(Brittle) Fracture Mechanics

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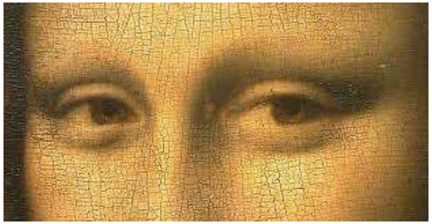
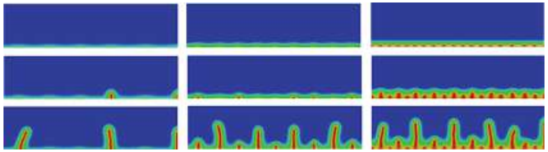
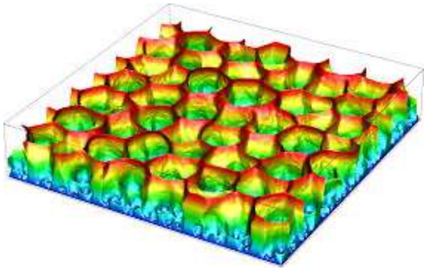






Acknowledgement: a large part of these slides is taken from the course of J.J. Marigo and K.Danas (Ecole Polytechnique), see the references

Tentative program (to update)

Lecture 1	Intro to Fracture, Stress concentrations, singularities (anti-plane)	21/9
Lecture 2	Stress singularities in plane elasticity, fracture modes, fracture toughness, Irwin criterion	28/9
Lecture 3	Energetic (variational) approach to fracture — Griffith's Theory: static problem	5/10
Lecture 4	Energetic (variational) approach to fracture — Griffith's Theory, Quasi-static evolution	12/10
Lecture 5	Numerical computation of the stress intensity factors I	19/10
Lecture 6	Numerical computation of the stress intensity factors II	26/10
Lecture 7	Examples	09/11
Lecture 8	Examples/Seminar	23/11
Final Exam (written)		30/11

I will probably give one Homework project at the end of october to do in groups of two students and final note will be calculate as

max (100% final exam 80% final exam + 20 % homework)

The **quasi-static problem** on a pre-assigned path

Content of Lecture 4

- Quasi-static evolutions on a crack on a preassigned crack path
- Example: Double Cantilever Beam (DCB)
- Structural hardening/softening
- Multiple-parameter crack sets (multiple crack tips)

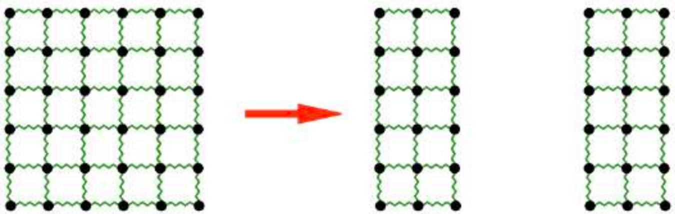
At the end of Lecture 4 you should be able to

- Formulate the quasi-static evolution problem for a single crack
- Apply the variational approach to simple cases with a single crack as DBC under displacement and force control
- Recognise cases with structural hardening or softening and their consequences on the crack propagation
- Advanced: solve cases with two or more cracks on preassigned paths

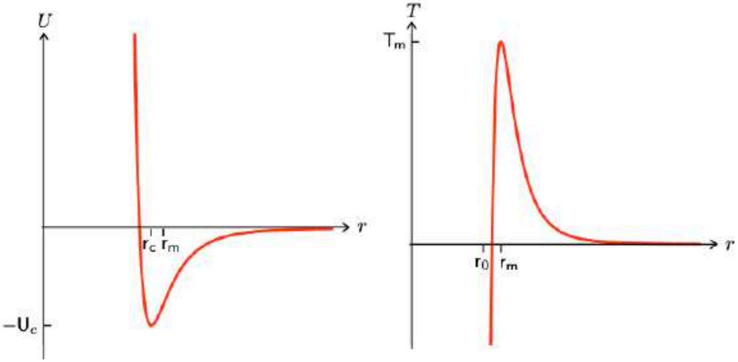
Quasi-static evolutions for a crack on a preassigned crack path

Griffith 1921, Francfort-Marigo 1998

Creation of a new surface by breaking an atomic lattice



Lennard-Jones interaction potential



Lennard-Jones potential and the associated surface energy

$$U(r) = 4U_c\left(\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^{6}\right)$$

$$\boxed{G_c = \frac{U_c}{r_c^2}}$$

In reality the dissipation is much larger because:

1. The real crack length is much larger

2. There are other dissipative effects around the crack, that are proportional to the crack surface (e.g. plasticity)

- Total energy and minimality principle for the static problem

POTENTIAL ENERGY

$$\boxed{\min_{\ell \geq \ell_0} \mathsf{E}(\ell), \quad \mathsf{E}(\ell) = \mathsf{P}(\ell) + G_c \ell}$$

SURFACE ENERGY

- First order optimality conditions (Karush-Kuhn-Tucker, KKT):

$G(\ell) = -P'(\ell)$

$$\mathsf{E}'(\ell) \geq 0, \quad \ell - \ell_0 \geq 0, \quad \mathsf{E}'(\ell)(\ell - \ell_0) = 0$$

$\underbrace{P'(\ell)}_{-G(\ell)} + G_c \geq 0$ $\qquad \Updownarrow \qquad$ $(\underbrace{P'(\ell)}_{-G(\ell)} + G_c)(\ell - \ell_0) = 0$

$$\boxed{\mathsf{G}(\ell) \leq G_c, \quad \ell - \ell_0 \geq 0, \quad (\mathsf{G}(\ell) - G_c)(\ell - \ell_0) = 0}$$

Griffith propagation criterion

$$\mathsf{G}(\ell) := -\mathsf{P}'(\ell)$$

Energy release rate

$G(\ell) < G_c \Rightarrow \ell = \ell_0$ NO PROPAG.

$\ell > \ell_0 \Rightarrow \boxed{G = G_c}$

Comparison between Irwin's and Griffith's criteria

$K_I \leq K_{Ic}$ IRWIN

$G \leq G_c$ GRIFFITH

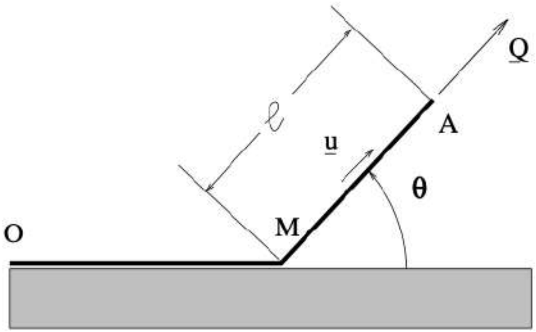
- If the material is isotropic and crack loaded in Mode I
  - The Irwin's and Griffith's criteria coincide
  - We can identify G_c by use of Irwin's criterion and experiments that measure the toughness (CT specimen, 3-point bending etc…)

$$\boxed{G_c = \frac{1-\nu^2}{E} K_{Ic}^2}$$

Irwin's formula in mode I
(to prove in a following lesson)

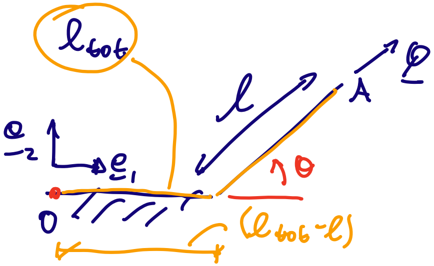
- If the material is anisotropic, mixed mode cracks, interface cracks…
  - Irwin's criterion is not applicable
  - Griffith's criterion is still applicable (but need for specific experiments to identify G_c in the case of anisotropic materials or interface cracks)

