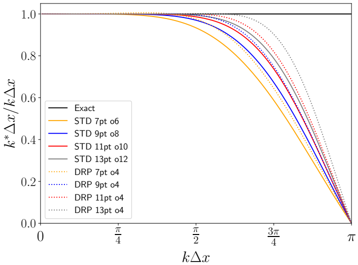
$$E = \int_{\ln(k\Delta x)_{\mathrm{lo}}}^{\ln(k\Delta x)_{\mathrm{hi}}} |k^\star \Delta x - k\Delta x| \, \mathrm{d}[\ln(k\Delta x)] \implies \frac{\partial E}{\partial a_j} = 0$$

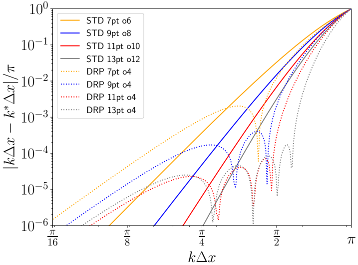
- $(k\Delta x)_{\mathrm{lo}}$ and $(k\Delta x)_{\mathrm{hi}}$ to be chosen
- To obtain an optimized scheme on $2N+1$ points of order $2M$ ($M < N$), one can use:
  - M relations to cancel Taylor terms up to Δx^{2M-1}
  - $M-N$ relations of type $\partial E/\partial a_j = 0$ for $j = 1, .., M-N$
  - Solve a system of N equations with N unknowns a_j

Example (Bogey and Bailly, 2004)
Optimized scheme on 11 points and order 4:

$$\begin{cases} \sum_{j=1}^{N} 2ja_j = 1 \\ \sum_{j=1}^{N} j^3 a_j = 0 \\ \dfrac{\partial E}{\partial a_1} = 0 \\ \dfrac{\partial E}{\partial a_2} = 0 \\ \dfrac{\partial E}{\partial a_3} = 0 \end{cases} \quad \text{with} \quad \begin{cases} (k\Delta x)_{\mathrm{lo}} = \dfrac{\pi}{16} \\ (k\Delta x)_{\mathrm{hi}} = \dfrac{\pi}{2} \end{cases}$$







Compact directional approximations (I)

- Standard way to obtain high-order accuracy is **adding more points**, but large stencils can be obtained (problems on the boundaries)
- Alternative: **compact schemes**. Starting from Taylor series expansion:

$$f_{j+1} = f_j + \Delta x f_j' + \frac{\Delta x^2}{2} f_j'' + \frac{\Delta x^3}{6} f_j''' + \frac{\Delta x^4}{24} f_j^{iv} + \mathcal{O}(\Delta x^5) \quad (1)$$

$$f_{j-1} = f_j - \Delta x f_j' + \frac{\Delta x^2}{2} f_j'' - \frac{\Delta x^3}{6} f_j''' + \frac{\Delta x^4}{24} f_j^{iv} + \mathcal{O}(\Delta x^5) \quad (2)$$

First derivative

1. Adding (1) + (2) and taking first derivative:

$$f_{j+1}' + f_{j-1}' = 2f_j' + \Delta x^2 f_j''' + \frac{\Delta x^4}{12} f_j^{v} + \mathcal{O}(\Delta x^6)$$

2. Subtracting (2) from (1):

$$f_{j+1} - f_{j-1} = 2\Delta x f_j' + 2\frac{\Delta x^3}{6} f_j''' + \mathcal{O}(\Delta x^5)$$

3. Eliminating f_j''':

$$f_{j+1}' + 4f_j' + f_{j-1}' = \frac{3}{\Delta x}(f_{j+1} - f_{j-1}) + \mathcal{O}(\Delta x^4)$$

Second derivative

1. Adding (1) + (2):

$$f_{j+1} + f_{j-1} = 2f_j + \Delta x^2 f_j'' + \frac{\Delta x^4}{12} f_j^{iv} + \mathcal{O}(\Delta x^6)$$

2. Taking second derivative:

$$f_{j+1}'' + f_{j-1}'' = 2f_j'' + \Delta x^2 f_j^{iv} + \frac{\Delta x^4}{12} f_j^{vi} + \mathcal{O}(\Delta x^6)$$

3. Eliminating f_j^{iv}:

$$f_{j+1}'' + 10 f_j'' + f_{j-1}'' = \frac{12}{\Delta x^2}(f_{j+1} - 2f_j + f_{j-1}) + \mathcal{O}(\Delta x^4)$$

Find f_j' and f_j'' **solving a tridiagonal system**

Compact directional approximations (II)

From Lele (1992): $\beta f'_{i-2} + \alpha f'_{i-1} + f'_i + \alpha f'_{i+1} + \beta f'_{i+2} = a\,\dfrac{f_{i+1}-f_{i-1}}{2\Delta x} + b\,\dfrac{f_{i+2}-f_{i-2}}{4\Delta x} + c\,\dfrac{f_{i+3}-f_{i-3}}{6\Delta x}$

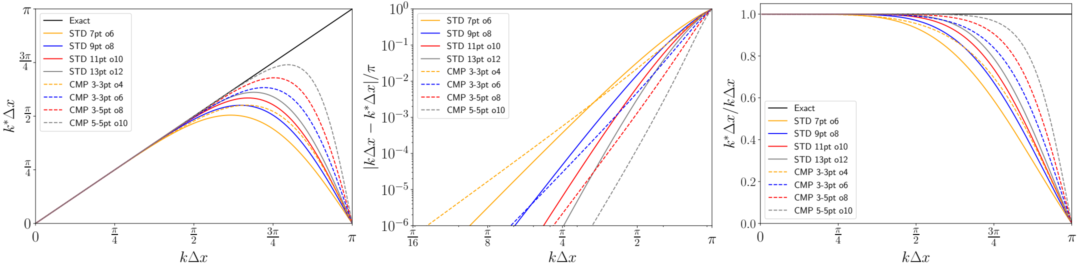
The terms of the Taylor expansion are canceled:

$$a+b+c = 1+2\alpha+2\beta \qquad \text{(order 2)}$$
$$a+2^2b+3^2c = 2\frac{3!}{2!}(\alpha+2^2\beta) \qquad \text{(order 4)}$$
$$a+2^4b+3^4c = 2\frac{5!}{4!}(\alpha+2^4\beta) \qquad \text{(order 6)}$$
$$a+2^6b+3^6c = 2\frac{7!}{6!}(\alpha+2^6\beta) \qquad \text{(order 8)}$$
$$a+2^8b+3^8c = 2\frac{9!}{8!}(\alpha+2^8\beta) \qquad \text{(order 10)}$$

