Class Note 2: The Pitch And Plunge Airfoil (PAPA) problem

Jean-Camille Chassaing, ∂'Alembert Institute[1] (released, 22. Sep. 2021)

PURPOSE: *The Lagrange's equations of motion are employed to derive the governing equations of an elastically mounted airfoil in pitch and plunge degrees of freedom. Such canonical aeroelastic problem will be used in next class notes to investigate, without loss of generality compared to more realistic lifting surfaces and aircraft, most common aeroelastic phenomena like divergence, flutter, LCO and response to atmospheric gusts.*

1 Aeroelastic configuration

It is assumed that the aeroelastic behavior of a straight wing of infinite span can be studied by representing the dynamics of an airfoil section by means of **coupling** *pitch* and *heave* motions. The elastic effects of the wing are then modeled by springs in torsion and flexion respectively.

1.1 Assumptions

H1 : The motion of the profile will be carried out under the hypothesis of **small displacements**.

H2 : The restoring forces of the springs are linear, as well as the aerodynamic forces acting on the airfoil.

H3 : We will be interested in the aeroelastic response of a profile for an **incompressible** flow.

Note: These hypotheses can be applied without any loss of generality on the mathematical formulation of the aeroelastic problem and its solution.

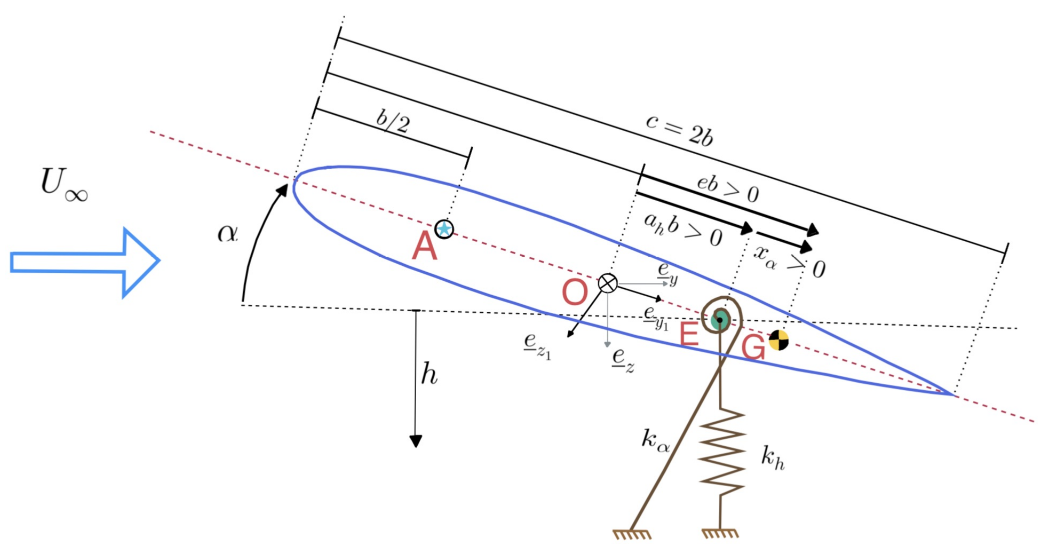

Figure 1: Schematic of the Pitch And Plunge Aeroelastic (PAPA) problem

[1]Email: jean-camille.chassaing@sorbonne-universite.fr – web: http://www.dalembert.upmc.fr/home/chassaing

1.2 Aeromechanical parameters

The behavior of this **aeroelastic** system will be entirely determined by:

- The mass m of the profile.
- Aerodynamic parameters: flow speed U_∞, air density ρ_∞.
- The stiffness of springs in bending (or heave) k_h ($[N.m^{-1}]$) and in torsion (or pitching) k_α ([N.m.rad $^{-1}$]).
- The chord of the airfoil c (distance between the leading edge and the trailing edge). Note b the half-chord.
- The geometry of the airfoil and the position of the aerodynamic center **A**, which is located at the first quarter of the chord in the case of an incompressible flow.
- The position eb between the middle of the profile (point **O** and the center of mass **G**. Note e the corresponding scaled distance with $e > 0$ when **G** points to the trailing edge.
- The position $a_h b$ between the point **O** and the elastic center **E** with $a_h > 0$ when **E** points to the trailing edge.

Note: It is very convenient to introduce the static unbalance coefficient $x_\alpha = e - a_h$, as $\underline{EG} = x_\alpha b \underline{e}_{y_1}$ with $x_\alpha > 0$ when the elastic center is located between the mid-airfoil section and the center of mass.

