Assignement 3 (SF2521) - 4^{th} order Equation

SF2521

Well-posedness and stability of multidimensional equation

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1 Introduction

In this report we will present the results for the Homework Assignment 3, dealing as well with stability, well-posedness, function's properties' analysis, and so on.

We will therefore present a first half showing the analyses of the points presented just above, and a second half treating those using numerical analysis.

2 Well-posedness / von Neumann

We will throughout this section consider the following 2-π Cauchy problem :

$$u_t = \alpha u_{xx} + \beta u_{xxxx} \qquad \forall x \in \Omega = (0, 2\pi)^2, \forall t > 0 \tag{1}$$
$$u(x,0) = sin(x) \qquad \forall x \in \Omega = (0, 2 * \pi)^2, t = 0 \tag{2}$$

2.1 Norm L_2

We take a look at the L2 norm cosidering $\alpha > 0$ and $\beta = 0$.
We thereby have the following equality that stands :

$$u_t = \alpha u_{xx}$$

which is a parabolic equation.

To show that a problem is well posed in norm L2, we have to check the followings characteristics :

- Existence of the solution ;
- Uniqueness of the solution ;
- Stability of the solution ;

On our part we will consider the first point validated and use the Fourier transform to show that the two others stand.

By applying the Fourier transform with respect to the variable x, we get :

$$
\begin{aligned}
F(\frac{\partial u}{\partial t}) &= F(\alpha\frac{\partial^2 u}{\partial t^2}) \\
\int_\Omega \frac{\partial u}{\partial t} e^{-i\omega x} dx &= \int_\Omega \alpha\frac{\partial^2 u}{\partial x^2} e^{-i\omega x} dx \\
\frac{\partial}{\partial t}\int_\Omega u * e^{-i\omega x} dx &= \int_\Omega \alpha\frac{\partial^2 u}{\partial x^2} e^{-i\omega x} \\
->\frac{\partial \tilde{u}}{\partial t} &= \int_\Omega \alpha\frac{\partial^2 u}{\partial x^2} e^{-i\omega x} dx
\end{aligned}
$$

We now take a better look to the right hand side's term of this last equality and we apply the principle of integration by parts :

$$\alpha \int_\Omega \frac{\partial^2 u}{\partial x^2} e^{-i\omega x} dx = \alpha[\frac{\partial u}{\partial x} * e^{-i\omega x}]_{\partial\Omega} - \alpha \int_\Omega -i\omega \frac{\partial u}{\partial x} e^{-i\omega x} dx$$

Yet, the term $[\frac{\partial u}{\partial x} * e^{-i\omega x}]_{\partial\Omega}$ is null as the integration of the exponential term at $\partial\Omega$ is zero.
Indeed :

$$
\begin{aligned}
[\frac{\partial u}{\partial x} * e^{-i\omega x}]_{\partial\Omega} &= \frac{\partial u(2*\pi)}{\partial x} \times e^{-i\omega\times 2*\pi} - \frac{\partial u(0)}{\partial x} \times e^{-i\omega \times 0} && (3) \\
\frac{\partial u(2*\pi)}{\partial x} &= \frac{\partial u(0)}{\partial x} \quad \text{(from the periodicity of the function)} && (4) \\
e^{-i\omega\times 2*\pi} &= e^{-i\omega\times 0} \quad \text{as well} && (5) \\
->[\frac{\partial u}{\partial x} * e^{-i\omega x}]_{\partial\Omega} &= \frac{\partial u(2*\pi)}{\partial x}(e^{-i\omega\times 2*\pi} - e^{-i\omega\times 0}) = 0 && (6)
\end{aligned}
$$

The previous equality then leads to (using once again integration by parts):

$$
\begin{aligned}
\alpha\int_\Omega \frac{\partial^2 u}{\partial x^2} e^{-i\omega x} dx &= \alpha i\omega \int_\Omega \frac{\partial u}{\partial x} e^{-i\omega x} dx && (7) \\
\alpha i\omega \int_\Omega \frac{\partial u}{\partial x} e^{-i\omega x} dx &= \alpha[u * e^{-i\omega x}]_{\partial\Omega} - \alpha i\omega \int_\Omega -i\omega u e^{-i\omega x} dx && (8) \\
-\alpha i\omega \int_\Omega -i\omega u e^{-i\omega x} dx &= +\alpha i^2\omega^2\tilde{u} && (9)
\end{aligned}
$$

NOTE : we once again used the periodicity of the function as $u_(2*\pi) = u_(0)$

By gathering the two transforms we have, we have :

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} &= +\alpha i^2\omega^2\tilde{u} && (10) \\
\frac{\partial \tilde{u}}{\partial t} &= -\alpha\omega^2\tilde{u} && (11)
\end{aligned}
$$

We thus have to solve a differential equation of order 1, leading to :

$$\tilde{u}(\omega,t) = C * e^{-\alpha\omega^2} \text{ with C} \in \mathbb{R}$$

Yet, $\tilde{u}(\omega,0) = C$, so we finally obtain :

$$\tilde{u}(\omega,t) = \tilde{u}(\omega,0) * e^{-\alpha\omega^2}$$

We then apply the L2 norm and as we have this equality that holds :

$$\begin{aligned}
||\tilde{u}(\omega,t)||_{L_2(\Omega)} &= ||\tilde{u}(\omega,0) * e^{-\alpha\omega^2}||_{L_2(\Omega)} && (12)\\
||\tilde{u}(\omega,t)||_{L_2(\Omega)} &\leq ||\tilde{u}(\omega,0)||_{L_2(\Omega)} \times ||e^{-\alpha\omega^2}||_{L_2(\Omega)} && (13)\\
-\alpha < 0 & \;-> e^{-\alpha\omega^2} < 1, \quad \forall\omega \in \Omega && (14)\\
-> ||e^{-\alpha\omega^2}||_{L_2(\Omega)} &= 1 && (15)\\
-> ||\tilde{u}(\omega,t)||_{L_2(\Omega)} &\leq ||\tilde{u}(\omega,0)||_{L_2(\Omega)} && (16)
\end{aligned}$$