Example (TD): Peeling test



- Assume an adhesion surface energy Gc. Assume that the adhesive is straight is the debonded part and that it is linear elastic with Young modulus E and thickness e. We work with load-control
- Determine the total energy and the energy release rate
- Determine the critical peeling force for debonding
- Determine the “best peeling angle theta” to debond a scotch-tape with a minimal effort

Elastic energy : $W(\ell) = \int_\Omega \frac{P^2}{2e^2E}\, d\Omega = \frac{P^2}{2e^2E}\cdot e\ell$



$\sigma = \frac{P}{e}$, $\varepsilon = \frac{P}{Ee}$

$\underline{P} = P\cos\theta\,\underline{e}_1 + P\sin\theta\,\underline{e}_2$

$\underline{OA} = (\ell_{tot}-\ell)\underline{e}_1 + \ell\cos\theta\,\underline{e}_1 + \ell\sin\theta\,\underline{e}_2$

$W_{ext} = \underline{P}\cdot\underline{OA}$

$= P\cos\theta\,(\ell_{tot} - \ell(1-\cos\theta)) + P\ell\sin^2\theta$

$= P\ell + P\ell_{tot}\cos\theta - P\ell\cos\theta$

$W_{ext} = \underbrace{P\ell_{tot}\cos\theta}_{Cst} + P\ell(1-\cos\theta)$

$$P(\ell) = \frac{1}{2}\frac{P^2}{eE}\ell - P\ell(1-\cos\theta) + \text{cst}$$

↳ POTENTIAL ENERGY

$$G(\ell) = -P'(\ell) = P(1-\cos\theta) - \frac{P^2}{2Ee}$$

↳ ERR

GRIFFITH CRITERION

$G(\ell) < G_c$ NO DEBONDING

$G(\ell) = G_c$ POSSIBLE DEBONDING

$$\boxed{P_c(1-\cos\theta) - \frac{P_c^2}{2Ee} = G_c}$$

↳ EQUATION FOR THE CRITICAL FORCE

The smallest solution is

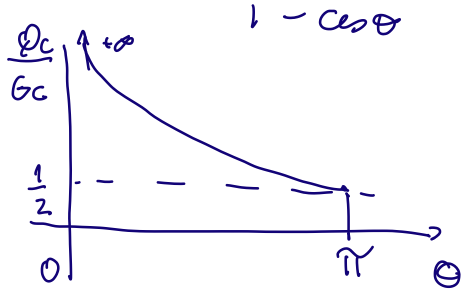
$$P_c = \frac{2G_c}{1-\cos\theta + \sqrt{\underbrace{(1-\cos\theta)^2 - \frac{2G_c}{Ee}}_{\geqslant 0}}}$$

CONDITION NECESSAIRE:

$$G_c \leq \frac{Ee}{2}(1-\cos\theta)^2$$

INEXTENSIBLE CASE $E \to \infty$

$$P_c \simeq \frac{G_c}{1-\cos\theta}$$



To answer a question asked in class

Note on the solution of the quadratic equation:

$a = 1$ $b = -2Ee(1-\cos\theta)$ $c = 2EeG_c$

$$P_c^2 - 2Ee(1-\cos\theta)P_c + 2EeG_c = 0$$

The smallest solution is

$$P_c = -\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$$

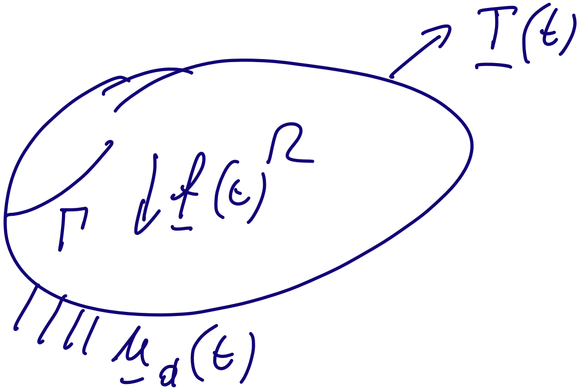
However since the equation is singular for $E \to \infty$, it is more interesting to rewrite the solution as follow

$$P_c = \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}\right)\frac{\left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}\right)}{\left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}\right)}$$

$$= \frac{\cancel{-\frac{b}{2}} + \cancel{\frac{b}{2}} + c}{-\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}}$$

which show clearly the limit for $E \to \infty$

$$P_c = \frac{G_c}{(1-\cos\theta) + \sqrt{(1-\cos\theta)^2 - \frac{G_c 2}{Ee}}}$$



ONE-PARAMETER LOADING

$\underline{T}(t), \underline{f}(t), \underline{u}_d(t)$

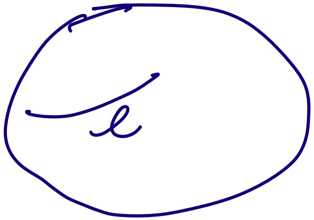
$\in$ paramètre scalaire "temps"

$$\mathcal{P}(\underline{u}, \Gamma, t) = \int_{\Omega/\Gamma} \frac{1}{2} \underline{\underline{\varepsilon}}^T(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u})\,dx - \int_{\Omega/\Gamma} \underline{f}(t) \cdot \underline{u}\,dx - \int_{\partial\Omega_N} \underline{T}(t) \cdot \underline{u}\,ds$$

$$\mathcal{C}(\Gamma, t) = \left\{ \underline{u} = \underline{u}_d(t) \text{ on } \partial\Omega_D/\Gamma,\ \underline{u} \in H^1(\Omega/\Gamma, \mathbb{R}^n) \right\}$$

$$P(\Gamma, t) = \inf_{\underline{u} \in \mathcal{C}(\Gamma, t)} \mathcal{P}(\underline{u}, \Gamma, t)$$

$\Gamma(\ell)$: One-parameter crack set



$P(\ell, t) := P(\Gamma(\ell), t)$ POTENTIAL ENERGY

$D(\ell) := G_c\, \ell$ DISSIPATED ENERGY

The potential energy for load control— structural stiffness

$$\underline{u}_d(t) = t\,\bar{\underline{u}} \quad , \quad \underline{f}(t) = \underline{0} \quad , \quad \underline{T}(t) = \underline{0}$$

$$P(\ell, t) = \min_{\underline{u} \in \mathcal{C}(\ell, t)} \underbrace{\int_{\Omega_\ell} \frac{1}{2}\, \underline{\underline{\varepsilon}}^T(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u})\, dx \; + \; 0}_{P(\underline{u}, \ell, t)}$$

$$\underline{u}(t) = \arg\min_{u \in \mathcal{C}(t)} P(\underline{u}, \ell, t) = t\; \underline{u}(1)$$

$$P(\ell, t) = \frac{t^2}{2}\, C(\ell) \quad , \quad C(\ell) = 2\, P(\ell, 1)$$

Clayperon's theorem and the potential energy for force control

$\underline{u}_d(t) = \underline{0}$, $(\underline{f}(t), \underline{T}(t)) \neq 0$

