AGT

Time dependent pertubation theory:

- Pictures:

-> Schrödinger picture: $|\psi\rangle_S = |\psi, t_0; t\rangle$, $\hat{A}_S \hat{=} \hat{A}_S(t_0)$ (independent of time)

-> Heisenberg picture: $|\psi\rangle_H = |\psi, t_0\rangle_H$ (independent of time), $\hat{A}_H = \hat{A}_H(t_0; t)$

$\hookrightarrow |\psi, t_0; t\rangle_S = \hat{U}(t_0; t)|\psi, t_0\rangle_H$

or $|\psi, t_0\rangle_H = \hat{U}^\dagger(t_0, t)|\psi, t_0; t\rangle_S$

with $\hat{U}(t_0; t) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]$

$\hookrightarrow \hat{A}_H(t_0; t) = \hat{U}^\dagger(t_0; t)\hat{A}_S(t_0)\,\hat{U}(t_0, t)$

or $\hat{A}_S(t_0) = \hat{U}(t_0, t)\hat{A}_H(t_0; t)\,\hat{U}^\dagger(t_0, t)$

-> Interaction picture: $|\psi\rangle_I = |\psi, t_0; t\rangle_I$ evo. by V_I, $\hat{A}_I = \hat{A}_I(t_0; t)$ evo. by $\hat{H}_0$

$\hookrightarrow |\psi, t_0; t\rangle_I = \hat{U}^{\dagger(0)}(t_0; t)|\psi, t_0; t\rangle_S$,

$\hookrightarrow \hat{A}_I(t_0, t) = \hat{U}^{\dagger(0)}(t_0; t)\hat{A}_S(t_0)\,\hat{U}^{(0)}(t_0, t)$

with $\hat{U}^{(0)}(t_0; t) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{H}^{(0)}(t')\,dt'\right] = \exp\left[-\frac{i}{\hbar}\hat{H}_0(t - t_0)\right]$,

since $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ with $\hat{H}^0(t) \hat{=} \hat{H}_0$

$\Rightarrow \hat{V}_I = \hat{U}^{\dagger(0)}\hat{V}\hat{U}^{(0)}$

$$\Rightarrow i\hbar\partial_t|\psi\rangle_I = i\hbar\partial_t(\hat{U}^{\dagger(0)}|\psi\rangle_S) \overset{S.C.}{=} -H_0\hat{U}^{\dagger(0)}|\psi\rangle_S + \hat{U}^{\dagger(0)}\hat{H}(t)|\psi\rangle_S = \hat{U}^{\dagger(0)}V\underbrace{\hat{U}^{(0)}\hat{U}^{\dagger(0)}}_{\mathbb{1}}|\psi\rangle_S$$

$= V_I|\psi\rangle_I$

- Pertubation (Dyson Series):

-> persume: $|\psi, t_0; t\rangle_I = \hat{U}_I(t_0; t)|\psi, t_0; t_0\rangle_I$, What is $\hat{U}_I$?

$\Rightarrow i\hbar\partial_t U_I = V_I U_I(t_0; t)$

At t_0 there is no change: $\hat{U}_I(t_0, t_0) = \mathbb{1}$

$$\Rightarrow \int_{\hat{U}_I(t_0, t_0)}^{\hat{U}_I(t_0; t)} d\hat{U}_I = -\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I(t')\hat{U}_I(t_0, t')dt'$$

$$\Rightarrow 1: \hat{U}_I^{(n)}(t_0; t) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\hat{U}_I^{(n-1)}(t', t_0)dt'$$

Starting from $\hat{U}_I^{(0)}$ and inserting into $\hat{U}_I^{(1)}, \ldots$ inserting $\hat{U}^{(n-1)}$ into $\hat{U}^{(n)}$.