Besides, in L_2 norm, we have the following equality standing :

$$||\tilde{u}(\omega,t)||_{L_2(\Omega)} \equiv ||u(.,t)||_{L_2(\Omega)}$$

and so

$$||u(.,t)||_{L_2(\Omega)} \leq ||u(.,0)||_{L_2(\Omega)}$$

This inequality shows both the uniqueness and the stability (stability as u is a maximized function and uniqueness as replacing u by the difference between two functions solution of our problem leads to zero).

2.2 Ill-posedness

We now consider the initial system with $\beta > 0$.

To show ill-posedness, we search for a couple solution of the system and aim in showing that its data growth rate is unbounded, i.e. $t \to \infty \Rightarrow u(.,t) \to \infty$.

We consider the following family solution with unity L^2 norm :

$$u_p(x) = e^{-(\sigma t + ikx)}$$

We will now see what information this particular solution is providing by implementing it into our system :

$$\begin{aligned}
\partial_t e^{-(\sigma t+ikx)} &= \alpha\partial_{xx} e^{-(\sigma t+ikx)} && + \beta\partial_{xxxx} e^{-(\sigma t+ikx)} && (17)\\
\to -\sigma &= \alpha(-ik)^2 && + \beta(-ik)^4 && (18)\\
-\sigma &= \alpha i^2 \times k^2 && + \beta i^4 \times k^4 && (19)\\
\sigma &= -\alpha i^2 \times k^2 && - \beta i^4 \times k^4 && (20)\\
\sigma &= +\alpha k^2 && - \beta k^4 \quad \text{as } i^2 = -1 \text{ and } i^4 = 1 && (21)
\end{aligned}$$

If as mentioned in the subject we consider that the term β can be the only one considered to conclude on the stability of the system, we then have the sign of σ by looking to the term in β :

$$
\begin{aligned}
\text{Sign of } \sigma &\equiv \text{Sign of } -\beta k^4 && (22)\\
\text{Yet } \beta &> 0 && (23)\\
\rightarrow -\beta k^4 &< 0 && (24)\\
\rightarrow \sigma &< 0 && (25)
\end{aligned}
$$

In fact, to get a ill-posed problem, it is necessary to have $\sigma < 0$. Indeed, we would then have an exponential solution whose growth is unbounded :

$$\lim_{t \rightarrow +\infty} e^{-(\sigma t + ikx)} \rightarrow \infty \tag{26}$$

To sum up, we assumed that β was providing the information on stability for the system and we see that we get a negative σ, which leads as seen to an unbounded function.

Moreover, with $\sigma < 0$ and by considering the previous equality :

$$
\begin{aligned}
\sigma &= -\alpha i^2 \times k^2 \quad -\beta i^4 \times k^4 && (27)\\
\rightarrow \alpha k^2 &\leq \beta k^4 && (28)\\
\rightarrow \alpha &\leq \beta k^2 && (29)\\
& && (30)
\end{aligned}
$$

The above equality is fulfilled without dependence on the sign of α, i.e. we have a ill-posed problem for $\alpha > 0$ as well as for $\alpha < 0$.

We can conclude that the problem is ill-posed when considering $\beta > 0$.

2.3 Max norm stability

2.3.1 Case $\beta = 0$

We consider $\alpha > 0$ and $\beta = 0$.

A numerical scheme is stable in the max norm if :

$$max|u_j^{n+1}| \leq max|u_j^n|$$

Yet, with respect to our conditions on α and β we have :

$$u_t = \alpha u_{xx}$$

We use a central difference in space for the approximation of u_{xx} and forward Euler in time. We thus get :

$$\frac{u_j^{n+1} - u_j^n}{\Delta_t} = \alpha \frac{u_{j+1}^n - 2 * u_j^n + u_{j-1}^n}{\Delta_x^2} \tag{31}$$

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta_t}{\Delta_x^2}(u_{j+1}^n - 2 * u_j^n + u_{j-1}^n) \tag{32}$$

$$\rightarrow u_j^{n+1} = u_j^n(1 - 2\alpha \frac{\Delta_t}{\Delta_x^2}) + \frac{\alpha \Delta_t}{\Delta_x^2}(u_{j+1}^n + u_{j-1}^n) \tag{33}$$

We will start deducing the general formula for U^{n+1} based on U0, that we can know developing the first terms:

$$u_j^1 = u_j^0(1 - 2\alpha \frac{\Delta_t}{\Delta_x^2}) + \frac{\alpha \Delta_t}{\Delta_x^2}(u_{j+1}^0 + u_{j-1}^0) \tag{34}$$

$$u_j^0 = sin(x_j), u_{j+1}^0 = sin(x_j + \Delta_x) \text{and} u_{j-1}^0 = sin(x_j - \Delta_x) \tag{35}$$

$$\rightarrow u_{j+1}^0 + u_{j-1}^0 = sin(x_j + \Delta_x) + sin(x_j - \Delta_x) = 2 sin(x_j) cos(\Delta_x) \tag{36}$$

$$\rightarrow u_j^1 = sin(x_j)(1 - \frac{2\alpha \Delta_t}{\Delta_x^2}(1 - cos(\Delta_x))) \tag{37}$$