" $\underline{f}(1)\, t$, " $\underline{T}(1)\, t$

$$P(\ell, t) = \min_{\underline{u} \in \mathcal{C}(\ell)} \underbrace{\int_{\Omega_\ell} \frac{1}{2} \underline{\underline{\varepsilon}}^T(\underline{u}) : \mathbb{C} : \underline{\underline{\varepsilon}}(\underline{u})\, dx}_{\underbrace{\frac{1}{2} a(\underline{u}, \underline{u})}_{\text{elastic energy}}} - \underbrace{\int_{\Omega_\ell} \underline{f}(t) \cdot \underline{u}\, dx - \int_{\partial_N \Omega_\ell} \underline{T}(t) \cdot \underline{u}\, ds}_{\underbrace{\ell(\underline{u}, t)}_{\text{work of external force}}}$$

u solution : $\underline{u} \in \mathcal{C}(\ell)$;

$$a(\underline{u}, \underline{v}) = \ell(\underline{v}, t) \quad \forall\, \underline{v} \in \mathcal{C}(\ell)$$

Hence, for $\underline{v} = \underline{u}$

$$\overbrace{a(\underline{u},\underline{u})}^{2\times\text{ELASTIC ENERGY}} = \overbrace{\ell(\underline{u},t)}^{\text{WORK OF EXTERNAL FORCES}}$$

$$P(\underline{u},t) = \frac{1}{2}a(\underline{u},\underline{u}) - \ell(\underline{u},t)$$

$$D_u P(\underline{u},t)(\underline{v}) := \frac{d}{dh}P(\underline{u}+h\,\underline{v},t)\Big|_{h=0}$$

EXTERNAL WORK = 2 ELASTIC ENERGY

(Clapeyron's Theorem)

$$P(\underline{u},t) = \frac{1}{2}\underline{a}(\underline{u},\underline{u}) - \underline{\ell}(\underline{u},t)$$

$$= 0 - \frac{a(\underline{u},\underline{u})}{2}$$

For $\underline{u}$, solution of the elastic problem

$$\underline{u}(t) = t\,\underline{u}(1)$$

↳ Linearity of the problem

Hence: $P(\underline{u}(t),t) = -\frac{1}{2}a(t\,\underline{u}(1),\, t\,\underline{u}(1))$

$$= -\frac{t^2}{2}a(\underline{u}(1),\underline{u}(1))$$

Potential energy in displacement or force control (example of 1 loading parameter)

- **Displacement control**
  - definition of the potential energy [redacted]

    $$\mathsf{P}(t,\ell) = \frac{1}{2}\mathsf{C}(\ell)q(t)^2, \qquad \mathsf{C}: \text{effective stiffness}$$

  - resulting force

    $$Q = \mathsf{C}(\ell)q(t).$$

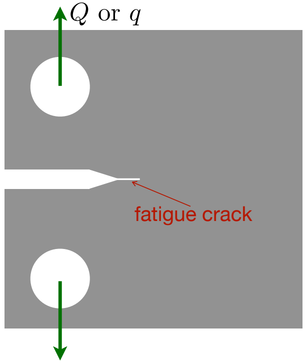
$-\frac{1}{2} C(\ell)\, q^2(\ell)$

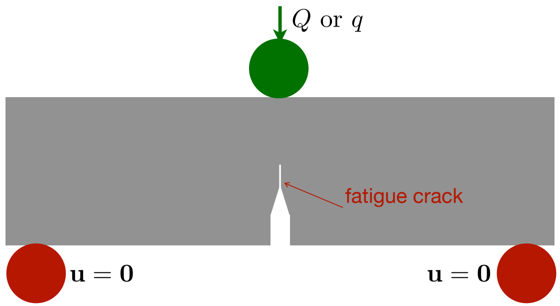
- **Force control**
  - definition of the potential energy (Clapeyron formula)

    $$\mathsf{P}(t,\ell) = \frac{1}{2}\mathsf{C}(\ell)q^2 - Q(t)q = -\frac{1}{2}\mathsf{S}(\ell)Q(t)^2, \qquad \mathsf{S}: \text{effective compliance}$$

  - resulting displacement

    $$q = \mathsf{S}(\ell)Q(t), \qquad \mathsf{S} = \mathsf{C}^{-1}$$



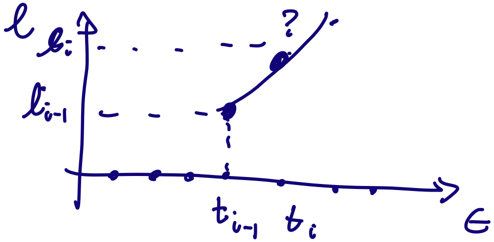


Time-incremental quasi-static evolution (important for numerics)

We assume here one-parameter loading devices and one-parameter crack sets

- Time-discrete evolution

$$\{t_0, t_1, \dots t_i, t_{i+1}, \dots t_n\}$$



- Minimality principle at each time step

$$l_i := \underset{l \geq l_{i-1}}{\operatorname{argmin}} \left(P(l, t_i) + G_c\, l \right)$$

$\Rightarrow$ KKT: $l_i \geq l_{i-1}$, $G(l_i; t_i) \leq G_c$, $(G(l_i; t_i) - G_c)(l_i - l_{i-1}) = 0$

this can be used for the numerical solution

The three "principles" of the time-continuous quasi-static Griffith's propagation law

$t :\to \ell(t)$,

We assume here one-parameter loading devices and one-parameter crack sets

we assume the solution régulière en t

I. Irreversibility of fracture:

The crack set cannot decrease in time $\boxed{\dot{\ell}(t) \geq 0}$ → IRREVERSIBILITY

II. Stability criterion of the cracked state

The stable cracked state is a unilateral local minimum of the total energy for all admissible (small) variations of the state $E(\ell,t) = P(\ell,t) + G_c \ell$

stability criterion

$\forall t,\ \exists h>0 : \forall\, 0 \leq \Delta\ell \leq h \quad E(\ell(t),t) \leq E(\ell(t)+\Delta\ell, t)$

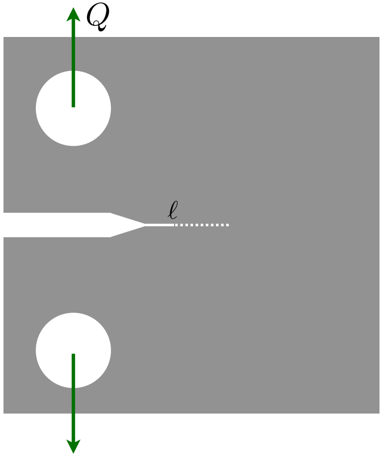
At first order in $\Delta\ell$: $E(\ell(t)+\Delta\ell,t) - E(\ell(t),t) \simeq \frac{\partial E}{\partial \ell}(\ell(t),t)\cdot\Delta\ell \geq 0,\ \forall \Delta\ell \geq 0 \Rightarrow \frac{\partial E}{\partial \ell} = \frac{\partial P}{\partial \ell} + G_c \geq 0 \Rightarrow \boxed{G(\ell(t),t) \leq G_c}$

III. Energy ~~conservation~~ balance

As the crack propagates, the total energy is conserved. All the elastic energy released during the crack propagation is dissipated in the creation of the fracture surface

when $\dot{\ell}(t) > 0 \Rightarrow \underbrace{\frac{\partial P(\ell(t),t)}{\partial \ell}}_{} = \frac{\partial D(\ell(t))}{\partial \ell} \Rightarrow \boxed{\left(G(\ell(t),t) - G_c\right)\dot{\ell}(t) = 0}$ → ENERGY BALANCE

