Lecture 7

Convergence Theory for Linear Methods - Part 1

Checking stability

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Checking stability of a scheme is usually the most difficult part when proving convergence.

Several different approaches, e.g.

1. CFL condition
   (necessary condition)
2. von Neumann analysis
   (sufficient condition, constant coefficients)
3. Energy method
   (sufficient condition, variable coefficients)

- We discuss the first two.
- L^1 version of energy method is briefly explained in Leveque 8.3.4.
- Von Neumann analysis can only handle periodic boundary conditions or no boundaries.
- Energy method can handle more general boundary conditions.

Checking stability

CFL condition

CFL condition

Consider a constant coefficient advection equation

$$\partial_t u + \mathbf{a}\partial_x u = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(0,x) = v_0(x).$$

and an explicit "3-point method", i.e.

$$Q_j^{n+1} = c_{-1}\, Q_{j-1}^n + c_0\, Q_j^n + c_1\, Q_{j+1}^n,$$

for some coefficients c_j.

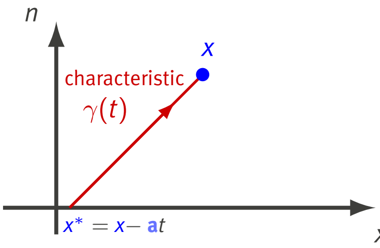
CFL condition in most simple form: "if the scheme is consistent, then

$$|\mathbf{a}|\frac{\Delta t}{\Delta x} \leq 1,$$

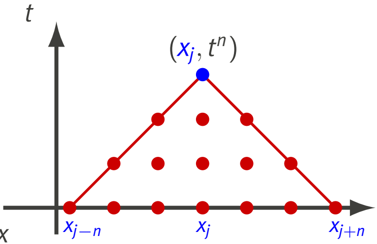
is a necessary condition for stability."

Note: all methods considered so far are consistent 3-point methods.

CFL condition - Intuitive explanation



Continuous problem



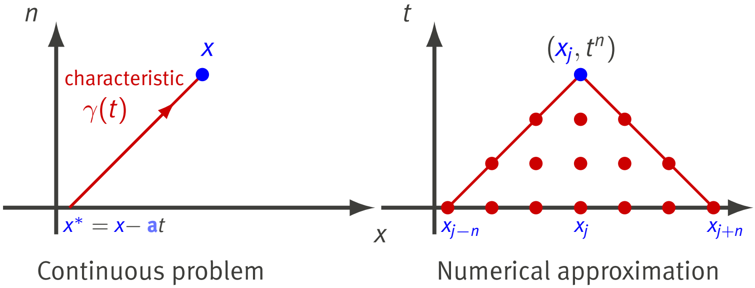
Numerical approximation

Solution u in point (x, t) only depends on initial data $v_0(x)$ in $x^* = x - at$.

"Domain of dependence" of (x, t) is $D(x, t) = \{x^*\} = \{x - at\}$.

Numerical case: each value at time level n depends on the three surrounding values at time level $n - 1$. Induction: numerical solution Q_j^n at (x_j, t_n) depends on initial data $v_0(x)$ evaluated in $x_{j-n}, \ldots, x_{j+n}$.

CFL condition - Intuitive explanation



Numerical case: each value at time level n depends on the three surrounding values at time level $n-1$. Induction: numerical solution Q_j^n at (x_j, t_n) depends on initial data $v_0(x)$ evaluated in $x_{j-n}, \ldots, x_{j+n}$.

"Numerical domain of dependence": $D_{\text{num}}(x_j, t_n) = [x_{j-n}, x_{j+n}]$.

For all sufficiently small $\Delta t, \Delta x \to 0$ we require $D(x_j, t_n) \subset D_{\text{num}}(x_j, t_n)$.

CFL condition - Intuitive explanation

General CFL condition then says:

> *A consistent method can only be stable if the continuous domain of dependence $D(x_j, t_n)$ is a subset of the numerical domain of dependence $D_{\text{num}}(x_j, t_n)$.*

- natural condition, because if $D(x_j, t_n) \not\subset D_{\text{num}}(x_j, t_n)$ then the numerical method cannot "know" the exact solution in (x_j, t_n). Hence, no hope to get a convergent method.
- Note again: CFL condition is only a necessary condition, i.e.
- the scheme may still be unstable if the condition is satisfied.

CFL condition - Intuitive explanation

In our example $D(x_j, t_n) \subset D_{\text{num}}(x_j, t_n)$ is equivalent to

$$x_{j-n} = x_j - n\Delta x \;\leq\; \underbrace{x_j - \mathbf{a}t_n}_{=D(x_j,t_n)} \;\leq\; x_j + n\Delta x = x_{j+n}.$$

This implies

$$|\mathbf{a}|t_n \leq n\Delta x.$$

With $t_n = n\Delta t$ this implies the CFL condition

$$|\mathbf{a}| \leq \frac{\Delta x}{\Delta t}.$$

CFL condition for systems

Recall: for a system of m equations,

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = 0,$$

with $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times m}$, the domain of dependence is

$$D(x,t) = \{x - \lambda_p t \mid 1 \leq p \leq m\},$$

where λ_p are the eigenvalues of $\mathbf{A}$.

Same arguments as before lead to the CFL condition

$$|\lambda_p| \frac{\Delta t}{\Delta x} \leq 1, \qquad p = 1, \ldots, m.$$