$$u_j^2 = u_j^1(1 - 2\alpha \frac{\Delta_t}{\Delta_x^2}) + \frac{\alpha \Delta_t}{\Delta_x^2}(u_{j+1}^1 + u_{j-1}^1) \tag{38}$$

$$u_j^2 = sin(x_j)(1 - \frac{2\alpha \Delta_t}{\Delta_x^2}(1 - cos(\Delta_x)))(1 - 2\alpha \frac{\Delta_t}{\Delta_x^2}) + \frac{\alpha \Delta_t}{\Delta_x^2}(1 - \frac{2\alpha \Delta_t}{\Delta_x^2}(1 - cos(\Delta_x)))(sin(x_j + \Delta_x) + sin(x_j - \Delta_x)) \tag{39}$$

$$u_j^2 = sin(x_j)(1 - \frac{2\alpha \Delta_t}{\Delta_x^2}(1 - cos(\Delta_x)))^2 \tag{40}$$

$$etc \tag{41}$$

NOTE : We have $u_j^0 = sin(x_j)$ from the initial condition.

We eventually see that :

$$u_j^{n+1} = sin(x_j)(1 - \frac{2\alpha \Delta_t}{\Delta_x^2}(1 - cos(\Delta_x)))^{n+1} \tag{42}$$

$$\rightarrow u_j^{n+1} = sin(x_j)(1 - \frac{4\alpha \Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2}))^{n+1} \tag{43}$$

$$\rightarrow u_j^{n+1} = u_j^n \times (1 - \frac{4\alpha \Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2})) \tag{44}$$

$$\tag{45}$$

We will now assume that the max norm is fulfilled and we will determine the condition on Δ_t ensuring it. We recall that if we check the max norm, we must check :

$$
\begin{aligned}
&max|u_j^{n+1}| &&\leq max|u_j^n| && (46)\\
&max|u_j^n \times (1 - \frac{4\alpha\Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2}))| &&\leq max|u_j^n| && (47)\\
\rightarrow &max|u_j^n| \times |(1 - \frac{4\alpha\Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2}))| &&\leq max|u_j^n| && (48)\\
\Rightarrow &|(1 - \frac{4\alpha\Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2}))| &&\leq 1 && (49)
\end{aligned}
$$

NOTE : we have no maximum norm for the term $(1 - \frac{4\alpha\Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2}))$ as it doesn't depend on j.

Before simplifying, as we know that $0 \leq sin^2 \leq 1$ we can assume that $\frac{4\alpha\Delta_t}{\Delta_x^2} sin^2(\frac{\Delta_x}{2}) \approx \frac{4\alpha\Delta_t}{\Delta_x^2}$ Thus, we finally have :

$$
\begin{aligned}
& -1 && \leq 1 - \frac{4\alpha\Delta_t}{\Delta_x^2} && \leq 1 && (50)\\
& -2 && \leq - \frac{4\alpha\Delta_t}{\Delta_x^2} && \leq 0 && (51)\\
& +2 && \geq + \frac{4\alpha\Delta_t}{\Delta_x^2} && \geq 0 && (52)\\
\rightarrow \quad & +2 && \geq + \frac{4\alpha\Delta_t}{\Delta_x^2} && \geq 0 && (53)\\
& && && && (54)\\
\Rightarrow \quad & 0 && \leq \Delta_t && \leq \frac{\Delta_x^2}{2\alpha} && (55)
\end{aligned}
$$

We did manage to get a condition on Δ_t *such that the numerical scheme is stable in max norm* :

$$\Delta_t \leq \frac{\Delta_x^2}{2\alpha}$$

2.3.2 Case $\beta \neq 0$

We now want to see if this method is usable for other condition on α and β. We thus want to see the case for which $\beta \neq 0$.

To the question : "Is it possible to have a second order finite difference approximation of u_{xxxx}", we will now see that the answer is yes.

In fact, to have a second order finite difference implies that the rest for the approximation will be of order 2. In our case, the expansion of u_{xxxx} using Taylor's series leads to :

$$u(x+h) = u(x) + \Delta_x \frac{u(x)^{(1)}}{1!} + \Delta_x^2 \frac{u(x)^{(2)}}{2!} + \Delta_x^3 \frac{u(x)^{(3)}}{3!} + \Delta_x^4 \frac{u(x)^{(4)}}{4!} + \Delta_x^5 \frac{u(x)^{(5)}}{5!} + \Delta_x^6 \frac{u(x)^{(6)}}{6!}$$

$$u(x-h) = u(x) - \Delta_x \frac{u(x)^{(1)}}{1!} + \Delta_x^2 \frac{u(x)^{(2)}}{2!} - \Delta_x^3 \frac{u(x)^{(3)}}{3!} + \Delta_x^4 \frac{u(x)^{(4)}}{4!} - \Delta_x^5 \frac{u(x)^{(5)}}{5!} + \Delta_x^6 \frac{u(x)^{(6)}}{6!}$$

$$u(x+2h) =$$
$$u(x) + 2\Delta_x \frac{u(x)^{(1)}}{1!} + 4\Delta_x^2 \frac{u(x)^{(2)}}{2!} + 8\Delta_x^3 \frac{u(x)^{(3)}}{3!} + 16\Delta_x^4 \frac{u(x)^{(4)}}{4!} + 32\Delta_x^5 \frac{u(x)^{(5)}}{5!} + 64\Delta_x^6 \frac{u(x)^{(6)}}{6!}$$

$$u(x-2h) =$$
$$u(x) - 2\Delta_x \frac{u(x)^{(1)}}{1!} + 4\Delta_x^2 \frac{u(x)^{(2)}}{2!} - 8\Delta_x^3 \frac{u(x)^{(3)}}{3!} + 16\Delta_x^4 \frac{u(x)^{(4)}}{4!} - 32\Delta_x^5 \frac{u(x)^{(5)}}{5!} + 64\Delta_x^6 \frac{u(x)^{(6)}}{6!}$$

By combining these equality, we finally get :

$$u^{(4)} = \frac{u(x+2h) - 4*u(x+h) + 6*u(x) - 4*u(x-h) + u(x-2h)}{\Delta_x^4} - \frac{\Delta_x^2 * u^{(6)}}{6}$$

We finally see that we do have as expected a rest of order 2, leading in the fact that it is possible to approximate a derivative of fourth order using a second order finite difference approximation.