VARIATION OF THE POTENTIAL ENERGY = VARIATION FRACTURE ENERGY



- Assumptions
  - Crack path is known
  - Crack (2D or 3D) depends only on a single parameter ℓ
  - The elastic energy and fracture energy are regular functions of ℓ
  - One-parameter loading
  - Isotropic homogeneous linear elastic solid

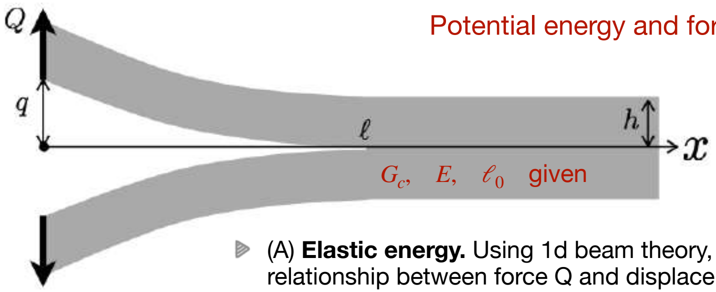
- Griffith's law

$$\boxed{G(t,\ell) = -\frac{\partial \mathsf{P}}{\partial \ell}, \quad \mathsf{D}'(\ell) = \mathsf{G}_c} \implies \boxed{\begin{cases} \dot{\ell}(t) \geq 0 & \text{(irreversibility)}, \\ \mathsf{G} \leq \mathsf{G_c} & \text{(}\cancel{\text{propagation and}}\text{ stability)}, \\ (\mathsf{G} - \mathsf{G_c})\,\dot{\ell}(t) = 0 & \text{(energy } \overset{\text{Balance}}{\cancel{\text{conservation}}}\text{)}. \end{cases}}$$

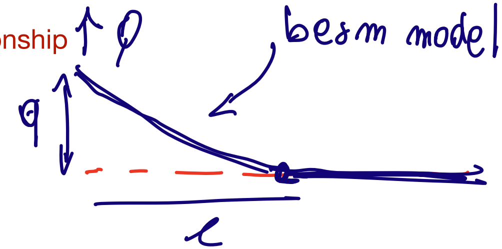
$\ell(0) = \ell_0$ initial condition

Example: Double Cantilever Beam (DCB)

TD



Potential energy and force-displacement relationship



- (A) **Elastic energy.** Using 1d beam theory, determine the elastic energy as a function of the crack length and the relationship between force Q and displacement q.

- (B) **Displacement control** (*hard device*) for a monotonically increasing loading q(t) = t :

  (a) Write the potential energy P(ℓ, t) and the energy release rate G(ℓ, t).

  (b) Being $\ell(0) = \ell_0$, determine the evolution of the crack length $\ell(t)$ and plot the force-displacement Q(q) diagram. Determine the critical displacement and force q_c, Q_c for the beginning of the crack propagation. Is the propagation stable?

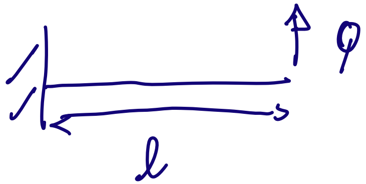
- (C) **Force control** (*soft device*), Q(t) = t :

  (a) Write the potential energy P(ℓ, t) and the energy release rate G(ℓ, t)

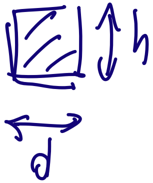
  (b) Being $\ell(0) = \ell_0$, determine the evolution of the crack length $\ell(t)$. Determine the critical displacement and force q_c, Q_c for the beginning of the crack propagation. Is the propagation stable?

- (D) **Cyclic loading** with imposed displacement. Discuss the structural response under cyclic loading

(A)



↝ CANTILEVER BEAM



$$E_Y I = \frac{E_Y h^3 d}{12}$$ BENDING STIFFNESS

ELASTIC ENERGY

$$W(\varphi, \ell) = \frac{\varphi^2 \ell^3}{6EI} \quad , \quad \varphi = \frac{3EI}{\ell^3} q = \frac{E h^3 d}{4 \ell^3} q$$

$$\int_0^\ell \frac{M^2(x)}{2EI} dx$$

DISSIPATED ENERGY

$$D(\ell) = G_c \, \ell \, d$$

(B) DISPLACEMENT CONTROL $\quad q = t$

$$\mathcal{P}(t,\ell) = \mathcal{W}(\varphi(t),\ell) = \frac{E h^3}{4\ell^3} d\, t^2$$

$$G(t,\ell) = -\frac{\partial \mathcal{P}(t,\ell)}{\partial \ell} = \frac{3}{4}\frac{E h^3 t^2}{\ell^4} d$$

$G(t,\ell) \leq G_c d$

irreversibility: $\quad \dot{\ell}(t) \geq 0$

stability : $\quad 4 G_c \ell(t)^4 \geq 3 E h^3 t^2 \quad (G \leq G_c)$

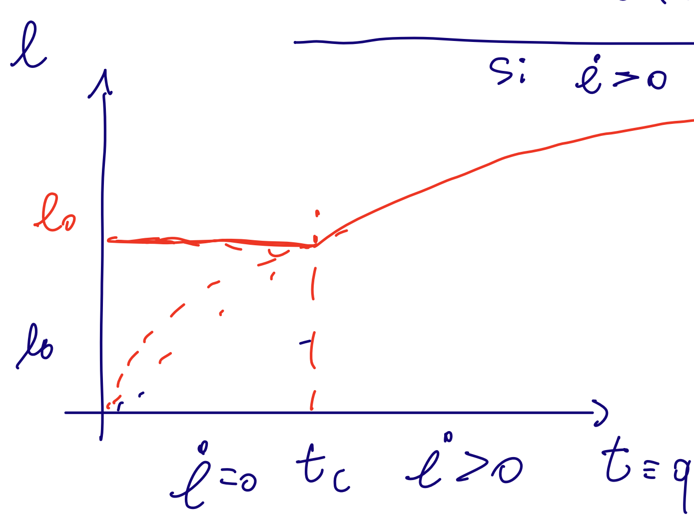
energy balance : $\quad \left(4 G_c \ell(t)^4 - 3 E h^3 t^2\right) \dot{\ell}(t) = 0$

$\ell(0) = \ell_0 \quad$, for $\quad t < t_c = \sqrt{\dfrac{4 G_c}{3 E h^3}}\, \ell_0^2$

