Präsenzaufgabe 9)

1)(i) $\int_{-\infty}^{+\infty} f_\alpha(t)\, e^{-ixt}\, dt = \int_{-\infty}^{+\infty} f(\alpha t)\, e^{-ixt}\, dt$

wenn $\alpha > 0$
$u = \alpha t$

$$= \frac{1}{\alpha}\int_{-\infty}^{+\infty} f(u)\, e^{-i\frac{x}{\alpha}u}\, du$$

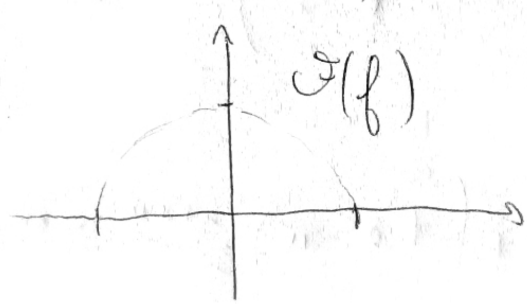
und wenn $\alpha < 0$, dann

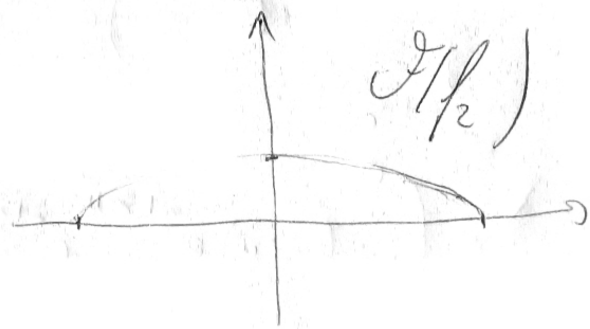
$$\int_{-\infty}^{+\infty} f(\alpha t)\, e^{-ixt}\, dt = \frac{1}{\alpha}\int_{+\infty}^{-\infty} f(u)\, e^{-i\frac{x}{\alpha}t}\, dt$$

$$= \frac{1}{(-\alpha)}\int_{-\infty}^{+\infty} f(u)\, e^{-i\frac{x}{\alpha}t}\, dt.$$

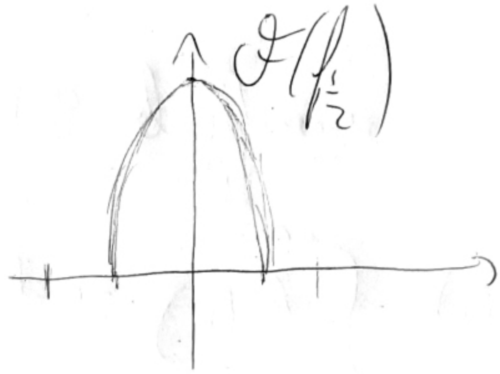
$$\Rightarrow \mathcal{F}(f_\alpha)(x) = \frac{1}{|\alpha|}\mathcal{F}(f)\left(\frac{x}{\alpha}\right)$$

(ii)





(iii)



2) $$\mathcal{F}(f_\alpha)(x) = \int_{-\infty}^{+\infty} f(\alpha+t)\, e^{-ixt}\, dt$$

$u = \alpha + t$

$$\overset{\downarrow}{=} \int_{-\infty}^{+\infty} f(u)\, e^{-ixu}\, e^{ix\alpha}\, du$$

$$= e^{ix\alpha}\, \mathcal{F}(f)(x)$$

3) Falls $x \neq 0$

$$\mathcal{F}(f)(x) = \int_{-a+x_0}^{+a+x_0} \frac{1}{a^2}\left(a - |t-x_0|\right) e^{-ixt}\, dt$$

$$= \int_{-a+x_0}^{x_0} \frac{1}{a^2}(a+t-x_0)\, e^{-ixt}\, dt + \int_{x_0}^{a+x_0} \frac{1}{a^2}(a-t+x_0)\, e^{-ixt}\, dt$$

$u = t - x_0$ $\qquad$ $u = t - x_0$

$$\overset{\downarrow}{=} \int_{-a}^{0} \frac{(a+u)}{a^2}\, e^{-ixu}\, e^{-ixx_0}\, du + \int_{0}^{a} \frac{(a-u)}{a^2}\, e^{-ixu}\, e^{-ixx_0}\, du$$

$u = -v$ für die erste Integral.

$$= e^{-ixx_0} \int_0^a \frac{(a-u)}{a^2} \cdot 2\cos(xu)\, du = \frac{e^{-ixx_0}}{a} \int_0^a 2\cos(xu)\, du - \frac{e^{-ixx_0}}{a^2}\, 2 \int_0^a \underset{\downarrow 1}{u}\, \overset{\frac{\sin(xu)}{x}}{\cos(xu)}\, du$$

$$= \frac{e^{-ix_0x}}{a} 2 \int_0^a \cos(xu)\,du - \frac{e^{ixx_0}2}{a^2} \left[u \frac{\sin(xu)}{x} \right]_0^a$$

$$+ \frac{e^{-ixx_0}}{a^2} \times 2 \int_0^a \frac{\sin(xu)\,du}{x}$$

$$= \frac{e^{-ix_0x}}{a} \times 2 \cancel{\left[\frac{\sin(xu)}{x} \right]_0^a} - \frac{e^{ixx_0}2}{a^2} \cancel{\left[u \frac{\sin(xu)}{x} \right]_0^a}$$

$$+ \frac{e^{-ixx_0}}{a^2} \times \frac{2}{x} \left[\frac{-\cos(xu)}{x} \right]_0^a$$

$$= \frac{2e^{-ixx_0}}{a^2x^2} \Big(\underbrace{1 - \cos(xa)}_{= 2\sin^2\left(\frac{xa}{2}\right)} \Big) = e^{-ixx_0} \frac{\sin^2\left(\frac{xa}{2}\right)}{\left(\frac{xa}{2}\right)^2}$$

Wenn $x = 0$, $\displaystyle\int_{-a}^{a} \frac{1}{a^2} \left(a - |t - x_0|\right) dt = 1$ (Leicht)