Modified wavenumber:

$$k^\star\Delta x = \frac{a\sin(k\Delta x) + (b/2)\sin(2k\Delta x) + (c/3)\sin(3k\Delta x)}{1+2\alpha\cos(k\Delta x)+2\beta\cos(2k\Delta x)}$$

- Standard schemes retrieved for $\alpha=\beta=0$
- Discretization stencil more compact
- Pentadiagonal ($\beta\neq 0$) or tridiagonal ($\beta=0$) system has to be solved



Spectral analysis for second derivative

The same analysis may be performed for **second derivatives**. Following the same study, one has

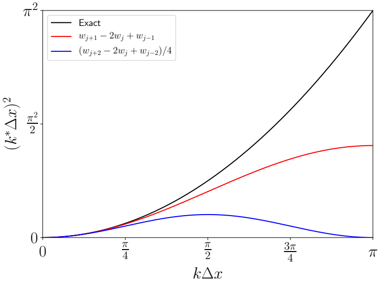
- **Exact** and **numerical** representation of second derivative:

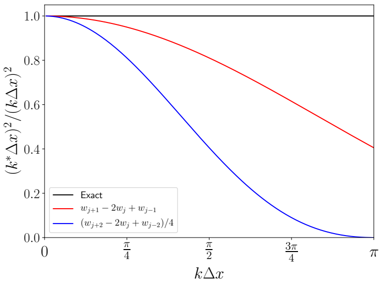
$$\widehat{\frac{\partial^2 w}{\partial x^2}} = -k^2 \widehat{w} \quad \text{and} \quad \widehat{\frac{\partial^2 w}{\partial x^2}} = \left[\frac{1}{(\Delta x)^2}\sum_{j=-N}^{N} a_j e^{ikj\Delta x}\right]\widehat{w} = -k^{\star 2}\widehat{w} \quad \Longrightarrow \quad k^{\star 2} = -\frac{1}{(\Delta x)^2}\sum_{j=-N}^{N} a_j e^{ikj\Delta x}$$

Consider 2 approximations, obtained by Taylor series and successive applications of 1$^{\text{st}}$ der., respectively:

$$\frac{\partial^2 w_j}{\partial x^2} = \frac{\delta^2 w_j}{\Delta x^2} = \frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} \qquad \frac{\partial^2 w_j}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial w_j}{\partial x}\right] = \frac{(\delta\mu)^2 w_j}{\Delta x^2} = \frac{w_{j+2} - 2w_j + w_{j-2}}{4\Delta x^2}$$

$$\widehat{\frac{\partial^2 w}{\partial x^2}} = 2[1 - \cos(k\Delta x)] \qquad \widehat{\frac{\partial^2 w}{\partial x^2}} = \sin^2(k\Delta x)$$





- Both 2$^{\text{nd}}$-order accurate, but different spectral responses!
- For the latter, effective diff. coeff. $\to 0$ for $k\Delta x \to \pi$
  - Does not stabilize the solution
  - Effective dissipation $\to 0$ whatever the ν_{sgs} model is

  Not acceptable for LES!

High-order dissipation term

- Appropriate values for a_m and b_l allow to:
  - **Maximize formal accuracy** (the minimum truncation error being $\mathcal{O}(\Delta x^{2(L+M)})$)
  - **Shape the spectral response** of the scheme and improve the representation of the Fourier modes with the highest wavenumbers supported by the grid

$$\boxed{\sum_{m=-M}^{M} a_m Df_{j+m} = \frac{1}{\Delta x} \sum_{l=-L}^{L} b_l f_{j+l}}$$

- Centred FD for $b_{-l}=b_l$ and $a_{-m}=a_m \implies$ null dissipation error in linear setting
  - Grid-to-grid oscillations or **wiggles** (every 2 points, *i.e.* $k\Delta x = \pi$) are not resolved by centred FD
  - $\hookrightarrow$ Can appear near stiff velocity gradients or discontinuities (such as BCs) and contaminate the solution

How to control them?

1. **Selective filtering**: use of a centred (thus non dispersive) filter to dissipate only high-frequencies (Gaitonde and Visbal, 2000; Bogey and Bailly, 2004):

$$f^{\text{filtered}}(x_0) = f(x_0) - \sigma_d D_f(x_0) \quad \text{with} \quad 0 \leq \sigma_d \leq 1 \quad \text{and} \quad D_f(x_0) = \sum_{j=-N}^{N} d_j f(x_0 + j\Delta x)$$

2. **Artificial dissipation** (Jameson et al., 1981; Kim and Lee, 2001): $\dfrac{\partial w}{\partial t} + \dfrac{\delta F}{\Delta x} = 0$ with $F = \overbrace{H}^{\text{centred approx.}} - \overbrace{D}^{\text{dissipative term}}$

3. **Skew-symmetric formulations**: Employ energy-consistent schemes using a skew-symmetric splitting for the convective term (Blaisdell et al., 1996; Ducros et al., 2000; Honein and Moin, 2004; Pirozzoli, 2011)

$$\frac{\partial (f u_j)}{\partial x_j} = \frac{1}{2}\frac{\partial (f u_j)}{\partial x_j} + \frac{1}{2} u_j \frac{\partial f}{\partial x_j} + \frac{1}{2} f \frac{\partial u_j}{\partial x_j}$$