$\Rightarrow \dot{\ell}(t) = 0 \quad \forall t \in (0, t_c)$

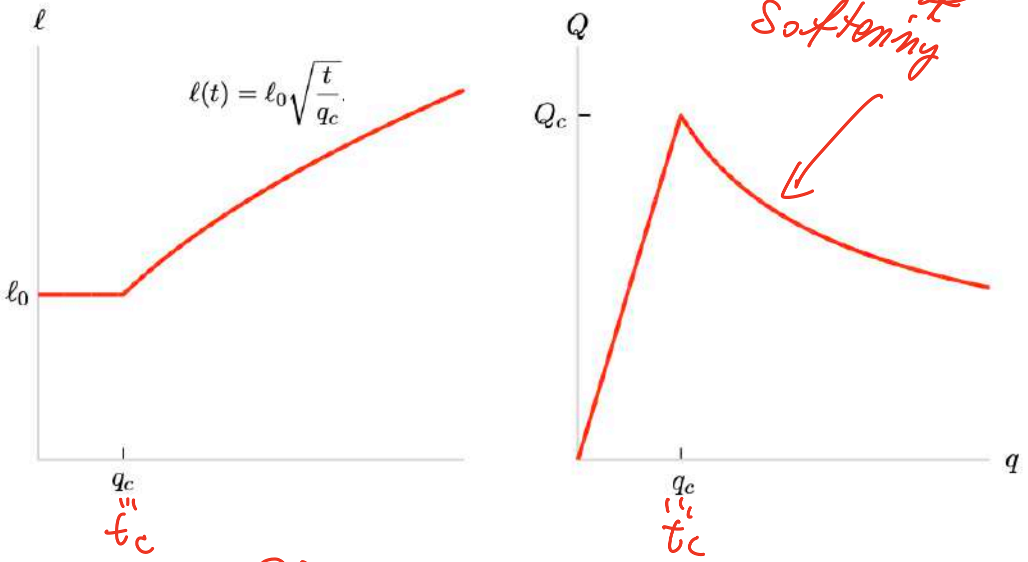
Si $\dot{\ell} > 0 \Rightarrow$

$$\ell(t) = \sqrt[4]{\frac{3 E h^3}{4 G_c}} \sqrt{t}$$



The DCB example: displacement control

Structural Softening



$$Q_c = \sqrt{\frac{\mathsf{G}_c E h^3 d^2}{12\ell_0^2}}$$

$$q_c = \sqrt{\frac{4\mathsf{G}_c \ell_0^4}{3Eh^3}}$$

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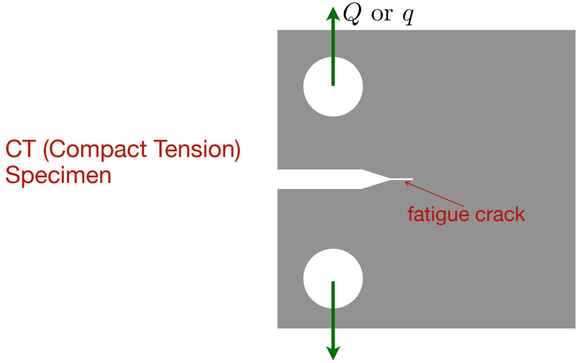
The DCB example: force control

Structural hardening/softening

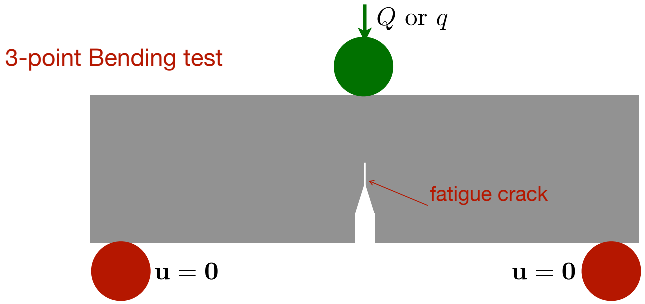
Controlled crack propagation and unstable crack propagation

- Assumptions
  - Crack depends only on one parameter ℓ
  - Energies are regular functions of ℓ
  - Loading depends only on one parameter



- Griffith's law

$$\begin{cases} \dot{\ell}(t) \geq 0 & \text{(irreversibility)} \\ -\frac{\partial \mathsf{P}}{\partial \ell}(t, \ell(t)) \leq G_c & \text{(stability)} \\ \left(\frac{\partial \mathsf{P}}{\partial \ell}(t, \ell(t)) + G_c\right) \dot{\ell}(t) = 0 & \text{(energy balance)} \end{cases}$$



with

$$\text{Displacement Control}: \quad \mathsf{P}(t, \ell) = \frac{1}{2}\mathsf{C}(\ell) q(t)^2$$

$$\text{Force Control}: \quad \mathsf{P}(t, \ell) = -\frac{1}{2}\mathsf{S}(\ell) Q(t)^2$$

- Griffith's propagation law

$$\begin{cases} \dot{\ell}(t) \geq 0 & \text{(irreversibility)} \\ -\frac{\partial \mathsf{P}}{\partial \ell}(t, \ell(t)) \leq G_c & \text{(stability)} \\ \left(\frac{\partial \mathsf{P}}{\partial \ell}(t, \ell(t)) + G_c\right) \dot{\ell}(t) = 0 & \text{(energy balance)} \end{cases}$$

- Griffith's propagation law using the homogeneity of the potential energy:
  - Displacement control, *t*=*q*: $\mathsf{P}(t,\ell) = \frac{1}{2}\mathsf{C}(\ell)t^2$
  - Force control, *t*=*Q*: $\mathsf{P}(t,\ell) = -\frac{1}{2}\mathsf{S}(\ell)t^2$

$$\boxed{\mathsf{P}(t,\ell(t)) = t^2\, P(\ell(t))}$$

DISPLACEMENT

$C'(\ell) \neq 0$

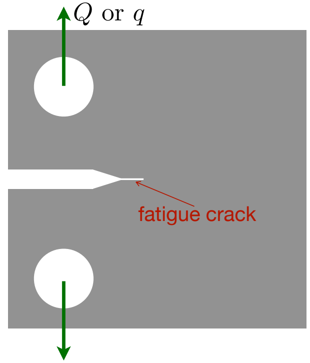
$$-\frac{\partial P}{\partial \ell} = -C'(\ell)\, t^2 \leq G_c \Leftrightarrow t^2 \leq -G_c / C'(\ell)$$

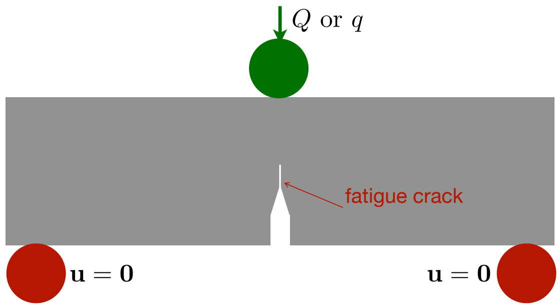
FORCE:

$S'(\ell) \neq 0$

$$-\frac{\partial P}{\partial \ell} = S'(\ell)\, t^2 \leq G_c \Leftrightarrow t^2 \leq G_c / S'(\ell)$$

with initial condition: $\ell(0) = \ell_0$





Crack propagation under increasing loads — displacement control

$\dot{\ell} = 0$

- Start of crack propagation at instant t_c :

$t \leq \sqrt{-\frac{G_c}{\mathcal{C}(\ell_0)}} = t_c$

$$t = t_c = \sqrt{\frac{G_c}{-P'(\ell_0)}}$$

$\dot{\ell} > 0$

- Condition of controlled propagation for increasing t :

The evolution problem of Griffith admits a continuous solution $\ell(t)$, if and only if

$\dot{\ell} \geq 0 \Leftarrow\!\!\!\!\!\!\!\!\not\;\Rightarrow P''(\ell) > 0$

$\quad \| \quad \mathcal{C}''(\ell) > 0$

- Then a unique solution, if it exists, is such that

$$\ell(t) = \begin{cases} \ell_0 & \text{if} \quad 0 \leq t \leq t_c = \sqrt{\frac{G_c}{-P'(\ell_0)}} \\ \ell(t) \leftarrow -t^2 P'(\ell(t)) = G_c, & \text{if} \quad t > t_c \end{cases}$$

$\dot{\ell}(t) =$

$-2t P'(\ell) - t^2 P'' \dot{\ell} \Rightarrow \dot{\ell} = -\frac{2P'}{P''} = \frac{2G(\ell)}{P''(\ell)}$

