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Aufgabe 1:

b)

i)

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-a(x^2+y^2)} = \int_0^{2\pi} d\varphi \int_0^{\infty} dr \cdot r \cdot e^{-ar^2} = 2\pi \int_0^{\infty} du\, \tfrac{1}{2} e^{-au}$$

$r^2 = u \Rightarrow \frac{du}{dr} = 2r$

$$= \pi \left[\tfrac{1}{-a} e^{-au} \right]_0^{\infty}$$

$\to a$

$$= \frac{\pi}{a}$$

$$\left(\int_{-\infty}^{\infty} dx\, e^{-ax^2} \right)^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-a(x^2+y^2)} = \frac{\pi}{a}$$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$u = \frac{x}{\sqrt{a}} - \frac{J}{\sqrt{2a}} \Rightarrow du = \frac{1}{\sqrt{a}} dx$

ii)

$$Z(J) = \int_{-\infty}^{\infty} e^{-ax^2 + Jx} dx = \int_{-\infty}^{\infty} dx\, e^{-(ax^2 - Jx + \frac{J^2}{2a} - \frac{J^2}{2a})} = \int_{-\infty}^{\infty} dx\, e^{-(\frac{x}{\sqrt{a}} - \frac{J}{\sqrt{2a}})^2 + \frac{J^2}{2a}} = e^{\frac{J^2}{2a}} \cdot \int_{-\infty}^{\infty} a^{-u^2} \cdot \sqrt{a}\, du$$

$$= e^{\frac{J^2}{2a}} \sqrt{a\pi}$$

iii)

$$\left.\frac{\partial^n Z(J)}{\partial J^n}\right|_{J=0} = \left.\frac{\partial^n}{\partial J^n} e^{\frac{J^2}{2a}} \sqrt{a\pi}\right|_{J=0} = \sqrt{a\pi}$$

$$\frac{\partial Z(J)}{\partial J} = \left.\sqrt{a\pi} \left(\frac{J}{a} e^{\frac{J^2}{2a}} \right)\right|_{J=0} = 0$$

$$\frac{\partial^2 Z(J)}{\partial J^2} = \left.\sqrt{a\pi}\, \exp(J^2/2a) \left(\frac{J^2}{a} + \frac{1}{a} \right)\right|_{J=0} = \sqrt{\frac{\pi}{a}}$$

$$\frac{\partial^3 Z(J)}{\partial J^2} = \left.\sqrt{a\pi}\, \exp(J^2/2a) \left(\frac{J^3}{a^2} + \frac{J}{a^2} + 2\frac{J}{a} \right)\right|_{J=0} = 0$$

$$\frac{\partial Z(J)}{\partial J^4} = \left.\sqrt{a\pi}\, \exp(J^2/2a) \left(\frac{J^4}{a^3} + \frac{J^2}{a^3} + 2\frac{J}{a} + 3\frac{J^2}{a^2} + \frac{1}{a^2} + \frac{2}{a} \right)\right|_{J=0} = \sqrt{a\pi} \left(\frac{1}{a^2} + \frac{2}{a} \right)$$

sym. Interval & asym. Funktion

$$\langle X \rangle = \int_{\mathbb{R}} e^{-ax^2} \cdot x\, dx = 0$$

$$(\Delta x)^2 = \int_{\mathbb{R}} e^{-ax^2} x^2 dx - 0 = \int_{\mathbb{R}} \frac{\partial}{\partial a} (-e^{-ax^2}) dx = -\frac{\partial}{\partial a} \sqrt{\frac{\pi}{a}} = \frac{1}{2} \left(\frac{\pi}{a} \right)^{-\frac{3}{2}}$$

$$\langle X^4\rangle = \int_{\mathbb{R}} e^{-ax^2} x^4 dx = \int_{\mathbb{R}} \frac{\partial^2}{\partial a^2} e^{-ax^2} dx = \frac{\partial^2}{\partial a^2}\sqrt{\frac{\pi}{a}} = \frac{\partial}{\partial a^2}\frac{1}{2}\left(\frac{\pi}{a}\right)^{-\frac{3}{2}} = -\frac{3}{4}\left(\frac{\pi}{a}\right)^{-\frac{5}{2}}$$

