AGT; Sheet 10; Marc Hauer, Angelo Brade; 15.12.2024

Exercise 1:

4.) Only if the amplitude of the Wave function $|\psi(\vec{x}=0)| \ll 1$, our potential is concentrated at the center $\vec{x}=0$, i.e. we can approximate our outgoing wave as sperical symmetrical. Thus it is a necessity for computing a spherical wave, starting from a plane wave.

$$\psi^{Born}_{scat}(\vec{x}) = -\frac{2\mu}{\hbar}\int d^3x' \frac{e^{i|\vec{k}||\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} V(\vec{x}') e^{i\vec{k}\cdot\vec{x}'} \quad \text{with} \quad V(\vec{x}') = \begin{cases} -V_0, & r \le r_0 \\ 0, & r > r_0 \end{cases}$$

$$|\psi^{Born}_{scat}(\vec{x}=0)| \ll 1$$

$$\Rightarrow 1 \gg \left|\frac{2\mu}{\hbar}\int_{-1}^{1}\int_0^{2\pi}\int_0^{r_0} d^3x' \frac{e^{i|\vec{k}||\vec{x}'|}}{4\pi|\vec{x}'|} V_0 e^{i\vec{k}\cdot\vec{x}'}\right|$$

$$\gg \left|\frac{2\mu}{\hbar}\int_0^{2\pi} d\varphi' \int_{-1}^{1} d(\cos\theta') \int_0^{r_0} dr'\, r'^2 \frac{e^{i|\vec{k}|r'(1+\cos\theta')}}{4\pi r'} V_0\right|$$

$$\gg \left|\frac{\mu V_0}{\hbar}\int_{-1}^{1} d(\cos\theta)\int_0^{r_0} dr'\, r'\, e^{i|\vec{k}|r'(1+\cos\theta)}\right|$$

$$\gg \left|\frac{\mu V_0}{\hbar}\int_0^{r_0} dr' \frac{r'}{i|\vec{k}|r'}\left(e^{2i|\vec{k}|r'}-1\right)\right|$$

$$\gg \left|\frac{\mu V_0}{\hbar}\left(\frac{e^{2i|\vec{k}|r_0}-1}{-2|\vec{k}|^2} - \frac{r_0}{i|\vec{k}|}\right)\right| \quad \Bigg| \quad |\vec{k}|r_0 \ll 1 \Rightarrow e^{2i|\vec{k}|r_0} \simeq 1$$

$$\gg \left|\frac{\mu V_0}{\hbar}\frac{r_0}{|\vec{k}|}\right| \quad \Big| \quad v_0 = \frac{2\mu V_0 r_0^2}{\hbar}$$

$$\gg \left|\frac{\hbar v_0}{2|\vec{k}|r_0}\right|$$

Since $|\vec{k}|r_0 \ll 1$ and $\hbar \ll 1 \Rightarrow \frac{\hbar}{|\vec{k}|r_0} \simeq 1$ | i dont actually think thats how it works lol

$$\Rightarrow \frac{v_0}{2} \ll 1$$

$\square$

5.)

$$\delta_0 = \arctan\left[\frac{k}{q}\tan(qr_0)\right] - kr_0 \quad \Big| \quad \frac{d^2}{dx^2}\arctan x = \frac{d}{dx}\frac{1}{x^2+1} = \frac{-2x}{1+2x^2+x^4}$$

$$= \left(0 + \frac{k}{q}\tan(qr_0) + 0 + \mathcal{O}(x^3)\right) - kr_0$$

$$\simeq \frac{k}{q}\tan(qr_0) - kr_0 \quad \Big| \quad \frac{d^2}{dx^2}\tan(x) = \frac{d}{dx}\frac{1}{\cos^2 x} = \frac{2\sin x}{\cos^3 x} = 2\frac{\tan x}{\cos^2 x}$$

$$= \frac{k}{q}\left(0 + qr_0 + 0 + \mathcal{O}(x^3)\right) - kr_0$$

$$\simeq 0$$

?

Quickies:

Q1) i) $\sigma^B \sim |f^B(\theta)|^2 \sim |V(r)|^2 \quad \Rightarrow \quad \sigma^B$ does not depend on the sign of V

ii) Bound states (with energy E): $V < E < 0 \quad \rightarrow$ only possible for $V < 0$

iii) Resonance occurs only if a bound state exists. Because the existence of bound states depends on the sign of V and the Born approximation does not, it is not possible to predict resonance via Born approximation.

