Hausaufgaben Blatt 9

1) Sei $u(x,y) = f_1(y)\cos(x) + f_{100}(y)\cos(100x)$

$$\partial_{xx}u + \partial_{yy}u = 0 \iff -\left[f_1(y)\cos(x) + 100^2 f_{100}(y)\cos(100y)\right]$$
$$+ \left[f_1''(y)\cos(x) + f_{100}''(y)\cos(100x)\right] = 0$$

($\cos(\cdot) \perp\!\!\!\perp \cos(100\,\cdot)$ siehe Präsenzaufgabe 2 von Blatt 8)

$$\Rightarrow \begin{cases} f_1''(y) - f_1(y) = 0 \\ f_{100}''(y) - 100^2 f_{100}(y) = 0 \end{cases} \Rightarrow \begin{cases} f_1(y) = A_1 e^{y} + B_1 e^{-y} \\ f_{100}(y) = A_{100} e^{100y} + B_{100} e^{-100y} \end{cases}$$

und
$$\begin{cases} u(x,0) = \cos(x) + \cos(100x) \\ \lim\limits_{y\to+\infty} u(x,y) = 0 \end{cases} \Rightarrow \begin{cases} (A_1 + B_1)\cos(x) + (A_{100} + B_{100})\cos(100x) \\ \quad = \cos(x) + \cos(100x) \\ \lim\limits_{y\to+\infty} \left(A_1 e^{y}\cos(x) + A_{100} e^{100y}\cos(100x)\right) = 0 \end{cases}$$

$\cos(\cdot) \perp\!\!\!\perp \cos(100\,\cdot)$

$$\Rightarrow \begin{cases} A_1 + B_1 = 1 \\ A_{100} + B_{100} = 1 \\ A_1 = 0 \\ A_{100} = 0 \end{cases}$$

$$\Rightarrow u(x,y) = e^{-y}\cos(x) + e^{-100y}\cos(100x)$$

Wir prüfen leicht, dass u eine Lösung ist.

2) i) $\hat{f}_k = \int_0^{2\pi} e^{-ikt} |\sin(t)| \, dt = \int_0^{\pi} e^{-ikt} \sin(t) \, dt - \int_{\pi}^{2\pi} e^{-ikt} \sin(t) \, dt$

Aber $\displaystyle\int_{\pi}^{2\pi} e^{-ikt} \sin(t) \, dt \overset{u = t-\pi}{=} e^{-i\pi k} \int_0^{\pi} e^{-iku} \sin(u+\pi) \, du = -\underbrace{e^{-\pi i k}}_{(-1)^k} \int_0^{\pi} e^{-iku} \sin(u) \, du$

$$\Rightarrow \int_0^{2\pi} e^{-ikt} |\sin(t)| \, dt = \left[1 + (-1)^k\right] \int_0^{\pi} e^{-ikt} \sin(t) \, dt$$

Für $k = 0$, $\displaystyle\int_0^{2\pi} e^{-ikt} |\sin(t)| \, dt = \int_0^{\pi} \sin(t) \, dt - \int_{\pi}^{2\pi} \sin(t) \, dt = 2 \int_0^{\pi} \sin(t) \, dt = 2\,[1+1] = 4$

Sonst $\displaystyle\int_0^{2\pi} e^{-ikt} |\sin(t)| \, dt = \left[1 + (-1)^k\right] \left[\underbrace{\int_0^{\pi} \cos(kt) \sin(t) \, dt}_{(\mathrm{I})_k} - i \underbrace{\int_0^{\pi} \sin(kt) \sin(t) \, dt}_{(\mathrm{II})} \right]$

Falls k ungerade, die Integrale ist 0.
Falls k gerade,

$$(\mathrm{I})_k = \int_0^{\pi} \underset{\downarrow \atop -k\sin(kt)}{\cos(kt)} \, \overset{-\cos(t) \atop \uparrow}{\sin(t)} \, dt = \underbrace{-\left[\cos(t)\cos(kt)\right]_0^{\pi}}_{= -[-1-1] = 2} - k \int_0^{\pi} \overset{\sin(t) \atop \uparrow}{\cos(t)} \, \underset{\downarrow \atop k\cos(kt)}{\sin(kt)} \, dt$$

$$= 2 - k \underbrace{\left[\sin(t)\sin(tk)\right]_0^{\pi}}_{=0} + k^2 \int_0^{\pi} \sin(t) \cos(kt) \, dt$$