NOTE : the term h corresponds to Δ_x

To finish with this sub-part, we are willing to apply the max norm again on this new case ($\beta \neq 0$). We will in a first time determine the general form of U_j^n with respect to our initial condition ($u(x,0) = sin(x)$) :

$$u_j^{n+1} =$$
$$u_j^n + \frac{\Delta_t}{\Delta_x^2}(\alpha(u_{j+1}^n - 2*u_j^n + u_{j-1}^n) + \beta(\frac{u(x+2\Delta_x) - 4*u(x+\Delta_x) + 6*u(x) - 4*u(x-\Delta_x) + u(x-2\Delta_x)}{\Delta_x^4}))$$

$$\rightarrow u_j^{n+1} = u_j^n \times (1 - 2\frac{\alpha\Delta_t}{\Delta_x^2} + \frac{6\beta\Delta_t}{\Delta_x^4}) + (u_{j+1}^n + u_{j-1}^n) \times (\frac{\alpha\Delta_t}{\Delta_x^2} - \frac{4\beta\Delta_t}{\Delta_x^4}) + (u_{j+2}^n + u_{j-2}^n) \times (\frac{\beta\Delta_t}{\Delta_x^4})$$

$$u_j^1 = u_j^0 \times (1 - 2\frac{\alpha\Delta_t}{\Delta_x^2} + \frac{6\beta\Delta_t}{\Delta_x^4}) + (u_{j+1}^0 + u_{j-1}^0) \times (\frac{\alpha\Delta_t}{\Delta_x^2} - \frac{4\beta\Delta_t}{\Delta_x^4}) + (u_{j+2}^0 + u_{j-2}^0) \times (\frac{\beta\Delta_t}{\Delta_x^4})$$

So with : $u_j^0 = sin(x_j)$, $u_{j+1}^0 = sin(x_j + \Delta_x)$, $u_{j-1}^0 = sin(x_j - \Delta_x)$, $u_{j+2}^0 = sin(x_j + 2\Delta_x)$ and $u_{j-2}^0 = sin(x_j - 2\Delta_x)$

We have $u_j^1 = sin(x_j)(1 - \frac{2\Delta_t}{\Delta_x^2}(\alpha(1 - cos(\Delta_x)) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x))))$

In the same way and by iterating one more time :

$$u_j^2 = u_j^1 \times (1 - 2\frac{\alpha\Delta_t}{\Delta_x^2} + \frac{6\beta\Delta_t}{\Delta_x^4}) + (u_{j+1}^1 + u_{j-1}^1) \times (\frac{\alpha\Delta_t}{\Delta_x^2} - \frac{4\beta\Delta_t}{\Delta_x^4}) + (u_{j+2}^1 + u_{j-2}^1) \times (\frac{\beta\Delta_t}{\Delta_x^4})$$

So with

$$u_j^1 = sin(x_j) \times (1 - \frac{2\Delta_t}{\Delta_x^2}(\alpha(1 - cos(\Delta_x)) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x)))) = sin(x_j) \times \tau$$

We have $u_j^2 = sin(x_j) \times \tau \times (1 - 2\frac{\alpha\Delta_t}{\Delta_x^2} + \frac{6\beta\Delta_t}{\Delta_x^4}) + (sin(x_j + \Delta_x) \times \tau + sin(x_j - \Delta_x) \times \tau) \times (\frac{\alpha\Delta_t}{\Delta_x^2} - \frac{4\beta\Delta_t}{\Delta_x^4}) + (sin(x_j + 2\Delta_x) \times \tau + sin(x_j - 2\Delta_x) \times \tau) \times (\frac{\beta\Delta_t}{\Delta_x^4})$

And finally
$\rightarrow u_j^2 = sin(x_j) \times \tau \times (1 - \frac{2\Delta_t}{\Delta_x^2}(\alpha(1 - cos(\Delta_x)) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x)))) = sin(x_j) \times \tau^2$

Regarding the two iterations we just did and similarly to what we had before we have :

$$u_j^{n+1} = sin(x_j) \times \tau^{n+1} \text{ and } u_j^n = sin(x_j) \times \tau^n$$

$$\text{So : } u_j^{n+1} = sin(x_j) \times \tau^{n+1} = sin(x_j) \times \tau^n \times \tau = u_j^n \times \tau$$

Subsequently we have considering the max norm :

$$max|u_j^{n+1}| \le max|u_j^n| \quad (56)$$
$$max|u_j^n \times \tau| \le max|u_j^n| \quad (57)$$
$$max|u_j^n| \times |\tau| \le max|u_j^n| \quad (58)$$

Which eventually lead us to :

$$-1 \le 1 - \frac{2\Delta_t}{\Delta_x^2}(\alpha(1 - cos(\Delta_x)) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x))) \le 1$$
$$-2 \le -\frac{2\Delta_t}{\Delta_x^2}(\alpha(1 - cos(\Delta_x)) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x))) \le 0$$

$$+2 \ge \frac{2\Delta_t}{\Delta_x^2}(\alpha(1 - cos(\Delta_x)) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x))) \ge 0$$
$$\text{So with} \quad 1 - cos(\Delta_x) = 2 * sin^2(\frac{\Delta_x}{2})$$
$$\frac{2\Delta_x^2}{2} \ge \Delta_t(2\alpha sin^2(\frac{\Delta_x}{2}) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x))) \ge 0$$
$$\Rightarrow \Delta_t \le \frac{\Delta_x^2}{(2\alpha sin^2(\frac{\Delta_x}{2}) - \frac{\beta}{\Delta_x^2}(3 + 4cos(\Delta_x) + cos(2\Delta_x)))}$$

We got an other condition on $Delta_t$, here ensuring the max norm in the case where $\beta \neq 0$. We note that for $\beta = 0$ and by doing the same consideration on $sin^2(\frac{h}{2})$, we get back the first condition we derived before.

