

Bestimmung des Quark-Antiquark Potentials in der Hamiltonischen Formulierung der kompakten U(1) Eichtheorie.

Angelo Brade^{1*}

^{1*}Rheinische Friedrich–Wilhelms–Universität, Bonn.

Corresponding author(s). E-mail(s): s72abrad@uni-bonn.de;

1 Introduction

The hamiltonian formulation of lattice gauge theory was first regarded as too costly for reasonable calculations. With the upcoming of quantum computers, we now expect a significant speedup of calculations that can be represented with states. The use of entanglement of these states promises an exponential speedup. This fact moves the hamiltonian formulation in a point of interest again. For that reason we will revisit this formulation and implement the theory on a classical computer.

I aim to give a good introduction such that other new students of the field will have condensed understanding. The setup will be in the compact U(1) theory, truncated appropriately and reduced from complete configuration space to a physical configuration space with Gauss's Law.

For the numerical results we will take a look at the expectation value for the plaquette operator and finally calculate the $q\bar{q}$ potential.

2 Theory

2.1 Discretisation

We start by discretising the positional space into sites and links. The sites are spaced by the lattice spacing a . To obtain the continuum theory in the limit of $a \rightarrow 0$ we need to introduce the coupling constant g [?].

We associate each site with a corresponding positional vector $\vec{r} = (r_x, r_y)$. On the sites the static

charges $Q_{\vec{r}}$ positioned. We associate each link with a corresponding position \vec{r} and a direction μ , where $-\mu$ is its opposite direction. For example with $\vec{r} = (1, 2)$ and $\mu = x$ we would denote the link that points from the site $(1, 2)$ to the x direction. We could denote the same link with $\vec{r} = (2, 2)$ and $\mu = -x$.

On the links are the electric field operators $\hat{E}_{\vec{r},\mu}$ positioned. Each electric field $\hat{E}_{\vec{r},\mu}$ has the vector potential $\hat{A}_{\vec{r},\mu}$ as its canonical conjugate variable. Thus

$$[\hat{E}_{\vec{r}_i,\mu}, \hat{A}_{\vec{r}_j,\nu}] = i\delta^{(3)}(\vec{r}_i - \vec{r}_j)\delta_{\mu\nu}. \quad (1)$$

We construct a unitary operator $\hat{U}_{\vec{r},\nu}$ (also called link operator) by using the vector potential as the generator in the Lie algebra \mathfrak{g} . Thus through the exponential mapping we get the elements of the Lie group G :

$$\hat{U}_{\vec{r},\mu} = e^{iag\hat{A}_{\vec{r},\mu}}. \quad (2)$$

The generated Lie group $G = U(n)$ has the complex matrix elements $\hat{U}_{\vec{r},\nu}$ of dimension $n \times n$. By taking $n = 1$ we choose the U(1) theory, where the Phase, thus our vector potential, is an element of \mathbb{R} . Through equation 1 and 2 we arrive at

$$[\hat{E}_{\vec{r}_i,\mu}, \hat{U}_{\vec{r}_j,\nu}] = \delta^{(3)}(\vec{r}_i - \vec{r}_j)\delta_{\mu\nu}\hat{U}_{\vec{r}_j,\nu}, \quad (3)$$

$$[\hat{E}_{\vec{r}_i,\mu}, \hat{U}_{\vec{r}_j,\nu}^\dagger] = -\delta^{(3)}(\vec{r}_i - \vec{r}_j)\delta_{\mu\nu}\hat{U}_{\vec{r}_j,\nu}^\dagger. \quad (4)$$

To explore the effect of the operators on the states, we choose to be in the electric basis with the basis states $|e_{\vec{r},\mu}\rangle$. The basis is made up of all possible states at the links. We obtain

$$\hat{E}_{\vec{r},\mu} |e_{\vec{r},\mu}\rangle = e_{\vec{r},\mu} |e_{\vec{r},\mu}\rangle. \quad (5)$$

Furthermore with equation 3, 4 and 5 we get

$$\hat{U}_{\vec{r},\mu} |e_{\vec{r},\mu}\rangle = |e_{\vec{r},\mu} + 1\rangle \text{ and} \quad (6)$$

$$\hat{U}_{\vec{r},\mu}^\dagger |e_{\vec{r},\mu}\rangle = |e_{\vec{r},\mu} - 1\rangle. \quad (7)$$

We see the unitary operators, also called linked operators, act as ladder operators.