Continuous crack propagation: force vs. displacement control

- Displacement control: $\mathsf{P}(q,\ell) = q^2\,P(\ell), \quad P(\ell) = \frac{1}{2}\mathsf{C}(\ell), \quad W_\mathsf{s}(\ell) = \mathsf{G_c}\ell d$

Propagation under increasing displacement control $\Longleftrightarrow$ $P'(\ell) \nearrow \Rightarrow \mathsf{C}' \nearrow$ $\Longleftrightarrow$ Stiffness strictly convex: $\mathsf{C}''(\ell) > 0$

- Force control: $\mathsf{P}(Q,\ell) = Q^2\,P(\ell), \quad P(\ell) = -\frac{1}{2}\mathsf{S}(\ell), \quad W_\mathsf{s}(\ell) = \mathsf{G_c}\ell d$

Propagation under increasing force control $\Longleftrightarrow$ $\overset{\ell}{\cancel{P'(\ell)}} \nearrow \Rightarrow \mathsf{S}' \searrow$ $\Longleftrightarrow$ Compliance strictly concave: $\mathsf{S}''(\ell) < 0$

- Comparison

Controlled propagation in force control $\Longrightarrow$ Controlled propagation in displacement control

$$\mathsf{C} = \frac{1}{\mathsf{S}} \quad \Rightarrow \quad \mathsf{C}' = -\frac{\mathsf{S}'}{\mathsf{S}^2} \quad \Rightarrow \quad \mathsf{C}'' = -\frac{\mathsf{S}''}{\mathsf{S}^2} + \frac{2\,\mathsf{S}'}{\mathsf{S}^3}, \quad \mathsf{S} > 0, \quad \mathsf{S}' > 0 \ (\text{since } \mathsf{G} > 0)$$

$$\boxed{\mathsf{S}'' < 0 \quad \Longrightarrow \quad \mathsf{C}'' > 0}$$

Attention: The reciprocal is not true !!!

- Analysis

$$\begin{cases} Q = \mathsf{C}(\ell) q, & \text{(force-displacement response)} \\ -\dfrac{1}{2}\mathsf{C}'(\ell) q^2 \leq \mathsf{G_c}\, d, & \text{(Griffith's criterion)} \end{cases}$$

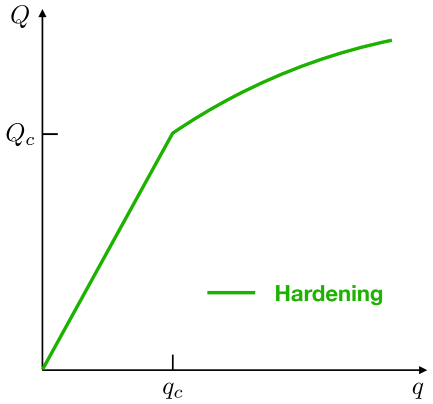
1. State before crack propagation

$$\boxed{q \leq q_c = \sqrt{\frac{2\mathsf{G_c} d}{-\mathsf{C}'(\ell_0)}},} \qquad \ell = \ell_0, \qquad Q = \mathsf{C}(\ell_0) q$$

2. State after crack propagation

$$q = \sqrt{\frac{2\mathsf{G_c} d}{-\mathsf{C}'(\ell)}}, \qquad Q = \mathsf{C}(\ell) q = \sqrt{\frac{2\mathsf{G_c} d}{\mathsf{S}'(\ell)}}, \qquad (\text{since } \mathsf{S}' = -\mathsf{C}'/\mathsf{C}^2)$$

- Force-Displacement curve



(i) Case where: $\mathsf{S}' \searrow \implies \mathsf{C}',\ q,\ Q \nearrow \implies$ **Hardening**

- Analysis

$$\begin{cases} Q = \mathsf{C}(\ell)q, & \text{(force-displacement response)} \\ -\dfrac{1}{2}\mathsf{C}'(\ell)q^2 \leq \mathsf{G_c}\, d, & \text{(Griffith's criterion)} \end{cases}$$

1. State before crack propagation

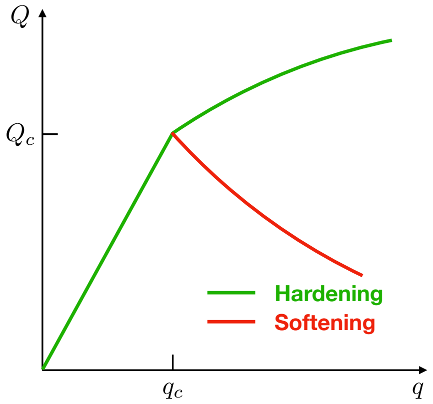
$$\boxed{q \leq q_c = \sqrt{\frac{2\mathsf{G_c}d}{-\mathsf{C}'(\ell_0)}},} \qquad \ell = \ell_0, \qquad Q = \mathsf{C}(\ell_0)q$$

2. State during crack propagation

$$q = \sqrt{\frac{2\mathsf{G_c}d}{-\mathsf{C}'(\ell)}}, \qquad Q = \sqrt{\frac{2\mathsf{G_c}d}{\mathsf{S}'(\ell)}}$$

Structural hardening *(durcissement)* vs structural softening *(adoucissement)*

- Force-Displacement curve



(ii) Case where: $\mathsf{S}' \nearrow$ & $\mathsf{C}' \nearrow \Longrightarrow$, $q \nearrow$, and $Q \searrow$ $\Longrightarrow$ **Softening**

- Analysis

$$\begin{cases} Q = \mathsf{C}(\ell) q, & \text{(force-displacement response)} \\ -\frac{1}{2}\mathsf{C}'(\ell) q^2 \leq \mathsf{G_c}\, d, & \text{(Griffith's criterion)} \end{cases}$$

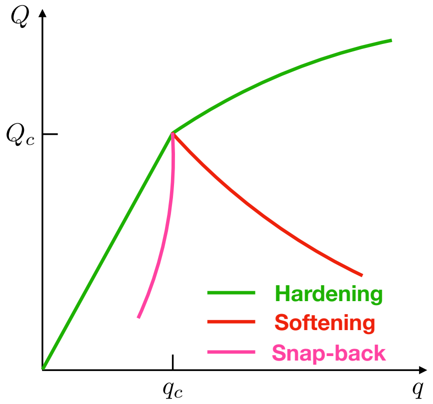
1. State before crack propagation

$$\boxed{q \leq q_c = \sqrt{\frac{2\mathsf{G_c} d}{-\mathsf{C}'(\ell_0)}},} \qquad \ell = \ell_0, \qquad Q = \mathsf{C}(\ell_0) q$$

2. State during crack propagation

$$q = \sqrt{\frac{2\mathsf{G_c} d}{-\mathsf{C}'(\ell)}}, \qquad Q = \sqrt{\frac{2\mathsf{G_c} d}{\mathsf{S}'(\ell)}}$$

- Force-Displacement curve



(iii) Case where: $\mathsf{C}' \searrow \Longrightarrow \quad q,$ and $Q \searrow \quad \Longrightarrow$ **Snap-back**

- Analysis

$$\begin{cases} Q = \mathsf{C}(\ell) q, & \text{(force-displacement response)} \\ -\dfrac{1}{2}\mathsf{C}'(\ell) q^2 \leq \mathsf{G_c}\, d, & \text{(Griffith's criterion)} \end{cases}$$

