Numerical solutions of differential equations

Patrick Henning

pathe@kth.se

Division of Numerical Analysis, KTH, Stockholm

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Finite Volumes Schemes of Higher Order

Limiter

Limiter Schemes

Idea: Only modify the scheme close to shocks and keep the second order scheme everywhere else.

Definition (Limiter for higher order schemes)

For the linear problem

$$\partial_t u + \mathbf{a}\partial_x u = 0, \qquad \text{for } \mathbf{a} > 0,$$

the Limiter Scheme with Limiter $\phi : \mathbb{R} \to \mathbb{R}$ is given by:

$$Q_j^{n+1} = Q_j^n - \lambda \mathbf{a}\, \Delta_- Q_j^n - \frac{\lambda \mathbf{a}}{2}(1 - \lambda \mathbf{a})\, \Delta_- \left(\phi(r_j)\Delta_+ Q_j^n\right) \quad \text{with } r_j := \frac{\Delta_- Q_j^n}{\Delta_+ Q_j^n}.$$

Note: scheme will be <u>only second order away from the shocks</u>.

Limiter Schemes

Definition (Limiter for higher order schemes)

For the linear problem

$$\partial_t u + a\partial_x u = 0, \qquad \text{for } a > 0,$$

the Limiter Scheme with Limiter $\phi : \mathbb{R} \to [0, 1]$ is given by:

$$Q_j^{n+1} = Q_j^n - \lambda a \, \Delta_- Q_j^n - \frac{\lambda a}{2}(1 - \lambda a)\, \Delta_- \left(\phi(r_j)\Delta_+ Q_j^n\right) \quad \text{with } r_j := \frac{\Delta_- Q_j^n}{\Delta_+ Q_j^n}.$$

1. r_j is an indicator for oscillations. For $r_j < 0$, the terms $\Delta_- Q_j^n$ and $\Delta_+ Q_j^n$ have different signs, which implies oscillations of the numerical solution. Contrary, in monoton regions it holds $r_j \geq 0$.
2. Hence: enforce $\phi(r_j) = 0$ for $r_j < 0$.

Example: Beam-Warming Scheme

Limiter Scheme with Limiter $\phi : \mathbb{R} \to [0,1]$ is given by:

$$Q_j^{n+1} = Q_j^n - \lambda a\, \Delta_- Q_j^n - \frac{\lambda a}{2}(1-\lambda a)\, \Delta_- \left(\phi(r_j)\Delta_+ Q_j^n\right) \quad \text{with } r_j := \frac{\Delta_- Q_j^n}{\Delta_+ Q_j^n}.$$

Example:

1. The Lax-Wendroff scheme is obtained for $\phi(r) = 1$. Hence
$$\Delta_- \left(\phi(r_j)\Delta_+ Q_j^n\right) = \Delta_-\Delta_+ Q_j^n = (Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n).$$
2. The Limiter $\phi(r) = r$ yields the Beam-Warming limiter scheme with
$$\Delta_- \left(\phi(r_j)\Delta_+ Q_j^n\right) = \Delta_-\Delta_- Q_j^n = (Q_j^n - 2Q_{j-1}^n + Q_{j-2}^n).$$

Can limiter schemes be "better" and how can we measure this?

TVD Schemes

One way to express that a scheme suppresses oscillations is to measure the Total Variation at each time t_n.

Goal: spatial oscillations shall not become stronger/more with time (as this is unphysical).

We call a scheme TVD (total variation diminishing) if for all n

$$\mathrm{TV}(\mathbf{Q}^{n+1}) \leq \mathrm{TV}(\mathbf{Q}^{n}).$$

Here, recall that

$$\mathrm{TV}(\mathbf{Q}^{n}) := \sum_{j\in\mathbb{Z}} |Q_{j+1}^{n} - Q_{j}^{n}|.$$

TVD Schemes

Remarks:

- We want schemes that are TVD, because they cannot hence oscillations.
- TVD schemes are necessary for convergence to the entropy solution (but not sufficient).
- Monotone scheme $\Rightarrow$ TVD scheme.
- But a TVD scheme is not necessarily a monotone scheme.

TVD Schemes

Goal:

We wish to state conditions for the Limiter ϕ such that the Limiter Scheme is

1. of consistency order 2 away from extrema
   (i.e. maxima or minima of the solution)
2. and TVD.