$$\Rightarrow \hat{U}_I^{(0)}(t_0; t) = \hat{U}_I(t_0; t_0) = \mathbb{1},\ \hat{U}^{(1)}(t_0, t) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\,dt',$$

$$\hat{U}^{(2)}(t_0, t) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\left(\mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t'')dt''\right)dt',\ \hat{U}^{(3)}(t_0, t) = \ldots$$

- Transition Probability:

→ For any known initial state $|i\rangle$: $|i, t_0; t_0\rangle_S = \hat{U}^{(0)}(0, t_0)|i\rangle$

$$\Rightarrow |i, t_0; t_0\rangle_I = \hat{U}^{\dagger(0)}(t_0, t_0)\, |i, t_0, t_0\rangle_S = \underbrace{\hat{U}^{\dagger(0)}(t_0, t_0)\hat{U}^{(0)}(t_0, t_0)|i\rangle}_{=|i\rangle}$$

Since $|i, t_0; t\rangle = \hat{U}_I(t, t_0)|i\rangle$ and $|i, t_0; t\rangle = \sum_n c_n(t)|n\rangle$.

$$\Rightarrow c_f(t) = \langle f|\hat{U}_I(t, t_0)|i\rangle$$

$$\Rightarrow c_f^{(0)}(t) = \langle f|\hat{U}_I^{(0)}|i\rangle = \langle f|i\rangle = \delta_{fi}$$

for $c_f^{(n)}$ with $n > 0$, $\hat{V} \neq 0 \Rightarrow |i\rangle \neq |f\rangle \Rightarrow \delta_{fi} = 0$.

$$c_f^{(1)}(t) = \underbrace{\langle f|i\rangle}_{0} - \frac{i}{\hbar}\int_{t_0}^{t}\langle f|\hat{V}_I(t')|i\rangle dt' = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{fi}t'}V_{fi}(t')dt'$$

$\vdots$

$$c_f^{(n)}(t) = \dots$$

with $\omega_{fi} = \dfrac{E_f - E_i}{\hbar}$ and $\hat{V}_I = \hat{U}^{\dagger(0)}\hat{V}\hat{U}^{(0)} = e^{i\hat{H}_0 t/\hbar}\hat{V}e^{-i\hat{H}_0 t/\hbar}$

$$\Rightarrow P_{fi} = |c_f^{(0)}(t) + c_f^{(1)}(t) + \dots|^2$$

Scattering Theory:

- Initial wave ψ_i: $\psi_i(\vec{x}, t_0) = \displaystyle\int \frac{d^3k}{(2\pi)^3} a(\vec{k})e^{i\vec{k}\cdot\vec{x}}$ $\quad (\vec{k} = \frac{\vec{p}}{\hbar})$

- Scattered wave: $\psi_k(\vec{x}) \xrightarrow[|\vec{x}| \to \infty]{} e^{i\vec{k}\cdot\vec{x}} + \dfrac{e^{i|\vec{k}||\vec{x}|}}{|\vec{x}|} f_{\vec{k}}(\theta, \varphi)$

with scattering amplitude
$$f_k(\theta, \varphi) = -\frac{m}{2\pi\hbar^2}\int d^3x' e^{-i\vec{k}'\cdot\vec{x}'}V(\vec{x}')\psi_{\vec{k}}(\vec{x}')$$
$$= -\frac{m}{2\pi\hbar^2}F^3[V(\vec{x}')\psi_{\vec{k}}(\vec{x}')](\vec{k}')$$

with $\vec{k}' = |\vec{k}|\frac{\vec{x}}{|\vec{x}|}$ (Plane wave $\Rightarrow \vec{k} \parallel \vec{x}$)

- Differential Scattering Cross Section:
$$\frac{d\sigma}{d\Omega} = |f_{\vec{k}}(\theta, \varphi)|^2$$
$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega$$