c)

i) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$$\Gamma(n+1) = \int_{\mathbb{R}^+} x^{(n+1)-1} e^{-x} dx = \int_{\mathbb{R}^+} x^n \cdot e^{-x} dx = \underbrace{[-x^n e^{-x}]_{\mathbb{R}^+}}_{\longrightarrow 0} + \int_{\mathbb{R}^+} n \cdot x^{n-1} e^{-x} dx$$

$$= n \cdot \Gamma(n)$$

$\square$

$$\Gamma(1) = \int_{\mathbb{R}^+} e^{-x} dx = [-e^{-x}]_0^\infty = -0 + 1 = 1$$

$\square$

$$\Gamma(1/2) = \int_{\mathbb{R}^+} x^{-1/2} e^{-x} dx \quad \left| \begin{array}{l} u^2 = x \Rightarrow \frac{dx}{du} = 2u \\ \Rightarrow dx = 2u \cdot du \end{array} \right.$$

$$= \int_{\mathbb{R}^+} u^{-1} e^{-u^2} \cdot 2 \cdot u \, du$$

$$= \int_{\mathbb{R}^+} e^{-u^2} du$$

$$= \frac{1}{2}\int_{\mathbb{R}} e^{-u^2} du$$

$$= \frac{1}{2}\sqrt{\pi}$$

?

$\square$

ii)

z: $\langle X^n\rangle = \frac{1}{\sqrt{\pi}} \sum_{k \text{ gerade}}^{n} \binom{n}{k} x_0^{n-k} (2\sigma)^{\frac{k}{2}} \Gamma\left(\frac{k+1}{2}\right)$

Aufgabe 2:

a) Potentielle Energie:

$$U = \sum_i m_i \cdot g \cdot h_i = \sum_i \rho \cdot s_i \cdot g \cdot z_i \qquad z_i \equiv h_i,\ m_i = \rho \cdot s_i$$

$$= g \cdot \rho \int_{s_p}^{s_\alpha} z(s)\, ds \quad \Big|\ ds = s - s_0 = \sqrt{\underbrace{(x-x_0)}_{dx}{}^2 + \underbrace{(z-z_0)}_{dz}{}^2}$$

$$\Rightarrow ds = \sqrt{dx^2 + dz^2} = \sqrt{1 + \Big(\underbrace{\frac{dz}{dx}}_{z'}\Big)^2}\, dx = \sqrt{1 + z'^2}\, dx$$

$$= g \cdot \rho \int_{x_p}^{x_\alpha} z \sqrt{1 + z'^2}\, dx$$

Einschränkung: $L = \text{const.}$

$$L = \int_{s_0}^{s_1} ds = \int_{x_p}^{x_\alpha} \sqrt{1 + z'^2}\, dx$$

Variation von Potential: $U \to U - \lambda \cdot L$

$$U = \rho \cdot g \int_{x_p}^{x_\alpha} (z - \lambda) \sqrt{1 + z'^2}\, dx$$

$$= \rho \cdot g \int_{x_p}^{x_\alpha} dx\, F(z, z', x) \quad \text{mit } F(z, z', x) = (z - \lambda)\sqrt{1 + z'^2} \text{ und } z' = \frac{dz}{dx}$$

b)

