(Brittle) Fracture Mechanics

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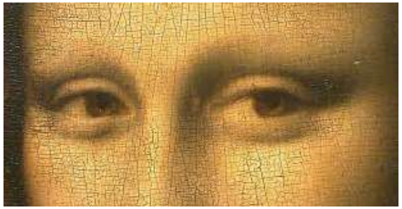
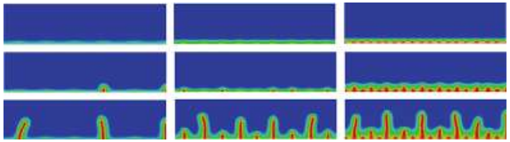
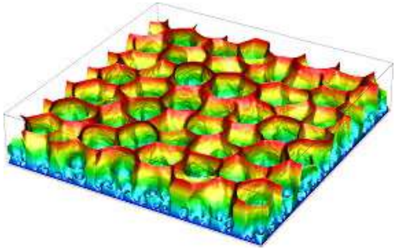






Acknowledgement: a large part of these slides is taken from the course of J.J. Marigo and K.Danas (Ecole Polytechnique), see the references

Tentative program (to update)

Lecture 1	Intro to Fracture, Stress concentrations, singularities (anti-plane)	21/9
Lecture 2	Stress singularities in plane elasticity, fracture modes, fracture toughness, Irwin criterion	28/9
Lecture 3	Energetic (variational) approach to fracture, Griffith's theory I	5/10
Lecture 4	Energetic (variational) approach to fracture, Griffith's theory II	12/10
Lecture 5	Numerical computation of the stress intensity factors I	19/10
Lecture 6	Numerical computation of the stress intensity factors II	26/10
Lecture 7	Examples	09/11
Lecture 8	Examples/Seminar	23/11
Final Exam (written)		30/11

I will probably give one Homework project at the end of october to do in groups of two students and final note will be calculate as

max (100% final examen, 80% final exam + 20 % homework)

Stress singularities in plane elasticity, fracture modes, fracture toughness, Irwin criterion

Content of Lecture 2

- Singularities in 2d elasticity elasticity
- The 3 cracks modes
- Irwin’s criterion and fracture toughness
- Exercises on Irwin’s and stress criteria

At the end of Lecture 2 you should be able to

- Calculate stress concentration in anti-plane elasticity
- Known the properties of singularities in plane elasticity and the 3 cracks modes
- Apply the Irwin’s criterion and the stress criterion to structural design

Stress concentrations in linear elasticity
Fracture modes and Stress Intensity Factors (SIF)

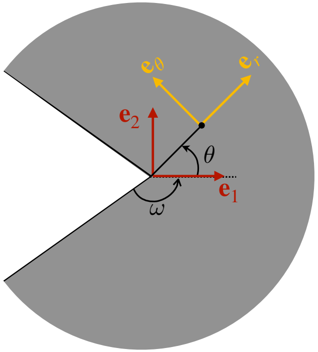
Notch problem in plane deformations: stress method (Airy stress function)

- Airy stress function: $\Psi(x_1, x_2)$

$\psi : \underline{x} \in \mathbb{R}^2 \rightarrow \psi(\underline{x}) \in \mathbb{R}$

- Equilibrium:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + f_1 &= 0 \\ \dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + f_2 &= 0 \end{cases} \iff \boxed{\sigma_{11} = \frac{\partial^2 \Psi}{\partial x_2^2}, \qquad \sigma_{22} = \frac{\partial^2 \Psi}{\partial x_1^2}, \qquad \sigma_{12} = -\frac{\partial^2 \Psi}{\partial x_1 \partial x_2}}$$



- Constitutive law (only in-plane 1-2 components matter):

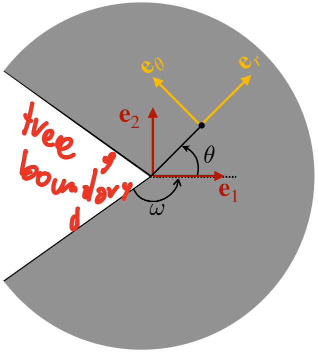
$$E\varepsilon_{11} = (1-\nu^2)\frac{\partial^2 \Psi}{\partial x_2^2} - \nu(1+\nu)\frac{\partial^2 \Psi}{\partial x_1^2}, \qquad E\varepsilon_{22} = (1-\nu^2)\frac{\partial^2 \Psi}{\partial x_1^2} - \nu(1+\nu)\frac{\partial^2 \Psi}{\partial x_2^2}, \qquad E\varepsilon_{12} = -(1+\nu)\frac{\partial^2 \Psi}{\partial x_1 \partial x_2}$$

- Compatibility:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \iff \boxed{\Delta^2 \Psi = 0}$$

+ BC's in terms of Ψ

Notch problem in plane deformations: stress method (Airy stress function) in polar coordinates



- Airy stress function in polar coordinates: $\Psi(r,\theta)$

- **Equilibrium** (we neglect bulk force in the singularity analysis by assuming that they are regular):

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{0} \quad \Leftrightarrow \quad \sigma_{rr} = \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\Psi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)$$

- **Compatibility** (in terms of stresses, after substitution of the constitutive equations):

$$\boxed{\Delta^2\Psi = \Delta\Delta\Psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right) = 0}$$