As we must have $\Delta_t > 0$ and that $\Delta_x^2 > 0$, the above relation provides another one with respect to the variables α and β :

$$
\begin{aligned}
&2\alpha sin^2(\frac{\Delta_x}{2}) - \frac{\beta}{\Delta_x^2}(3+4cos(\Delta_x)+cos(2\Delta_x)) &&>0 && (59)\\
\rightarrow &2\alpha sin^2(\frac{\Delta_x}{2}) &&>\frac{\beta}{\Delta_x^2}(3+4cos(\Delta_x)+cos(2\Delta_x)) && (60)\\
&\frac{2\alpha sin^2(\frac{\Delta_x}{2})}{(3+4cos(\Delta_x)+cos(2\Delta_x))} &&>\frac{\beta}{\Delta_x^2} && (61)\\
&\frac{2\alpha sin^2(\frac{\Delta_x}{2})}{(2+4cos(\Delta_x)+2\frac{(1+cos(2\Delta_x))}{2})} &&>\frac{\beta}{\Delta_x^2} && (62)\\
&\frac{2\alpha sin^2(\frac{\Delta_x}{2})}{2(1+2cos(\Delta_x)+cos^2(\Delta_x))} &&>\frac{\beta}{\Delta_x^2} && (63)\\
\rightarrow &\frac{\alpha sin^2(\frac{\Delta_x}{2})}{(1+cos(\Delta_x))^2} &&>\frac{\beta}{\Delta_x^2} && (64)\\
\Rightarrow &\alpha(\frac{\Delta_x sin(\frac{\Delta_x}{2})}{(1+cos(\Delta_x))})^2 &&>\beta && (65)
\end{aligned}
$$

To discuss a bit this information, if we ensure $\beta < 0$ such that the problem isn't ill-posed, we see that we can have α either positive or negative and so we have no restriction else than the one seen before.
We have been able to derive conditions on both Δ_t and Δ_x, as well as on β and α.

2.4 Von Neumann Analysis

We consider a finite volume scheme by using a finite difference approximation with central differences in space and forward difference in time. We are willing to work on a method ensure the scheme to be stable and not only a necessary condition. We will in the followings consider consequently the well and ill posed cases.

2.4.1 Well-posed case

In a first time we assume that we are in the well-posed case, i.e. that we have $\beta = 0$ and $\alpha > 0$.

We thus have the following equality that stands regarding our initial system and the equation already depicted above :

$$\begin{aligned}
u_j^{n+1} =& u_j^n + \frac{\Delta_t}{\Delta_x^2}(\alpha(u_{j+1}^n - 2 * u_j^n + u_{j-1}^n) + && (66)\\
& \beta(\frac{u(x+2\Delta_x) - 4 * u(x+\Delta_x) + 6 * u(x) - 4 * u(x-\Delta_x) + u(x-2\Delta_x)}{\Delta_x^4})) && (67)\\
\rightarrow u_j^{n+1} =& u_j^n \times (1 - 2\frac{\alpha\Delta_t}{\Delta_x^2} + \frac{6\beta\Delta_t}{\Delta_x^4}) + (u_{j+1}^n + u_{j-1}^n) \times (\frac{\alpha\Delta_t}{\Delta_x^2} - \frac{4\beta\Delta_t}{\Delta_x^4}) + && (68)\\
& (u_{j+2}^n + u_{j-2}^n) \times (\frac{\beta\Delta_t}{\Delta_x^4}) && (69)
\end{aligned}$$

The Von Neumann analysis lead us in seeking for u_j^{n+1} under the form :

$$u_j^{n+1} = \sum_{l=-m}^{M} b_l(\Delta_t, \Delta_x) u_{j+l}^n \tag{70}$$

Yet, once applied the Fourier transform :

$$u_j^n = \sum_{k=-N/2}^{N/2-1} \tilde{u_k}^n e^{+ikx_j} \tag{71}$$

$$\tilde{u_k}^n = \frac{1}{N} \sum_{j=0}^{N-1} u_j^n e^{-ikx_j} \tag{72}$$

We can rewrite the expression as follows :

$$\tilde{u_k}^{n+1} = g_k(\Delta_t, \Delta_x) \times \tilde{u_k}^n \tag{73}$$

$$\text{With} \quad g_k(\Delta_t, \Delta_x) = \sum_{l=-m}^{M} b_l(\Delta_t, \Delta_x) e^{+ikl\Delta_x} \tag{74}$$

Our purpose will be to have g_k as low as possible and we will see what condition on Δ_t and Δ_x must be set to ensure so.

