The Heat Equation

Well-posedness

The Heat Equation - Simplified setting in 2d

Find $u = u(x, y, t) : \Omega \times [0, \infty) \to \mathbb{R}$ with

$$
\begin{aligned}
\partial_t u - \nabla \cdot (k \nabla u) &= S && \text{in } \Omega \times [0, \infty); && \text{(PDE)} \\
u(\cdot, 0) &= v && \text{for } (x, y) \in \Omega; && \text{(initial value)} \\
\partial_{\mathbf{n}} u + hu &= u_e && \text{for } (x, y) \in \Omega. && \text{(boundary value)}
\end{aligned}
$$

We make the following simplifications (to verify existence):

- $k \equiv 1$
- $S = h = u_e = 0$
- $\Omega = (0, \pi) \times (0, \pi)$

  (pick a domain that can be easily extended by periodicity: this simplifies enforcement of boundary conditions and allows use of Fourier transforms).

The Heat Equation - Simplified setting in 2d

Hence, we seek $u(x,y,t)$ with

$$\begin{aligned} \partial_t u - \partial_{xx} u - \partial_{yy} u &= 0 && \text{in } (0,\pi)\times(0,\pi)\times[0,\infty);\\ u(\cdot,0) &= v && \text{in } (0,\pi)\times(0,\pi)\\ \partial_{\mathbf{n}} u &= 0 && \text{on } \partial(0,\pi)^2\times[0,\infty). \end{aligned}$$

Fourier Ansatz / Cosine series and separation of variables:

$$u(x,y,t) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \hat{u}_{k\ell}(t)\ \cos(kx)\ \cos(\ell y)$$

where with $c_{k\ell} = \frac{1+\max\{0,k-1\}}{k}\,\frac{1+\max\{0,\ell-1\}}{\ell}$ (note $c_{k\ell} = 1$ if $k,\ell \geq 1$)

$$\hat{u}_{k\ell}(t) = \frac{c_{k\ell}}{4\pi^2}\int_0^{\pi}\int_0^{\pi} u(x,y,t)\cos(kx)\cos(\ell y)\,dx\,dy.$$

The Heat Equation - Simplified setting in 2d

We seek $u(x, y, t)$ with

$$\begin{aligned} \partial_t u - \partial_{xx} u - \partial_{yy} u &= 0 && \text{in } (0, \pi) \times (0, \pi) \times [0, \infty); \\ u(\cdot, 0) &= v && \text{in } (0, \pi) \times (0, \pi) \\ \partial_{\mathbf{n}} u &= 0 && \text{on } \partial(0, \pi)^2 \times [0, \infty). \end{aligned}$$

Fourier Ansatz / Cosine series and separation of variables:

$$u(x, y, t) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \hat{u}_{k\ell}(t) \; \cos(kx) \; \cos(\ell y).$$

Function obviously fulfills initial condition since

$$\hat{u}_{k\ell}(0) = \frac{c_{k\ell}}{4\pi^2} \int_0^{\pi} \int_0^{\pi} u(x, y, 0) \cos(kx) \cos(\ell y) \, dx \, dy = \hat{v}_{k\ell},$$

which is just the cosine series of v.

The Heat Equation - Simplified setting in 2d

We seek $u(x, y, t)$ with

$$\begin{aligned} \partial_t u - \partial_{xx} u - \partial_{yy} u &= 0 && \text{in } (0,\pi)\times(0,\pi)\times[0,\infty); \\ u(\cdot, 0) &= v && \text{in } (0,\pi)\times(0,\pi) \\ \partial_{\mathbf{n}} u &= 0 && \text{on } \partial(0,\pi)^2 \times [0,\infty). \end{aligned}$$

Fourier Ansatz / Cosine series and separation of variables:

$$u(x, y, t) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \hat{u}_{k\ell}(t)\ \cos(kx)\ \cos(\ell y).$$

Function also fulfills boundary condition since for $0 < x, y < \pi$ and $t \geq 0$

$$\partial_x u(0, y, t) = \partial_x u(\pi, y, t) = \partial_y u(x, 0, t) = \partial_y u(x, \pi, t) = 0.$$

$$\text{Hence:} \qquad \nabla u \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega.$$

