Decompositions of Cartesian Products of Cycles

Moriah Aberle, Sarah Gold, Rivkah Moshe

Denison University, Haverford College, Boston University

Advisor: David Offner, Carnegie Mellon University

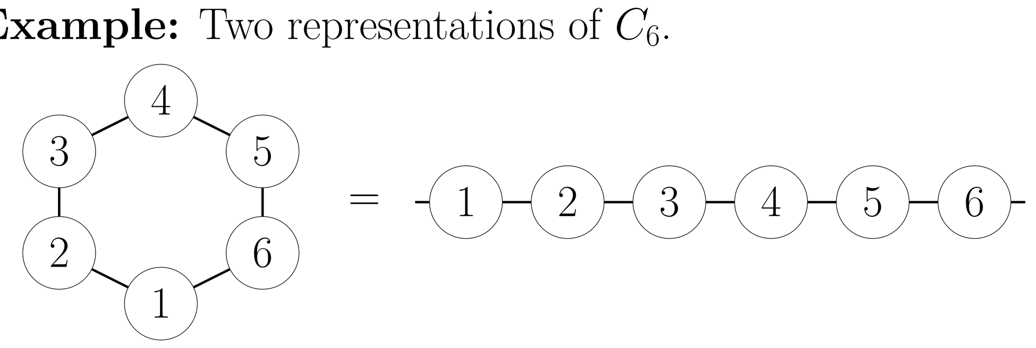
suamilogo.png

Graph Decompositions

Definition 1 (Graph) *A* graph G *is a set of vertices* $V(G)$ *along with a set of edges* $E(G)$ *where each edge "connects" two vertices.*

Definition 2 (Cycle) *A* cycle *of length* k*, denoted* C_k *has* k *vertices, ordered cyclically, with edges between consecutive vertices.*

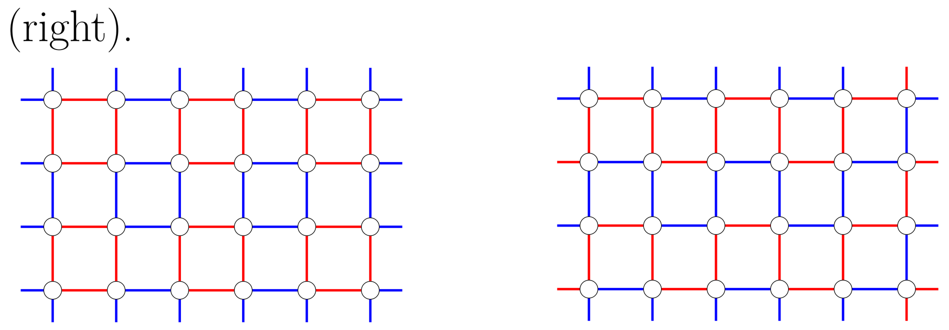
Example: Two representations of C_6.



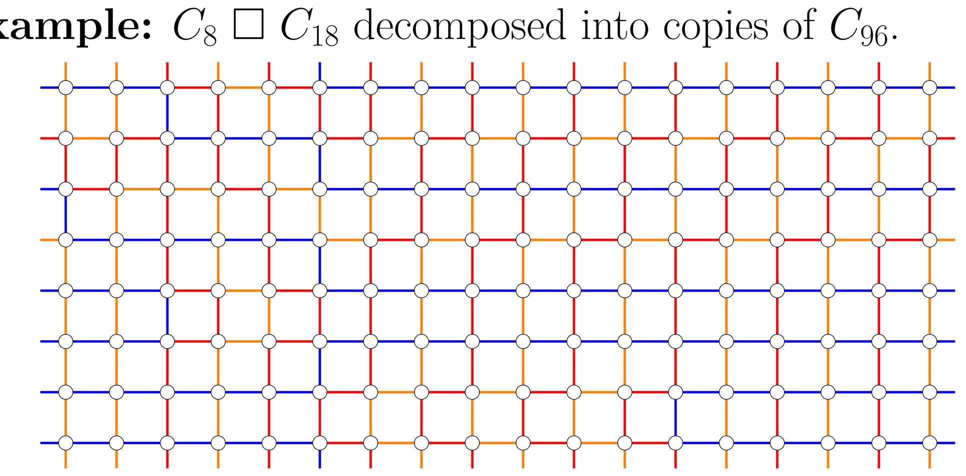
Definition 3 (Cartesian product) *The* Cartesian product *of two graphs* G *and* H*, written* $G \square H$*, is the graph with vertex set* $V(G) \times V(H)$*, where an edge* $e = (u,v)(u',v') \in E(G \square H)$ *if* $u = u'$ *and* $vv' \in E(H)$*, or if* $v = v'$ *and* $uu' \in E(G)$.