High-order selective standard filters

The terms of the Taylor expansion are canceled until the order Δx^{2N-1}:

$$D_f(x_0) = d_0 f(x_0) + \sum_{j=1}^{N} d_j \left[f(x_0 + j\Delta x) + f(x_0 - j\Delta x)\right]$$
$$= d_0 f(x_0) + \sum_{j=1}^{N} d_j \left[2f(x_0) + j^2 \Delta x^2 f''(x_0) + \frac{2j^4\Delta x^4}{4!} f^{(4)}(x_0) + \frac{2j^6\Delta x^6}{6!} f^{(6)}(x_0) + \dots\right]$$

$$\begin{cases} d_0 + 2\sum_{j=1}^{N} d_j = 0 \\ \sum_{j=1}^{N} j^2 d_j = 0 \\ \vdots \\ \sum_{j=1}^{N} j^{2N-2} d_j = 0 \end{cases}$$

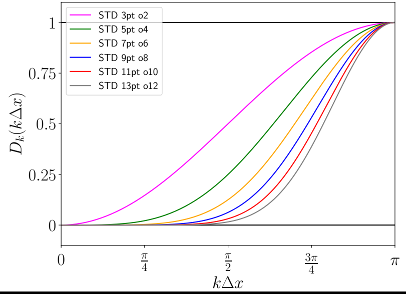
N **relations** + in Fourier's space:

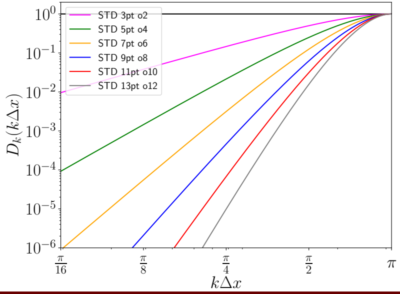
- $\left(D_k(0) = 0 \implies d_0 + 2\sum_{j=1}^{N} d_j = 0\right)$ (Redundant, condition already used)
- $D_k(\pi) = 1 \implies d_0 + 2\sum (-1)^j d_j = 1$

Damping function of centred exp. flt.:

$$D_k(k\Delta x) = d_0 + \sum_{j=1}^{N} 2d_j \cos(jk\Delta x)$$

$2N+1$ **points,** $N+1$ **relations order** $2N$

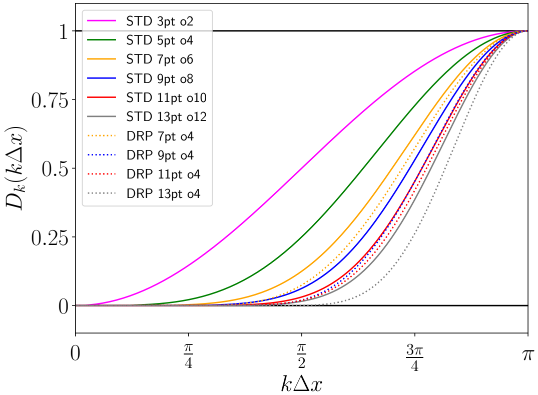


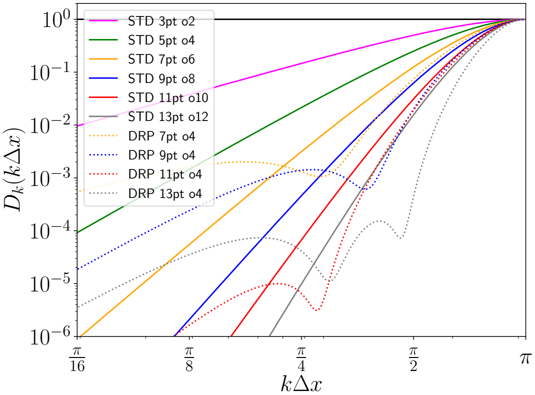


High-order selective filters (2): optimized filters

Optimized filters: minimization of the dissipation (Bogey and Bailly, 2004)

$$E = \int_{\ln(\pi/16)}^{\ln(\pi/2)} D_k\,(k\Delta x)\ \mathrm{d}[\ln(k\Delta x)]$$

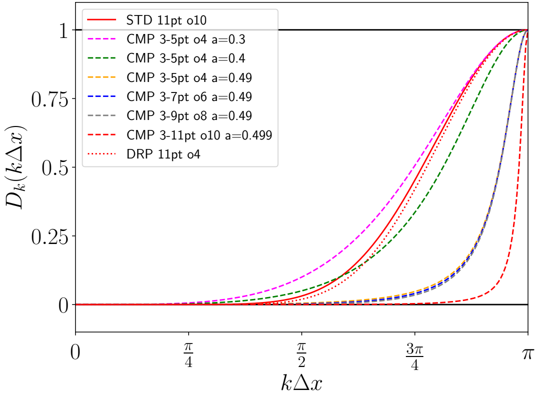


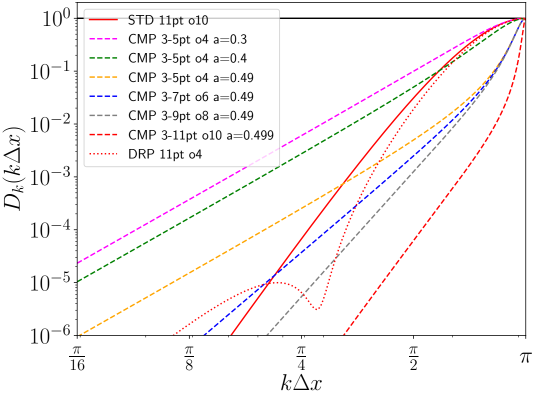


High-order selective filters (3): compact filters

$$\beta f^f_{i-2} + \alpha f^f_{i-1} + f^f_i + \alpha f^f_{i+1} + \beta f^f_{i+2} = a f_i + \frac{b}{2}\left(f_{i+1} + f_{i-1}\right) + \frac{c}{2}\left(f_{i+2} + f_{i-2}\right) + \frac{d}{2}\left(f_{i+3} + f_{i-3}\right)$$