- **Boundary conditions** (free boundary):

$$\boxed{\text{For} \quad \theta = \pm\omega, \ \forall r > 0: \qquad \sigma_{\theta\theta} = \frac{\partial^2\Psi}{\partial r^2} = 0, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right) = 0}$$

With the Airy stress function, the problem is reduced to solve the scalar bi-Laplacian equation with the above BC's

Notch problem in plane deformations: singular solution

- Let us look for (the most singular) solutions in the form: $\boxed{\Psi(r,\theta) = K r^{\lambda+1} f(\theta)}$

- Compatibility condition

$\neq 0\ \forall\, \lambda$ "assez petit"

$$\Delta^2\Psi = K r^{\lambda-3}\left(f^{(4)}(\theta) + ((1+\lambda)^2 + (1-\lambda)^2) f''(\theta) + (1+\lambda)^2(1-\lambda)^2 f(\theta)\right) = 0$$

$$\boxed{f(\theta) = A\cos(1+\lambda)\theta + B\sin(1+\lambda)\theta + C\cos(1-\lambda)\theta + D\sin(1-\lambda)\theta}$$

- Boundary conditions

$$\begin{cases} \sigma_{rr} &= \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} = K r^{\lambda-1}(f''(\theta) + (\lambda+1)f(\theta)), \\ \sigma_{\theta\theta} &= \frac{\partial^2\Psi}{\partial r^2} = K r^{\lambda-1}(\lambda)(\lambda+1)f(\theta) \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right) = -K r^{\lambda-1}(\lambda)f'(\theta) \end{cases}$$

$$\boxed{f(\pm\omega) = 0, \quad f'(\pm\omega) = 0}$$

$$\boxed{\begin{bmatrix} \cos(1+\lambda)\omega & \cos(1+\lambda)\omega \\ (1+\lambda)\sin(1+\lambda)\omega & (1-\lambda)\sin(1-\lambda)\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} \sin(1+\lambda)\omega & \sin(1-\lambda)\omega \\ (1+\lambda)\cos(1+\lambda)\omega & (1-\lambda)\cos(1-\lambda)\omega \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

Notch problem in plane deformations: singular solution

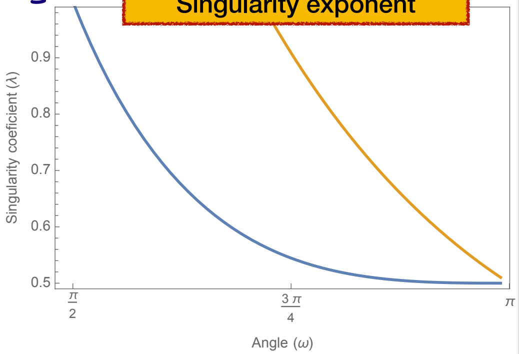
$\omega = \pi \Rightarrow \sin 2\lambda\pi = 0 \Rightarrow \lambda = 1$

Conditions for non-zero solutions

$$\begin{bmatrix} \cos(1+\lambda)\omega & \cos(1-\lambda)\omega \\ (1+\lambda)\sin(1+\lambda)\omega & (1-\lambda)\sin(1-\lambda)\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\sin(2\lambda\omega) + \lambda\sin(2\omega) = 0}$$

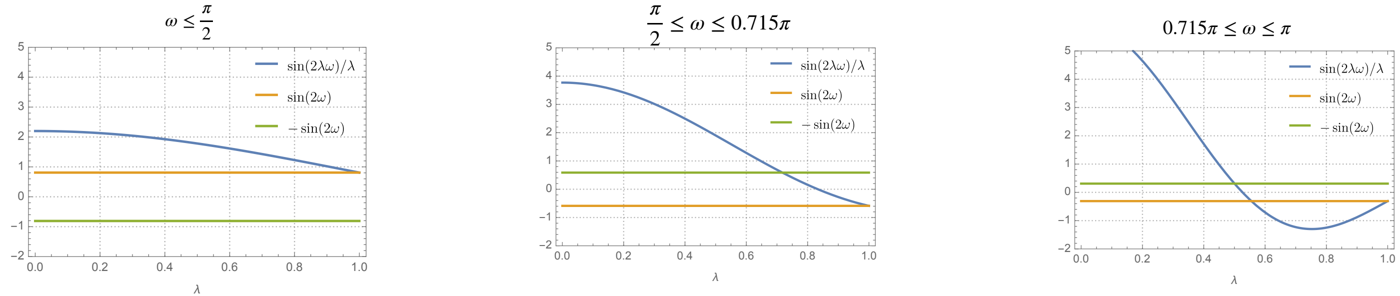
$$\begin{bmatrix} \sin(1+\lambda)\omega & \sin(1-\lambda)\omega \\ (1+\lambda)\cos(1+\lambda)\omega & (1-\lambda)\cos(1-\lambda)\omega \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\sin(2\lambda\omega) - \lambda\sin(2\omega) = 0}$$

Singularity exponent

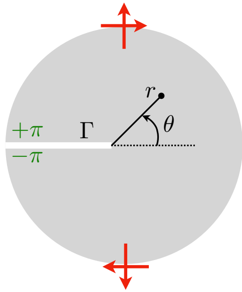


$$\frac{1}{\lambda}\sin(2\lambda\omega) = \pm \quad \sin(2\omega)$$

Graphical solution of the characteristic equations (we look for finite-energy singularities: $0 < \lambda < 1$)