2.2 Truncation

Since each field can be any real value, there are infinitely many possible fields for each link, thus constructing an infinitely dimensional basis. To be computable we truncate the possible values, such that the basis is finite.

Revisiting the elements of $U(1)$ we see, that they live on the unit circle in the complex plane. By limiting the phase of our exponential mapping in equation 2 to $[-l, l]$, the phase is now an element of \mathbb{Z}_{2l+1} . This also truncates the electric field, such that $e_{\vec{r},\mu} \in [-l, l]$. The unitary operators, which as we saw act as ladder operators to the eigenstates, can now also be expressed as

$$\hat{U}_{\vec{r},\mu} \mapsto \begin{bmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ 0 & \ddots & \vdots & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix}, \hat{U}_{\vec{r},\mu}^\dagger \mapsto \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & \dots & \dots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}.$$

The unitarity $\hat{U}_{\vec{r},\mu} \hat{U}_{\vec{r},\mu}^\dagger \neq \mathbb{1}$ for this is not, but can be recovered in the continuum limit of $l \rightarrow \infty$. It is important to point out, that we annihilate the first (last) state when using $\hat{U}_{\vec{r}}^\dagger$ ($\hat{U}_{\vec{r}}$).

2.3 Hamiltonian

The Hamiltonian for the system was originally formulated by Kogut and Susskind [?] and thus since known as the so called Kogut Susskind Hamiltonian. It reads

$$\hat{H} = \hat{H}_E + \hat{H}_B + \hat{H}_m + \hat{H}_{\text{kin}} \text{ with}$$

$$\begin{aligned} \hat{H}_E &= \frac{g^2}{2} \sum_{\vec{r}} \left(\hat{E}_{\vec{r},x}^2 + \hat{E}_{\vec{r},y}^2 \right), \\ \hat{H}_B &= -\frac{1}{2a^2 g^2} \sum_{\vec{r}} \left(\hat{P}_{\vec{r}} + \hat{P}_{\vec{r}}^\dagger \right), \\ \hat{H}_m &= m \sum_{\vec{r}} (-1)^{r_x+r_y} \hat{\phi}_{\vec{r}}^\dagger \hat{\phi}_{\vec{r}} \text{ and} \\ \hat{H}_{\text{kin}} &= \frac{i}{2a} \sum_{\vec{r}} \left(\hat{\phi}_{\vec{r}}^\dagger \hat{U}_{\vec{r},x} \hat{\phi}_{\vec{r}+x} - \text{h.c.} \right) \\ &\quad - \frac{(-1)^{r_x+r_y}}{2a} \sum_{\vec{r}} \left(\hat{\phi}_{\vec{r}}^\dagger \hat{U}_{\vec{r},y} \hat{\phi}_{\vec{r}+y} + \text{h.c.} \right). \end{aligned}$$

Firstly, let's get rid of the part we don't need. At the beginning we introduced the static charges $Q_{\vec{r}}$. They are called static, since they can't move. We say that they have infinite mass, which implies they won't introduce any field and can not be moved. Thus the fermionic fields $\hat{\phi}_{\vec{r}}$ vanish at all \vec{r} . This premiss yields

$$\hat{H}_m = \hat{H}_{\text{kin}} = 0.$$

The computation can also be done with dynamic charges $\hat{q}_{\vec{r}}$ where the mass and kinetic Hamiltonian will make contributions. In this case a phenomena called doubling problem will rise. A common approach for this problem are staggered fermions. We won't take a look at this and stay in the pure gauge case, where only gauge fields exist.