Damping function: $$D_k\left(k\Delta x\right) = \frac{a + b\cos\left(k\Delta x\right) + c\cos\left(2k\Delta x\right) + d\cos\left(3k\Delta x\right)}{1 + 2\alpha\cos\left(k\Delta x\right) + 2\beta\cos\left(2k\Delta x\right)}$$



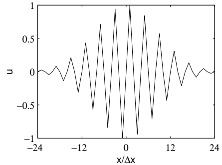


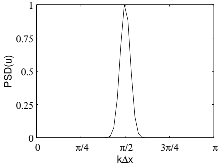
- Tridiagonal filters on $2N+1$ points of order $2N$ (Gaitonde and Visbal, 2000): $\beta = 0, \quad 0.3 \leq \alpha_f < 0.5$
- Optimized pentadiagonal filter (Lele, 1992), 6th-order + 2 constraints: $\frac{\mathrm{d}^2 D_k(\pi)}{\mathrm{d}(k\Delta x)^2} = 0$ and $\frac{\mathrm{d}^4 D_k(\pi)}{\mathrm{d}(k\Delta x)^4} = 0$

Test case: Solution of the advection equation

From Bogey and Bailly (2004):

- Advection equation: $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$ with $a = 1$ and $\Delta t = \text{CFL}\,\dfrac{\Delta x}{a}$
- Initial perturbation: $u(x) = \sin\left(\dfrac{2\pi x}{4\Delta x}\right)\exp\left(-\ln 2\left(\dfrac{x}{9\Delta x}\right)^2\right)$
  - principal wavenumber for $k\Delta x = \dfrac{\pi}{2}$ $(\lambda_0 = 4\Delta x)$
- Propagation over a large distance: $200\Delta x = 50\lambda_0$
- Numerical error: $e_{\text{num}} = \left(\sum (u_{\text{calc}} - u_{\text{exact}})^2 / \sum u_{\text{exact}}^2\right)^{1/2}$



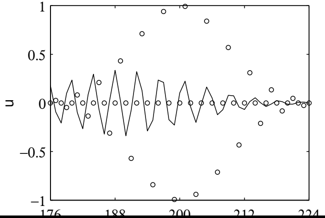


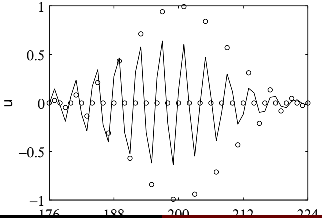
Solutions obtained with optimized Runge-Kutta scheme with 6 substeps and:

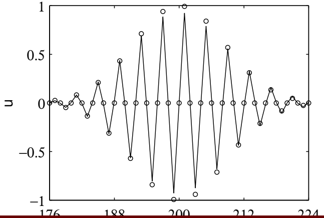
opt. 9 pts FD + opt. 9 pts SF
$\Longrightarrow$ $e_{\text{num}} = 0.905$

opt. 11 pts FD + opt. 11 pts SF
$\Longrightarrow$ $e_{\text{num}} = 0.488$

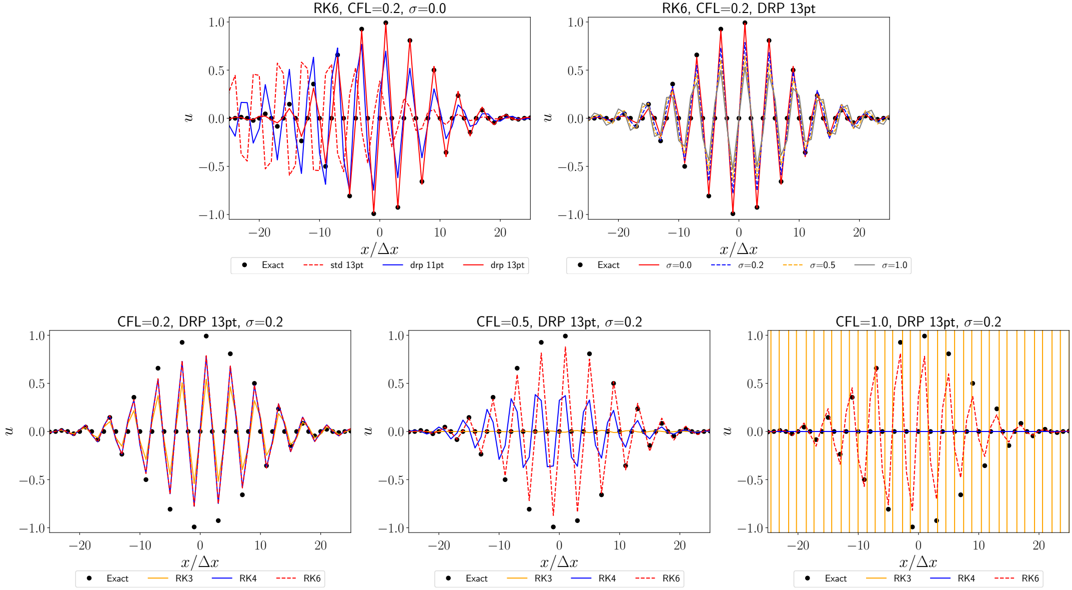
opt. 13 pts FD + opt. 13 pts SF
$\Longrightarrow$ $e_{\text{num}} = 0.077$







Test case: Solution of the advection equation (2)



Artificial Dissipation: DNC schemes

Dissipative DNC schemes may be constructed via an **upwind recursive correction** of the truncation error using the flux of a first-order (dissipative) scheme:

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} + \frac{\delta F}{\Delta x} = 0 \quad \text{with} \quad F = \underbrace{H}_{\substack{\textit{central}\\\textit{approx}}} - \underbrace{D}_{\substack{\textit{dissipative}\\\textit{term}}} = \mu f - \frac{1}{2} Q \delta w \qquad Q \text{ dissipation matrix}$$

By doing that, one loses one order but **introduce dissipation**. General formula for dissipative DNC schemes:

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} + \left(\mathcal{I} - \sum_{p=0}^{P} (-1)^p a_p \delta^{2p+2}\right) \frac{\delta \mu f_j}{\Delta x} = \frac{\delta D}{\Delta x} = \left((-1)^{P+1} \frac{a_P}{2} |Q| \delta^{2P+3}\right) \frac{\delta w}{\Delta x}$$

- Scheme of order $2P+3$, using $2(P+2)+1$ points in each direction. Examples for $P = 0, 1, 2, 3$:

$$F_{j+\frac{1}{2}} = \left[\left(\mathcal{I} - \frac{1}{6}\delta^2\right)\mu f + \frac{1}{12}|Q|\delta^3 w\right]_{j+\frac{1}{2}} \qquad \text{(order 3, } P{=}0\text{)}$$

$$F_{j+\frac{1}{2}} = \left[\left(\mathcal{I} - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4\right)\mu f - \frac{1}{60}|Q|\delta^5 w\right]_{j+\frac{1}{2}} \qquad \text{(order 5, } P{=}1\text{)}$$

$$F_{j+\frac{1}{2}} = \left[\left(\mathcal{I} - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4 - \frac{1}{140}\delta^6\right)\mu f + \frac{1}{280}|Q|\delta^7 w\right]_{j+\frac{1}{2}} \qquad \text{(order 7, } P{=}2\text{)}$$

$$F_{j+\frac{1}{2}} = \left[\left(\mathcal{I} - \frac{1}{6}\delta^2 + \frac{1}{30}\delta^4 - \frac{1}{140}\delta^6 + \frac{1}{630}\delta^8\right)\mu f - \frac{1}{1260}|Q|\delta^9 w\right]_{j+\frac{1}{2}} \qquad \text{(order 9, } P{=}3\text{)}$$

Energy-consistent schemes: conservation properties

$$(\star) \quad \frac{\partial w}{\partial t} + \frac{\partial F(w)}{\partial x} = \frac{\partial w}{\partial t} + a(w)\frac{\partial w}{\partial x} = \nu \frac{\partial^2 w}{\partial x^2} \quad \text{with } F \text{ vanishing at } x \to \pm\infty$$

Invariants in the inviscid limit:

1. **Primary conservation property**:

$$\int_{-\infty}^{+\infty} (\star)\,\mathrm{d}x \implies \frac{\mathrm{d}\int w\,\mathrm{d}x}{\mathrm{d}t} = 0 \implies w \text{ conserved}$$

- **Not affected** by discontinuous w or finite viscosity

2. **Secondary conservation property**:

$$\int_{-\infty}^{+\infty} w \times (\star)\,\mathrm{d}x \implies \frac{\mathrm{d}\int \frac{w^2}{2}\,\mathrm{d}x}{\mathrm{d}t} = 0 \implies \frac{w^2}{2} \text{ conserved}$$

- **Destroyed** by discontinuous w or finite viscosity

- If the numerical discretization can be recast in the locally conservative form:

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\left(\widehat{f}_{j+1/2} - \widehat{f}_{j-1/2}\right)$$

  - The primary conservation property holds
  - The numerical flux respects the **telescopic property**, *i.e.* at the semi-discrete level:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_j w_j = 0$$

- Similar considerations for the second invariant:

$$\frac{\mathrm{d}\frac{w_j}{2}}{\mathrm{d}t} = -\frac{1}{\Delta x}\left(\widehat{g}_{j+1/2} - \widehat{g}_{j-1/2}\right)$$

Conservation of Kinetic energy (I)

Consider the semi-discrete inviscid Burgers equation $\boxed{\dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} = 0}$ in advective and conservative form:

$$\frac{\partial w}{\partial t} = -w\frac{\partial w}{\partial x} \approx -w_j\frac{(w_{j+1} - w_{j-1})}{2\Delta x} \qquad \frac{\partial w}{\partial t} = -\frac{1}{2}\frac{\partial w^2}{\partial x} \approx -\frac{1}{2}\frac{(w_{j+1}^2 - w_{j-1}^2)}{2\Delta x}$$

- Both can be cast in locally conservative form, by defining: $\widehat{f}_{j+1/2} = \dfrac{w_j w_{j+1}}{2}$ and $\widehat{f}_{j+1/2} = \dfrac{w_j^2 + w_{j+1}^2}{4}$

$\Longrightarrow$ **Both** satisfy the primary conservation property (w conserved)

- But **neither of them** satisfy the second conservation property (w^2 not conserved)
  - It is not possible to identify a locally conservative flux form. One has indeed that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{w_j^2}{2}\,\mathrm{d}x &= \sum w_j\left[-w\frac{\partial w}{\partial x}\right]_j \Delta x \\ &= \sum\left[w_j^2\left(w_{j-1} - w_{j+1}\right)\right] \\ &= ..+w_{j-1}^2\left(w_{j-2} - w_j\right) \\ &\quad + w_j^2\left(w_{j-1} - w_{j+1}\right) + .. \neq 0\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{w_j^2}{2}\,\mathrm{d}x &= \sum w_j\left[-\frac{1}{2}\frac{\partial w^2}{\partial x}\right]_j \Delta x \\ &= \sum\left[\frac{w_j}{2}\left(w_{j-1}^2 - w_{j+1}^2\right)\right] \\ &= ..+\frac{w_{j-1}}{2}\left(w_{j-2}^2 - w_j^2\right) \\ &\quad + \frac{w_j}{2}\left(w_{j-1}^2 - w_{j+1}^2\right) + .. \neq 0\end{aligned}$$

- And if we consider a **linear combination** of the two approximations?