Crack: $\boxed{\omega = \pi}$ $f(\theta) = A\cos(1+\lambda)\theta + B\sin(1+\lambda)\theta + C\cos(1-\lambda)\theta + D\sin(1-\alpha)\theta$



Conditions for non-zero solutions

$$\begin{bmatrix} \cos(1+\lambda)\pi & \cos(1-\lambda)\pi \\ (1+\lambda)\sin(1+\lambda)\pi & (1-\lambda)\sin(1-\lambda)\pi \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\sin(2\lambda\pi) = \lambda\sin(2\pi)}$$

$$\begin{bmatrix} \cos(1+\lambda)\omega & \cos(1-\lambda)\omega \\ (1+\lambda)\sin(1+\lambda)\omega & (1-\lambda)\sin(1-\lambda)\omega \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\sin(2\lambda\pi) = -\lambda\sin(2\pi)}$$

Strongest singularity exponent

Modes

$$\sin(2\lambda\pi) = 0 \quad \Rightarrow \quad \boxed{\lambda = \frac{1}{2}} \rightarrow \boxed{A = -C \qquad B = -\frac{D}{3}}$$

Setting $D = \frac{K_{\mathrm{I}}}{\sqrt{2\pi}}$, $C = \frac{K_{\mathrm{II}}}{\sqrt{2\pi}}$:

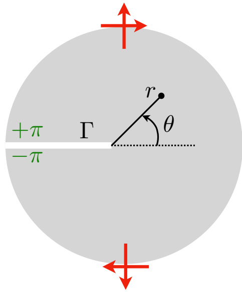
$$\Psi(r,\theta) = \frac{K_{\mathrm{I}}}{3\sqrt{2\pi}} r^{\frac{3}{2}} \left(\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2} \right) - \frac{K_{\mathrm{II}}}{\sqrt{2\pi}} r^{\frac{3}{2}} \left(\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2} \right) + \mathcal{O}(r^{5/2})$$

Plane deformations: singular crack solution in Modes I & II

$\omega = \pi$

- Final solution

$$\Psi(r,\theta) = \frac{\mathsf{K_I}}{3\sqrt{2\pi}} r^{\frac{3}{2}} \left(\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2} \right) - \frac{\mathsf{K_{II}}}{\sqrt{2\pi}} r^{\frac{3}{2}} \left(\sin\frac{3\theta}{2} + 3\sin\frac{\theta}{2} \right) + \mathcal{O}(r^{5/2})$$



leads to a singular stress field at r=0:

$\sigma \sim \frac{1}{\sqrt{r}}$

$$\begin{cases} \sigma_{rr}(r,\theta) = \boxed{\frac{\mathsf{K_I}}{4\sqrt{2\pi\, r}}} \left(5\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right) + \boxed{\frac{\mathsf{K_{II}}}{4\sqrt{2\pi\, r}}} \left(-5\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2} \right) + \mathcal{O}(r^0), \\ \sigma_{\theta\theta}(r,\theta) = \boxed{\frac{\mathsf{K_I}}{4\sqrt{2\pi\, r}}} \left(3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \right) + \boxed{\frac{\mathsf{K_{II}}}{4\sqrt{2\pi\, r}}} \left(-3\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2} \right) + \mathcal{O}(r^0), \\ \sigma_{r\theta}(r,\theta) = \boxed{\frac{\mathsf{K_I}}{4\sqrt{2\pi\, r}}} \left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right) + \boxed{\frac{\mathsf{K_{II}}}{4\sqrt{2\pi\, r}}} \left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} \right) + \mathcal{O}(r^0). \end{cases}$$

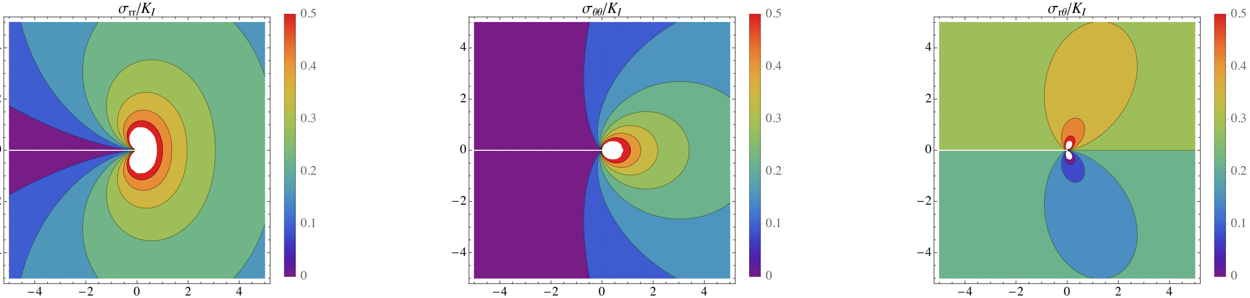
Units of SIF : $\mathrm{MPa}\sqrt{\mathrm{m}}$

and a non-singular displacement field at r=0:

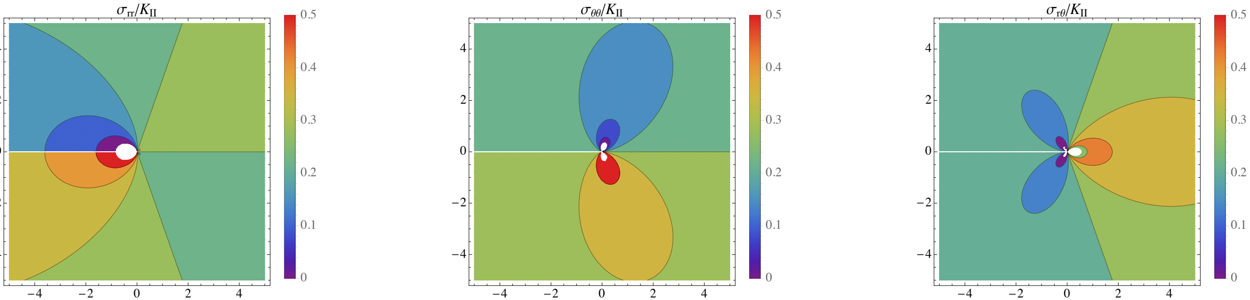
$$\begin{cases} u_r(r,\theta) = \frac{\mathsf{K_I}}{4\mu}\sqrt{\frac{r}{2\pi}} \left((2k-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right) + \frac{\mathsf{K_{II}}}{4\mu}\sqrt{\frac{r}{2\pi}} \left(-(2k-1)\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2} \right) + \mathcal{O}(r), \\ u_\theta(r,\theta) = \frac{\mathsf{K_I}}{4\mu}\sqrt{\frac{r}{2\pi}} \left(-(2k+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right) + \frac{\mathsf{K_{II}}}{4\mu}\sqrt{\frac{r}{2\pi}} \left(-(2k+1)\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} \right) + \mathcal{O}(r). \end{cases}$$

with $k = \begin{cases} 3-4\nu, & \text{(plane strain)} \\ \frac{3-\nu}{1+\nu}, & \text{(plane stress)}. \end{cases}$

Mode I

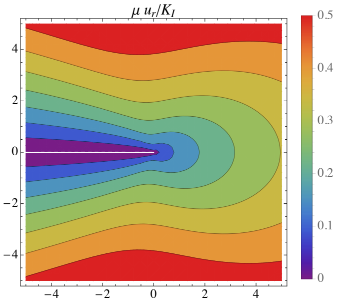


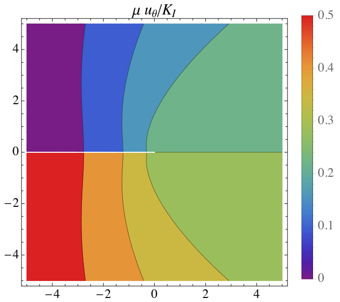
Mode II



Graphical illustration of the displacement fields: Modes I & II

Mode I

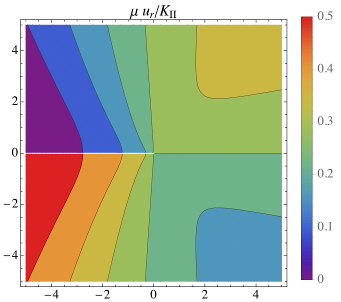


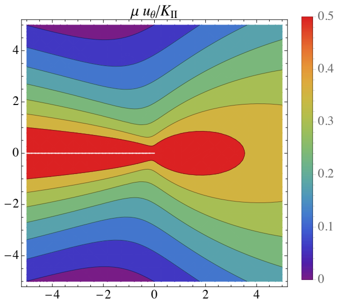


jump in normal component

Mode II

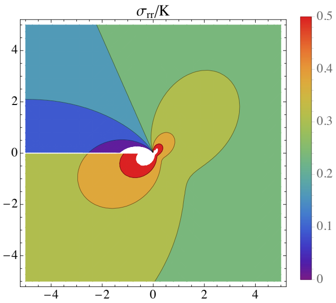
jump in tangential component

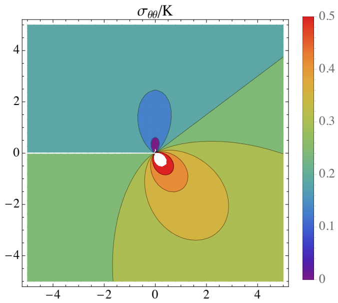


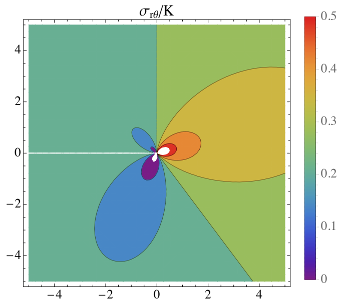


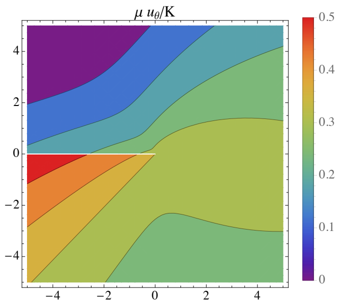
Graphical illustration of the stress fields: Mixed Modes I & II