Now we take a look at the two remaining terms. The so called electric Hamiltonian \hat{H}_E and the magnetic Hamiltonian \hat{H}_B . The latter originates from taking the real part of the plaquette operator $\hat{P}_{\vec{r}}$. It is the smallest Wilson loop (a closed loop through the lattice on the link operators) and the oriented product of the link operators, i.e.

$$\hat{P}_{\vec{r}} = \hat{U}_{\vec{r},x} \hat{U}_{\vec{r}+x,y} \hat{U}_{\vec{r}+x,y}^\dagger \hat{U}_{\vec{r},y}^\dagger, \quad (8)$$

where we denote e.g. $\vec{r} + x = (r_x + 1, r_y)$. We take the hermitian conjugate of the two latter link operators since all link operators only show in positive x or y direction and when going through the plaquette we are going against the orientation of the two last link operators. We see that by taking the hermitian conjugate of the plaquette operator, we just go through the plaquette the other way around.

The heuristic explanation of the form of the magnetic Hamiltonian is the following: The

Plaquettes, as they are closed loops, are essentially conductor loops that introduce a magnetic moment by Farady's law, that is then measured by the magnetic hamiltonian.

Last but not least we take a look at the electric Hamiltonian. It is essentially the sum over each lattice site over the square of the electric field. Here we use the electric field operators $\hat{E}_{\vec{r},\mu}^2$. But not all of them are free. Gauss's law constraints about the half of them such that the others are truly free. The formulation reads

$$\left[\sum_{\mu=x,y} \left(\hat{E}_{\vec{r},\mu} - \hat{E}_{\vec{r}-\mu,\mu} \right) - \hat{q}_{\vec{r}} - Q_{\vec{r}} \right] |\Psi\rangle = 0. \quad (9)$$

Here we sum over all electric field operators that are linked to a site \vec{r} and are expecting them to be the same as the charge deposited on the site. Only those lattice states $|\Psi\rangle$, which produce this relation, are thus physicaly possible. They live in the physical space \mathcal{H}_{ph} :

$$|\Psi\rangle \in \mathcal{H}_{\text{ph}}, \quad (10)$$

where we denote by \mathcal{H} the space in which a lattice configuration lives. Here the physical Hamiltonian is a subspace of the the complete Hamiltonian.

3 Implementation

3.1

For the implementation we construct the lattice and place all operators on thier links. Then we calculate each entry

$$\begin{aligned} \langle i | \hat{H} | j \rangle &= \frac{g^2}{2} \sum_{\vec{r}} \left(e_{\vec{r},x}^2 + e_{\vec{r},y}^2 \right) \delta_{ij} \\ &\quad - \frac{1}{2a^2 g^2} \sum_{\vec{r}} \langle i | \left(\hat{P}_{\vec{r}} + \hat{P}_{\vec{r}}^\dagger \right) | j \rangle \end{aligned}$$

with $|i\rangle \in \mathcal{H}$. Let us calculate $\langle i | \hat{P}_{\vec{r}} | j \rangle$:

$$\begin{aligned} \langle i | \hat{P}_{\vec{r}} | j \rangle &= \langle i | \hat{U}_{\vec{r},x} \hat{U}_{\vec{r}+x,y} \hat{U}_{\vec{r}+y,x}^\dagger \hat{U}_{\vec{r},y}^\dagger | j \rangle \\ &= \langle i | |e_{\vec{r},x}^{(j)} + 1\rangle \otimes |e_{\vec{r}+x,y}^{(j)} + 1\rangle \\ &\quad \otimes |e_{\vec{r}+y,x}^{(j)} - 1\rangle \otimes |e_{\vec{r},y}^{(j)} - 1\rangle \\ &\quad \bigotimes_{\text{rest links}} |e_{\vec{r}',\mu'}^{(j)}\rangle \end{aligned}$$

$$\begin{aligned} &= \langle i | k \rangle \\ &= \delta_{ik} \end{aligned}$$

The formulation reads as following: State $|j\rangle$ will be transformed by the plaquette operator $\hat{P}_{\vec{r}}$ into some state $|k\rangle$. Thus $\hat{P}_{\vec{r}}$ gets an entry at row $\langle i |$ and collumn $|j\rangle$ if, and only if, state $|j\rangle$ is transformed into state $|i\rangle$. Now knowing $\hat{P}_{\vec{r}}$ in matrix representation, trivially yealds $\hat{P}_{\vec{r}}^\dagger = \left(\hat{P}_{\vec{r}}^* \right)^T$ and with it the matrix representation of the magnetic hamiltonian.

