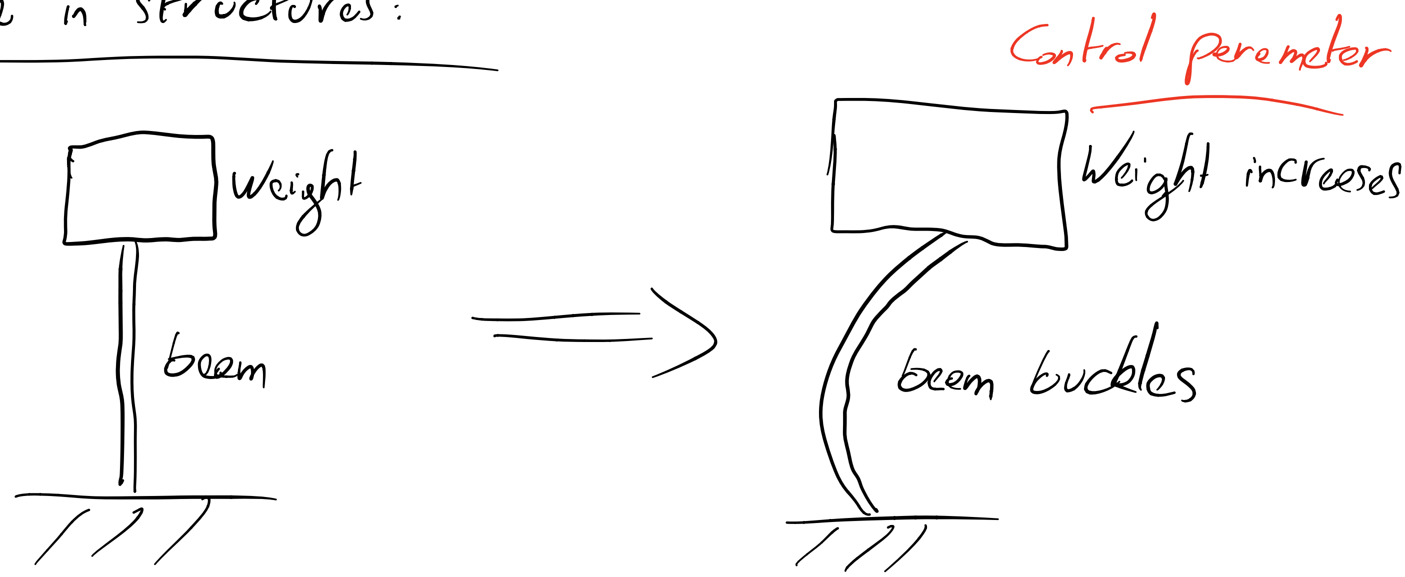
2. Bifurcations :

In the case where the dynamical system depends on one control parameter, qualitative structure of the flow can change as parameters are varied: Fixed points can be created or destroyed, or their stability can change.
These qualitative changes are called bifurcations, and the parameter values at which they occur are called bifurcation points.

Example in structures:

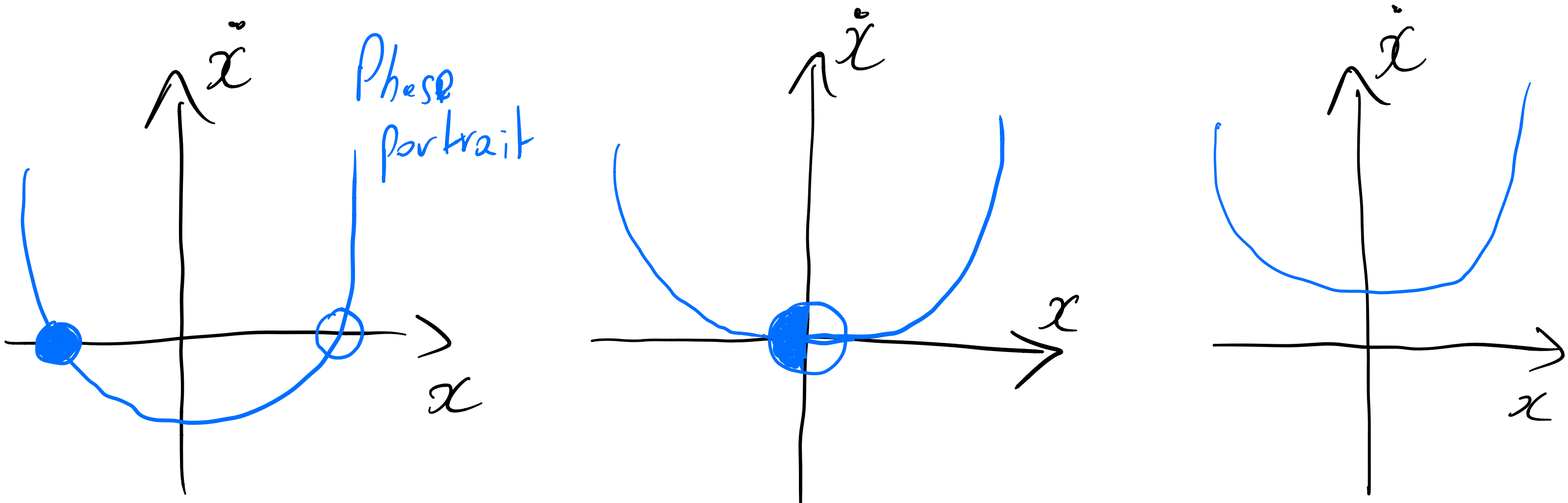


2.1 Saddle-node bifurcation : (point selle)

SN bifurcation is the basic mechanism by which fixed points are creeted or destroyed. As a peremeter is veried, two fixed points move toward each others, collide and mutually anihilate.

Prototypel example of a Seddle-Node bifurcation is given by : $\boxed{\dot{x} = r + x^2}$

where r is the control peremeter and $r \in \mathbb{R}$.



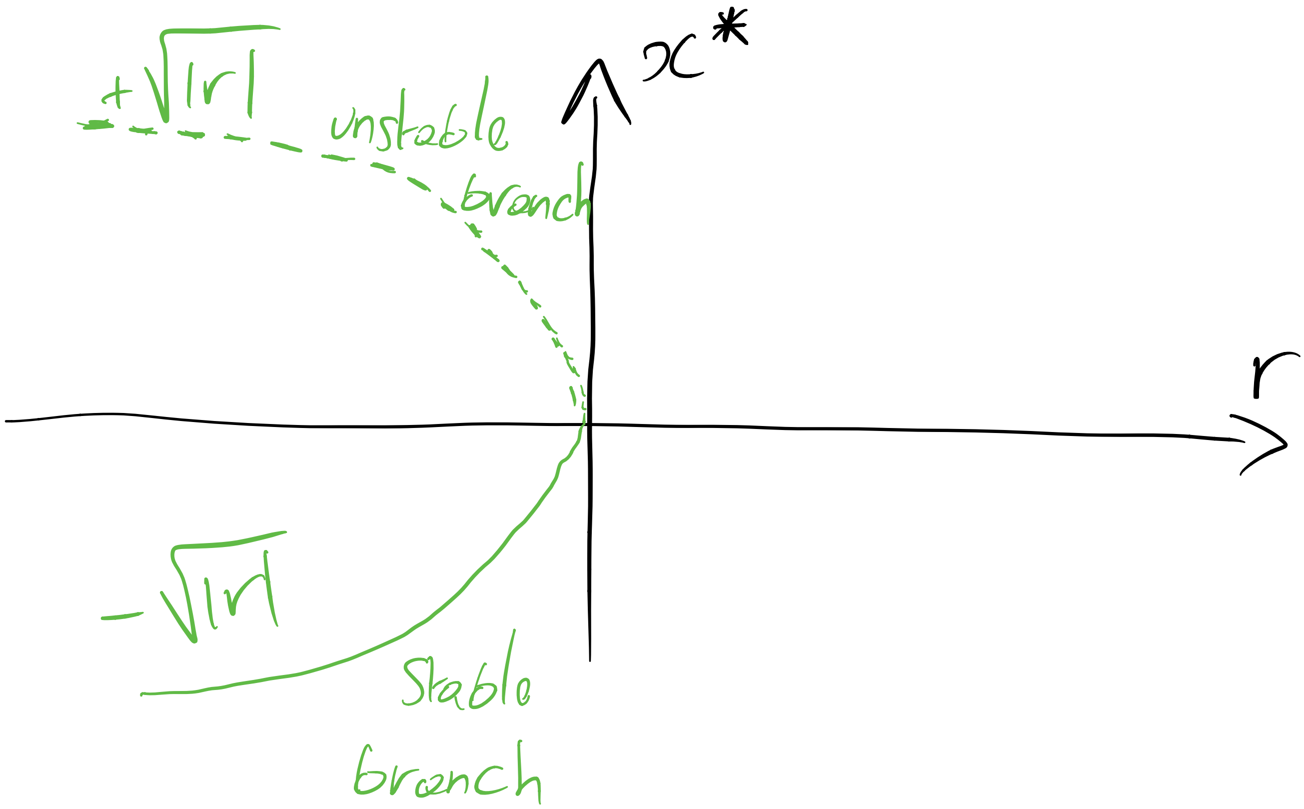
$r < 0$ — Two fixed points, one stable, one unstable

$r_c = r = 0$ (bifurcation point) — One half-stable fixed point

$r > 0$ — No more fixed point!