1. State before crack propagation

$$\boxed{q \leq q_c = \sqrt{\frac{2\mathsf{G_c} d}{-\mathsf{C}'(\ell_0)}},} \qquad \ell = \ell_0, \qquad Q = \mathsf{C}(\ell_0) q$$

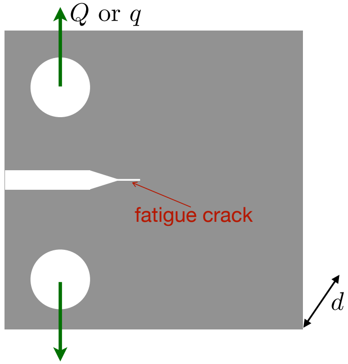
2. State during crack propagation

$$q = \sqrt{\frac{2\mathsf{G_c} d}{-\mathsf{C}'(\ell)}}, \qquad Q = \sqrt{\frac{2\mathsf{G_c} d}{\mathsf{S}'(\ell)}}$$

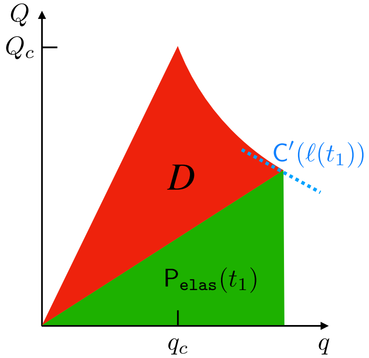
An experimental method to evaluate $\mathsf{G_c}$

Energies

1. Work of external forces: $W_{\mathtt{ext}}(t_1) = \int_0^{t_1} Q(t)q(t)\mathrm{d}t$
2. Elastic energy: $\mathsf{P}_{\mathtt{elas}}(t_1) = \frac{1}{2}Q(t_1)\,q(t_1) = \frac{1}{2}\mathsf{C}(\ell(t_1))\,q(t_1)^2$
3. Surface Energy: $W_{\mathsf{s}}(t_1) = \mathsf{G_c}\,\ell(t_1)\,d$
4. Energy balance (in the absence of snap-back): $W_{\mathtt{ext}}(t_1) = \mathsf{P}_{\mathtt{elas}}(t_1) + W_{\mathsf{s}}(t_1)$

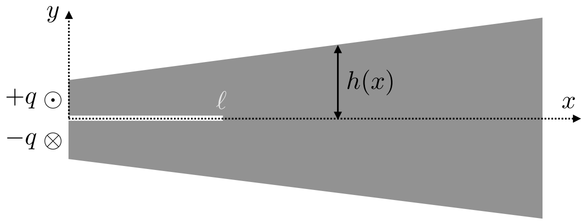


Experimental measurement



$$\begin{cases} \dot{W}_{\mathtt{ext}}(t_1) = Q\,\dot{q} = \mathsf{C}(\ell)\,q\,\dot{q} \\ \dot{\mathsf{P}}_{\mathtt{elas}}(t_1) = \frac{1}{2}\mathsf{C}'(\ell)\dot{\ell}q^2 - \mathsf{C}(\ell)q\dot{q} \\ \dot{W}_{\mathsf{s}}(t_1) = \mathsf{G_c}\,d\,\dot{\ell} \end{cases} \overset{\dot{W}_{\mathrm{ext}}=\dot{\mathsf{P}}_{\mathrm{elas}}+\dot{W}_{\mathrm{s}}}{\Longrightarrow} \boxed{\mathsf{G_c} = \frac{1}{2\,d}\mathsf{C}'(\ell(t_1))\,q(t_1)^2}$$

Remark: The $\mathsf{G_c}$ evaluation depends on the point t_1 and not on the previous state



d: out-of-plane thickness

- Calculation of potential energy

1. 1D approximation of the displacement field

$$\mathbf{u}(\mathbf{x}) = q\,w(x)\,\mathrm{sign}(y)\,\mathbf{e}_z, \quad w(0)=1, \quad w(x)=0 \quad \text{if} \quad x \geq \ell$$

2. Associated potential energy

$$\mathcal{P}(w) = \mu q^2 d \int_0^\ell w'(x)^2\, h(x)\mathrm{d}x$$

(there is a 2 factor included since both upper and bottom beams bend)

3. Minimizing the potential energy

$$\mathsf{P}(w) = \min_{w:\, w(0)=1,\, w(\ell)=0,\, x\in[0,\ell]} \mathcal{P}(w)$$

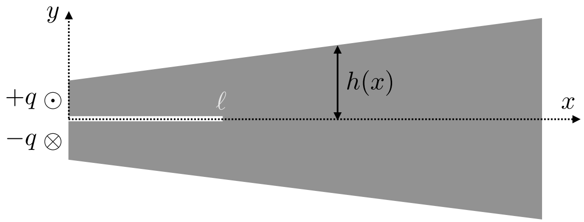
4. Which leads to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\frac{\mathrm{d}w}{\mathrm{d}x}\right) = 0$$

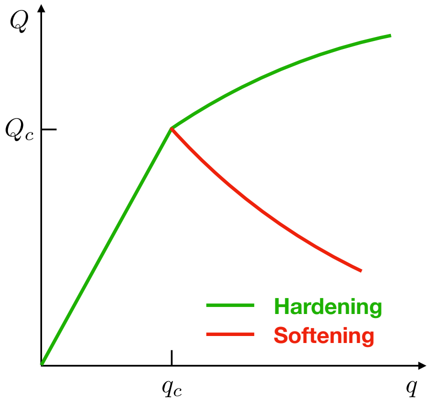
5. Which leads to the final result

$$\mathsf{P}(q,\ell) = \frac{1}{2}\mathsf{C}(\ell)\,q^2, \quad \text{with} \quad \boxed{\mathsf{C}(\ell) = \frac{2\mu\, d}{\displaystyle\int_0^\ell \frac{\mathrm{d}x}{h(x)}}}$$

Softening or hardening of the structure depends on geometry



d: out-of-plane thickness



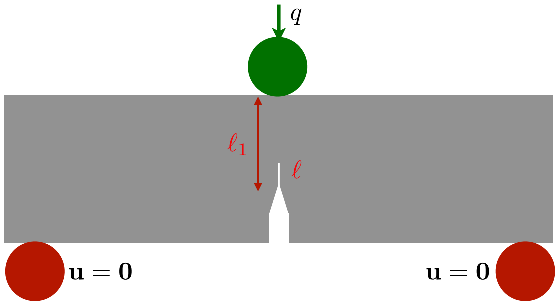
- Stability conditions for propagation

$$\mathsf{C}(\ell) = \frac{2\mu\, d}{\displaystyle\int_0^\ell \frac{\mathrm{d}x}{h(x)}}, \qquad \mathsf{S}(\ell) = \mathsf{C}(\ell)^{-1} = \frac{1}{2\mu\, d}\int_0^\ell \frac{\mathrm{d}x}{h(x)}$$

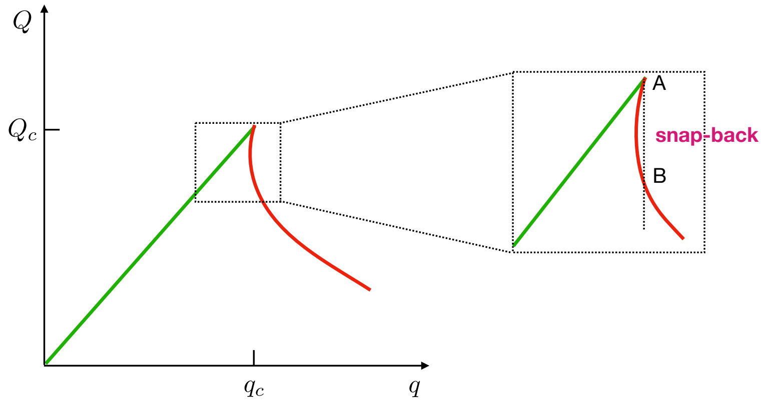
$$\text{Hardening} \iff \mathsf{S}'(\ell) \searrow \iff h(x) \nearrow$$