Sufficient condition for TVD

Sufficient condition for a limiter so that the scheme is TVD (without proof):

Lemma

Suppose that for the Limiter scheme it holds the CFL condition

$$\lambda a \leq 1, \qquad \text{where } \lambda = \frac{\Delta t}{\Delta x}.$$

If the limiter ϕ is such that

$$\phi(r) = 0 \qquad \text{for } r < 0,$$

and

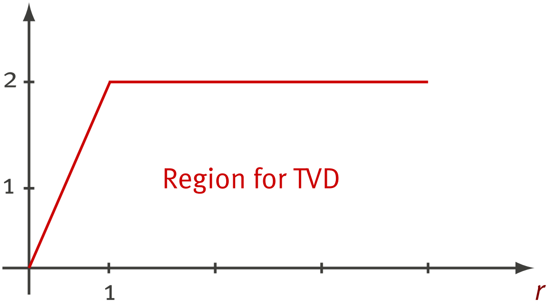
$$0 \leq \max\left(\frac{\phi(r)}{r}, \phi(r)\right) \leq 2 \qquad \text{for } r \geq 0,$$

then the limiter scheme is TVD.

Sufficient condition for TVD

Hence, a sufficient condition for a TVD scheme reads

$$
\begin{aligned}
0 \leq \phi(r) \leq 2r \quad & \text{for } 0 < r \leq 1 \\
0 \leq \phi(r) \leq 2 \quad & \text{for } r \geq 1.
\end{aligned}
$$



Sufficient condition for 2nd order

Sufficient condition for the limiter to obtain schemes of 2*nd* order (without proof):

Lemma

If the limiter such that

$$\phi(r) = (1 - \Theta(r)) + \Theta(r) \cdot r$$

for a Lipschitz-continuous function $\Theta : \mathbb{R} \to [0, 1]$, then

the scheme has consistency order 2 away from local extrema (i.e. for $u' \neq 0$).

Limiters

Recall:

All the sufficient conditions for TVD and second order only refer to linear problems!

Sufficient condition for a 2nd order TVD scheme

Theorem

For the linear problem $\partial_t u + \mathbf{a}\partial_x u = 0$ for $\mathbf{a} > 0$, we consider the Limiter Scheme. Suppose that the CFL condition

$$\lambda \mathbf{a} \leq 1, \qquad \text{with } \lambda = \frac{\Delta t}{\Delta x}$$

holds and that the limiter ϕ is such that

$$\phi(r) = 0 \qquad \text{for } r < 0,$$

$$0 \leq \max\left(\frac{\phi(r)}{r}, \phi(r)\right) \leq 2 \qquad \text{for } r \geq 0,$$

and $\quad \phi(r) = (1 - \Theta(r)) + \Theta(r) \cdot r \quad$ for Lipschitz-continuous $\Theta : \mathbb{R} \to [0, 1]$.

Then the Limiter Scheme is TVD and for $r_j > 0$ of 2nd order.

Sufficient condition for a 2nd order TVD scheme

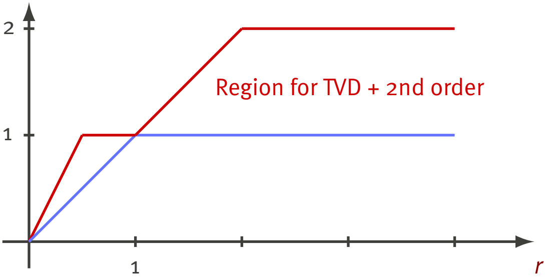
Conclusion: For the Limiter Scheme assume that the CFL-condition holds and that the limiter is continuous and such that

$$\phi(r) = 0 \text{ for } r < 0,$$

$$r \leq \phi(r) \leq \min\{2r, 1\} \text{ for } 0 \leq r \leq 1,$$

$$1 \leq \phi(r) \leq \min\{2, r\} \text{ for } r \geq 1.$$

Then the Limiter Scheme is TVD and for $r > 0$ of 2nd order.



Examples for admissible limiters

Example

1. Minmod-Limiter: $\phi(r) = \max\{0, \min\{r, 1\}\}$
2. Superbee-Limiter: $\phi(r) = \max\{0, \min\{2r, 1\}, \min\{r, 2\}\}$
3. Von Leer-Limiter: $\phi(r) = \frac{|r|+r}{|r|+1}$
4. Van Albada-Limiter: $\phi(r) = \frac{r^2+r}{r^2+1}$
5. Chakravarthy and Osher:
   $\phi(r) = \max\{0, \min\{r, \beta\}\}$ with $1 \leq \beta \leq 2$

Note: For each limiter we always assume $\phi(r) = 0$ for $r < 0$.