We can represent this bifurcation with a bifurcation diagram. We represent the fixed points of the system as a function of the control peremeter.

For a Saddle Node bifurcation: $(\dot{x}^* = 0 = r + x^{*2})$

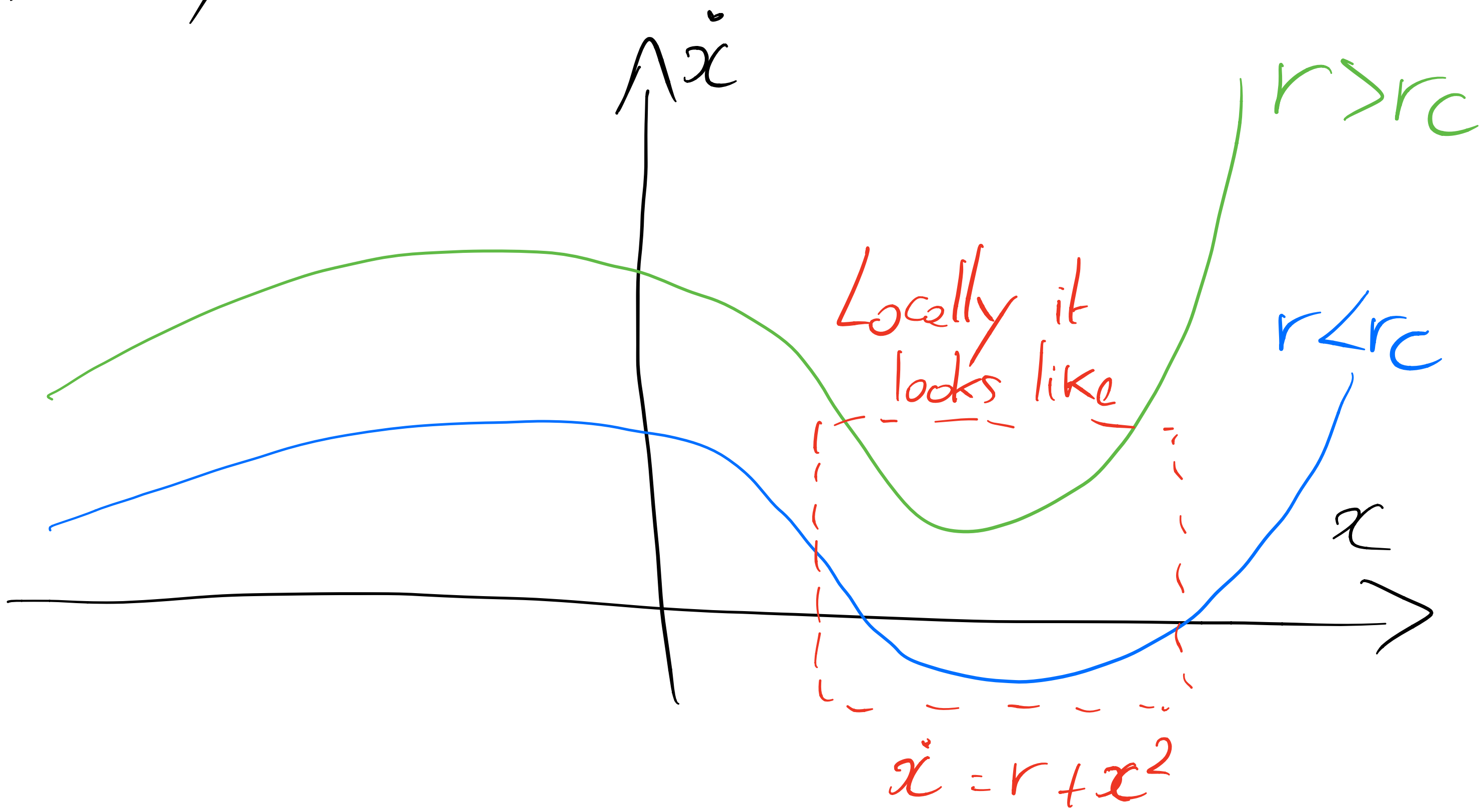


SN bifurcations are sometimes called fold bifurcation or turning point bifurcations.

