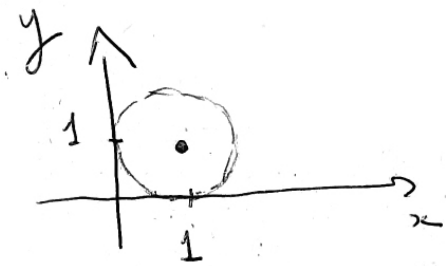
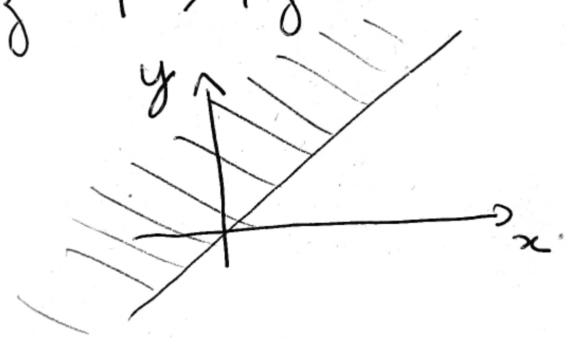
Präsenzaufgabenblatt 1

1) (i) $|z-1-i| = 1 \Leftrightarrow (x-1)^2 + (y-1)^2 = 1$



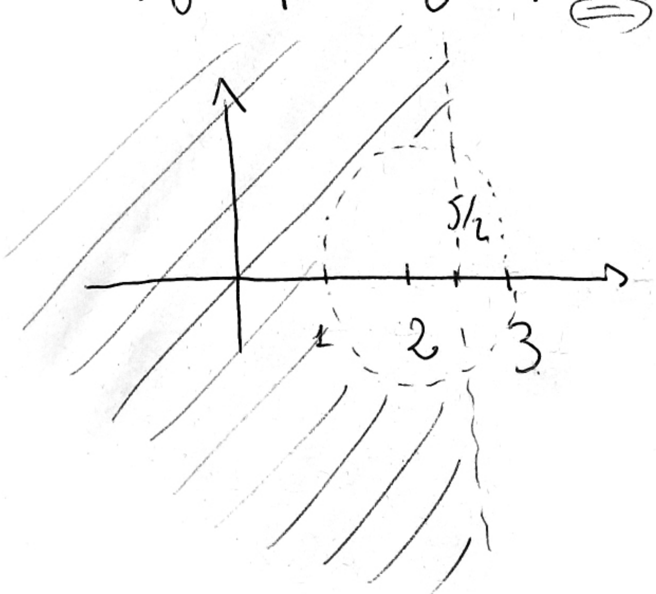
(ii) $|z-1| \geq |z-i| \Leftrightarrow y \geq x$



(iii) $|z-2|^2 = (x-2)^2 + y^2 \geq 1$

$|z-2| < |z-3| \Leftrightarrow (x-2)^2 + y^2 < (x-3)^2 + y^2$

$\Leftrightarrow x < \frac{5}{2}$

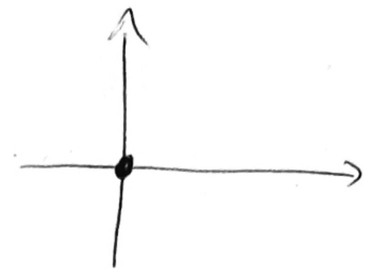


(iv) $|z-i|^2 + |z+i|^2 = 2$

$\Leftrightarrow x^2 + (y-1)^2 + x^2 + (y+1)^2 = 2$

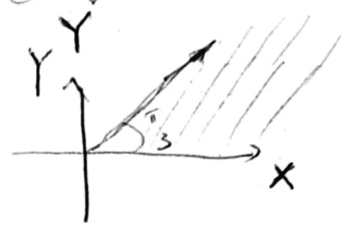
$\Leftrightarrow x^2 + y^2 = 0$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

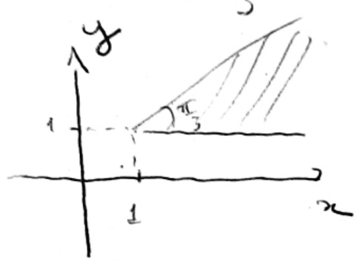


(v) $z - i - 1 = \underbrace{(x-1)}_{=X} + i\underbrace{(y-1)}_{Y}$

$0 < \arg(X + iY) < \frac{\pi}{3} \Leftrightarrow$



$\rightarrow$



2) (i) $e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

(ii) $\frac{1}{1+i} = \frac{(1-i)}{(1+i)(1-i)} = \frac{1}{2} - \frac{1}{2}i$

(iii) $\frac{1+i}{2+i} = \frac{(1+i)(2-i)}{(2+i)(2-i)} = \frac{1}{5}(3+i) = \frac{3}{5} + \frac{1}{5}i$