Conservation of Kinetic energy (II)

Combine both schemes and find the value of α for which w^2 is preserved:

$$\frac{dw_j}{dt} = -\frac{\alpha}{2\Delta x}\alpha w_j(w_{j+1} - w_{j-1}) - \frac{1-\alpha}{4\Delta x}\left(w_{j+1}^2 - w_{j-1}^2\right)$$

$$\frac{d}{dt}\int \frac{w_j^2}{2}\,dx = -\frac{1}{2}\sum\left[\alpha w_j^2(w_{j+1} - w_{j-1}) + \frac{1-\alpha}{2}w_j\left(w_{j+1}^2 - w_{j-1}^2\right)\right]$$

$$= .. + \alpha w_{j-1}^2(w_j - w_{j-2}) + \frac{1-\alpha}{2}w_{j-1}\left(w_j^2 - w_{j-2}^2\right)$$

$$+ \alpha w_j^2(w_{j+1} - w_{j-1}) + \frac{1-\alpha}{2}w_j\left(w_{j+1}^2 - w_{j-1}^2\right) + .. = 0$$

$$= .. + \alpha(w_{j-1}^2 w_j - w_{j-1}^2 w_{j-2}) + \frac{1-\alpha}{2}\left(w_{j-1}w_j^2 - w_{j-1}w_{j-2}^2\right)$$

$$+ \alpha(w_j^2 w_{j+1} - w_j^2 w_{j-1}) + \frac{1-\alpha}{2}\left(w_j w_{j+1}^2 - w_j w_{j-1}^2\right) + .. = 0 \implies \alpha + \frac{1-\alpha}{2} = 0 \implies \boxed{\alpha = \frac{1}{3}}$$

$$\frac{dw_j}{dt} = -\frac{1}{6\Delta x}\left[w_j(w_{j+1} - w_{j-1}) - \left(w_{j+1}^2 - w_{j-1}^2\right)\right] = \boxed{-\frac{1}{2\Delta x}\left[\frac{w_{j+1} + w_j + w_{j-1}}{3}(w_{j+1} - w_{j-1})\right]}$$

This conserves **both** w and w^2!

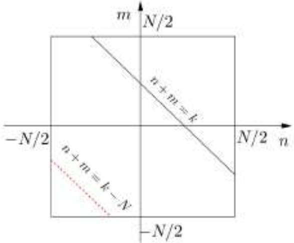
- A numerical flux can be written: $\widehat{g}_{j+1/2} = \dfrac{w_j^2 + w_j w_{j+1} + w_{j+1}^2}{6}$ such that the telescopic property holds
- **KE preservation** $\implies$ the numerical solution cannot diverge in finite time
- This is valid also for a **generic central difference operator** $\dfrac{dw_j}{dt} = -\alpha w_j D w_j - \dfrac{1-\alpha}{2}(Dw^2)_j$

Aliasing and nonlinear stability

Aliasing arises whenever two functions are multiplied on a discrete grid in physical space. Consider the Fourier expansions of u and v

$$u_j = \sum_{n=-N/2}^{N/2-1} \widehat{u}_n e^{i(2\pi j/N)n} \qquad v_j = \sum_{m=-N/2}^{N/2-1} \widehat{v}_m e^{i(2\pi j/N)m}$$

Their pointwise product is $\widehat{w}_k = \sum_{n+m=k} \widehat{u}_n \widehat{v}_m + \sum_{n+m=k\pm N} \widehat{u}_n \widehat{v}_m$



- Generation of high-freq modes that cannot be resolved on the mesh $\Longrightarrow$ **aliased** to lower-frequency modes
  - The error is large only when the modes near the highest resolvable wavenumber carry significant energy (case for LES and partially DNS)
- **Aliasing errors modified by truncation errors** (*i.e.*, Fourier modes multiplied by modified wavenumbers)
  - Spectral methods: de-aliasing techniques mandatory
  - FD: $k^\star$ decrease at high k $\Longrightarrow$ problem mitigated
- Consider two approximations:

$$N_1 = \frac{\mathrm{d}uv}{\mathrm{d}x} \qquad N_2 = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  - It can be shown that aliasing errors of N_1 and N_2 are of opposite sign!
  - $\Longrightarrow$ **Skew-symmetric form:**

$$N_3 = \frac{N_1 + N_2}{2} = \frac{1}{2}\frac{\mathrm{d}uv}{\mathrm{d}x} + \frac{1}{2}u\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{1}{2}v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  well-behaved even without de-aliasing!
  - Aliasing errors and KE preservation intimately related!

Skew-symmetric form for compressible flows

$$
\begin{aligned}
(I) \qquad & \frac{\partial \rho u_i \varphi}{\partial x_i} = \frac{1}{2}\frac{\partial \rho u_i \varphi}{\partial x_i} + \frac{1}{2}\varphi\frac{\partial \rho u_i}{\partial x_i} + \frac{1}{2}\rho u_i \frac{\partial \varphi}{\partial x_i} \\
(II) \qquad & \frac{\partial \rho u_i \varphi}{\partial x_i} = \frac{1}{2}\frac{\partial \rho u_i \varphi}{\partial x_i} + \frac{1}{2}u_i\frac{\partial \rho \varphi}{\partial x_i} + \frac{1}{2}\rho \varphi \frac{\partial u_i}{\partial x_i} \\
(III) \qquad & \frac{\partial \rho u_i \varphi}{\partial x_i} = \alpha\frac{\partial \rho u_i \varphi}{\partial x_i} + \beta\left[u_i\frac{\partial \rho \varphi}{\partial x_i} + \rho\frac{\partial u_i \varphi}{\partial x_i} + \varphi\frac{\partial \rho u_i}{\partial x_i}\right] \\
& \qquad + (1-2\alpha-2\beta)\left[\rho u_i\frac{\partial \varphi}{\partial x_i} + \rho\varphi\frac{\partial u_i}{\partial x_i} + u_i\varphi\frac{\partial \rho}{\partial x_i}\right]
\end{aligned}
$$