Definition 4 (Decomposition) *A* decomposition *of a graph is a partition of the edges of the graph into copies of a fixed subgraph.*

Example: $C_4 \square C_6$ decomposed into copies of C_4 (left) and C_{24} (right).



Example: $C_8 \square C_{18}$ decomposed into copies of C_{96}.



Motivating Theorem (Kotzig, 1973)
C_{mn} decomposes $C_m \square C_n$.

Research Questions:

- Given m and n, what cycles C_k decompose $C_m \square C_n$?
- Given a cycle length k, what Cartesian products $C_m \square C_n$ does C_k decompose?

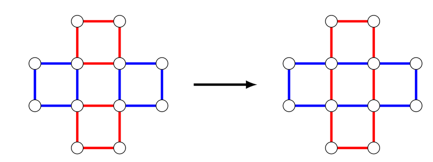
Theorem 5 (Divisibility Criterion) *If C_k decomposes $C_m \square C_n$, then k divides the number of edges, $2 \cdot m \cdot n$.*

Theorem 6 *C_4 decomposes $C_m \square C_n \iff 2 \mid m, n$.*

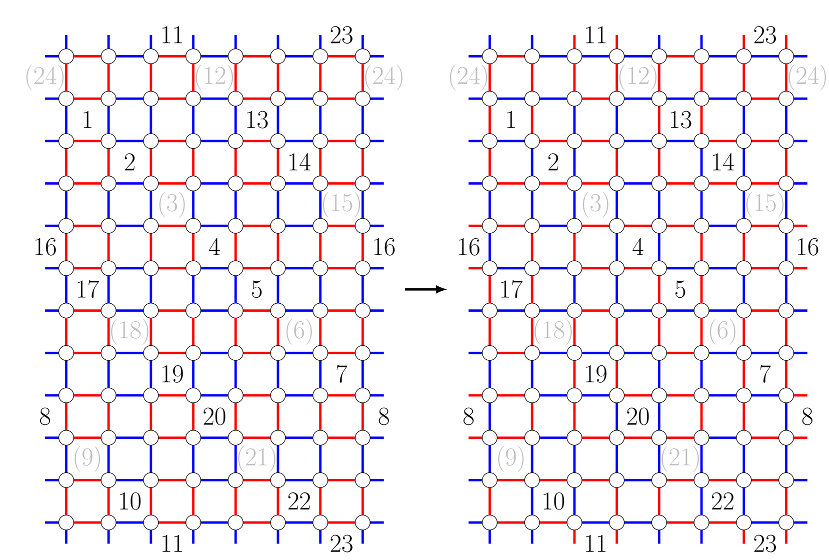
Decomposing where $4 \mid m$ and $4 \mid n$

Theorem 7 *Let $G = C_m \square C_n$ where $4 \mid m$ and $4 \mid n$. If $4 \mid k$ and $k \mid mn$, then C_k decomposes G.*