1.3 Generalised Coordinates

The airfoil is assumed to be **rigid** (no deformations in camber) with a **solid** rotationnal motion around the **elastic center** E. The motion of the airfoil is then fully characterized by the following 2 degrees of freedom:

- **The pitch angle** α between the direction of the incident flow and the position of the airfoil.
- The **vertical displacement** h of the elastic center.

Sign convention:

- α will be positive when the leading edge of the profile is above the horizontal axis.
- h will be positive when its displacement vector is directed to the downward vertical axis.

2 Governing Equations

The equations of motion of this aeroelastic mechanical system are deduced from the **Lagrange** equations, which for a non-conservative n degrees of freedom are written:

$$\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial \dot{q}_i}\right) - \frac{\partial(T-U)}{\partial q_i} = Q_i \qquad i = 1,2 \tag{1}$$

where $q_i = (h, \alpha)$ represents the generalized coordinates and U denotes the potential energy, T the kinetic energy and Q_i the generalized aerodynamic forces.

2.1 Potential energy

We immediately have:

$$U = \frac{1}{2}k_h h^2 + \frac{1}{2}k_\alpha \alpha^2 \tag{2}$$

2.2 Kinetic energy

The velocity of the elastic center E reads: $\underline{V}_E = \dot{h}\underline{e}_z$ with ($\dot{h} = dh/dt$). Then, can then deduce the velocity at the center of mass given the relation between velocity vectors of two points belonging to the same solid, namely:

$$\underline{V}_G = \underline{V}_E + \underline{GE} \wedge \underline{\Omega}_{Solide/R_0} \tag{3}$$

with

$$\underline{\Omega}_{Solide/R_0} = \dot{\alpha}\ \underline{e}_x \quad \text{and} \quad \underline{GE} = -bx_\alpha \underline{e}_{y_1}$$

By convention, the static disequilibrium coefficient x_α which denotes the distance between the center of mass and the elastic center is positive when the center of mass is situated after the elastic center.

Assuming that angles are small (ie $cos(\alpha) \sim 1$ and $sin(\alpha) \sim 0$), we have $-\underline{e}_{y_1} \wedge \underline{e}_x \sim \underline{e}_z$. Under these conditions, the velocity of the center of mass is equal to:

$$\underline{V}_G = \dot{h}\underline{e}_z + bx_\alpha\dot{\alpha}\underline{e}_z \tag{4}$$

Let I_G the moment of inertia of the airfoil around the center of mass ($[I_G] = kg.m^{-1}$), the kinetic energy T of the PAPA system is obtained by:

$$T = \frac{1}{2}m\underline{V}_G.\underline{V}_G + \frac{1}{2}I_G\dot{\alpha}^2 \tag{5}$$

Then

$$T \quad = \frac{1}{2}m\left(\dot{h}^2 + 2\dot{h}b\dot{\alpha}x_\alpha + x_\alpha^2 b^2\dot{\alpha}^2\right) + \frac{1}{2}I_G\dot{\alpha}^2 = \frac{1}{2}m\left(\dot{h}^2 + 2\dot{h}b\dot{\alpha}x_\alpha\right) + \frac{1}{2}I_E\dot{\alpha}^2 \tag{6}$$

where $I_E = I_G + mb^2x_\alpha^2$ denotes the moment of inertia of the airfoil around the elastic center.

2.3 Aerodynamic loads

The expressions of generalized forces for the two degrees of freedom h and α are obtained using the principle of **virtual work**. Consider first the expression of the velocity of the aerodynamic center (point A):

$$\underline{V}_A = \underline{V}_E + \underline{AE} \wedge \underline{\Omega}_{Solide/R_0} = \dot{h}\underline{e}_z + b\dot{\alpha}\left(a_h + \frac{1}{2}\right)\underbrace{\underline{e}_{y_1} \wedge \underline{e}_x}_{\approx -\underline{e}_z} \tag{7}$$

The virtual displacement of the aerodynamic focus $\delta\underline{d}_A$ is evaluated by replacing the time derivative operator d/dt by the symbol δ for each unknown in the previous equation. We thus obtain:

$$\delta\underline{d}_A = \delta h\ \underline{e}_z - b\ \delta\alpha\ \left(a_h + \frac{1}{2}\right)\underline{e}_z \tag{8}$$

Then, the virtual work of the lift L for a virtual displacement $\delta \underline{d}_A$ of the aerodynamic center and of the aerodynamic moment around the elastic center E for a virtual movement of solid rotation is:

$$\delta W = -L\underline{e}_z.\delta \underline{d}_A + M_A \delta\alpha = \left[-\delta h + b\left(a_h + \frac{1}{2}\right)\delta\alpha\right] L + M_{FA}\delta\alpha \tag{9}$$