- (I) semidiscrete KE preservation (Honein and Moin, 2004)
- (II) minimization of the aliasing error (Blaisdell et al., 1996)
- (III) semidiscrete KE preservation for $\alpha = \beta = \frac{1}{4}$ (Kennedy and Gruber, 2008) and additional robustness

✔ All formulation yield locally conservative schemes with explicit central formulas (Pirozzoli, 2010)

✘ Recall that discrete energy conservation only applies for **smooth solutions**

- Energy is dissipated in the presence of shocks

  $\Longrightarrow$ Applications to shocked flows may yield **stable but unphysical** numerical solutions

1 Introduction

2 Methods for smooth flows
- High-order centred derivatives
- Stabilization for smooth flows
- Energy-consistent schemes

3 Methods for non smooth flows

Shock-capturing schemes

- Spurious Gibbs oscillations near shock jumps may lead to nonlinear instabilities. Two strategies available:
  1. **Shock-fitting approaches**: shock are genuine discontinuities, governed by their own set of algebraic equation and uses RH relation as BCs on the two sides
     - ✔ Very accurate
     - ✘ Only feasible for steady shocks and simple geometries
  2. **Shock-capturing approaches**: use the same discretization everywhere and achieve regularization by addition of numerical dissipation.
     - 2.1 **Hybrid Schemes** (WENO + centred differences): switch based on a smoothness sensor
     - 2.2 **Nonlinear Selective Filtering**: add a low-order filter
     - 2.3 **Artificial viscosity**: add a low-order artificial flux or artificial transport properties
     - 2.4 **Flux limiters**: well-suited for strong shocks, but costly and convergence issues

Properties of shock-capturing schemes:

- Analysis difficult because of their **inherent nonlinearity**
  - Classical Fourier analysis cannot be applied generally
  - Comparison possible only on a case-by-case basis
- Major flaw: **reduction of accuracy near shocks**!
  Even high-order schemes yield first-order accurate solutions downstream of moving shocks
- Paramount importance of the **shock-detector function**

Weighted Essentially Non-Oscillatory (WENO) Methods

Idea: determine the numerical flux from a high-order reconstruction over an adaptive stencil that is selected to avoid as much interpolation across discontinuities as possible.

- Construct the flux by convex linear combination of lower-order polynomial reconstructions, with weights selected to achieve maximum formal order of accuracy in smooth regions.

Example: WENO3 ($L = 3$)

L points $\to L+1$ substencils: $\widehat{f}_{j+\frac{1}{2}} = \sum_{l=0}^{L} \omega_l \widehat{f}^l_{j+\frac{1}{2}}$

with $\widehat{f}^l_{j+\frac{1}{2}}$ the numerical flux resulting from polynomial

reconstruction over the stencil S_l: $\widehat{f}^l_{j+\frac{1}{2}} = \sum_{m=0}^{L-1} c_{lm} f_{j-L+1+l+m}$

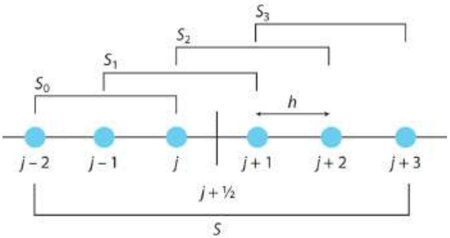
and the weights defined as

$$\omega_l = \frac{\alpha l}{\sum_{m=0}^{L} \alpha_m} \qquad \alpha_l = \frac{d_l}{(\varepsilon + \beta_l)^2}$$

are functions of the smoothness measurements associated with the substencils

$$\beta_l = \sum_{m=1}^{L-1} \left[\sum_{n=1}^{L-1} \gamma_{lmn} f_{j-L+1+l+n} \right]^2$$

- Choose d_l s.t. maximum order of accuracy (2L) is obtained
- Choose ω_l s.t. it nullifies is S_l contains a jump



✔ Well established, very robust method

✔ No tuning parameters

✘ Highly dissipative

$\Longrightarrow$ hybridization with central schemes

✘ Characteristic variable transform needed

$\Longrightarrow$ high computational cost

Artificial viscosity: Jameson's approach

From Jameson et al. (1981); Kim and Lee (2001):

- **Explicit addition** of a dissipative low-order term
- Starting from the dissipative flux of DNC schemes:

$$D_{j+\frac{1}{2}} = \rho(A)_{j+\frac{1}{2}} \left(\varepsilon_2 \delta w - \varepsilon_4 \delta^3 w\right)_{j+\frac{1}{2}}$$

$$\varepsilon_{2,j+\frac{1}{2}} = k_2 \max(\varphi_j, \varphi_{j+1}) \quad \varepsilon_{4,j+\frac{1}{2}} = \max(0, k_4 - \varepsilon_{2,j+\frac{1}{2}})$$

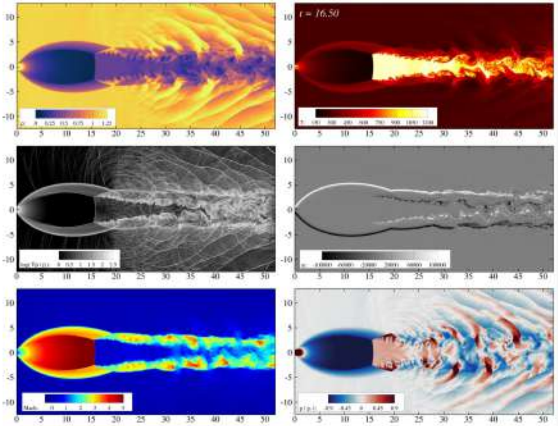
with $\varphi_j \propto \left|\dfrac{\partial^2 \phi_j}{\partial x^2}\right|$ a scalar discontinuity sensor; e.g.,