We thus seek for the expression of $g_k(\Delta_t, \Delta_x)$.
By identification with our finite volume scheme we first get the b_l terms :

$$u_j^{n+1} = \sum_{l=-m}^{M} b_l(\Delta_t, \Delta_x) u_{j+l}^n \tag{75}$$

$$\Rightarrow \qquad m = 2 \qquad \text{and} \qquad M = 2 \tag{76}$$

$$\tag{77}$$

$$\rightarrow \qquad b_{-2} = b_{+2} = \frac{\beta \Delta_t}{\Delta_x^4} \tag{78}$$

$$\rightarrow \qquad b_{-1} = b_{+1} = \frac{\alpha \Delta_t}{\Delta_x^2} - \frac{4\beta \Delta_t}{\Delta_x^4} \tag{79}$$

$$\rightarrow \qquad b_0 = 1 - 2\frac{\alpha \Delta_t}{\Delta_x^2} + \frac{6\beta \Delta_t}{\Delta_x^4} \tag{80}$$

To symplify the above expressions we now on note $\lambda_1 = \frac{\alpha \Delta_t}{\Delta_x^2}$ and $\lambda_2 = \frac{\beta \Delta_t}{\Delta_x^4}$

For the expression of $g_k(\Delta_t, \Delta_x)$ we subsequently have :

$$g_k(\Delta_t, \Delta_x) = b_{-2} \times e^{-2ik\Delta_x} + b_{-1} \times e^{-ik\Delta_x} + b_0 + b_{+1} \times e^{+ik\Delta_x} + b_{+2} \times e^{+2ik\Delta_x} \tag{81}$$

$$g_k(\Delta_t, \Delta_x) = b_{-2} \times (e^{-2ik\Delta_x} + e^{+2ik\Delta_x}) + b_{-1} \times (e^{-ik\Delta_x} + e^{+ik\Delta_x}) + b_0 \tag{82}$$

$$\rightarrow |g_k(\Delta_t, \Delta_x)| = |b_{-2} \times (e^{-2ik\Delta_x} + e^{+2ik\Delta_x}) + b_{-1} \times (e^{-ik\Delta_x} + e^{+ik\Delta_x}) + b_0| \tag{83}$$

$$\rightarrow |g_k(\Delta_t, \Delta_x)| \leq |b_{-2}| \times |(e^{-2ik\Delta_x} + e^{+2ik\Delta_x})| + |b_{-1}| \times |(e^{-ik\Delta_x} + e^{+ik\Delta_x})| + |b_0| \tag{84}$$

$$|g_k(\Delta_t, \Delta_x)| \leq |b_{-2}| \times |(e^{-2ik\Delta_x} + e^{+2ik\Delta_x})| + |b_{-1}| \times |(e^{-ik\Delta_x} + e^{+ik\Delta_x})| + |b_0| \tag{85}$$

We simplify the exponential terms and we replace the b_l by their definition :

$$|g_k(\Delta_t, \Delta_x)| \leq |\lambda_2| \times |2cos(2k\Delta_x)| + |\lambda_1 - 4\lambda_2| \times |2cos(k\Delta_x)| + |1 - 2\lambda_1 + 6\lambda_2| \tag{86}$$

Finally, as $|cos(2k\Delta_x)| = |cos(k\Delta_x)| = 1$, and knowing the signs of α and β, we have :

$$|g_k(\Delta_t, \Delta_x)| \leq 2|\lambda_2| + 2|\lambda_1 - 4\lambda_2| + |1 - 2\lambda_1 + 6\lambda_2| \tag{87}$$

$$|g_k(\Delta_t, \Delta_x)| \leq 2\lambda_1 + |1 - 2\lambda_1| \tag{88}$$

$$\tag{89}$$

Yet, we have seen that we had a relation bounding Δ_t to Δ_x which is :

$$
\begin{aligned}
\Delta_t &\leq \frac{\Delta_x^2}{2\alpha} && (90)\\
\rightarrow 0 \leq 2\alpha\frac{\Delta_t}{\Delta_x^2} \leq 1 &&& (91)\\
\lambda_1 = \frac{\alpha\Delta_t}{\Delta_x^2} &&& (92)\\
\rightarrow 0 \leq \lambda_1 \leq \frac{1}{2} &&& (93)\\
\rightarrow 1 \geq 1 - 2\times\lambda_1 =\geq 0 &&& (94)\\
\Rightarrow 2\times\lambda_1 + |1-2\times\lambda_1| = 1 &&& (95)\\
\Rightarrow |g_k(\Delta_t,\Delta_x)| \leq 1 &&& (96)\\
&&& (97)
\end{aligned}
$$

This relation is the fact of a strong condition ensuring that the solution doesn't grow exponentially.

2.4.2 Ill-posed cases

CASE 1 : $\beta = 0$ and $\alpha < 0$ (and so $\lambda_2 = 0$)

We have the same first inequality verified by the term $g_k(\Delta_t, \Delta_x)$ which is :

$$
\begin{aligned}
|g_k(\Delta_t,\Delta_x)| &\leq 2|\lambda_2| + 2|\lambda_1 - 4\lambda_2| + |1 - 2\lambda_1 + 6\lambda_2| && (98)\\
|g_k(\Delta_t,\Delta_x)| &\leq 2|\lambda_1| + |1-2\lambda_1| && (99)\\
&&& (100)
\end{aligned}
$$

We now that if $\alpha < 0$, then :

$$
\begin{aligned}
|g_k(\Delta_t,\Delta_x)| &\leq 2|\lambda_1| + |1-2\lambda_1| && = 2|\lambda_1| + |1 - 2\frac{\alpha\Delta_t}{\Delta_x^2}| && (101)\\
|g_k(\Delta_t,\Delta_x)| &\leq -2\frac{\alpha\Delta_t}{\Delta_x^2} + (1 - 2\frac{\alpha\Delta_t}{\Delta_x^2}) && = 1 - 4\frac{\alpha\Delta_t}{\Delta_x^2} && (102)\\
&&&&& (103)
\end{aligned}
$$

Besides, we have :

$$
\begin{aligned}
&\alpha < 0 && (104)\\
&4\frac{\alpha\Delta_t}{\Delta_x^2} < 0 && (105)\\
&1 - 4\frac{\alpha\Delta_t}{\Delta_x^2} > 1 && (106)\\
&|g_k(\Delta_t,\Delta_x)| > 1 && (107)
\end{aligned}
$$

This last inequality is the effect of a solution whose growth is unbounded, i.e. a solution growing exponentially.

