AGT; Sheet 13; Marc Hanc, Angelo Brade; 19.01.2025

Quickies:

1.: $[a_i^\dagger; a_j] = \delta_{ij} = [b_i^\dagger; b_j]$

$[a_i^\dagger; b_j] = 0$

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2.: $\hat{a}_i^\dagger |n_1, \dots, n_i, \dots, n_N\rangle = \sqrt{n_i+1}\, |n_1, \dots, n_i+1, \dots, n_N\rangle$

$\hat{b}_i^\dagger |n_1, \dots, n_i, \dots, n_N\rangle = (1-n_i)(-1)^{\sum_{k<i} n_k} |n_1, \dots, n_i, \dots, n_N\rangle$

3.: $\hat{n}_i := \hat{a}_i \hat{a}_i^\dagger$, $\hat{n}_i := \hat{b}_i \hat{b}_i^\dagger$:

$\hat{n}_i |\dots, n_i, \dots\rangle = n_i |\dots, n_i, \dots\rangle$

Exercise 1:

1.: $$\frac{\partial}{\partial x'_\mu} \frac{\partial}{\partial x'^\mu} = \partial'^\mu \partial'_\mu$$
$$= \partial'^\mu g_{\mu\nu} \partial'^\nu$$
$$= \Lambda^\mu_\alpha \partial^\alpha g_{\mu\nu} \Lambda^\nu_\beta \partial^\beta$$
$$= \partial^\alpha \Lambda^\mu_\alpha g_{\mu\nu} \Lambda^\nu_\beta \partial^\beta$$
$$= \partial^\alpha g'_{\alpha\beta} \partial^\beta \quad | \; g'_{\alpha\beta} = g_{\alpha\beta}$$
$$= \partial^\alpha \partial_\alpha \quad | \text{ rename } \alpha \to \mu$$
$$= \partial^\mu \partial_\mu$$
$$= \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu}$$

$\square$

2.: $$\frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^2} = \partial_\mu \partial^\mu \quad | \text{ shown in } 1$$

$$= \partial'_\mu \partial'^\mu$$

$$= \frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^2}$$

$\Rightarrow$ invariant (per definition) $\square$

$$\frac{\partial^2}{\partial(ct)^2} + \frac{\partial^2}{\partial x^2} = \partial_\mu \partial_\mu$$

$$= \partial_\alpha \Lambda^\alpha_\mu g_{\mu\nu} \Lambda^\nu_\beta \partial^\beta \quad | \ \exists \Lambda : \Lambda^\alpha_\mu g_{\mu\nu} \Lambda^\nu_\beta \neq g'_{\alpha\beta} = g_{\alpha\beta}$$

$$\neq \partial'_\alpha g_{\alpha\beta} \partial'^\beta$$

$$= \partial'_\alpha \partial'_\alpha$$

$$= \partial'_\mu \partial'_\mu$$

$$\Rightarrow \frac{\partial^2}{\partial(ct)^2} + \frac{\partial^2}{\partial x^2} = \partial_\mu \partial_\mu \neq \partial'_\mu \partial'_\mu \quad \square$$

3.: $$\psi(x_\mu) = e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$$

$$= e^{-ik^\mu x_\mu}$$

Boost $\psi(x_\mu) \to \psi'(x_\mu)$:

$$\psi'(x_\mu) = e^{-ik'^\mu x'_\mu} \quad | \text{ Shown on last sheet: } k'^\mu x'_\mu = k^\mu x_\mu$$

$$= e^{-ik^\mu x_\mu}$$

$$= \psi(x_\mu)$$

$\Rightarrow$ invariant

$\square$

4.: $$\int_\Omega \psi(p)\, d\vec{p}^{\,3} = \int_a^b \psi(\gamma(t)) \, \|\dot{\gamma}(t)\|_2 \, dt$$

Exercise 2:

1.:

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \mathbb{1}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

2.:

$$[\sigma_l, \sigma_k] = \sigma_l \sigma_k - \sigma_k \sigma_l$$

$$\{\sigma_k, \sigma_l\} = 0 \Rightarrow \sigma_k \sigma_l = -\sigma_l \sigma_k$$

$$\Rightarrow \sigma_l \sigma_k - \sigma_k \sigma_l = \sigma_l \sigma_k + \sigma_l \sigma_k$$

$$\Rightarrow [\sigma_l, \sigma_k] = 2 \sigma_l \sigma_k$$

3.:

$$l = k: \ [\sigma_l, \sigma_k] = 0, \ \varepsilon_{lkj} = 0 \ \forall j \in \{1, 2, 3\}$$

$$l \neq k: \ [\sigma_l, \sigma_k] = 2i \sum_{j=1}^{3} \varepsilon_{lkj} \sigma_j \quad \forall l, k \in \{1, 2, 3\}$$

$$\left| \begin{array}{l} \text{if } l = j \Rightarrow \varepsilon_{lkj} = 0 \\ \text{if } k = j \Rightarrow \varepsilon_{lkj} = 0 \\ \text{if } l \neq j \neq k \Rightarrow \varepsilon_{lkj} = 1 \end{array} \right.$$

$$= 0 + 0 + 2i \varepsilon_{lkj} \sigma_j$$

$\Rightarrow$ Only one j contributes to the sum.

4.:

$$k = l: \ \sigma_k \sigma_l \delta_{kl} = \sigma_k^2 = \mathbb{1}$$

$$\delta_{kl} \mathbb{1} \delta_{kl} + i \sum_{j=1}^{3} \varepsilon_{kkj} \sigma_j \delta_{kl} = \mathbb{1} = i \sum_{j=1}^{3} \varepsilon_{kkj} \sigma_j = \mathbb{1}$$

$\square$

$$k \neq l: \ \sigma_k \sigma_l = i \sum_{j=1}^{3} \varepsilon_{klj} \sigma_j \qquad 1 - \varepsilon_{lkj} = \varepsilon_{klj}$$

$$= -\frac{1}{2}\left(2i \sum_{j=}^{3} \varepsilon_{lkj} \sigma_j\right) = -\frac{1}{2}[\sigma_l, \sigma_k] = -\frac{1}{2} 2 \sigma_l \sigma_k = \sigma_k \sigma_l$$

$\square$

5:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \left(\sum_{i=1}^{3} a_i \sigma_i\right)\cdot\left(\sum_{i=1}^{3} b_i \sigma_i\right)$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i \sigma_i b_j \sigma_j$$

$$= \sum_{i,j}^{3} a_i b_j \left(\delta_{ij} \mathbb{1} + i\sum_{k=1}^{3} \varepsilon_{ijk} \sigma_k\right)$$

$$= \vec{a}\cdot\vec{b}\,\mathbb{1} + \sum_{j,i}^{3} a_i b_j \, i \sum_{k=1}^{3} \varepsilon_{ijk} \sigma_k$$

$$= \vec{a}\cdot\vec{b}\,\mathbb{1} + i\sum_{k=1}^{3} \sigma_k \sum_{ij}^{3} \varepsilon_{ijk} a_i b_j \quad \Big| \quad \sum_{ij}^{3} \varepsilon_{ijk} a_i b_j = (\vec{a}\times\vec{b})_k$$

$$= \vec{a}\,\vec{b}\,\mathbb{1} + i\sum_{k=1}^{3} \sigma_k (\vec{a}\times\vec{b})_k$$

$$= \vec{a}\,\vec{b}\,\mathbb{1} + i\,\vec{\sigma}(\vec{a}\times\vec{b})$$