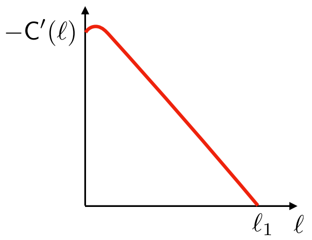
$$\text{Softening or snap-back} \iff h(x) \searrow$$

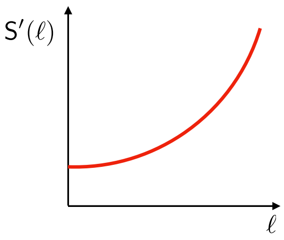
Example: 3-point bending



- Finite element calculations







Multiple-parameter crack sets (multiple crack tips)

Quasi-static Griffith's propagation law with multiple crack tips:

We assume here one-parameter loading devices and one-parameter crack sets

I. Irreversibility of fracture:

The crack set cannot decrease in time

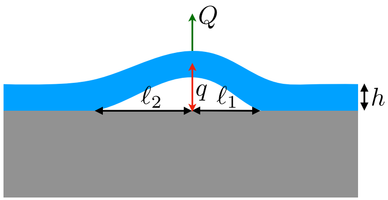
II. Stability criterion of the cracked state

The stable cracked state is a unilateral local minimum of the total energy for all admissible (small) variations of the state

III. Energy conservation

As the crack propagates, the total energy is conserved. All the elastic energy released during the crack propagation is dissipated in the creation of the fracture surface

Crack propagation at the interface of two materials: the case of peeling



controlled displacement

$$\boxed{q = t}$$

- **Remarks**
  - Crack at the interface/ boundary
  - Singularities are different than those of an internal crack
  - Griffith's law is still possible to use

- **Potential energy and Surface energy**

Approximation of the potential energy via use of beam theory (valid for thin layers)

$$Q = \frac{1}{4} E h^3 d \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right)^3 q$$

$$\boxed{\mathcal{P}(q, \ell_1, \ell_2) = \frac{1}{8} E h^3 d \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right)^3 q^2, \qquad W_{\mathsf{s}}(\ell_1, \ell_2) = \mathsf{G}_{\mathsf{c}} d \left(\ell_1 + \ell_2 \right)}$$

- **Energy release rate G**

$$\boxed{\mathsf{G}_i(q, \ell_1, \ell_2) = \frac{3}{8} E h^3 \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right)^3 \frac{q^2}{\ell_i^2}, \quad i = 1, 2}$$

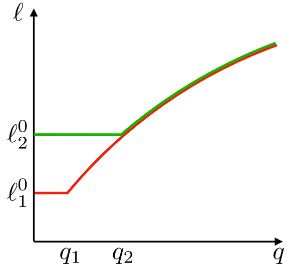
- **Evolution problem**

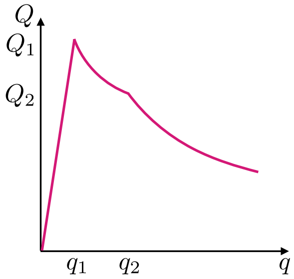
$$\begin{cases} \dot{\ell}_1(t) \geq 0, \quad \dot{\ell}_2(t) \geq 0 \quad \text{(irreversibility)}, \\ \mathsf{G}_1(q, \ell_1, \ell_2) \leq \mathsf{G}_{\mathsf{c}}, \quad \mathsf{G}_2(q, \ell_1, \ell_2) \leq \mathsf{G}_{\mathsf{c}} \quad \text{(propagation and stability)}, \\ \left(\mathsf{G}_1(q, \ell_1, \ell_2) - \mathsf{G}_{\mathsf{c}} \right) \dot{\ell}_1 = \left(\mathsf{G}_2(q, \ell_1, \ell_2) - \mathsf{G}_{\mathsf{c}} \right) \dot{\ell}_2 = 0 \quad \text{(energy conservation)}. \end{cases}$$

with the initial condition $\ell_1(0) = \ell_1^0 < \ell_2^0 = \ell_2(0)$

$$\boxed{\mathsf{G}_i(q, \ell_1, \ell_2) = \frac{3}{8} E h^3 \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right)^3 \frac{q^2}{\ell_i^2}}$$

$$\frac{G_1}{G_2} = \left(\frac{\ell_2}{\ell_2} \right)^2$$





- Propagation initiation

1. Since $\ell_1^0 < \ell_2^0$, we have $\mathsf{G}_1 > \mathsf{G}_2$as long as the crack tips do not propagate

2. The propagation threshold is attained by G_1, when $q = q_1$:

$$q_1 = \sqrt{\frac{8\mathsf{G_c}}{3Eh^3}} \frac{\ell_1^0}{\dfrac{1}{\ell_1^0} + \dfrac{1}{\ell_2^0}}$$

3. Phase of propagation of only the crack tip 1

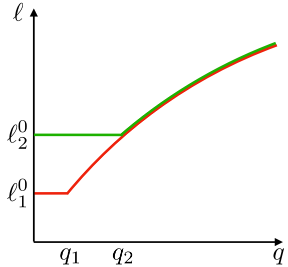
Since $\mathsf{G}_2 < \mathsf{G}_c$ when $q = q_1$, crack tip 2 remains fixed, whereas crack tip 1 propagates. The position of crack tip 1 is given by $\mathsf{G}_1 = \mathsf{G_c}$:

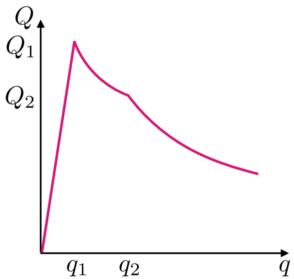
$$\ell_1 = \frac{2\ell_2^0}{\sqrt{1 + 4\dfrac{\ell_2^0}{\ell_1^0}\left(\dfrac{\ell_2^0}{\ell_1^0} + 1\right)\dfrac{q_1}{q}} - 1}$$

During this phase G_2 continues to grow but remains smaller than $\mathsf{G_c}$

$$\boxed{\mathsf{G}_i(q, \ell_1, \ell_2) = \frac{3}{8} E h^3 \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right)^3 \frac{q^2}{\ell_i^2}}$$

$$\frac{G_1}{G_2} = \left(\frac{\ell_2}{\ell_2} \right)^2$$





4. When $q = q_2$ we have $\mathsf{G}_1 = \mathsf{G}_2 = \mathsf{G_c}$

$$q_2 = \frac{\ell_2^0}{\ell_1^0} \left(\frac{\ell_2^0}{\ell_1^0} + 1 \right) \frac{q_1}{2}$$

5. Phase of propagation of both crack tips

   When $q > q_2$, the two crack tips propagate and their position is obtained from $\mathsf{G}_1 = \mathsf{G}_2 = \mathsf{G}_c$

$$\ell_1 = \ell_2 = \ell_2^0 \sqrt{\frac{q}{q_2}}, \qquad \text{if} \quad q > q_2$$