CASE 2 : $\beta > 0 \forall \alpha \in$ (and so $\lambda_2 \neq 0$)

We go back from the equation (98) to then rewrite g_k :

$$|g_k(\Delta_t, \Delta_x)| \leq |\lambda_2| \times |2cos(2k\Delta_x)| + |\lambda_1 - 4\lambda_2| \times |2cos(k\Delta_x)| + |1 - 2\lambda_1 + 6\lambda_2| \tag{108}$$

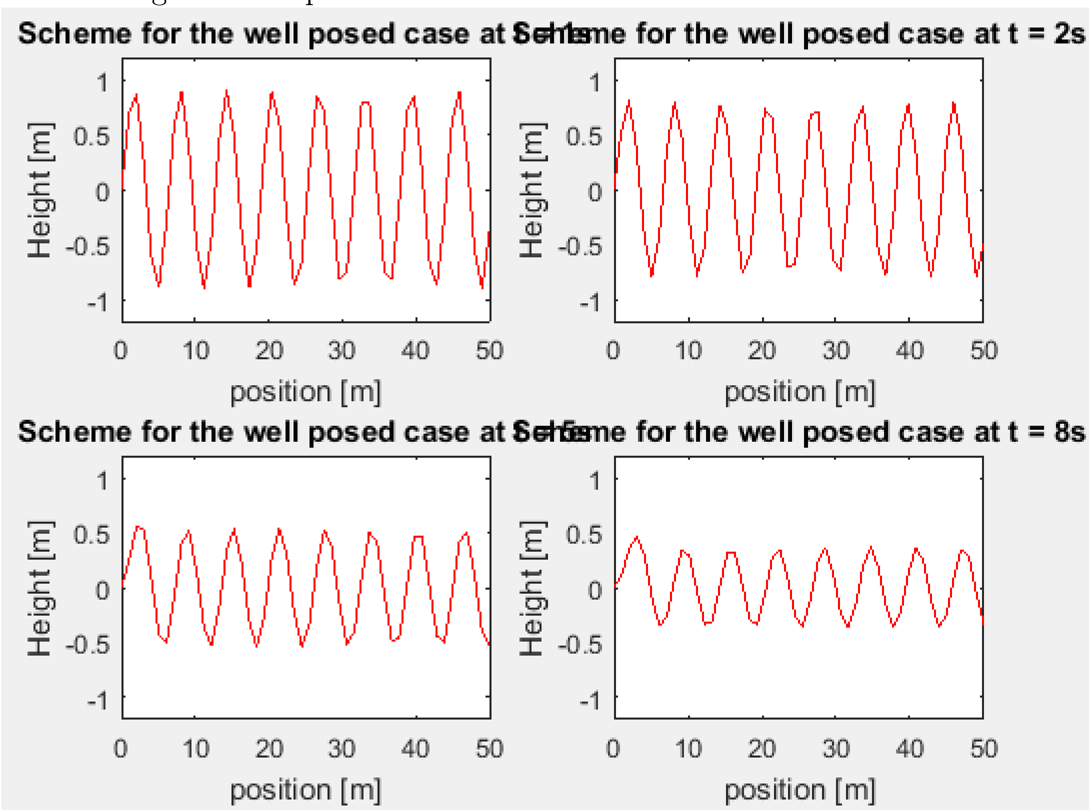
$$\tag{109}$$

2.5 Matlab implementation

We implement the scheme for the different cases seen above (well and ill posed cases).

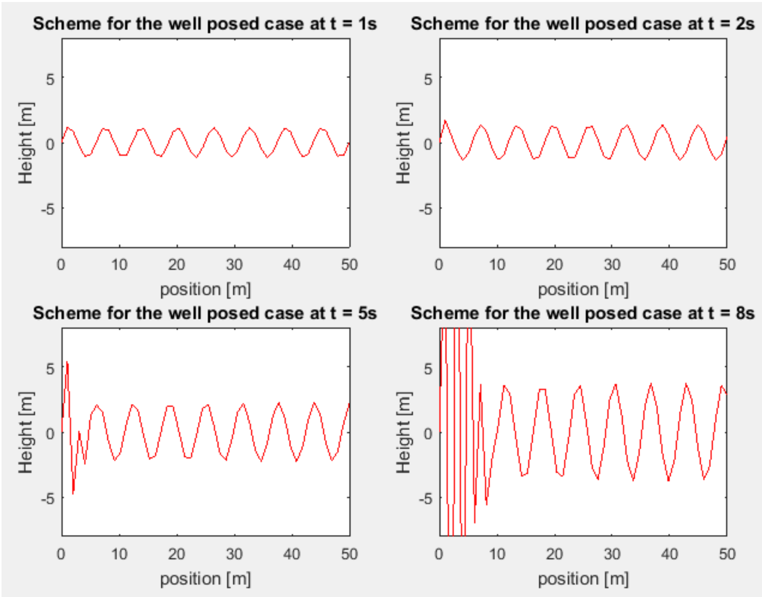
2.5.1 Well posed

Below, you can see some graphs showing how behaves the scheme. We can already see that it is stable and that the initial sinusoïd we had is progressively diminishing, and thus acting as a well posed function :



2.5.2 Ill posed

In the same way we display here the case of the ill posed case for which we put $\beta = 0$



As expected the functions is ill posed and grows exponentially.