By identification, we obtain the generalized forces associated with heave h and pitch α displacements:

$$\begin{cases} Q_h = -L \\ Q_\alpha = \underbrace{M_A + b\left(a_h + \frac{1}{2}\right)L}_{M_E} \end{cases} \tag{10}$$

2.4 Equation of motion

Injecting the expressions of potential energy (2), kinetic energy (3) and generalized aerodynamic forces (4d) in the Lagrange equations, we obtain the following system of two coupled differential equations:

$$\begin{cases} m\ddot{h}(t) + mbx_\alpha\ddot{\alpha}(t) + k_h h(t) = -L(t) \\ \\ mbx_\alpha\ddot{h}(t) + I_E\ddot{\alpha}(t) + k_\alpha\alpha(t) = M_A(t) + b\left(\frac{1}{2} + a_h\right)L(t) \end{cases} \tag{11}$$

As we will see later, the resolution of his equations can be done in the *time-domain* or in the *frequency-domain*. The first approach will make it possible to compute the time-history of (h and α) and then to deduce the **stable** (damped), **neutral** (harmonic) or **unstable** (divergning) oscillations of the airfoil. Solving the problem in the *frequency* domain will directly determine the critical speed of aeroelastic linear (classical) flutter.

Before solving the equations of the aeroelastic motion, it is convenient to make the vertical displacement of the elastic center non-dimensionalized by introducing as new degree of freedom $\xi = h/b$

Dividing the two equations of the system (5a) by mb and mb^2 respectively, we obtain

$$\begin{cases} \ddot{\xi} + x_\alpha\ddot{\alpha} + \dfrac{k_h}{m}\xi = -\dfrac{L(t)}{mb} \\ x_\alpha\ddot{\xi} + \dfrac{I_E}{mb^2}\ddot{\alpha} + \dfrac{k_\alpha}{mb^2}\alpha(t) = \dfrac{M_A}{mb^2} + \dfrac{L(t)}{mb}\left(\dfrac{1}{2} + a_h\right) \end{cases} \tag{12}$$

Let us introduce the following dimensionless aeroelastic parameters:

- $\omega_h = \sqrt{\dfrac{k_h}{m}}$: The uncoupled natural frequency in plunge.
- $\omega_\alpha = \sqrt{\dfrac{k_\alpha}{I_E}}$: The uncoupled natural frequency in pitch.
- $r_\alpha = \sqrt{\dfrac{I_E}{mb^2}}$: Radius of gyration of the airfoil according to the mid-chord.
- $\mu = \dfrac{m}{\pi\rho_\infty b^2}$: Mass ratio per unit span, between the mass of the wing on the mass of a cylindrical volume of diameter b.

• $\Omega = \dfrac{\omega_h}{\omega_\alpha}$: Ratio of the uncoupled natural frequency in plunge and pitch.

• $V = \dfrac{U_\infty}{b\omega_\alpha}$: Non-dimensionalized free-stream velocity by the uncoupled natural frequency in pitch.

The equations of motion (5c) then take the following form

$$\begin{cases} \ddot{\xi} + x_\alpha \ddot{\alpha} + \underbrace{\dfrac{k_h}{m}}_{=\omega_h^2} \xi = -\dfrac{L(t)}{mb} \\ x_\theta \ddot{\xi} + \dfrac{r_\alpha^2 mb^2}{mb^2} \ddot{\alpha} + \omega_\alpha^2 \underbrace{\dfrac{I_E}{mb^2}}_{r_\alpha^2} \alpha(t) = \underbrace{\dfrac{M_A}{mb^2} + \dfrac{L(t)}{mb}\left(\dfrac{1}{2} + a_h\right)}_{M_E/(mb^2)} \end{cases} \tag{13}$$

The previous system of differential equations can then be written in matrix form as

$$\underbrace{\begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix}}_{\text{Mass matrix } \mathbf{M_s}} \begin{bmatrix} \ddot{\xi} \\ \ddot{\alpha} \end{bmatrix} + \underbrace{\begin{bmatrix} \omega_h^2 & 0 \\ 0 & r_\alpha^2 \omega_\alpha^2 \end{bmatrix}}_{\text{Stiffness matrix } \mathbf{K_s}} \begin{bmatrix} \xi \\ \alpha \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{L}{mb} \\ \frac{M_E}{mb^2} \end{bmatrix}}_{\mathbf{F_{aero}}} \tag{14}$$

Either in a compact way

$$\mathbf{M_s}\ddot{\mathbf{q}} + \mathbf{K_s}\mathbf{q} = \mathbf{F_{aero}}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \tag{15}$$

where $\mathbf{q} = [\xi, \alpha]^T \in \mathbb{R}^2$ denotes the vector of *generalized coordinates* of the PAPA problem. Let us note that the complete determination of the equations of the dynamics of this aeroelastic system requires the explicit expression of the generalized aerodynamic forces $\mathbf{F_{aero}}$ as a function of the generalized coordinates $\mathbf{q}$ and its first and second order derivatives.