For instance for $(x, 0) \in \partial\Omega$ we have $\mathbf{n}(x, 0) = (0, -1)^\top$ and hence

$$\nabla u(x, 0, t) \cdot \mathbf{n}(x, 0) = -\partial_y u(x, 0, t) = 0.$$

The Heat Equation - Simplified setting in 2d

We seek $u(x, y, t)$ with

$$
\begin{aligned}
\partial_t u - \partial_{xx} u - \partial_{yy} u &= 0 && \text{in } (0, \pi) \times (0, \pi) \times [0, \infty);\\
u(\cdot, 0) &= v && \text{in } (0, \pi) \times (0, \pi)\\
\partial_{\mathbf{n}} u &= 0 && \text{on } \partial(0, \pi)^2 \times [0, \infty).
\end{aligned}
$$

Fourier Ansatz / Cosine series and separation of variables):

$$
u(x, y, t) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \hat{u}_{k\ell}(t) \; \cos(kx) \; \cos(\ell y).
$$

Concerning well-posedness.

- We verified initial condition,
- we verified boundary condition,
- remains to derive formula for $\hat{u}_{k\ell}(t)$ (that is independent of u) such that the PDE is fulfilled.

The Heat Equation - Simplified setting in $2d$

We have

$$\partial_t u(x,y,t) - \partial_{xx} u(x,y,t) - \partial_{yy} u(x,y,t) = 0.$$

Using this equation in the ansatz

$$u(x,y,t) = \sum_{k,\ell=0}^{\infty} \hat{u}_{k\ell}(t)\ \cos(kx)\ \cos(\ell y),$$

yields

$$\sum_{k,\ell=0}^{\infty} \partial_t \hat{u}_{k\ell}(t)\ \cos(kx)\ \cos(\ell y) + \sum_{k,\ell=0}^{\infty} \hat{u}_{k\ell}(t)\,(k^2+\ell^2)\cos(kx)\ \cos(\ell y) = 0.$$

Comparing the coefficients yields for all $k, \ell \in \mathbb{N}_0$

$$\partial_t \hat{u}_{k\ell}(t) + \hat{u}_{k\ell}(t)\,(k^2+\ell^2) = 0.$$

The Heat Equation - Simplified setting in $2d$

The ODE

$$\partial_t \hat{u}_{k\ell}(t) + \hat{u}_{k\ell}(t)\,(k^2 + \ell^2) = 0$$

with initial condition $\hat{u}_{k\ell}(t) = \hat{v}_{k\ell}$ has the solution

$$\hat{u}_{k\ell}(t) = \hat{v}_{k\ell}\; e^{-(k^2+\ell^2)t}.$$

We conclude that

$$u(x,y,t) = \sum_{k,\ell=0}^{\infty} \hat{v}_{k\ell}\; e^{-(k^2+\ell^2)t}\; \cos(kx)\; \cos(\ell y),$$

with

$$\hat{v}_{k\ell} = \frac{c_{k\ell}}{4\pi^2} \int_0^{\pi} \int_0^{\pi} v(x,y) \cos(kx) \cos(\ell y)\, dx\, dy$$

is a solution to our problem.

The Heat Equation - Simplified setting in $2d$

From the solution

$$u(x,y,t) = \sum_{k,\ell=0}^{\infty} \hat{v}_{k\ell}\, e^{-(k^2+\ell^2)t} \cos(kx)\, \cos(\ell y),$$

to the heat equation $\partial_t u - \triangle u = 0$ we see that

- high frequencies (large k, ℓ) damped fast ($e^{-(k^2+\ell^2)t}$-contribution)
- Backward heat equation

$$\partial_t u + \triangle u = 0 \qquad \Rightarrow \qquad e^{+(k^2+\ell^2)t}.$$

  Solution grows unbounded in time $\Rightarrow$ ill-posed problem
  (small perturbations - often large frequencies - are amplified more)
- More complicated to show existence in a general setting.

The Heat Equation - New setting in $2d$

We want to derive an energy estimate.

New simplified setting:

Find $u = u(x, y, t) : \Omega \times [0, \infty) \to \mathbb{R}$ with

$$
\begin{aligned}
\partial_t u - k \triangle u &= S && \text{in } \Omega \times [0, \infty); \\
u(\cdot, 0) &= v && \text{for } (x, y) \in \Omega; \\
\partial_{\mathbf{n}} u + h u &= 0 && \text{for } (x, y) \in \Omega.
\end{aligned}
$$

Here $k > 0$ and $h > 0$ are const.