- The Born Approximation: $\psi_{\vec{k}}(\vec{x}) \to \psi_{\vec{k}}^{(0)}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}$ (0th order result of integral)

$$\Rightarrow f_k^{(Born)}(\theta, \varphi) = -\frac{m}{2\pi\hbar^2}\int d^3x' e^{-i\vec{q}\cdot\vec{x}'}V(\vec{x}) \text{ with } \vec{q} = \vec{k}' - \vec{k}$$
$$= -\frac{m}{2\pi\hbar^2}F^3[V(\vec{x})](\vec{q})$$

If potential spherical symmetrical: $V(\vec{x}) = V(|\vec{x}|)$

$$\Rightarrow f_{\vec{q}}^{(Born)}(\theta,\varphi) = -\frac{2m}{\hbar^2 |\vec{q}|} \int_0^\infty r' \sin(|\vec{q}| r')\, V(r')\, dr'$$

Yukawa potential $V(r) = g\frac{e^{-\mu r}}{r}$ is helpfull for calculating $V \sim \frac{1}{r}$ with $\mu \to 0$.

<u>Transformation and Symmetries:</u>

- $\{A, B\} = \sum_k \left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k}\right)$
- Lagrange: $L = L(\vec{q}, \dot{\vec{q}})$ with $\frac{d}{dt}\left(\frac{\partial L(\vec{q},\dot{\vec{q}})}{\partial \dot{q}_i}\right) = \frac{\partial L(\vec{q},\dot{\vec{q}})}{\partial q_i}$
- Hamilton: $H(\vec{q},\vec{p}) = \dot{q}_i p_i - L(\vec{q},\dot{\vec{q}})$ with $p_i = \frac{\partial L(\vec{q},\dot{\vec{q}})}{\partial \dot{q}_i}$

  follows $\frac{\partial H(\vec{q},\vec{p})}{\partial q_i} = \dot{p}_i$ and $\frac{\partial H(\vec{q},\vec{p})}{\partial p_i} = \dot{q}_i$.

- Transformation: $q_i \to \bar{q}_i(\vec{q},\vec{p})$ and $p_i \to \bar{p}_i(\vec{q},\vec{p})$
  - $\to$ e.o.m. form invariant if canonical: $\{\bar{q}_i, \bar{q}_k\} = \{\bar{p}_i, \bar{p}_k\}$ and $\{\bar{q}_i, \bar{p}_k\} = \delta_{ik}$
    - $\hookrightarrow \{A(q_k,p_k), B(q_k,p_k)\}_{(q,p)} = \{A(\bar{q}_k,\bar{p}_k), B(\bar{q}_k,\bar{p}_k)\}_{(\bar{q},\bar{p})}$
  - $\to$ passive: refer to same physical point in phase space.
  - $\to$ regular: same range of values as original q_i and p_i: e.g. rotation or translation but not cartesian to cylindical coordinates
  - $\to$ active: regular and refer to different points in phase space
    - $\hookrightarrow$ A quantity $A(q_i,p_i)$ is invariant under active trafo. if $A(q_i,p_i) = A(\bar{q}_i,\bar{p}_i)$
  - $\to$ producing infinitesimal canonical trafo with smooth generating function g (Generator):

    $q_i \to \bar{q}_i = q_i + \varepsilon \delta q_i$; $p_i \to \bar{p}_i = p_i + \varepsilon \delta p_i$ with $\delta q_i = \frac{\partial g}{\partial p_i}$ and $\delta p_i = \frac{\partial g}{\partial q_i}$
  - $\to$ g is conserved if $H(q_i,p_i)$ invariant under active trafo.
  - $\to$ $(\bar{q}_i(t), \bar{p}_i(t))$ valid trajectory if H invariant under passive trafo and if $(q_i(t), p_i(t))$ valid trajectory
  - $\to$ e.g. $g = p_i$: $\delta q_i = \varepsilon \delta_{ik}$; $\delta p_i = 0$ (translation invariance $\Leftrightarrow$ conservation of linear mom.)

    $g = L_z$: rotation around z-axis $\Leftrightarrow$ conservation of angular mom.)

