Hausaufgabenblatt 8)

1) Sei $M = \sup_{z \in A_{0,1}(0)} |f(z)| < +\infty$. $f(z)$ läßt sich für alle $z \in A_{0,1}(0)$ als Laurent-Reihe darstellen:

$$f(z) = \sum_{k=-\infty}^{+\infty} b_k z^k$$

Man hat, dass für alle $\rho \in (0,1)$ und $k<0$

$$|b_k| = \left| \frac{1}{2\pi i} \int_{\partial B_\rho(0)} \frac{f(z)}{z^{k+1}} dz \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(\rho e^{it}) \rho i e^{it}}{(\rho e^{it})^{k+1}} dt \right|$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(\rho e^{it})}{\rho^k e^{itk}} dt \right| \leq \frac{1}{2\pi \rho^k} \int_0^{2\pi} |f(\rho e^{it})| dt$$

$$\leq \frac{M}{\rho^k} = M \rho^{|k|} \xrightarrow[\rho \to 0]{} 0$$

$$\Rightarrow b_k = 0 \ \forall k<0 \Rightarrow f(z) = \sum_{k=0}^{+\infty} b_k z^k \quad \forall z \in B_1(0)\setminus\{0\}.$$

2)(i) $$e^{z+\frac{1}{z}} = e^z e^{\frac{1}{z}} = \sum_{k_1=0}^{+\infty} \frac{z^{k_1}}{k_1!} \sum_{k_2=0}^{+\infty} \frac{z^{-k_2}}{k_2!}$$

$$= \sum_{k_1=0}^{+\infty} \sum_{k_2=0}^{+\infty} \frac{z^{k_1-k_2}}{k_1! k_2!} \sum_{j=-\infty}^{+\infty} \mathbb{1}_{\{j=k_1-k_2\}}$$

$$= \sum_{j=-\infty}^{+\infty} \sum_{k_1=0}^{+\infty} \sum_{k_2=0}^{+\infty} \frac{z^j}{k_1! k_2!} \mathbb{1}_{\{j=k_1-k_2\}} = \sum_{j=-\infty}^{+\infty} \left[\sum_{k_1 \geq \max(j,0)} \frac{1}{k_1!} \frac{1}{(k_1-j)!} \right] z^j$$

(ii) Die Singularitäten sind $\{z, \exists k \in \mathbb{Z}, z = \pi k\}$

$$\mathrm{Res}\left(\frac{\cos(z)}{z\sin(z)}, 0\right) = ?$$

$$\frac{\cos(z)}{z\sin(z)} = \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{2k!} \cdot \frac{1}{\left(z^2 - \frac{z^4}{3!} + O(z^6)\right)} = \frac{\left(1 - \frac{z^2}{2!} + O(z^4)\right)}{z^2\left(1 - \frac{z^2}{3!} + O(z^4)\right)}$$

$$= \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + O(z^4)\right)\left(1 + \frac{z^2}{3!} + O(z^4)\right)$$

$$= \frac{1}{z^2} + \text{cste} + \sum_{k=2}^{+\infty} \alpha_k z^k$$

$$\Rightarrow \mathrm{Res}\left(\frac{\cos(z)}{z\sin(z)}, 0\right) = 0$$

Für $k \neq 0$

$$\mathrm{Res}\left(\frac{\cos(z)}{z\sin(z)}, \pi k\right) = \lim_{z \to \pi k} (z - \pi k)\frac{\cos(z)}{z\sin(z)}$$

$$= \frac{\cos(\pi k)}{\sin(\pi k) + \pi k \cos(\pi k)} = \frac{1}{\pi k}$$

(iii) $$\frac{\cos(z)}{z\sin(z)^2} = \frac{\cos(z)}{z^3\left(1 - \frac{z^2}{3!} + O(z^4)\right)^2} = \frac{\cos(z)}{z^3}\left(1 + \frac{z^2}{3!} + O(z^4)\right)^2$$

$$= \frac{\left(1 - \frac{z^2}{2!} + O(z^4)\right)\left(1 + \frac{2z^2}{3!} + O(z^4)\right)}{z^3}$$

$$= \frac{1}{\ldots} - z^{-1}\ldots O(z^4) \quad \Rightarrow \mathrm{Res}\left(\frac{\cos(z)}{z\sin(z)^2}, 0\right) = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$$

3) $$f(x) = \sum_{k=-N}^{N} \gamma_k e^{ikx}$$

$$= \sum_{k=-N}^{N} \gamma_k \cos(kx) + i \sum_{k=-N}^{N} \gamma_k \sin(kx)$$

$$= \gamma_0 + \sum_{k=1}^{N} (\gamma_k + \gamma_{-k}) \cos(kx)$$

$$+ i \sum_{k=1}^{N} (\gamma_k - \gamma_{-k}) \sin(kx)$$

Wir wollen, dass

$$\underbrace{\sum_{k=0}^{N} \alpha_k \cos(kx) + \beta_k \sin(kx)}_{=(\mathrm{I})(x)} = \underbrace{\gamma_0 + \sum_{k=1}^{N} (\gamma_k + \gamma_{-k}) \cos(kx) + \sum_{k=1}^{N} i (\gamma_k - \gamma_{-k}) \sin(kx)}_{=(\mathrm{II})(x)}$$

$$\left(\int_0^{2\pi} (\mathrm{I})(x) \cos(k'x)\, dx = \int_0^{2\pi} (\mathrm{II})(x) \cos(l'x)\, dx \right)$$

Siehe letzte Woche $x \mapsto \cos(k'x)$ Basis Präsenz 8 Aufgabe 2

$$\Longrightarrow \begin{cases} \alpha_0 = \gamma_0 \\ \alpha_{k'} = \gamma_{k'} + \gamma_{-k'} \end{cases}$$

und $$\left(\int_0^{2\pi} (\mathrm{I}) \sin(k'x)\, dx = \int_0^{2\pi} (\mathrm{II}) \sin(k'x)\, dx \right) \Longrightarrow \left\{ \beta_{k'} = i(\gamma_{k'} - \gamma_{-k'}) \right.$$

$\Longleftarrow$ Falls $\begin{cases} \alpha_0 = \gamma_0 \\ \alpha_k = \gamma_k + \gamma_{-k} \\ \beta_k = i(\gamma_k - \gamma_{-k}) \end{cases}$ die Gleichheit folgt leicht.