$\Rightarrow (\mathrm{I})_k \left[1 - k^2\right] = 2 \quad \Rightarrow (\mathrm{I})_k = \dfrac{2}{(1-k^2)}$ für k gerade.

$$(\mathrm{II})_k = \int_0^{\pi} \underset{\downarrow \atop k\cos(kt)}{\sin(kt)} \, \overset{-\cos(t) \atop \uparrow}{\sin(t)} \, dt = \underbrace{-\left[\sin(kt)\cos(t)\right]_0^{\pi}}_{=0} + k \int_0^{\pi} \underset{\downarrow \atop -k\sin(kt)}{\cos(kt)} \, \overset{\sin(t) \atop \uparrow}{\cos(t)}$$

$$= k \underbrace{\left[\cos(kt)\sin(t)\right]_0^{\pi}}_{=0} + k^2 \int_0^{\pi} \sin(kt) \sin(t) \, dt$$

$\Rightarrow (\mathrm{II})_k (1 - k^2) = 0 \quad \Rightarrow (\mathrm{II})_k = 0 \quad \forall k$ gerade

$\Rightarrow \hat{f}_{2k} = \dfrac{4}{1-4k^2}$, $\hat{f}_{2k+1} = 0$.

Riesz-Fischer
$$\Rightarrow f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx} = 4 + \sum_{k=1}^{+\infty} \frac{4}{1-4k^2}\left(e^{2ikx} + e^{-2ikx}\right)$$

$$= 4 + 8\sum_{k=1}^{+\infty} \frac{\cos(2kx)}{1-4k^2}$$

2) (ii) $\boxed{M1}$ Man merkt, dass

$$\hat{u}_k(1-k^2) = \hat{f}_k \qquad \left(\mathcal{F}(u'')(x) = (ix)^2 \mathcal{F}(u)(x)\right)$$

$\forall k \notin \{-1,1\}$
$$\Rightarrow \hat{u}_k = \frac{\hat{f}_k}{(1-k^2)} \Rightarrow \begin{cases} u_{2k+1} = 0 & \forall k \notin \{-1,1\} \\ u_{2k} = \dfrac{\hat{f}_{2k}}{(1-4k^2)} \end{cases}$$

So wir nehmen ($u_1 = 0$, $u_{-1} = 0$)
$$u(x) := 8\sum_{k=1}^{+\infty} \frac{\cos(2kx)}{(1-4k^2)^2} + 4$$

und man prüft leicht, dass

$$u''(x) + u(x) = |\sin(x)|.$$

2)(ii) $\boxed{M2}$ (Allgemeine Methode aber die Lösung ist nicht periodisch)

Die Lösung von $u'' + u = 0$ $\forall x \in [0, 2\pi]$ ist einfach

$u(x) = A\cos(x) + B\sin(x)$.

Methode: Variation der Konstanten.

Wir suchen die Lösung $y''(x) + y'(x) = |\sin(x)|$ $\forall x \in [0, 2\pi]$

in der Form $y(x) = A(x)\cos(x) + B(x)\sin(x)$

- $y'(x) = A'(x)\cos(x) + B'(x)\sin(x) + A(x)[-\sin(x)] + B(x)\cos(x)$
- $y''(x) = A''(x)\cos(x) + B''(x)\sin(x) + A'(x)[-\sin(x)] + B'(x)\cos(x)$
$\quad + A'(x)[-\sin(x)] + B'(x)\cos(x) + A(x)[-\cos(x)] + B(x)[-\sin(x)]$

$\Rightarrow$ $A''(x)\cos(x) + B''(x)\sin(x) - 2A'(x)\sin(x) + 2B'\cos(x) = |\sin(x)|$

Wir können $A'(x) = -\frac{1}{2}$ wählen falls $x \in [0, \pi]$

und $A'(x) = \frac{1}{2}$ // // $x \in [\pi, 2\pi]$

und $B(x) = 0$.

So wir nehmen $y(x) = -\frac{x-\pi}{2}\cos(x)$ $x \in [0, \pi]$

$= \frac{x-\pi}{2}\cos(x)$ $x \in [\pi, 2\pi]$.

($-\frac{x}{2}\cos(x)$ ist auch Lösung $-\frac{x-\pi}{2}\cos(x)$ war wegen Stetigkeit)

Man prüft leicht, dass y eine Lösung ist.