    $g = H$: time translation $\Leftrightarrow$ conservation of energie ?

- Trafo in QM: $\langle \hat{q}_i \rangle \to \langle \hat{q}_i \rangle + \varepsilon \langle \frac{\partial g}{\partial p_i} \rangle$ and $\langle \hat{p}_i \rangle \to \langle \hat{p}_i \rangle - \varepsilon \langle \frac{\partial g}{\partial q_i} \rangle$
  - $\to$ active : $\psi \to \psi_g = \psi + \delta_g \psi$ and $\hat{A} \to \hat{A}$
  - $\to$ passive : $\psi \to \psi$ and $\hat{A} \to \hat{A}_g = \hat{A} + \delta_g \hat{A}$

  with $\psi_g = \hat{U}_g \psi$ for some unitary operator $\hat{U}_g^\dagger = \hat{U}_g^{-1}$

  and $\hat{A} \to \hat{A}_g = \hat{U}_g^\dagger \hat{A} \hat{U}_g$

  $\to \hat{U}_g(\xi) = e^{-\frac{i\xi}{\hbar}\hat{g}}$ with $\hat{G} = \hat{g}$

  $\hookrightarrow \hat{U}_g = 1 - \frac{i\varepsilon}{\hbar}\hat{G}$ for $\varepsilon \ll 1$

Gauge Invariance:

$\vec{B} = \vec{\nabla} \times \vec{A}$; $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$, $\frac{d\rho}{dt} + \vec{\nabla}\vec{j} = 0$ (continuity equation)

Gauge trafo.: $\vec{A} \to \vec{A} + \nabla \lambda$; $V \to V - \frac{\partial \lambda}{\partial t}$

$\hookrightarrow$ leaves $\vec{E}$ and $\vec{B}$ unchanged

$\Rightarrow$ e.o.m of classical electrodynamics are gauge invariant

Extention in QM: $\underline{\psi \to e^{\frac{iq}{\hbar}\lambda} \psi}$

Aharonov-Bohm Effect:

Phath integral formulation of QM:

- Every QM problem solved with $U(x_0,t_0;x,t)$
- consider all paths with weight $e^{\frac{i}{\hbar}S(x(t))}$ with $S(x(t))=\int_{t_0}^{t_N} L(x(t),\dot{x}(t),t)\,dt$

$\hookrightarrow U(x_0,t_0;x_N,t_N)=A\int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S(x(t))}$

$\hookrightarrow$ with discrete time $t_0,\dots,t_i,\dots,t_N$ with $t_N-t_0=\varepsilon\cdot N$ $\Rightarrow x(t_i)=x_i$:

$$S=\sum_{i=1}^{N-1} L\left(x_i,\frac{x_{i+1}-x_i}{\varepsilon},t_i\right)\varepsilon\,,\ \text{thus:}$$

$$U(x_0,t_0;x_N,t_N)=A\lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\prod_{i=1}^{N-1}\int_{-\infty}^{\infty}dx_i\ \exp\left[\sum_{i=1}^{N-1}L\left(x_i,\frac{x_{i+1}-x_i}{\varepsilon},t_i\right)\cdot\varepsilon\right]$$

Or with momentum:

$$U(x,x',t)=\int\mathcal{D}p\int\mathcal{D}x\ \exp\left[\frac{i}{\hbar}\int_0^t dt'\,(p\dot{x}-H(x,p))\right]$$

$$=\lim_{N\to\infty}\prod_{i=1}^{N-1}\int_{\mathbb{R}}dx_i\prod_{i=1}^{N}\int\frac{dp_i}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\sum_{i=1}^{N}\left(\frac{\varepsilon p_i^2}{2m}-p_i(x_{i+1}-x_i)+\varepsilon V(x_{i-1})\right)\right]$$

Relativistic QM:

- Klein-Gordon: $(m^2c^2-\hat{p}_\mu\hat{p}^\mu)\psi=0$ with $\hat{p}_\mu=i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t},-\vec{\nabla}\right)=i\hbar\partial_\mu$