Normal forms:

Examples like $\dot{x} = r + x^2$ are representative of all SN bifurcations, that's why we call them normal forms. Close to a SN bifurcation, the dynamics typically looks like $\dot{x} = r + x^2$.

Graphically: In order for two fixed points x^* of $\dot{x} = f(x)$ to collide and disappear as r is varied, $f(x)$ locally looks like a parabola:



Near the bifurcation, we can analyse the behavior of $\dot{x} = f(x, r)$ about $x = x^*$ at $r = r_c$ thanks to a Taylor expansion:

$$\dot{x} = f(x, r)$$

$$\dot{x} = \underbrace{f(x^*, r_c)}_{=0 \text{ because } x^* \text{ is a fixed point}} + (x - x^*) \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)}}_{0 \text{ at } r = r_c} + (r - r_c) \left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \frac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + O(r - r_c)^2 + O(x - x^*)^3$$

$$\boxed{\dot{x} = a(r-r_c) + b(x-x^*)^2}$$ Normal form of a SN

with $a = \frac{\partial f}{\partial r}\Big|_{(x^*, r_c)}$ and $b = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{(x^*, r_c)}$

Example: $\dot{x} = r - x - e^{-x}$ at the bifurcation point $x^* = 0$ and $r_c = 1$.

$$\left.\begin{array}{l} \frac{\partial f}{\partial r}\Big|_{(x^*, r_c)} = 1 \\ \frac{\partial^2 f}{\partial x^2}\Big|_{(x^*, r_c)} = -1 \end{array}\right\} \text{Approx}$$

$$\boxed{\dot{x} \simeq (r-1) - \frac{1}{2}x^2}$$

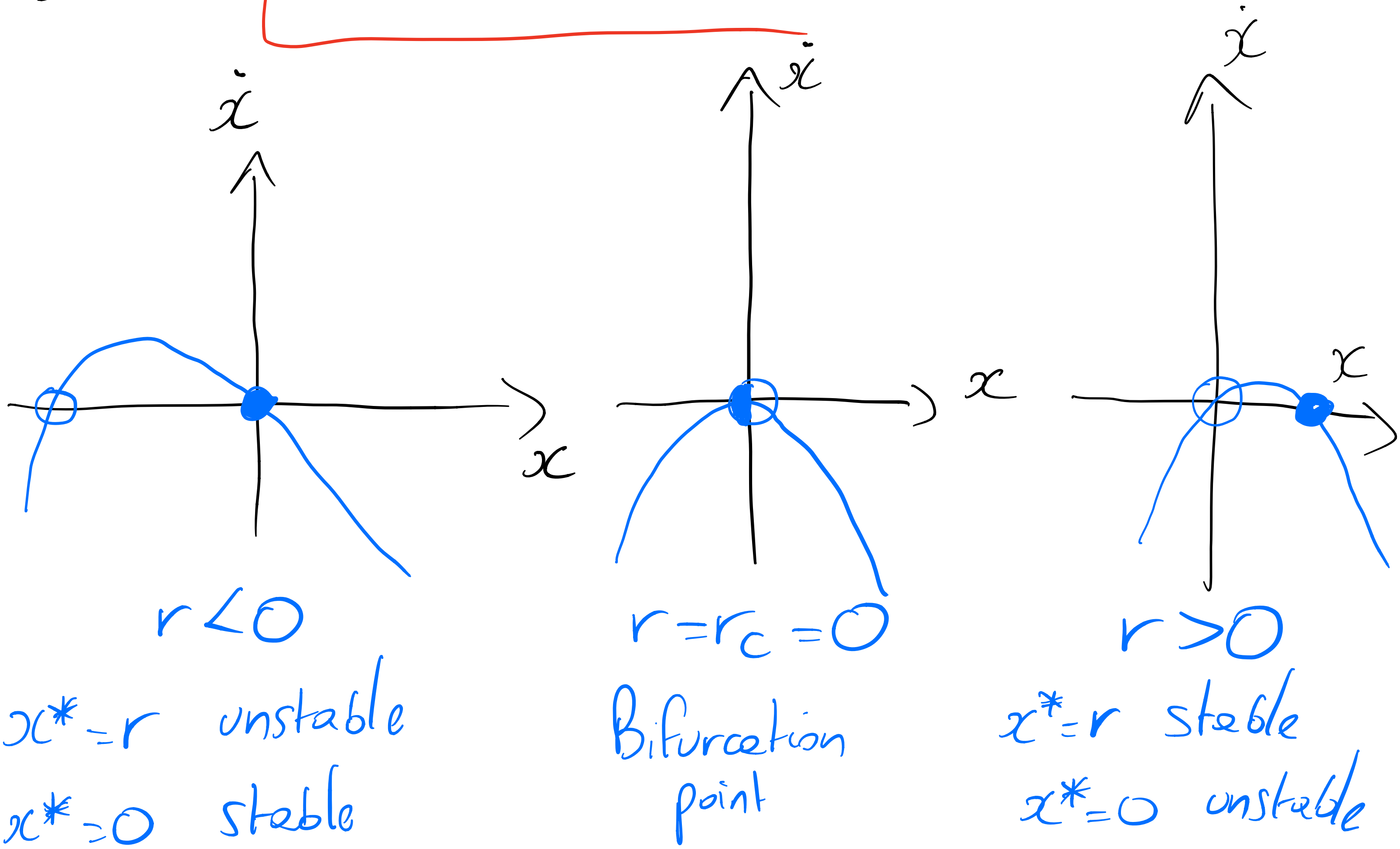
Normal form of a Saddle Node.

