Exercice 2.9 (Pope)

$$\frac{\partial \vec{U}}{\partial t} + \vec{U}\cdot\nabla\vec{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{U}$$

$$\vec{U}\cdot\nabla\vec{U} = \nabla\frac{|\vec{U}|^2}{2} + \vec{\omega}\times\vec{U} \quad \text{(Lagrange)}$$

$$\Rightarrow \frac{\partial \vec{U}}{\partial t} - \vec{U}\times\vec{\omega} + \nabla\frac{|\vec{U}|^2}{2} + \nabla\frac{p}{\rho} = \nu\nabla^2\vec{U}$$

$$\Rightarrow \frac{\partial \vec{U}}{\partial t} - \vec{U}\times\vec{\omega} + \nabla\left(\frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho}\right) = \nu\nabla^2\vec{U}$$

Inviscid, steady, constant density:

$$\frac{\partial \vec{U}}{\partial t} = 0 \qquad \nu\nabla^2\vec{U} = 0$$

Donc: $$\nabla\left(\frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho}\right) = \vec{U}\times\vec{\omega}$$

a) Intégration sur une ligne de courant :

$$\int_{\mathcal{C}} \nabla\left(\frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho}\right)\cdot d\vec{\ell} = \int_{\mathcal{C}} \vec{U}\times\vec{\omega}\cdot d\vec{\ell}$$

avec $d\vec{\ell} \,//\, \vec{U} \Rightarrow \vec{U}\times\vec{\omega}\cdot d\vec{\ell} = 0$

et
$$\int_{\mathcal{C}} \nabla\left(\frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho}\right)\cdot d\vec{\ell} = \int_{\mathcal{C}} d\left(\frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow \frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho} = H = \text{cont} \text{ le long de } \mathcal{C}$$

b) ligne de vorticité : même calcul mais cette fois $d\vec{\ell} \,//\, \vec{\omega}$: H cont le long de $\mathcal{C}$

c) Irrotationnel : $\vec{\omega} = 0$

$$\Rightarrow \int_{\mathcal{C}} d\left(\frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho}\right) = 0 \quad \forall \mathcal{C} \Rightarrow$$

$$\Rightarrow \frac{\vec{U}\cdot\vec{U}}{2} + \frac{p}{\rho} = H = \text{cont} \quad \text{partout}$$

2.10

Equation de Helmholtz :

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\vec{u} + \nu\nabla^2\vec{\omega}$$

On multiplie par $\vec{\omega}$:

$$\underbrace{\vec{\omega}\cdot\frac{D\vec{\omega}}{Dt}} = \underbrace{\vec{\omega}\cdot\vec{\omega}\cdot\nabla\vec{u}} + \underbrace{\nu\,\vec{\omega}\cdot\nabla^2\vec{\omega}}$$

$$\vec{\omega}\cdot\frac{D\vec{\omega}}{Dt} = \frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\vec{\omega}}{2}\right)$$

$$\vec{\omega}\cdot\vec{\omega}\cdot\nabla\vec{u} = \omega_i\,\omega_j\,\frac{\partial \vec{U}_i}{\partial x_j}$$

$$\vec{\omega}\cdot\nabla^2\vec{\omega} = \vec{\omega}\cdot\left[\nabla\cdot(\nabla\vec{\omega})\right] = \nabla\cdot(\vec{\omega}\cdot\nabla\vec{\omega}) - \nabla\vec{\omega}:\nabla\vec{\omega} =$$
$$= \nabla\cdot\left(\nabla\frac{\vec{\omega}\cdot\vec{\omega}}{2}\right) - \nabla\vec{\omega}:\nabla\vec{\omega} =$$
$$= \nabla^2\frac{\vec{\omega}\cdot\vec{\omega}}{2} - \nabla\vec{\omega}:\nabla\vec{\omega}$$