$\hookrightarrow\ (m^2c^2-\hat{p}_\mu\hat{p}^\mu)\psi=0$

$\Rightarrow (m^2c^2-i^2\hbar^2\partial_\mu\partial^\mu)\psi=0$

$\Rightarrow \left(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\right)\psi=0$

$\Rightarrow \left(\Box+\frac{m^2c^2}{\hbar^2}\right)\psi=0$

- Dirac Equation: $(\hat{\not{p}}-mc)\psi=0$ with $\hat{\not{p}}=\gamma_\mu\hat{p}^\mu$

$\hookrightarrow (\hat{\not{p}}-mc)\psi=0$

$(\gamma_\mu\hat{p}^\mu-mc)\psi=0$

$(i\hbar\gamma_\mu\partial^\mu-mc)\psi=0$

$\hookrightarrow \gamma_\mu=(\beta,\beta\vec{\alpha})$ with $\vec{\alpha}=(\alpha_i)_{1\le i\le 3}$ with $\alpha_i=\begin{pmatrix}0&\sigma_i\\\sigma_i&0\end{pmatrix}$ and $\beta=\begin{pmatrix}\mathbb{1}_{2\times2}&0\\0&-\mathbb{1}_{2\times2}\end{pmatrix}$

$\hookrightarrow \vec{\sigma}=\left[\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]$

Second Quantization:

- Systems of identical particles

-> Permutation operator $\hat{P}_{ik}\psi(\dots,i,\dots k,\dots) = \psi(\dots,k,\dots,i,\dots)$

=> $\hat{P}_{ik}^2 = 1$ => EV ± 1, $\hat{P}_{ik}^{+} = \hat{P}_{ik}^{-1}$

-> $\hat{H}(\dots,i,\dots,k,\dots) = \hat{H}(\dots,k,\dots,i,\dots)$

=> $[\hat{P}_{ik}, \hat{H}] = 0$ (as for all symmetrical operators $\hat{S}$)

=> $\hat{P}_{ik}$ must have no observable consequences => all observable operators symmetrical

-> for (anti-)symmetrical state $\psi_\pm$: $\hat{P}_{ik}\psi_\pm = \pm 1 \cdot \psi_\pm$

with:

1.) $(+) \hat{=}$ bosons: $S_z^{(b)} = n\hbar$

2.) $(-) \hat{=}$ fermions: $S_z^{(f)} = (n+\frac{1}{2})\hbar$

-> $\hat{P}$ permutates arbitrary many particles

=> $\hat{P}$ can be product of cyclical permutations

=> $\hat{P}\psi_\pm = (\pm 1)^P \psi_\pm$ with $P \hat{=}$ number of cyclic permutations

- Constructing completely (anti-) symmetrical states

-> $|i_1,\dots,i_N\rangle = |i_1\rangle_1 \dots |i_N\rangle_N$ with $|i_\alpha\rangle_\alpha$ being particle α in state i_α.

-> $\hat{H}$ totally symmetrical => propagator $\hat{U}$ is totally symmetrical

=> $\hat{P}\hat{U}|\psi\rangle = \hat{U}\hat{P}|\psi\rangle$

-> $\hat{S}_\pm|i_1,\dots,i_N\rangle = \frac{1}{\sqrt{N!}}\sum_P (\pm 1)^P \hat{P}|i_1,\dots,i_N\rangle$ generates (anti-) symmetrical states by permutating with $\hat{P}$.