2.2: Transcritical bifurcation:

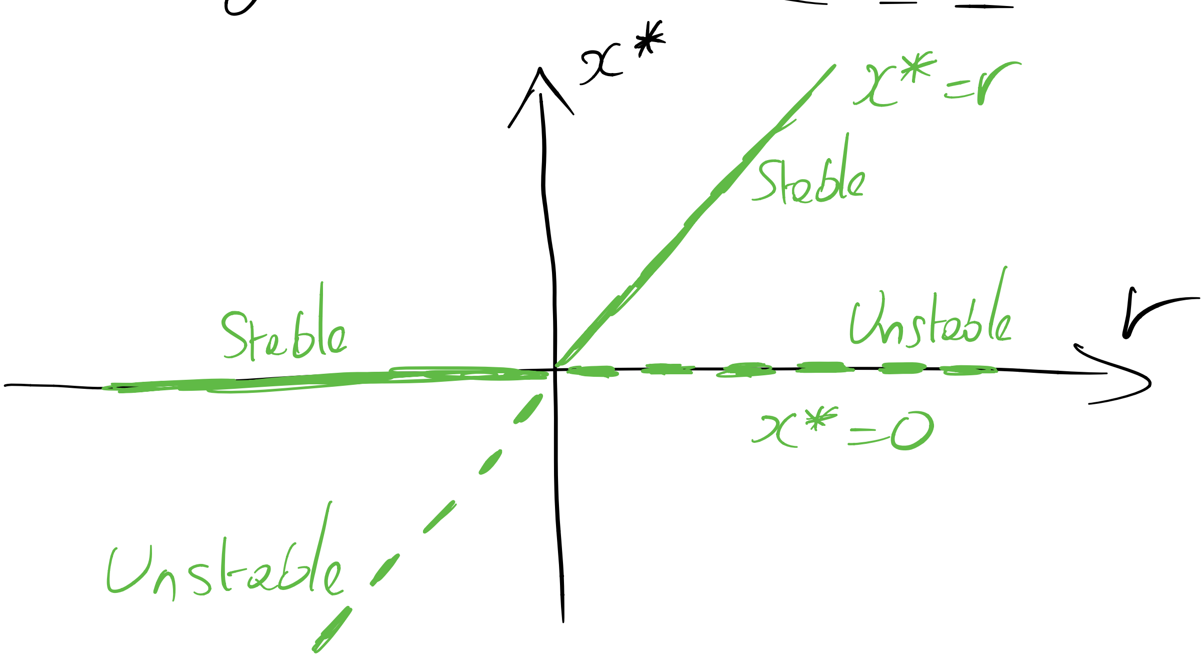
There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed.

However, such a fixed point may change its stability as the parameter is varied.

The normal form for a transcritical bifurcation is $\boxed{\dot{x} = rx - x^2}$



Bifurcation diagram of the normal form:



2.3: Pitchfork bifurcation: (Notebook)

This bifurcation is common in physical problems that have a symmetry, like the buckling beam example.

Two different types of pitchfork bifurcation.

A. Supercritical Pitchfork bifurcation:

The normal form is $\dot{x} = rx - x^3$.