Da F stationär in x ist, können wir F Legendre transformieren und erhalten die in x konstante Hamiltonfunktion, die wir h nennen:

$$\frac{\partial F}{\partial x}\Big|_{x = x_0} = 0$$

$$\Rightarrow h = F - z' p \qquad \Big|\ \text{definiere } p = \frac{\partial F}{\partial z'} \text{ analog zu } p = \frac{\partial L}{\partial \dot q}$$

$$= F - z' \frac{\partial F}{\partial z'} \qquad \Big|\ F = (z - \lambda)\sqrt{1 + z'^2}$$

$$= (z - \lambda)\sqrt{1 + z'^2} - z'^2 (z - \lambda) \frac{1}{\sqrt{1 + z'^2}}$$

$$= (z - \lambda)\sqrt{1 + z'^2}\left(1 - \frac{z'^2}{1 + z'^2}\right)$$

$$= (z - \lambda)\sqrt{1 + z'^2}\left(\frac{1 + z'^2 - z'^2}{1 + z'^2}\right)$$

$$= \frac{z - \lambda}{\sqrt{1 + z'^2}}$$

$$\Rightarrow z' = \sqrt{\left(\frac{z - \lambda}{h}\right)^2 - 1}$$

c)

$$\frac{dz}{dx} = \sqrt{\left(\frac{z-\lambda}{h}\right)^2 - 1}$$

$$\Rightarrow dx = \frac{dz}{\sqrt{\left(\frac{z-\lambda}{h}\right)^2 - 1}} \qquad \left| \begin{array}{l} \frac{z-\lambda}{h} = \cosh u \\ \Rightarrow \frac{dz}{du} = \frac{d}{du}(h\cosh u + \lambda) = h\sinh u \end{array} \right.$$

$$\Rightarrow \int dx = \int \frac{h \sinh u \, du}{\sqrt{\cosh^2 u - 1}} \qquad \left| \ \cosh^2 u - \sinh^2 u = 1 \right.$$

$$= \int h \, du$$

$$\Rightarrow x + a = u \cdot h + b \qquad \left| \ \frac{z-\lambda}{h} = \cosh u \Rightarrow \operatorname{arch}\left(\frac{z-\lambda}{h}\right) = u \right.$$

$$\Rightarrow x + a = \operatorname{arch}\left(\frac{z-\lambda}{h}\right) \cdot h + b$$

$$\Rightarrow z(x) = \cosh\left(\frac{x+a-b}{h}\right) \cdot h + \lambda \qquad \left| \ a - b \equiv c \right.$$

$$= \cosh\left(\frac{x+c}{h}\right) \cdot h + \lambda$$

$$z_p = z(x_p) \qquad \left| \ (x_p, z_p) = (-d, 0) \right.$$

$$\Rightarrow 0 = \cosh\left(\frac{-d+c}{h}\right) \cdot h + \lambda$$

$$\Rightarrow \lambda = -\cosh\left(\frac{c-d}{h}\right) \cdot h$$

$$\Rightarrow z(x) = h\cosh\left(\frac{x+c}{h}\right) - h\cosh\left(\frac{c-d}{h}\right)$$

$$z_a = z(x_a) \qquad \left| \ (x_a, z_a) = (d, 0) \right.$$

$$0 = h\cosh\left(\frac{d+c}{h}\right) - h\cosh\left(\frac{c-d}{h}\right) \qquad \left| \ d \overset{!}{\neq} 0, \ \cosh(-u) = \cosh(u) \right.$$

$$0 = h \cdot \cosh\left(\frac{d+c}{h}\right) - h\cosh\left(\frac{d-c}{h}\right)$$

$$\Rightarrow d + c = d - c$$

$$\Rightarrow c = 0$$

$$\Rightarrow z(x) = h\cosh\left(\frac{x}{h}\right) - h\cosh\left(\frac{d}{h}\right)$$

d)

$$L = \int_{x_p}^{x_a} \sqrt{1 + z'^2} \, dx = \int_{x_p}^{x_a} \sqrt{1 + \sinh^2\left(\frac{x}{h}\right)} \, dx = \int_{x_p}^{x_a} \cosh\frac{x}{h} \, dx = h\left[\sinh\frac{x}{h}\right]_{x_p}^{x_a} = h\left(\sinh\frac{x_a}{h} - \sinh\frac{x_p}{h}\right)$$