=> Pauliprinciple: $|\psi\rangle_- = \hat{S}_-|\psi\rangle$

$\hat{P}_{\alpha\beta}\hat{P} = \hat{P}'$ with $P' = P+1$ since we permutate once more

$$\Rightarrow \hat{P}_{\alpha\beta}\hat{S}_-|\psi\rangle = \frac{1}{\sqrt{N!}}\sum_P (-1)^P \hat{P}_{\alpha\beta}\hat{P}|\psi\rangle$$

$$= \frac{1}{\sqrt{N!}}\sum_{P'} (-1)^P \hat{P}'|\psi\rangle \quad | P' = P+1$$

$$= -\hat{S}_-|\psi\rangle = -|\psi\rangle_-$$

If now $|i_\alpha\rangle_\alpha$ and $|i_\beta\rangle_\beta$ were the same (permutating doesn't change state):

$|\psi\rangle_- = \hat{P}|\psi\rangle_- = -|\psi\rangle_-$ => $|\psi\rangle = 0$ (only possible for non existing states)

- Second quantization of bosons:

$\rightarrow \Psi_s = \frac{1}{\sqrt{n_1! \dots n_\infty!}} \hat{S}_+ |i_1, \dots, i_N\rangle = |n_1, n_2, \dots\rangle$

with occupationnumber n_i for states $i = 1, \dots, \infty$

$\rightarrow$ Bosonic creation and annihilation operators:

creation: $\hat{a}_i^+ |\dots, n_i, \dots\rangle = \sqrt{n_i + 1} |\dots, n_i + 1, \dots\rangle$

annihilation: $\hat{a}_i |\dots, n_i, \dots\rangle = \sqrt{n_i} |\dots, n_i - 1, \dots\rangle$ ($\hat{a}_i |0\rangle = 0$) with $|0\rangle \hat{=}$ vacuum state)

$[\hat{a}_i, \hat{a}_k] = 0$

$[\hat{a}_i^+, \hat{a}_k^+] = 0$

$[\hat{a}_i, \hat{a}_k^+] = \delta_{ik}$

number operator $\hat{n}_i := \hat{a}_i^+ \hat{a}_i \Rightarrow \hat{n}_i |\Psi\rangle = n_i |\Psi\rangle$

$\hookrightarrow$ Total number operator: $\hat{N} = \sum_i \hat{n}_i \Rightarrow \hat{N}|\Psi\rangle = \sum_i \hat{n}_i |\Psi\rangle = N|\Psi\rangle$

$\rightarrow$ Bosonic operators of

1.) single-particle operators: $\hat{T} = \sum_{\alpha=1}^{N} \hat{t}_\alpha = \sum_{ik} t_{ik} \hat{a}_i^+ \hat{a}_k$

with $t_{ik} = \langle i|\hat{t}|k\rangle$ and $\sum_{\alpha=1}^{N} |i\rangle_\alpha \langle k|_\alpha = \hat{a}_i^+ \hat{a}_k$

"one less particle in state k"

"one more particle in state i"

$\Rightarrow \hat{H}|\Psi\rangle = \sum_i \varepsilon_i n_i |\Psi\rangle$

2.) two-particle operators: $\hat{F} = \frac{1}{2} \sum_{\alpha \neq \beta} f(\vec{x}_\alpha, \vec{x}_\beta) = \frac{1}{2} \sum_{i,j,k,l} \langle i,j|\hat{f}|k,l\rangle \hat{a}_i^+ \hat{a}_j^+ \hat{a}_l \hat{a}_k$

$\rightarrow$ here: $\langle i,j|\hat{f}|k,l\rangle = \int d^3x \, d^3y \, \Psi_i^*(\vec{x}) \Psi_j^*(\vec{y}) \hat{f}(\vec{x}, \vec{y}) \Psi_k(\vec{x}) \Psi_l(\vec{y})$

- Second quantization of fermions

$\rightarrow$ fermionic creation and annihilation operators:

$\rightarrow$ antisymmetric nature requires: $\hat{b}_i^+ \hat{b}_k^+ = -\hat{b}_k^+ \hat{b}_i^+ \Leftrightarrow \{\hat{b}_i^+, \hat{b}_k^+\} = 0$