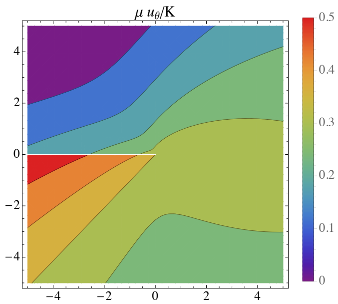
Mode I & II: K_I=K_{II}=K









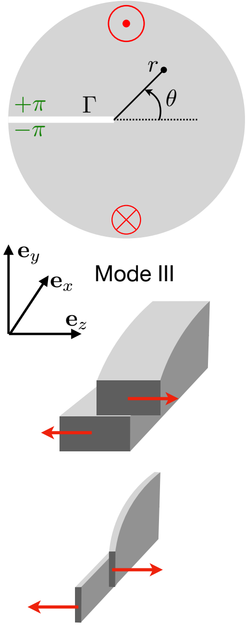


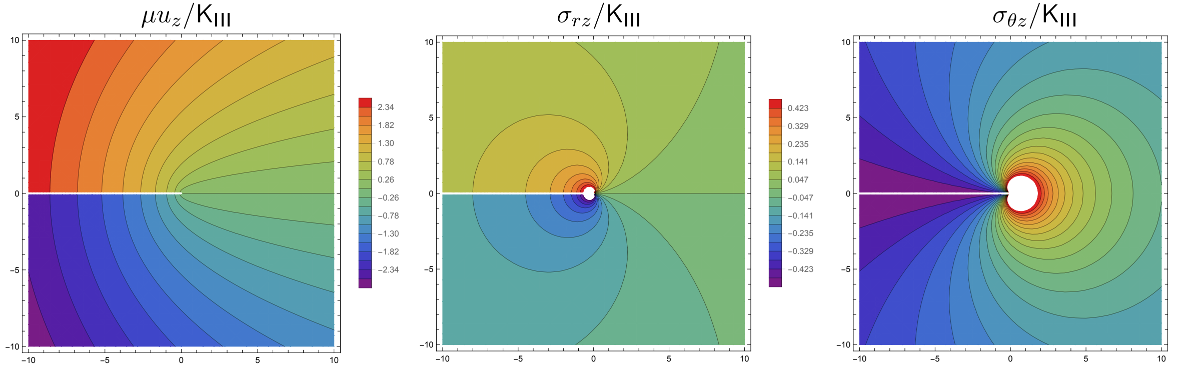
- Displacement field: $\boxed{\mathbf{u}(r,\theta) = u_z(r,\theta)\mathbf{e}_z}$

STRESS INTENSITY FACTOR (SIF): K_{III}

$$u_z = \frac{2K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2} + \mathcal{O}(r), \qquad \begin{pmatrix}\sigma_{rz}(r,\theta)\\ \sigma_{\theta z}(r,\theta)\end{pmatrix} = \boxed{\frac{K_{III}}{\sqrt{2\pi r}}}\begin{pmatrix}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} + \mathcal{O}(r^0)$$

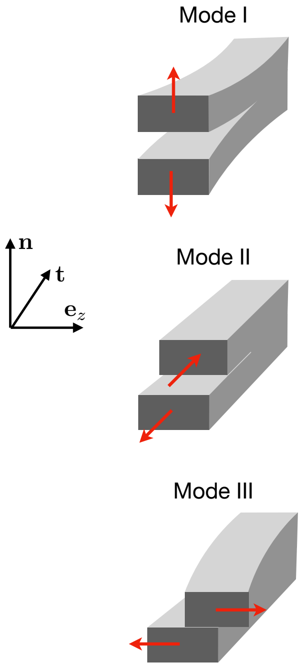
Units of SIF : MPa $\sqrt{\text{m}}$





We will work out this solution in the PC.

Singular modes at the crack tip and displacement jumps



- Mode I or “Opening” mode
  - normal discontinuity
  - no tangential discontinuity
  - non-interpenetration condition: $\mathsf{K}_{\mathsf{I}} \geq 0$

$$\boxed{[\![\mathbf{u}]\!] \cdot \mathbf{n} = \frac{\mathsf{K}_{\mathsf{I}}\,(1+k)}{\mu}\sqrt{\frac{r}{2\pi}}}$$

$$k = \begin{cases} 3 - 4\nu, & \text{(plane strain)} \\ \dfrac{3-\nu}{1+\nu}, & \text{(plane stress).} \end{cases}$$

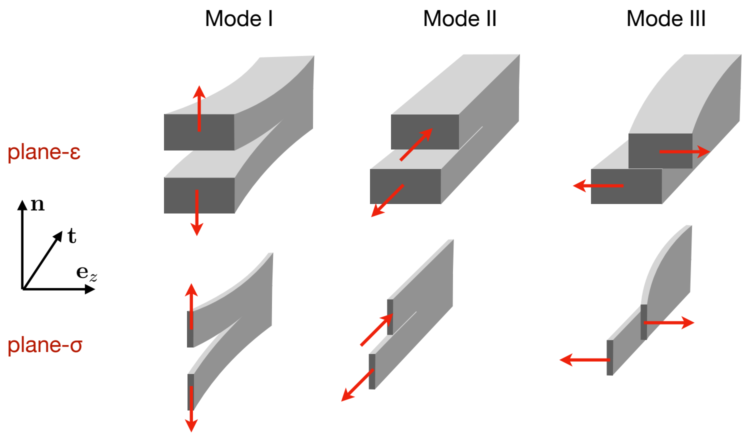
- Mode II or “Shearing” mode
  - no normal discontinuity
  - in-plane tangential discontinuity

$$\boxed{[\![\mathbf{u}]\!] \cdot \mathbf{t} = \frac{\mathsf{K}_{\mathsf{II}}\,(1+k)}{\mu}\sqrt{\frac{r}{2\pi}}}$$

