Präsenzaufgabenblatt 2 (Emotum)

2) $x = r\cos(\varphi)$
$y = r\sin(\varphi)$

(1) $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos(\varphi) + \frac{\partial u}{\partial y}\sin(\varphi)$$

(2) $$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos(\varphi) + \frac{\partial v}{\partial y}\sin(\varphi)$$

(3) $$\frac{1}{r}\frac{\partial v}{\partial \varphi} = \frac{1}{r}\frac{\partial v}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{1}{r}\frac{\partial v}{\partial y}\frac{\partial y}{\partial \varphi} = -\frac{\partial v}{\partial x}\sin(\varphi) + \frac{\partial v}{\partial y}\cos(\varphi)$$

(4) $$-\frac{1}{r}\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\sin(\varphi) - \frac{\partial u}{\partial y}\cos(\varphi)$$

- Wenn die C.R. Differentialgleichungen gelten, man hat

$$\frac{\partial u}{\partial r} \overset{(1)}{=} \frac{\partial u}{\partial x}\cos(\varphi) + \frac{\partial u}{\partial y}\sin(\varphi) \overset{C.R.}{=} \frac{\partial v}{\partial y}\cos(\varphi) + \left(-\frac{\partial v}{\partial x}\right)\sin(\varphi) \overset{(3)}{=} \frac{1}{r}\frac{\partial v}{\partial \varphi}$$

und

$$\frac{\partial v}{\partial r} \overset{(2)}{=} \frac{\partial v}{\partial x}\cos(\varphi) + \frac{\partial v}{\partial y}\sin(\varphi) \overset{C.R.}{=} -\frac{\partial u}{\partial y}\cos(\varphi) + \frac{\partial u}{\partial x}\sin(\varphi) \overset{(4)}{=} -\frac{1}{r}\frac{\partial u}{\partial \varphi}$$

- Wenn $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}$ und $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}$

dann
$$\begin{cases} \frac{\partial u}{\partial x}\cos(\varphi) + \frac{\partial u}{\partial y}\sin(\varphi) = -\frac{\partial v}{\partial x}\sin(\varphi) + \frac{\partial v}{\partial y}\cos(\varphi) \\ \frac{\partial v}{\partial x}\cos(\varphi) + \frac{\partial v}{\partial y}\sin(\varphi) = \frac{\partial u}{\partial x}\sin(\varphi) - \frac{\partial u}{\partial y}\cos(\varphi) \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \end{pmatrix}}_{=M} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Wenn M Invertible wäre, dann $\exists \varphi$, $\cos(\varphi) = 0$ und $\sin(\varphi) = 0$ $\lightning$

$\Rightarrow$ M ist nicht Invertierbar und $\det(M) = 0$

$$\Rightarrow \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 = 0$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$