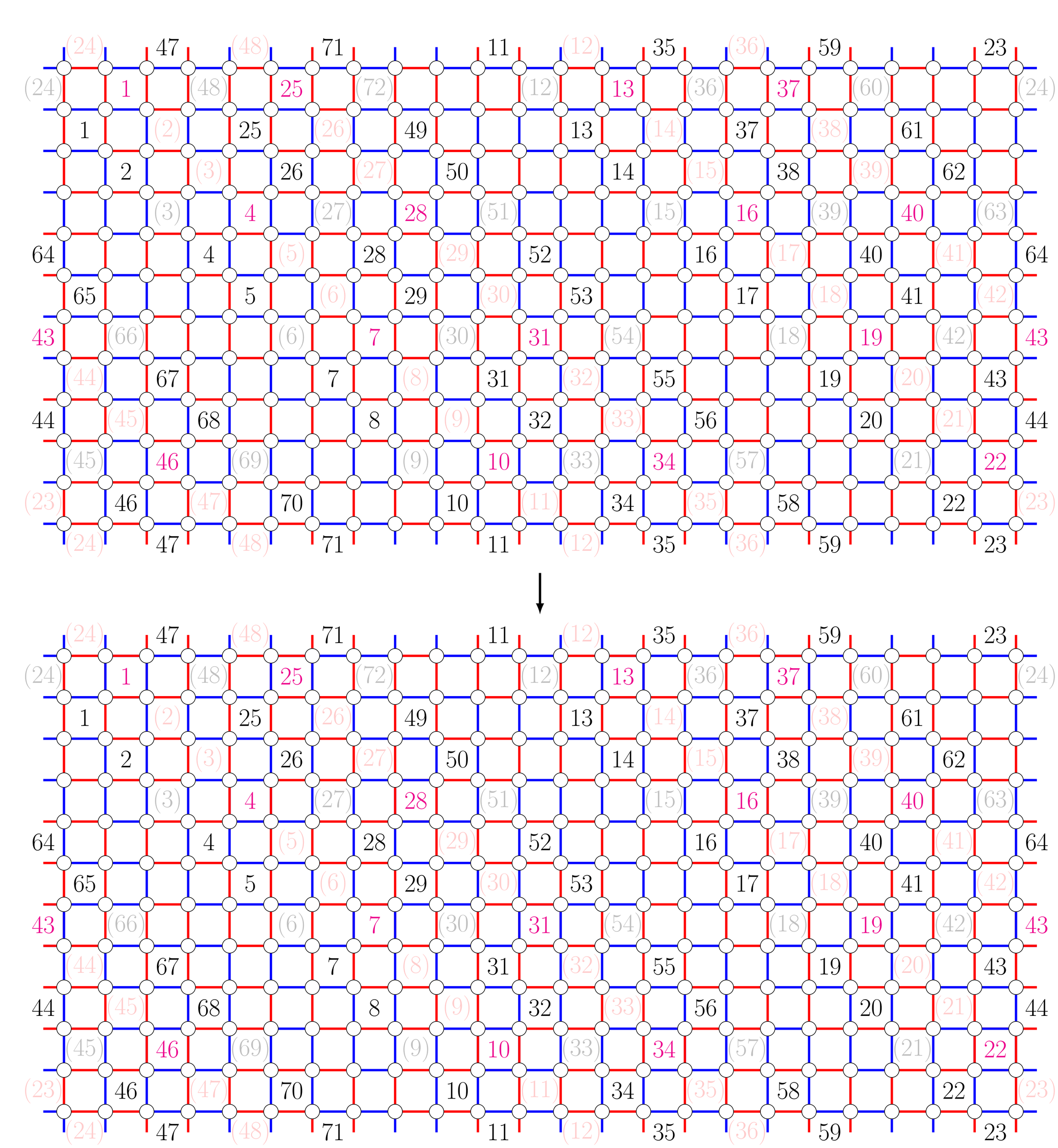
Proof: Cycle combination operation:



From a decomposition of $C_n \square C_m$ into copies of C_4, number selected locations where cycle combination operation can be applied, and perform operation. Demonstrated with a decomposition of $C_{12} \square C_8$ into copies of C_{12}.



If $\gcd(m,n) \neq 2^i$, for some cycle lengths it is necessary to perform a second phase of the cycle combination operation. Shown below: decomposition of $C_{12} \square C_{24}$ into copies of C_{36}.



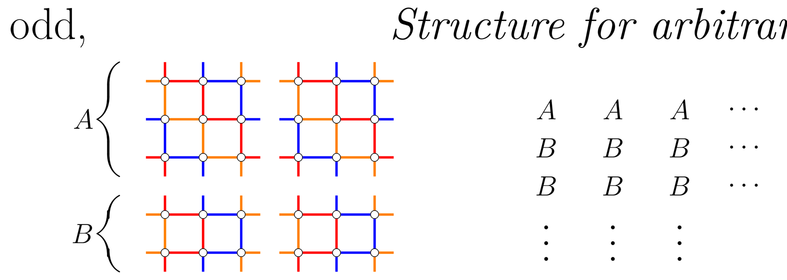
This method produces most of the possible decompositions of $C_m \square C_n$ when m and n are multiples of 4.