Concretely, this amounts to model the action of aerodynamic forces, generated by the *unsteady* flow, exerted on the *moving* airfoil. As we will see in the following chapters, different theories of unsteady aerodynamics can then be used, taking care to choose their level of physical meanming according to the aeroelastic problem considered.

On the basis that we have a model of the aerodynamic forces generated in the *time-domain*, it is then possible to describe the aerodynamic operator in the following matrix form

$$\mathbf{F_{aero}}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) = \mathbf{M_{aero}}\ddot{\mathbf{q}} + \mathbf{D_{aero}}\dot{\mathbf{q}} + \mathbf{K_{aero}}\mathbf{q} \tag{16}$$

where $\mathbf{M_{aero}}$ is the aerodynamic mass matrix, $\mathbf{D_{aero}}$ denotes the aeridynamic damping matrix and $\mathbf{K_{aero}}$ represent the aerodynamic stiffness matrix. The expressions of the coefficients of these matrices of $\mathbb{R}^2 \times 2$ will depend on the type of aerodynamic modeling chosen. Some types of historical models r currently used in engineering will be presented in next notebooks and the resolution of the associated flutter problem will be the subject of many application notebooks (see flutter problem 3.1 for instance).

In the case where there is no mathematical theory to explicitly obtain the aerodynamic matrices, the coefficients of the latter are obtained numerically from results of numerical simulations and the application of the flutter derivatives.

$$(\mathbf{M_s} - \mathbf{M_{aero}})\ddot{\mathbf{q}} + (\mathbf{D_s} - \mathbf{D_{aero}})\dot{\mathbf{q}} + (\mathbf{K_s} - \mathbf{K_{aero}})\mathbf{q} = \mathbf{0} \tag{17}$$

3 To go further

Taking into account to the viscous damping forces

$$\underbrace{\begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix}}_{\text{Mass matrix } \mathbf{M_s}} \begin{bmatrix} \ddot{\xi} \\ \ddot{\alpha} \end{bmatrix} + \underbrace{\begin{bmatrix} \omega_z^2 & 0 \\ 0 & r_\alpha^2 \omega_\alpha^2 \end{bmatrix}}_{\text{Stiffness matrix } \mathbf{K_s}} \begin{bmatrix} \xi \\ \alpha \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{L}{mb} \\ \frac{M_E}{mb^2} \end{bmatrix}}_{\mathbf{F_{aero}}} - \underbrace{\begin{bmatrix} 2\zeta_z \omega_z & 0 \\ 0 & 2\zeta_\alpha r_\alpha^2 \omega_\alpha \end{bmatrix}}_{\text{Damping matrix } \mathbf{D_s}} \begin{bmatrix} \dot{\xi} \\ \dot{\alpha} \end{bmatrix} \tag{18}$$

Demonstration left to students.

References

[1] D.H. Hodges, G. A. Pierce, "Introduction to Structural Dynamics and Aeroelasticity ", Cambridge Aerospace series, ISBN 0-521-80698-4, 2002

[2] D.A. Peters, S. Karunamoorthy, W.-M. Cao, “Finite State Induced Flow Models; Part I: Two-Dimensional Thin Airfoil,” Journal of Aircraft, Vol. 32, No. 2, Mar.-Apr. 1995, pp. 313–322.

[3] T. Theodorsen, "General Theory of Aerodynamic Instability and the Mechanism of Flutter", NACA TR 496, 1934.

[4] H.J. Hassig, “An Approximate True Damping Solution of the Flutter Equation by Determinant Iteration,” Journal of Aircraft, Vol. 8, No. 11, Nov. 1971, pp. 885 – 889.

[5] E.H. Dowell, E.F. Crawley, H.C. Curtiss, Jr., D.A. Peters, R.H. Scanlan and F. Sisto, A Modern Course in Aeroelasticity, 3rd ed., Kluwer Academic Publishers, 1995.

[6] R.L. Bisplinghoff, H. Ashley and R.L. Halfman, Aeroelasticity, Addison-Wesley Publishing Co., Inc., 1955.