$\Rightarrow (\hat{b}_1^+)^{n_1} (\hat{b}_2^+)^{n_2} = (-1)^{n_1 n_2} (\hat{b}_2^+)^{n_2} (\hat{b}_1^+)^{n_1}$

creation: $\hat{b}_i^+ |\dots, n_i, \dots\rangle = (1 - n_i)(-1)^{\sum_{k<i} n_k} |\dots, n_i + 1, \dots\rangle$

annihilation: $\hat{b}_i |\dots, n_i, \dots\rangle = n_i (-1)^{\sum_{k<i} n_k} |\dots, n_i - 1, \dots\rangle$

$\Rightarrow \{\hat{b}_i, \hat{b}_k^+\} = \delta_{ik}$

$\Rightarrow \hat{n}_i^{(f)} = \hat{b}_i^+ \hat{b}_i$

→ Bosonic operators of

1.) single-particle operators: $\hat{T} = \sum_{i,k} \langle i|\hat{f}|k\rangle \hat{b}_i^\dagger \hat{b}_k$

2.) two-particle operators: $\hat{F} = \frac{1}{2} \sum_{i,j,k,l} \langle i,j|\hat{f}|k,l\rangle \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l$

- Field Operators:

→ construct $\hat{a}_\lambda^\dagger = \sum_i \langle i|\lambda\rangle \hat{a}_i^\dagger$ with spectral theorem $|\lambda\rangle = \sum_i |i\rangle\langle i|\lambda\rangle$

in different basis with $\hat{a}_\lambda = (\hat{a}_\lambda^\dagger)^\dagger = \sum_i \langle\lambda|i\rangle \hat{a}_i$

$\Rightarrow \hat{a}_\lambda^\dagger |0\rangle = |\lambda\rangle$

→ applying on position basis: $\varphi_i(x) = \langle x|i\rangle$

$$\Rightarrow \begin{cases} \hat{\psi}(\vec{x}) = \sum_i \varphi_i(\vec{x}) \hat{a}_i & \text{creates one particle at } \vec{x} \\ \hat{\psi}^\dagger(\vec{x}) = \sum_i \varphi_i^*(\vec{x}) \hat{a}_i^\dagger & \text{annihilates one particle at } \vec{x} \end{cases}$$

$\Rightarrow [\hat{\psi}(\vec{x}), \hat{\psi}(\vec{x}')] = 0, \ [\hat{\psi}^\dagger(\vec{x}), \hat{\psi}^\dagger(\vec{x}')] = 0, \ [\hat{\psi}(\vec{x}), \hat{\psi}(\vec{x}')] = \delta(\vec{x} - \vec{x}')$

→ $\hat{n}(\vec{x}) = \sum_\alpha \delta^{(3)}(\vec{x} - \vec{x}_\alpha) = \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x})$ by expanding with $\varphi_i(\vec{x})$ basis

$\Rightarrow \hat{N} = \int d^3x \, \hat{n}(\vec{x})$

→ Kinetic energy: $\hat{T}$

$\hat{T} = \sum_{i,k} \hat{a}_i^\dagger T_{ik}^{(1)} \hat{a}_k$ with $T_{ik}^{(1)} = \langle i|T^{(1)}|k\rangle \hat{=}$ single particle matrix element

$= \frac{\hbar^2}{2m} \int d_x^3 [\vec{\nabla} \hat{\psi}^\dagger(\vec{x})][\vec{\nabla} \hat{\psi}(\vec{x})]$

→ External potential energy:

$\hat{U} = \sum_{i,k} \hat{a}_i^\dagger U_{ik} \hat{a}_k$

$= \int d_x^3 \, \hat{\psi}^\dagger(\vec{x}) U(x) \hat{\psi}(\vec{x})$

→ Two-Particle interaction:

$\hat{V} = \frac{1}{2} \int d_x^3 d_{x'}^3 \, \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{x}') V(\vec{x}, \vec{x}') \hat{\psi}(\vec{x}') \hat{\psi}(\vec{x})$