Decomposing into 3 Cycles

Theorem 8 *If $3 \mid m$ or $3 \mid n$, then it is possible to decompose $C_m \square C_n$ into three cycles of length $\frac{2mn}{3}$.*

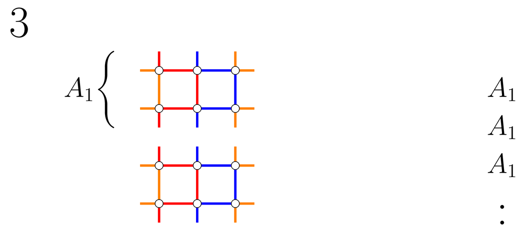
Proof: Without loss of generality, assume $3 \mid n$.

If m is odd, *Structure for arbitrary size:*



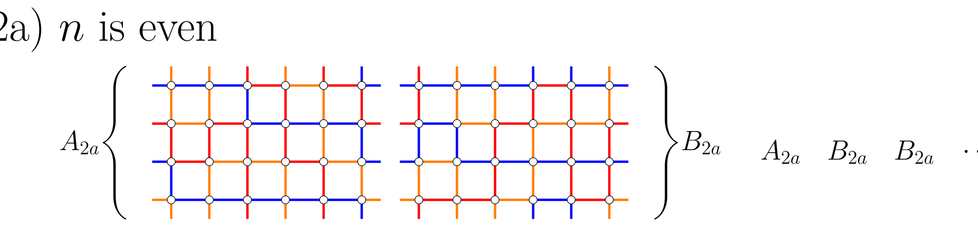
If m is even,

Case 1: $n = 3$

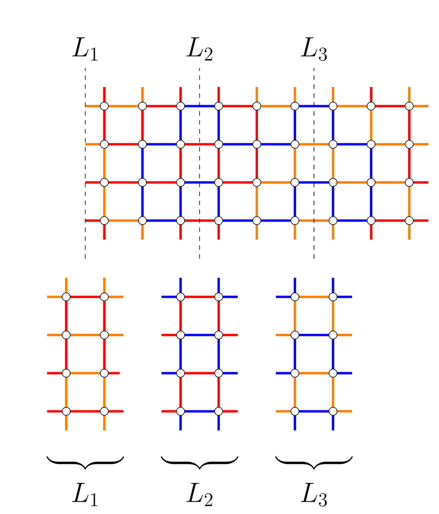


Case 2: $m = 4$

2a) n is even

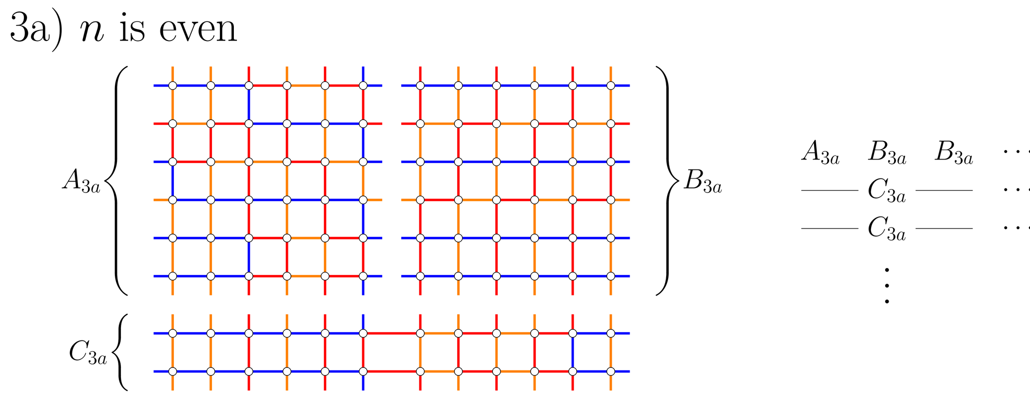


2b) n is odd



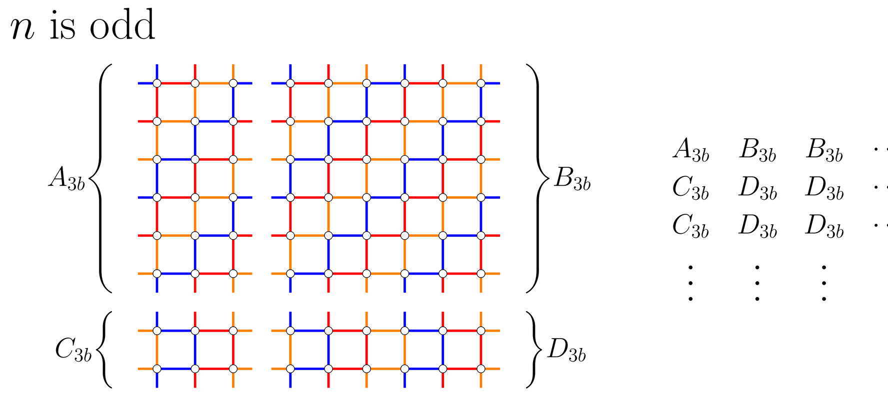
Case 3: $m \geq 6$

3a) n is even



- The $C_8 \square C_{18}$ decomposition into 3 cycles in Column 1 demonstrates Case 3a for a larger n.

3b) n is odd



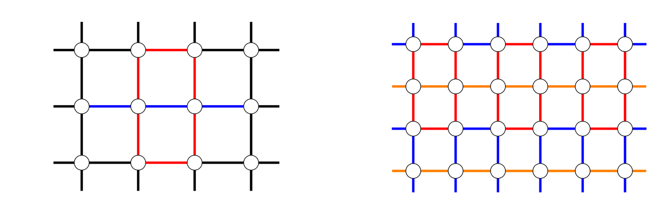
Decomposing into C_6

Theorem 9 *The graph $C_m \square C_n$ can be decomposed into copies of C_6 if and only if $m = n = 3$, or $n = 6$ and $4 \mid m$.*

Example:

Left: The blue edges cannot be a part of C_6.

Right: $C_4 \square C_6$ decomposed into copies of C_6.



"Wrapping" and Odd Cycles

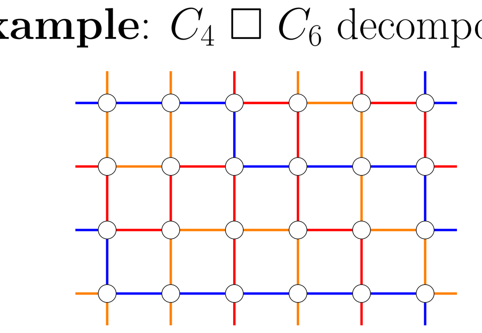
Theorem 10 *If C_k decomposes the graph $C_m \square C_n$, then*

$$k = 2\ell + mp + nq$$

for $p, q \in \mathbb{N}$.

For a given cycle, we interpret p to be the number of times the cycle "wraps around" the torus vertically and q the number of times the cycle wraps around the torus horizontally.

Example: $C_4 \square C_6$ decomposed into copies of C_{16}.

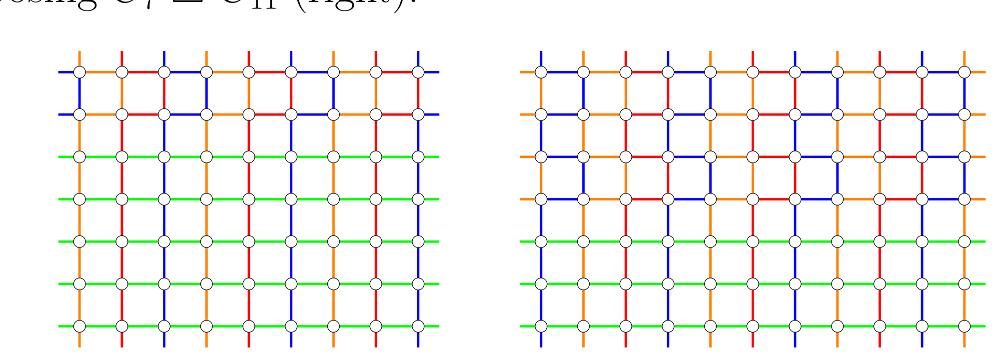


Color	ℓ	p	q
Blue	0	1	2
Red	1	2	1
Orange	3	1	1

Theorem 11 *If k is odd and $m, n > k$, then C_k does not decompose $C_m \square C_n$.*

Theorem 12 *If n and m are odd and $n < 2m$, then C_n decomposes $C_m \square C_n$.*

Example: C_9 decomposing $C_7 \square C_9$ (left) and C_{11} decomposing $C_7 \square C_{11}$ (right).



References & Acknowledgements

gibson_offner
kotzig