- Mode III or “Tearing” mode
  - no normal discontinuity
  - out-of-plane tangential discontinuity

$$\boxed{[\![\mathbf{u}]\!] \cdot \mathbf{e}_z = \frac{4\mathsf{K}_{\mathsf{III}}}{\mu}\sqrt{\frac{r}{2\pi}}}$$

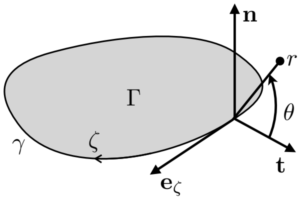
Plane and Anti-plane Deformations Modes (summary)



- [Assumptions]
  - Homogeneous, isotropic, linear elastic material
  - Crack faces are traction free
  - Plane deformations (plane-ε or plane-σ)

- [Method of calculation]
  - Airy stress function for Modes I & II
  - Displacement method for Mode III

3D crack SIF and fields



- 3D crack occupying a domain Γ with contour front γ

tangent vector to the front :	$\mathbf{e}_\zeta$
normal plane to the crack at the front :	$(\mathbf{t}, \mathbf{n})$
tangent plane to the crack at the front :	$(\mathbf{e}_\zeta, \mathbf{t})$
curvilinear coordinates :	(r, θ, ζ)
polar coordinates at the normal plane :	(r, θ)
arc-length parametrization of the front :	ζ

- Singularities

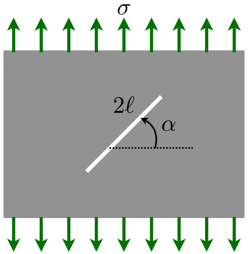
$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{N} \mathsf{K}_{\mathsf{i}}(\zeta) r^{\lambda_i} (U_r^i(\theta)\mathbf{e}_r + U_\theta^i(\theta)\mathbf{e}_\theta + U_\zeta^i(\theta)\mathbf{e}_\zeta) + \cdots$$

- the modes I and II are the same with those in plane deformations in the normal plane to the crack at the front
- the mode III is the same with that in the anti-plane deformation in the tangent direction to the front
- the SIF $K_i(\zeta)$ vary along the crack front

Few examples of SIF

$K_I < K_{Ic} \Rightarrow \sigma\sqrt{\ell} < K_{Ic}$

Straight crack in an infinite 2D solid



- Uniaxial tension at infinity: σ > 0
  - mixed mode I-II unless α=0 or π/2

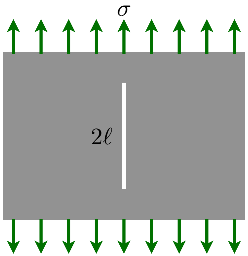
$$K_I = \sigma\sqrt{\pi\ell}\cos^2\alpha, \qquad K_{II} = \sigma\sqrt{\pi\ell}\cos\alpha\sin\alpha$$

$K \sim \sigma\sqrt{\ell}$

- Uniaxial compression at infinity: σ < 0
  - frictionless contact between crack faces
  - Pure Mode II unless α=0 in which case the crack is invisible

$$K_I = 0, \qquad K_{II} = \sigma\sqrt{\pi\ell}\cos\alpha\sin\alpha$$

$$K_I{}^{max} = \sigma\sqrt{\pi\ell} \quad \text{for} \quad \alpha = 0, \qquad K_{II}{}^{max} = \frac{1}{2}\sigma\sqrt{\pi\ell} \quad \text{for} \quad \alpha = \pi/4$$

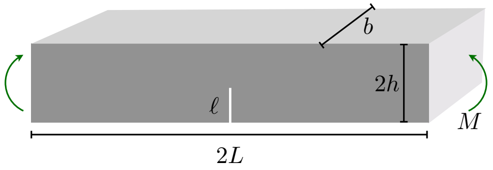


- Crack parallel to loading: σ >(<) 0
  - the crack is invisible both to tension and compression

$$K_I = 0, \qquad K_{II} = 0$$

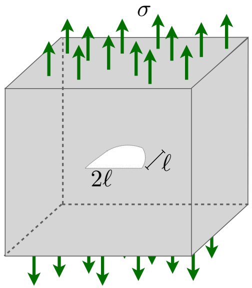
Note: Stress intensity factors obviously depend on the loading conditions and the crack geometry!

Numerical solutions for practical examples



- **Pure bending of a bar with a single crack**
  - bottom part under almost uniaxial tension
  - Simplifications done in slender structures not appropriate here
  - crack is at the center of the specimen

$$\boxed{\begin{array}{ll} \text{for } L/h > 4: & \mathsf{K_I} \approx f(\ell/2h)\sigma\sqrt{\pi\ell} \\ \sigma = \dfrac{3M}{2bh^2}, & f(t) = 1.122 - 1.4t + 6.33t^2 \end{array}}$$

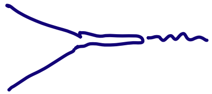


- **Uniaxial tension of a semi-circular 3D crack in an infinite medium**
  - the crack emerges from the front exterior surface
  - the crack is assumed to be shallow w.r.t. the thickness of the specimen
  - crack is in a pure Mode I state

$$\boxed{\mathsf{K_I} \approx 1.1\sigma\sqrt{\pi\ell}}$$

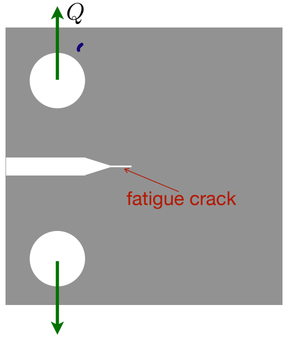
Toughness and Irwin's criterion

Fracture toughness and Irwin's criterion



- **Irwin's definition of fracture toughness**
  - Critical mode-I stress intensity factor for the propagation of a preexisting crack
  - Characteristic of the material
  - To be measured experimentally (experimental specimens to design with special care to 3D effects, size effects, plasticity)