On a side note, going through states $|i\rangle$ means, counting up in base $2l+1$ with the link states $|e_{\vec{r},\mu}\rangle$ being the "digits".

4 Results

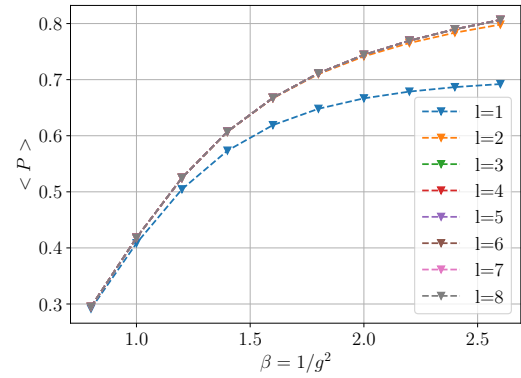


Abbildung 1 plaquette expectation values for a 2x2 lattice.

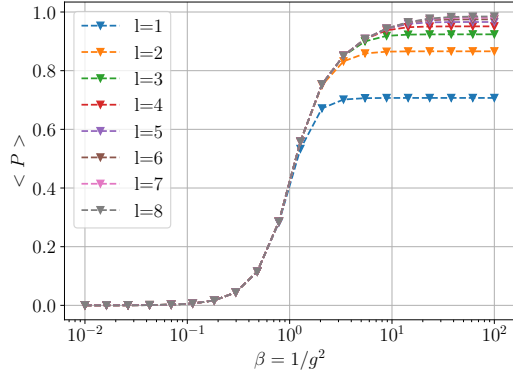


Abbildung 2 Plaquette expectation values for a 2x2 lattice in a logarithmic plot.

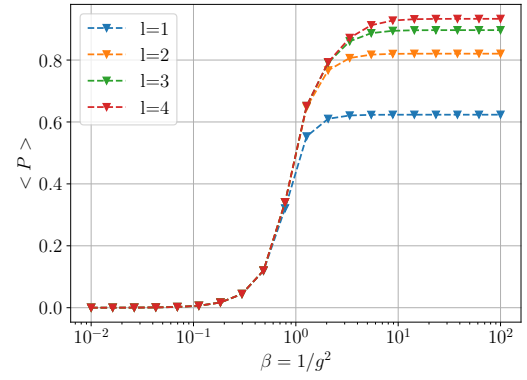


Abbildung 4 Plaquette expectation values for a 2x2 pbc lattice in a logarithmic plot.

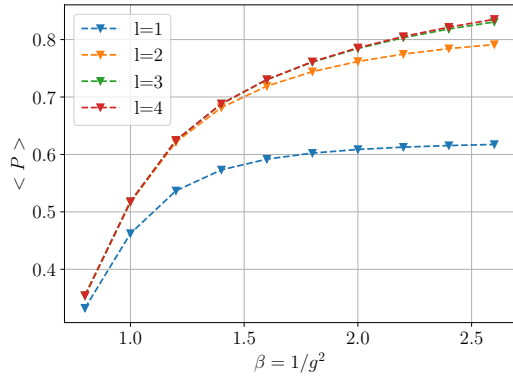


Abbildung 3 plaquette expectation values for a 2x2 pbc lattice.

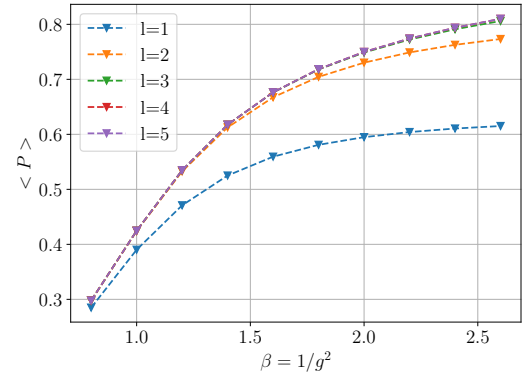


Abbildung 5 Plaquette expectation values for a 3x3 lattice.