3) (i) $i = 0 + 1i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$

(ii) $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$

(iii) $1-2i = \sqrt{5}\left(\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}i\right) = \sqrt{5}\,e^{-i\arccos\left(\frac{1}{\sqrt{5}}\right)}$

(sehe $\arccos: [-1,1] \rightarrow [0,\pi]$)

4) (i) $z^m = 1 \Rightarrow |z^m| = 1 \Rightarrow |z|^m = 1$

$\Rightarrow |z| = 1$

Wir suchen $z = e^{i\theta}$

Wir nehmen $z_k = e^{i\frac{2\pi k}{n}}$ für $k \in \{0, \dots, m-1\}$

$\frac{2\pi k}{m} \in [0, 2\pi[$ und $z_k^m = e^{\frac{i 2\pi k m}{m}} = e^{i 2\pi k} = 1$

Wir haben die n-Lösungen gefunden. (Polynom von Grad n).

(ii) $z^2 - 2iz + 1 = 0$.

Wir definieren $u := iz \Rightarrow -u^2 + 2u^2 + 1 = 0$

(Klassik)

$\Rightarrow u_1 = (1+\sqrt{2})$ $\Rightarrow \begin{cases} z_1 = i(1+\sqrt{2}) \\ z_2 = i(1-\sqrt{2}) \end{cases}$

$u_2 = (1-\sqrt{2})$

5) (i) $f(z) = u(z) + i v(z)$ wobei $v(z) = 0$

$u(z) = |z|^2$

aber $\partial_x u(x,y) = \partial_x(x^2+y^2) = 2x \neq \partial_y v(x,y) = 0$

$\Rightarrow f$ nicht holomorph.

(ii) $\frac{f(z+h) - f(z)}{h} = \frac{hi}{h} = i$ $\Rightarrow$ holomorph.

(iii) $f(z) = u(z) + i v(z)$ wobei $v(z) = 0$

$u(z) = Re(z)$

Aber $\partial_x u(x,y) = \partial_x x = 1 \neq \partial_y v(x,y) = 0$

(iv) $\frac{f\binom{x+h_1}{y+h_2} - f(x,y)}{h_1 + ih_2} = e^{x+iy} \frac{\left(e^{h_1+ih_2} - 1\right)}{h_1+ih_2} = e^{x+iy} \left(\frac{\left[e^{h_1}[\cos(h_2) + i\sin(h_2)] - 1\right]}{h_1+ih_2}\right)$

Taylor

$= e^{x+iy} \frac{\left[(1+h_1+o(h_1))(1-o(h_2)) - 1 + i(1+h_1+o(h_1))(h_2+o(h_2))\right](h_1-ih_2)}{h_1^2+h_2^2} \xrightarrow{h \to 0} 1 \times e^{x+iy}$

(v) $f(x+h_1, x+h_2) - f(x,y) = 2xh_1 + h_1^2 - h_2^2 - 2h_2 y$
$+ 2x h_2 i + 2h_1 y i + 2h_1 h_2 i$

$$\Rightarrow \frac{f(x+h_1, x+h_2) - f(x,y)}{h_1 + ih_2} = \frac{(2xh_1 + h_1^2 - h_2^2 - 2h_2 y + 2xh_2 i + 2h_1 y i + 2h_1 h_2 i)(h_1 - ih_2)}{h_1^2 + h_2^2}$$

$$= \big(2xh_1^2 + h_1^3 - h_2^2 h_1 - 2\cancel{h_2 h_1 y} + 2\cancel{xh_2 h_1 i} + 2h_1^2 y i + 2h_1^2 h_2 i$$
$$- 2\cancel{xih_1 h_2} - ih_2 h_1^2 + ih_2^3 + 2ih_2^2 y + 2xh_2^2 + 2\cancel{h_1 h_2 y} + 2h_1 h_2^2\big) / (h_1^2 + h_2^2)$$

$$= \frac{2x(h_1^2 + h_2^2) + i2y(h_1^2 + h_2^2) + o(h_1^2 + h_2^2)}{h_1^2 + h_2^2} \longrightarrow 2x + i2y$$

6) i) $\partial_x u = 3x^2 - 3y^2$

$\partial_y u = -6yx$

Es muss gelten $\partial_y v = \partial_x u = 3x^2 - 3y^2$

$\partial_x v = -\partial_y u = 6xy$

$$\Rightarrow \begin{cases} v = 3x^2 y - y^3 + f(x) \\ v = 3x^2 y + g(y) \end{cases}$$

$$\Rightarrow v = 3x^2 y - y^3 + c$$

ii) ~~es~~ ist nicht möglich : Es soll gelten

$$\begin{cases} \partial_y v = \partial_x u = 2x \\ \partial_x v = -\partial_y u = 0 \Rightarrow v \text{ unabhängig von } x \end{cases}$$

aber $\partial_y v = 2x$! ⚡