3 Shallow water equations, dissipation and dispersion

3.1 Lax-Friedrichs program for Homework 2

In this case we look at the same the program done for the previous assignment but now we change the values of $\frac{\Delta t}{\Delta x}$. As seen in the subject of the previous assignment

$$\alpha = \frac{\Delta x}{\Delta t} = \max |\mathbf{f}'(\mathbf{u})|$$

The optimum value for α after computing it a couple of times on `MATLAB` is 5.3792. Now, let's run the program for different value of $\frac{\Delta t}{\Delta x}$.

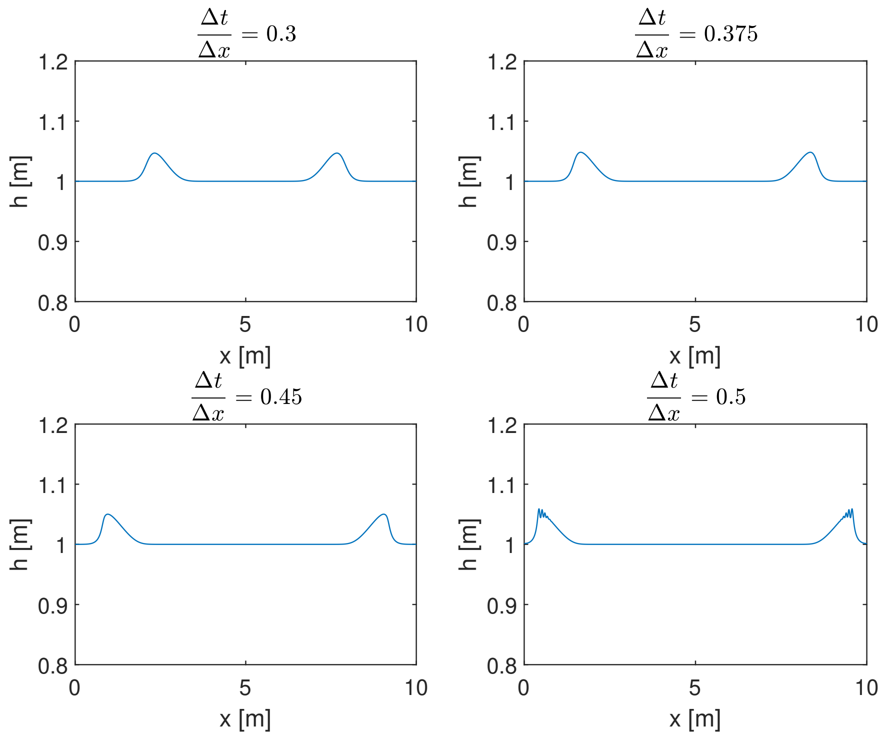


Figure 1: `MATLAB` plots of the height of the wave for different value of $\frac{\Delta t}{\Delta x}$ at $t = 2.7s$.

From figure 1, the damping of the wave seems to be larger for smaller values of Δt. The larger damping is not really noticeable however, greater values of Δt some dispersion appears it can be seen for $\Delta t = \frac{1}{2}\Delta x$. Another thing is worth mentioning, it is the fact that Δt has an influence on the speed of the modeled wave, the greater Δ, the more imprecise the propagation speed of the wave.

3.2 Two-step McCormack's scheme

In this part, we have to write a similar program that solves the problem by the two-step McCormack's scheme:

$$
\begin{aligned}
u_j^* &= u_j^n - \frac{\Delta t}{\Delta x}[f(u_{j+1}^n) - f(u_j^n)] \\
u_j^{n+1} &= \frac{1}{2}(u_j^n + u_j^*) - \frac{\Delta t}{2\Delta x}[f(u_j^*) - f(u_{j-1}^*)]
\end{aligned}
\tag{110}
$$

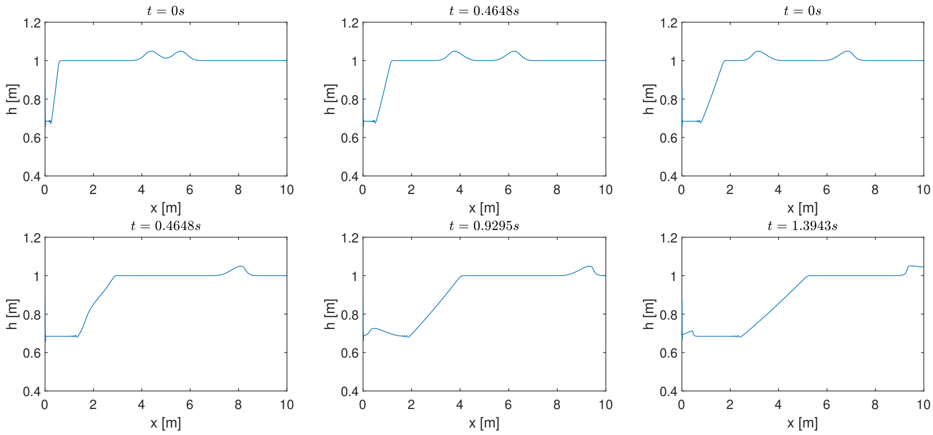


Figure 2: `MATLAB` plots of the height of the wave at different time using the two-step McCormack's scheme.

On figure 2, we can see some dispersion at the bottom of the curve. *However we do not know if this is the expected curve there is maybe a problem in the program.*

And by changing the spatial step, there is not a complete curve over the time. We cannot really compare these result with the Lax-Friedrich scheme, since they are totally different in our case.