CT (Compact Tension) Specimen

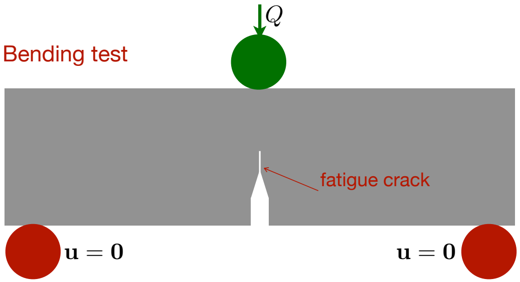


- **Irwin's (1957) criterion**
  - Exclusively for a pure Mode I crack in an isotropic material
  - In general loads it cannot be used!

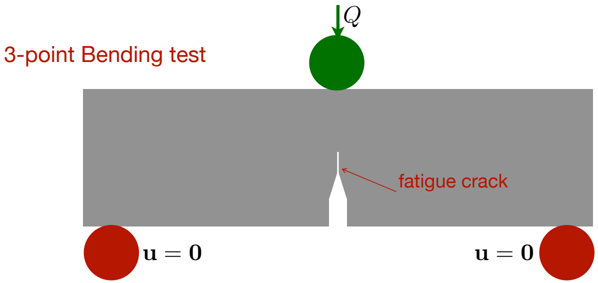
$$\begin{cases} \text{if } \mathsf{K}_\mathsf{I} < \mathsf{K}_\mathsf{Ic} & \Rightarrow \dot{\ell} = 0 : \quad \text{no propagation} \\ \text{if } \mathsf{K}_\mathsf{I} = \mathsf{K}_\mathsf{Ic} & \Rightarrow \dot{\ell} > 0 : \quad \text{possible propagation.} \end{cases}$$

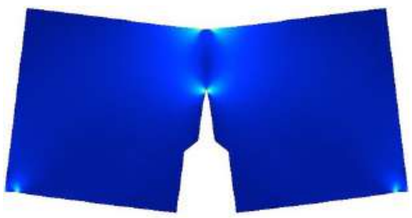
TENACITÉ → MATERIAL PROPERTY

3-point Bending test



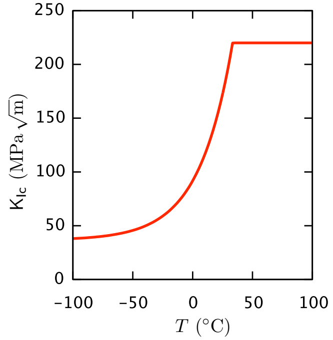
Fracture toughness measurements





- General procedure
  - Specimen dimensions internationally normalized
  - pre-cracking of the specimens via fatigue loading
  - monotonic loading and identification of the load at propagation
- Estimation of the SIF
  - Numerical evaluation of the SIF K_I by use of the finite element method (FEM) in plane deformation conditions and linear elasticity
  - By linearity K_I is proportional to Q
- Sources of errors
  - Calculations done in 2D, whereby the experiment is 3D
  - The value of KI is not constant in front of the crack (higher order terms affect the result)
  - Inelastic phenomena (plasticity, viscoelasticity, creep etc) near the crack tip add-up to the 3D issues

Fracture toughness measurements



Material	E (GPa)	K_{Ic} (MPa $m^{1/2}$)	G_c (J m^{-2})
Diamond	1000	4	15
Glass (Silica)	70	0.75	8
Carbon-fiber composites	200–400	20–25	1000-3000
Cement	20	0.5	10
Concrete	30	1–1.5	30–80
Steel	200	20–200	50–50000
Natural rubber (caoutchouc)	1e-3-0.1	0.5 - 2	100-1000

- Influence of temperature to toughness

- Metals (and polymers) are ductile at room and high temperatures but brittle at low temperatures
- They exhibit a brittle-ductile transition (see graph)
- This was a source of spectacular accidents (e.g., Liberty ships (construction 1939-1945, ~1000 fracture incidents and ~200 ships sank, Bridge of Sully sur Loire)

$$\boxed{G_c = \frac{1-\nu^2}{E} {K_{Ic}}^2}$$

Limits of the Irwin's criterion

- Necessity of extending Irwin's criterion (1957, Note dates…)

  ☑ Applies to cracks subjected to Mode I loads

  ☑ Applies to isotropic, linear elastic materials

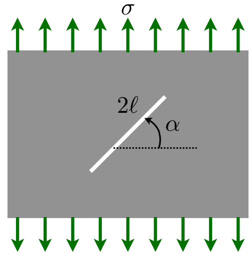
  x Does not apply to anisotropic materials

  x Does not apply to mixte mode cracks

  x Does not apply to interface or boundary cracks

  x It does not provide a complete propagation criterion (stable/unstable, branching etc…)

