

Advanced Quantum Theory (WS 24/25)

Homework no. 2 (October 14, 2024)

To be handed in by Sunday, October 20!

1 Canonical Transformations and Classical Trajectories

In classical Hamiltonian mechanics, a canonical transformation can be generated by a function $g(q_i, p_i)$, where the q_i are the generalized coordinates and the p_i the canonically conjugated momenta. A given function g generates an infinitesimal transformation

$$q_i \rightarrow \bar{q}_i = q_i + \delta q_i = q_i + \epsilon \frac{\partial g}{\partial p_i}, \quad p_i \rightarrow \bar{p}_i = p_i + \delta p_i = p_i - \epsilon \frac{\partial g}{\partial q_i}, \quad (1)$$

where $|\epsilon| \ll 1$ is an otherwise arbitrary constant.

Here we want to treat eq.(1) as an *active transformation*, which connects two *different* points (q_i, p_i) and (\bar{q}_i, \bar{p}_i) in phase space. The system under consideration, described by the Hamilton function H , is invariant under this transformation iff the Poisson bracket of g and H vanishes, $\{g, H\} = 0$.

1. Show that if $(q_i(t), p_i(t))$ describes a valid trajectory (i.e. satisfies the equations of motion), and $\{g, H\} = 0$, then the transformation (1) generates another valid trajectory, i.e. $(\bar{q}_i(t), \bar{p}_i(t))$ is another valid trajectory. [4P]
2. Now consider the simple case of a single particle. Convince yourself that finite transformations of one of the Cartesian coordinates, $x_k \rightarrow \bar{x}_k = x_k + \delta$ with arbitrary δ , generate valid trajectories $(\bar{x}_i(t), p_i(t))$ given a valid trajectory $(x_i(t), p_i(t))$, if this transformation leaves the Hamilton function invariant. *Hint:* What is the generator of this transformation? What does invariance under this transformation imply for the Hamilton function? [3P]

2 Canonical Transformation in Quantum Mechanics

We saw in class that the generating function g of a canonical transformation in classical mechanics defines a unitary quantum mechanical operator

$$\hat{U}_g(\xi) = \exp(-i\xi\hat{g}/\hbar), \quad (2)$$

so that a finite active transformation can be described by

$$\psi(q_i, t) \rightarrow \bar{\psi}(q_i, t) = \hat{U}_g(\xi)\psi(q_i, t). \quad (3)$$

Here ψ is the wave function of the system under consideration, the q_i are the generalized coordinates, and ξ is an arbitrary real constant.

1. The transformation (3) can be made a bit more explicit by expressing the wave function in terms of eigenfunctions of the hermitean operator \hat{g} ,

$$\psi(q_i, t) = \sum_n c_n(t) \psi_n(q_i), \quad (4)$$

with $\hat{g}\psi_n = g_n\psi_n$. The transformed wave function $\bar{\psi}(q_i, t)$ can be expressed analogously, with expansion coefficients $\bar{c}_n(t)$. How are the $\bar{c}_n(t)$ related to the original $c_n(t)$? [3P]

2. Now consider a single particle system, and $g = L_z$, the z component of orbital angular momentum. As shown in class, this generates rotations around the z -axis. Prove this result in the formalism of eq.(4). *Hint:* Use the explicit form of the eigenfunctions of \hat{L}_z . [3P]

3. Now consider a single particle system, and $g = L^2$, the square of the orbital angular momentum. Consider three cases: (i) The wave function is an eigenfunction of \hat{L}^2 and \hat{L}_z with fixed quantum numbers l and m ; (ii) the wave function is a superposition of eigenfunctions of \hat{L}^2 and \hat{L}_z , with fixed l but different values of m ; (iii) the wave function is a superposition of eigenfunctions of \hat{L}^2 and \hat{L}_z , where both l and m take different values. In which of these three cases does the active transformation $\psi \rightarrow \hat{U}_{L^2}(\xi)\psi$ corresponds to a *physical* change? *Hint:* Recall that the overall phase of the wave function has no physical significance. [3P]

3 Gauge Invariance in Classical Electrodynamics

Classical electrodynamics can be formulated in terms of the electric field \vec{E} and the magnetic field \vec{B} , or equivalently in terms of the scalar potential U and the vector potential \vec{A} . The two sets of quantities are related by

$$\vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t); \quad \vec{E}(\vec{x}, t) = -\vec{\nabla}U(\vec{x}, t) - \frac{\partial \vec{A}(\vec{x}, t)}{\partial t}. \quad (5)$$

A gauge transformation is defined by a real function $\lambda(\vec{x}, t)$, such that

$$\vec{A}(\vec{x}, t) \rightarrow \vec{A}(\vec{x}, t) + \vec{\nabla}\lambda(\vec{x}, t); \quad U(\vec{x}, t) \rightarrow U(\vec{x}, t) - \frac{\partial \lambda(\vec{x}, t)}{\partial t}. \quad (6)$$

Note that both \vec{A} and U have to be transformed simultaneously. We are using SI units in this problem.

1. Show that the gauge transformation (6) leaves the fields \vec{B} , \vec{E} defined in eq.(5) unchanged. This is the basis of gauge invariance. [3P]
2. Show that the homogeneous (source-independent) Maxwell equations,

$$\vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0; \quad \vec{\nabla} \times \vec{E}(\vec{x}, t) = -\frac{\partial \vec{B}(\vec{x}, t)}{\partial t}, \quad (7)$$

are satisfied *automatically* if the fields are expressed as in (5). [4P]

3. The ‘‘Lorenz gauge’’ is defined by

$$\vec{\nabla} \cdot \vec{A}(\vec{x}, t) = -\mu_0 \epsilon_0 \frac{\partial U(\vec{x}, t)}{\partial t}. \quad (8)$$

Show that this decouples the two inhomogeneous Maxwell equations,

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \rho(\vec{x}, t)/\epsilon_0; \quad \vec{\nabla} \times \vec{B}(\vec{x}, t) = \mu_0 \vec{j}(\vec{x}, t) + \mu_0 \epsilon_0 \frac{\partial \vec{E}(\vec{x}, t)}{\partial t}, \quad (9)$$

when the fields are expressed in terms of the potentials; here the charge density ρ and the current density \vec{j} are sources of the fields. [4P]

4. The Lagrange function of classical electrodynamics is given by

$$L = \int d^3x \left[\frac{\epsilon_0}{2} \vec{E} \cdot \vec{E} - \frac{1}{2\mu_0} \vec{B} \cdot \vec{B} - \rho U + \vec{j} \cdot \vec{A} \right]. \quad (10)$$

Show that the integrand of L (often called the Lagrange density) is *not* invariant under a gauge transformation (6), if the sources ρ and \vec{j} are assumed to be gauge invariant. However, show that L is gauge invariant, under the usual assumption that surface terms can be ignored. *Hint:* Use the fact that the current is conserved! [5P]