Q2) $k \ll 1 \quad \Rightarrow \quad \sigma_l \sim k^{4l}$ where $k = |\vec{k}|$

i) $\sigma_l \sim 4\pi(2l+1)|a_l|^2 \sim k^{4l} \quad \Rightarrow \quad |a_l|^2 \sim k^{4l} \quad \Rightarrow \quad |a_l| \sim k^{2l}$

ii) $a_l \sim \frac{\sin(\delta_l)}{k} \sim \frac{\delta_l}{k} \sim k^{2l} \quad \Rightarrow \quad \delta_l \sim k^{2l+1}$

Q3) i) electron-proton-scattering: $m_p > m_e \quad \Rightarrow$ for non-relativistic energies the formalism can be applied to this situation with the Proton as a fixed scattering center.

ii) neutron-nucleus-scattering: $m_{nuc.} > m_n \quad \Rightarrow$ for non-relativistic energies the formalism can be applied to this situation with the Nucleus as a fixed scattering center.

iii) proton-proton-scattering: $m_p = m_p \quad \Rightarrow$ because of the equal mass there is no fixed scattering center

$\hookrightarrow$ formalism can't be applied

1) Potential Well and Born Approximation

$$V(r) = \begin{cases} -V_0 , & r \leq r_0 \\ 0 , & r > r_0 \end{cases}$$

1) Show: $f_{\vec{k}}^B = \frac{2\mu V_0 r_0}{\hbar^2 |\vec{q}|^2} \left[\frac{1}{r_0 |\vec{q}|} \sin(r_0 |\vec{q}|) - \cos(r_0 |\vec{q}|) \right]$

$$f_{\vec{k}}^B = -\underbrace{\frac{2\mu}{\hbar^2 |\vec{q}|}}_{:= A} \int_0^\infty dr\, r\, V(r) \sin(|\vec{q}| r)$$

$$= A \cdot V_0 \int_0^{r_0} dr\, r \sin(qr) \qquad \text{where } |\vec{q}| = q$$

$$= A \cdot V_0 \left[-\frac{r}{q} \cos(qr) + \frac{1}{q} \int dr \cos(qr) \right]_0^{r_0}$$

$$= \frac{A V_0}{q} \left[-r\cos(qr) + \frac{1}{q} \sin(qr) \right]_0^{r_0}$$

$$= \frac{A V_0}{q} \left(\frac{1}{q} \sin(q r_0) - r_0 \cos(q r_0) \right)$$

$$= \frac{2\mu V_0 r_0}{\hbar^2 q^2} \left(\frac{1}{q r_0} \sin(q r_0) - \cos(q r_0) \right) \qquad \square$$

2) Show: $f_{\vec{k}}^B \simeq v_0 \frac{r_0}{3}$ (for $|\vec{q}| r_0 \ll 1$) where $v_0 = \frac{2\mu V_0 r_0^2}{\hbar^2}$

$$f_{\vec{k}}^B = \frac{2\mu V_0 r_0}{\hbar^2 q^2} \left(\frac{1}{q r_0} \sin(q r_0) - \cos(q r_0) \right) \qquad \Big| \quad \sin(x) = x - \frac{1}{6}x^3 + \mathcal{O}(x^5) \quad , \quad \cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}(x^4)$$

$$\Rightarrow \quad f_{\vec{k}}^B \simeq \frac{2\mu V_0 r_0}{\hbar q^2} \left[\frac{1}{q r_0} \left(q r_0 - \frac{1}{6}(q r_0)^3 \right) - \left(1 - \frac{1}{2}(q r_0)^2 \right) \right]$$

$$= \frac{2\mu V_0 r_0}{\hbar q^2} \left[1 - \frac{1}{6}(q r_0)^2 - 1 + \frac{1}{2}(q r_0)^2 \right]$$

$$= \frac{2\mu V_0 r_0^3}{3\hbar^2}$$

$$= v_0 \frac{r_0}{3} \qquad \text{where} \quad v_0 = \frac{2\mu V_0 r_0^2}{\hbar^2} \qquad \square$$

3) Show: $\sigma^B \simeq 4\pi r_0^2 \frac{v_0^2}{9}$

$$\sigma = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\, |f_k(\varphi, \theta)|^2$$

$$\Rightarrow \quad \sigma^B = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \left| \frac{v_0 r_0}{3} \right|^2 = \frac{v_0^2 r_0^2}{9} \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) = \frac{2}{9} v_0^2 r_0^2 \int_0^{2\pi} d\varphi = 4\pi r_0^2 \frac{v_0^2}{9} \qquad \square$$