The Heat Equation - Energy Estimate

Starting from

$$\partial_t u - k\triangle u = S \qquad \text{in } \Omega \times [0,\infty);$$

we multiply both sides with u and integrate over Ω:

$$\underbrace{\int_\Omega u\,\partial_t u}_{=:\mathrm{I}} - \underbrace{\int_\Omega u\,k\triangle u}_{=:\mathrm{II}} = \underbrace{\int_\Omega S\,u}_{=:\mathrm{III}}.$$

We have (with $\|v\|_{L^2(\Omega)} := \left(\int_\Omega v^2\right)^{1/2}$)

$$\mathrm{I} = \int_\Omega u\,\partial_t u = \int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}u^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2(\Omega)}$$

The Heat Equation - Energy Estimate

For the second term we use integration by parts (Green's identity) to see

$$
\begin{aligned}
\text{II} &= -\int_\Omega k \,\triangle u \, u \\
&\overset{\text{IP.}}{=} \int_\Omega k \nabla u \cdot \nabla u - \int_{\partial\Omega} k u \underbrace{\nabla u \cdot \mathbf{n}}_{=\partial_{\mathbf{n}} u = -hu} \qquad \text{(boundary condition)} \\
&= \int_\Omega k |\nabla u|^2 + \int_{\partial\Omega} k\, h\, |u|^2 \\
&= k \|\nabla u\|^2_{L^2(\Omega)} + kh \|u\|^2_{L^2(\partial\Omega)}
\end{aligned}
$$

The Heat Equation - Energy Estimate

For the third term we use the Cauchy-Schwarz inequality

$$(v, w)_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}$$

which implies

$$\mathrm{III} = \int_\Omega S\, u \leq \|S\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.$$

Combining I, II and III we obtain

$$\|u\|_{L^2(\Omega)} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2(\Omega)} + \underbrace{k\|\nabla u\|^2_{L^2(\Omega)} + kh\|u\|^2_{L^2(\partial\Omega)}}_{\geq 0} \leq \|S\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2(\Omega)} \leq \|S\|_{L^2(\Omega)} \quad \Rightarrow \quad \|u(t)\|_{L^2(\Omega)} \leq \int_0^t \|S(r)\|_{L^2(\Omega)}\, dr + \|v\|_{L^2(\Omega)}.$$

The Heat Equation - Energy Estimate

With

$$\|u\|_{L^2(\Omega)}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2(\Omega)} + k\|\nabla u\|^2_{L^2(\Omega)} + kh\|u\|^2_{L^2(\partial\Omega)} \leq \|S\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$$

and

$$\|u(t)\|_{L^2(\Omega)} \leq \int_0^t \|S(r)\|_{L^2(\Omega)}\,dr + \|v\|_{L^2(\Omega)}$$

we conclude that we also have an energy estimate with

$$\begin{aligned}
&\int_0^t \left(k\|\nabla u(r)\|^2_{L^2(\Omega)} + kh\|u(r)\|^2_{L^2(\partial\Omega)}\right)\,dr \\
&\leq \quad \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \|S(r)\|^2_{L^2(\Omega)}\,dr + \frac{1}{2}\int_0^t \|u(r)\|^2_{L^2(\Omega)}\,dr.
\end{aligned}$$

As before this implies

- a stable solution
- uniqueness.

Remark on integration by parts in $\mathbb{R}^d$

Let $\Omega \subset \mathbb{R}^d$ be a (smooth) domain.

Divergence theorem for (smooth) vector valued function $\mathbf{F} : \Omega \to \mathbb{R}^d$

$$\int_\Omega \nabla \cdot \mathbf{F} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}.$$

For (smooth) $v, u : \Omega \to \mathbb{R}$ we set $F := v\, k\, \nabla u : \Omega \to \mathbb{R}^d$ and conclude

$$\int_\Omega (\nabla \cdot k\nabla u)\, v + \int_\Omega k\nabla u \cdot \nabla v = \int_\Omega \nabla \cdot (k\nabla u\, v) = \int_{\partial\Omega} (k\nabla u \cdot \mathbf{n})\ v.$$

This is called Green's identity.

Example: for $d = 1$ we recover common integration by parts

$$\begin{aligned}\int_a^b (ku')'\, v + \int_a^b ku'v' &= \int_{\partial(a,b)} (ku' \cdot \mathbf{n})\ v \\ &= \big(k(a)u'(a) \cdot (-1)\big)\ v(a) + \big(k(b)u'(b) \cdot 1\big)\ v(b) \\ &= \big[ku'\, v\big]_a^b .\end{aligned}$$