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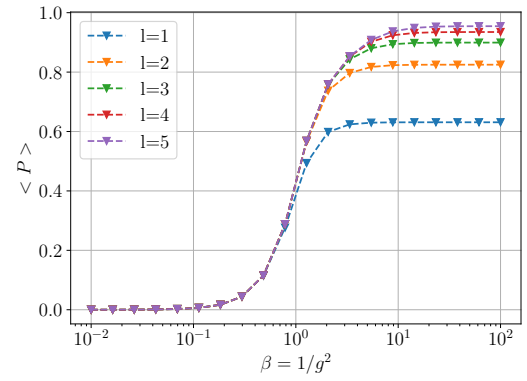


Abbildung 6 Plaquette expectation values for a 3x3 lattice in a logarithmic plot.

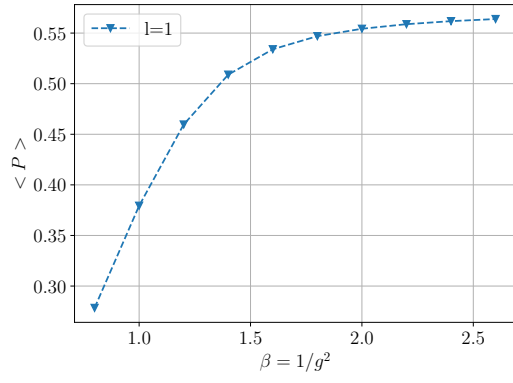


Abbildung 7 Plaquette expectation values for a 3x3 pbc lattice.

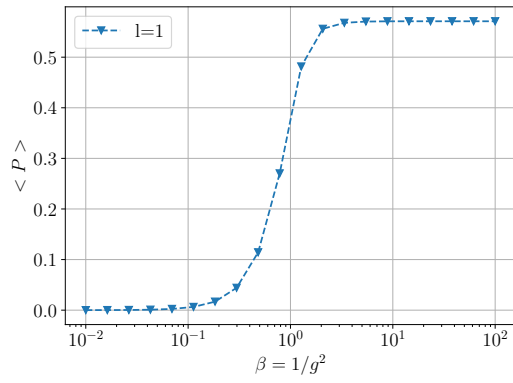


Abbildung 8 Plaquette expectation values for a 3x3 pbc lattice in a logarithmic plot.

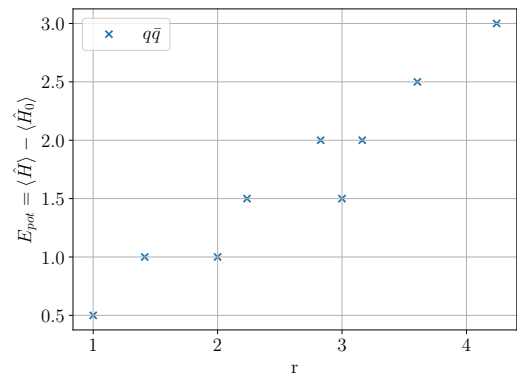


Abbildung 10 Quark-Antiquark potential.

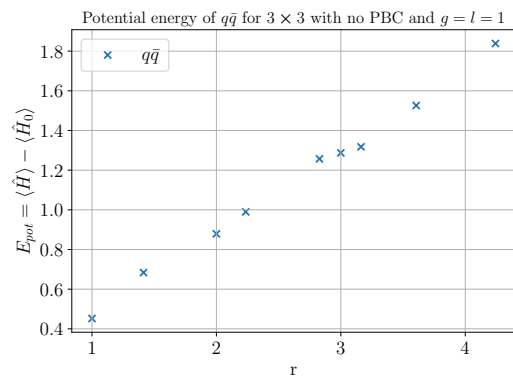


Abbildung 9 Quark-Antiquark potential.