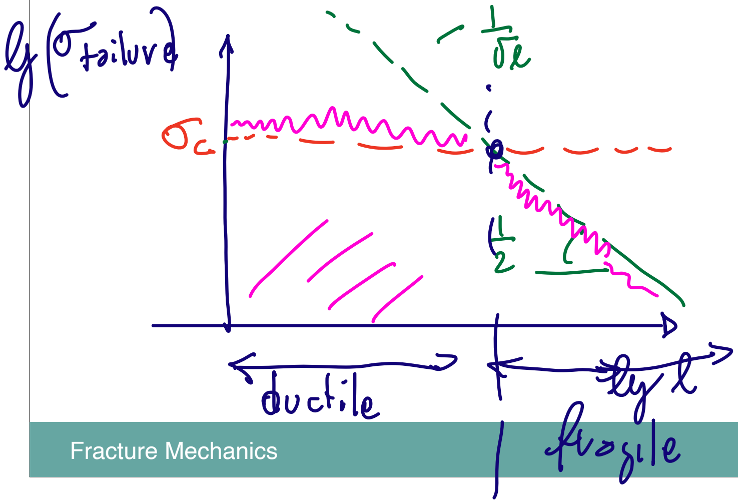
Exercise (TD): critical length for a crack in a plate



Consider a plate of width L with a preexisting crack of length 2ℓ loaded as in figure.

Assume that the crack length is much smaller that the dimension of the structure and that the crack propagate when $K_\mathrm{I} = K_\mathrm{Ic}$ and that the critical stress in the material is σ_c

- Determine the most dangerous crack orientation and position $\to \alpha = 0 \ \left(\max_\alpha K_I(\alpha)\right)$
- Plot qualitatively the critical load at failure as a function of the crack length (use a log scale)



stress criterion: $\sigma \leq \sigma_c$

toughness criterion: $K_I \leq K_{Ic}$

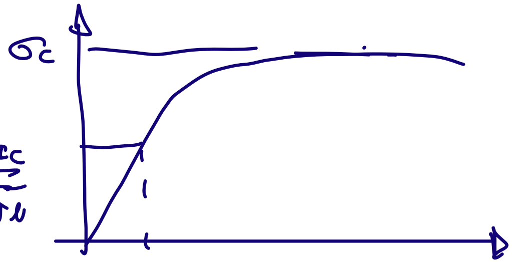
$$K_I = \sigma \sqrt{\pi \ell} \cos^2\alpha \leq K_{Ic}$$

$$\sigma \leq \frac{K_{Ic}}{\sqrt{\pi \ell}} \quad (\alpha = 0)$$

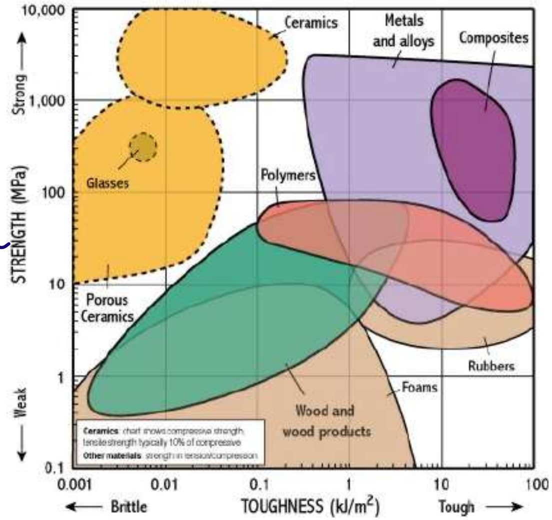
$$\sigma_c \sqrt{\pi \ell} = K_{Ic} \Rightarrow \ell_c \sim \frac{K_{Ic}^2}{\sigma_c^2}$$

Strength vs toughness material selection chart

$$\sigma = \frac{K_{Ic}}{\sqrt{\pi a}}$$



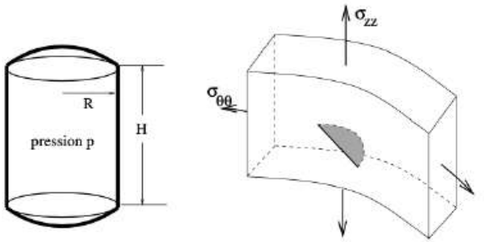
σ_c

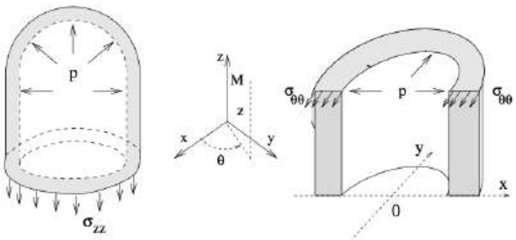


http://www-materials.eng.cam.ac.uk/mpsite/interactive_charts/strength-toughness/basic.html

Exercise (TD): pressure vessel design

See Suquet MEC551, pag 43





Select the best steel among the following for p = 50Mpa
and considering that we can detect fractures larger than 5 mm.
R = 2 m. Find the minimal wall thickness e.

$$\text{nuance A}: \quad \sigma_u = 1250\text{MPa} \quad K_{\text{Ic}} = 90\text{MPa}\sqrt{m},$$
$$\text{nuance B}: \quad \sigma_u = 900\text{MPa} \quad K_{\text{Ic}} = 120\text{MPa}\sqrt{m},$$
$$\text{nuance C}: \quad \sigma_u = 650\text{MPa} \quad K_{\text{Ic}} = 190\text{MPa}\sqrt{m}.$$