3) i) $g(x) = x^2$

$$g_k = \int_0^{2\pi} x^2 e^{-ikx}\,dx \Rightarrow g_0 = \int_0^{2\pi} x^2\,dx = \frac{(2\pi)^3}{3}$$

$\forall k \neq 0$, $$g_k = \int_0^{2\pi} x^2 e^{-ikx}\,dx = \left[\frac{e^{-ikx}}{-ik}x^2\right]_0^{2\pi} + \int_0^{2\pi} \frac{2x}{ik} e^{-ikx}$$

$$= \frac{4\pi^2}{-ik} + \frac{2}{ik}\int_0^{2\pi} x e^{-ikx}\,dx$$

$$= \frac{4\pi^2}{-ik} + \left[x e^{-ikx}\right]_0^{2\pi}\left(\frac{-2}{(ik)^2}\right) + \frac{2}{(ik)^2}\int_0^{2\pi} e^{-ikx}\,dx$$

$$= \frac{4\pi^2 i}{k} + \frac{4\pi}{k^2} + 0$$

$$\Rightarrow \sum_{k=-\infty}^{+\infty} g_k e^{ikx} = \frac{(2\pi)^3}{3} + \sum_{k=1}^{+\infty} \frac{4\pi^2 i}{k}\left(e^{ikx} - e^{-ikx}\right) + \sum_{k=1}^{+\infty} \frac{4\pi}{k^2}\left(e^{ikx} + e^{-ikx}\right)$$

$$= \frac{(2\pi)^3}{3} + 8\pi^2 \underbrace{i^2}_{=-1} \sum_{k=1}^{+\infty} \frac{\sin(kx)}{k} + 8\pi \sum_{k=1}^{+\infty} \frac{\cos(kx)}{k^2}$$

ii) $h(x) = x^3$

$$h_0 = \int_0^{2\pi} x^3\,dx = \frac{(2\pi)^4}{4}$$

$$\forall k \neq 0,\quad h_k = \int_0^{2\pi} \underset{\downarrow\ 3x^2}{x^3}\ \overset{\frac{1}{-ik}\ \uparrow}{e^{-ikx}}\, dx = \left[x^3 \frac{e^{-ikx}}{-ik} \right]_0^{2\pi} + \frac{3}{ik} \int_0^{2\pi} x^2 e^{-ikx}\, dx$$

$$= \frac{(2\pi)^3}{-ik} + \frac{3}{ik} \overset{i)}{\left[\frac{4\pi^2 i}{k} + \frac{4\pi}{k^2} \right]}$$

$$\frac{(2\pi)^3}{-ik} + \frac{12\pi^2}{k^2} + \frac{12\pi}{ik^3}$$

$$\Rightarrow \sum_{k=-\infty}^{+\infty} h_k e^{ikx} = \frac{(2\pi)^4}{4} + \frac{(2\pi)^3}{-i} \sum_{k=1}^{+\infty} \frac{1}{k} \left[e^{ikx} - e^{-ikx} \right]$$

$$+ 12\pi^2 \sum_{k=1}^{+\infty} \frac{1}{k^2} \left[e^{ikx} + e^{-ikx} \right] + \frac{12\pi}{i} \sum_{k=1}^{+\infty} \frac{1}{k^3} \left[e^{ikx} - e^{-ikx} \right]$$

$$= \frac{16\pi^4}{4} + 16\pi^3(-1) \sum_{k=1}^{+\infty} \frac{\sin(kx)}{k} + 24\pi^2 \sum_{k=1}^{+\infty} \frac{\cos(kx)}{k^2}$$

$$+ 24\pi(+1) \sum_{k=1}^{+\infty} \frac{\sin(kx)}{k^3}$$

$$= 4\pi^4 + 24\pi^2 \sum_{k=1}^{+\infty} \frac{\cos(kx)}{k^2} + \sum_{k=1}^{+\infty} \sin(kx) \left[-\frac{16\pi^3}{k} + \frac{24\pi}{k^3} \right]$$

$$\Rightarrow \int_0^{2\pi} e^{-ikt} |\sin(t)| \, dt = \frac{2 \times 2}{(1-k^2)} = \frac{4}{1-k^2}$$ für k gerade

und 0 sonst

$$\Rightarrow \sum_{k=-\infty}^{+\infty} c_k e^{ikx} = \sum_{k=-\infty}^{+\infty} \frac{4}{1-4k^2} e^{2ikx}$$

$$= 4 + \sum_{k=1}^{+\infty} \frac{8 \cos(2kx)}{1-4k^2}$$