Jameson's pressure sensor: $\varphi_j = \left|\dfrac{p_{j+1} - 2p_j + p_{j-1}}{p_{j+1} + 2p_j + p_{j-1}}\right|$

- The nonlinear term is $\mathcal{O}(\Delta x^3)$ in smooth regions and becomes $\mathcal{O}(\Delta x)$ close to discontinuities
- Straightforward extensions to higher orders by suitable modification of the term $\varepsilon_4 \delta^3 w$
- Other variables or combinations of variables can be used for the discontinuity sensor

Example: DNC-Jameson 9

- 2D underexpanded $N_2 - O_2$ jet
- Nozzle Pressure Ratio $NPR = 15$
- $L_x \times L_y = 50D \times 25D$, $\Delta x = 0.25$ mm, $16 \cdot 10^6$ pts



Sensor Example: DNC9

$$\varphi_j = \underbrace{\frac{1}{2}\left[1 - \tanh\left(2.5 + 10\frac{\Delta x}{a}\nabla\cdot\boldsymbol{u}\right)\right]}_{\text{(I) Bhagatwala \& Lele}} \times \underbrace{\frac{(\nabla\cdot\boldsymbol{u})^2}{(\nabla\cdot\boldsymbol{u})^2 + |\nabla\times\boldsymbol{u}|^2 + \epsilon}}_{\text{(II) Ducros}} \times \underbrace{\left|\frac{p_{j+1} - 2p_j + p_{j-1}}{p_{j+1} + 2p_j + p_{j-1}}\right|}_{\text{(III) Jameson}}$$

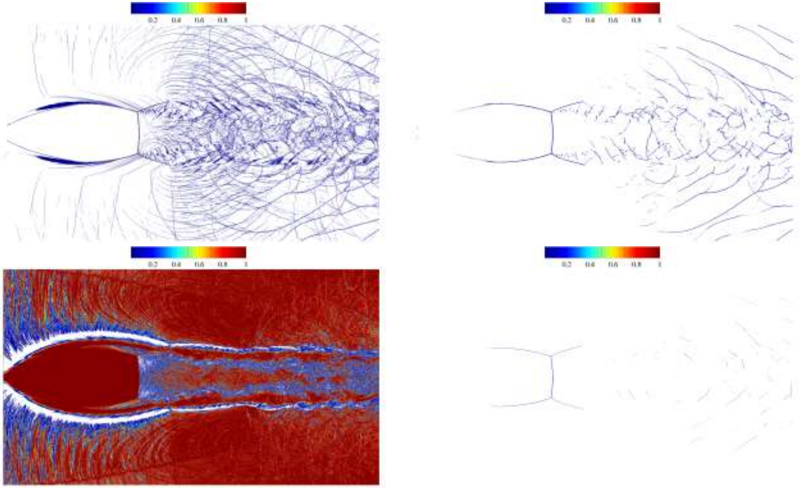
(I):
Excessive damping of acoustic waves

(II):
Always active in solenoidal regions

(III):
Damping of turbulent motions

(I)+(II)+(III):
Highly-selective combination

- ✔ Low computational cost
- ✔ Few tuning parameters
- ✘ Large stencils needed for high orders
- ✘ Not very robust



Top: I, Bottom: II — **Top: III, Bottom: I+II+III**

Localized Artificial Diffusivity (LAD)

- **Idea**: Add artificial transport properties to regularize equations (Kawai et al., 2010)
  - μ^* for unresolved sgs eddies
  - β^* for shock waves
  - κ^* for contact discontinuities

$$\mu^* = C_\mu \rho \overline{\left| \frac{\partial^r F_\mu}{\partial x^r} (\Delta x)^2 \right|} D_\mu^2$$

$$\beta^* = C_\beta \rho f_{sw} \overline{\left| \frac{\partial^r F_\beta}{\partial x^r} \left[\Delta x \frac{\nabla \rho}{|\nabla \rho|} \right]^2 \right|} D_\beta^2$$

$$\kappa^* = C_\kappa \frac{\rho a}{T} \overline{\left| \frac{\partial^r F_\kappa}{\partial x^r} \Delta x \frac{\nabla e}{|\nabla e|} \right|}$$

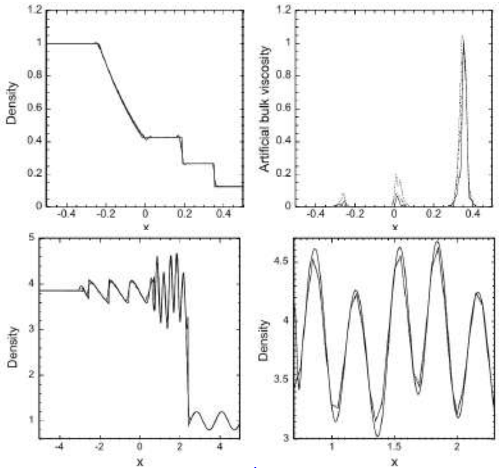
with

$$F_\mu = |S_{ij}|, \quad F_\kappa = e = \frac{p}{(\gamma - 1)\rho}, \quad F_\beta = \nabla \cdot \boldsymbol{u}$$

$$f_{sw} = H(-\nabla \cdot \boldsymbol{u}) \cdot \frac{(\nabla \cdot \boldsymbol{u})^2}{(\nabla \cdot \boldsymbol{u})^2 + |\nabla \times \boldsymbol{u}|^2 + \epsilon}$$

- D_μ, D_β wall-damping functions

Sod shock tube and Shu-Osher problem



- ✔ Low computational cost (not as cheap as Central)
- ✔ Spectral resolution achieved with compact FD
- ✘ Several tuning parameters
- ✘ High diffusivity reduce $(\Delta t)_{stab}$ on stretched grids

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