Angelo Brade, Marc Huber

1) Canonical transformation in Classical Trajectories

$$q_i \to \bar q_i = q_i + \delta q_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \quad ; \quad p_i \to \bar p_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \qquad \text{and} \quad g = g(q_i, p_i)$$

1.1) Show: $(q_i(t), p_i(t)) \wedge \{g, H\} = 0 \Rightarrow (\bar q_i(t), \bar p_i(t))$

· $\dot{\bar q}_i = \frac{\partial H}{\partial \bar p_i} = \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \bar p_i} + \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \bar p_i}$ | $p_i = \bar p_i + \varepsilon \frac{\partial g}{\partial q_i} \Rightarrow \frac{\partial p_i}{\partial \bar p_i} = 1 + \varepsilon \frac{\partial}{\partial \bar p_i}\frac{\partial g}{\partial q_i}$, $q_i = \bar q_i - \varepsilon \frac{\partial g}{\partial p_i} \Rightarrow \frac{\partial q_i}{\partial \bar p_i} = -\varepsilon \frac{\partial}{\partial \bar p_i}\frac{\partial g}{\partial p_i}$

$\Rightarrow \dot{\bar q}_i = \frac{\partial H}{\partial p_i}\left(1 + \varepsilon \frac{\partial}{\partial \bar p_i}\frac{\partial g}{\partial q_i}\right) + \frac{\partial H}{\partial q_i}\left(-\varepsilon \frac{\partial}{\partial \bar p_i}\frac{\partial g}{\partial p_i}\right)$

$= \frac{\partial H}{\partial p_i} + \varepsilon \frac{\partial}{\partial \bar p_i}\left(\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$ | $\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i} = \{g, H\} = 0$

$= \frac{\partial H}{\partial p_i} = \dot q_i$ $\qquad \dot{\bar q}_i = q_i = \frac{\partial H}{\partial p_i} \stackrel{?}{=} \frac{\partial H}{\partial \bar p_i}$ [yes, but not shown]

· $\dot{\bar p}_i = -\frac{\partial H}{\partial \bar q_i} = -\frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \bar q_i} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \bar q_i}$ | $\frac{\partial p_i}{\partial \bar q_i} = \varepsilon \frac{\partial}{\partial \bar q_i}\frac{\partial g}{\partial q_i}$, $\frac{\partial q_i}{\partial \bar q_i} = 1 - \varepsilon \frac{\partial}{\partial \bar q_i}\frac{\partial g}{\partial p_i}$

$\Rightarrow \dot{\bar p}_i = -\frac{\partial H}{\partial q_i}\left(1 - \varepsilon \frac{\partial}{\partial \bar q_i}\frac{\partial g}{\partial p_i}\right) - \frac{\partial H}{\partial p_i}\left(\varepsilon \frac{\partial}{\partial \bar q_i}\frac{\partial g}{\partial q_i}\right)$

$= -\frac{\partial H}{\partial q_i} + \varepsilon \frac{\partial}{\partial \bar q_i}\left(\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$ | $\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i} = \{H, g\} = -\{g, H\} = 0$

$= -\frac{\partial H}{\partial q_i} = \dot p_i$

↳ if $(q_i(t), p_i(t))$ satisfies the e.o.m. (⟺ valid trajectory), then also $(\dot{\bar q}(t), \dot{\bar p}(t))$ satisfies the e.o.m. □

1.2) $x_k \to \bar x_k = x_k + \delta$, $\delta \in \mathbb{R}$

$\Rightarrow g = p_k$, because $\bar x_k = x_k + \delta \frac{\partial p_k}{\partial p_k} = x_k + \delta$

· Hamilton function is invariant ⟺ $H(\bar x_i, p_i) = H(x_i, p_i)$ (*)

$\Rightarrow \dot{\bar q} = \frac{\partial H}{\partial \bar p_i} \stackrel{*}{=} \frac{\partial H}{\partial p_i} = q_i$

$\Rightarrow \dot{\bar p} = -\frac{\partial H}{\partial \bar q_i} \stackrel{*}{=} -\frac{\partial H}{\partial q_i} \neq p_i$

⇒ transformation generates valid trajectory $(q_i(t), p_i(t))$ □

$\dot{\bar p} = \{\bar p, H\} = \{g, H\} = 0$!

2) Canonical Transformation in Quantum Mechanics

$$\Psi(q_i, t) \to \bar\Psi(q_i, t) = \hat U_g(\xi)\,\Psi(q_i, t) \quad , \quad \hat U_g(\xi) = \exp\left(-\frac{i\xi}{\hbar}\hat g\right)$$

2.1) $\bar\Psi(q_i, t) = \hat U_g(\xi)\Psi(q_i, t)$

$\Rightarrow \sum_n \bar c_n(t)\Psi_n(q_i) = \hat U_g(\xi)\sum_n c_n(t)\Psi_n(q_i)$

$\Rightarrow \sum_n \bar c_n(t)\Psi_n(q_i) = \sum_n c_n(t)\cdot \exp\left(-\frac{i\xi}{\hbar}\hat g\right)\Psi_n(q_i) = \sum_n c_n \cdot \exp\left(-\frac{i\xi}{\hbar}g_n\right)\Psi_n(q_i)$

$\Rightarrow \bar c_n(t) = \exp\left(-\frac{i\xi}{\hbar}g_n\right)c_n(t)$

2.2) $\hat g = \hat L_z \to \hat U_{L_z} = \exp\left(-\frac{i\xi}{\hbar}\hat L_z\right)$

$\Rightarrow \bar\Psi(q_i, t) = \hat U_{L_z}\Psi(q_i, t) = \exp\left(-\frac{i\xi}{\hbar}\hat L_z\right)\sum_m c_m(t)\Psi_m(q_i)$ | $\hat L_z\Psi_m(q_i) = \hbar m\,\Psi_m(q_i)$

$= \sum_m c_m(t)\exp(-i\xi m)\Psi_m(q_i)$ | eigenfunctions $\Psi_m = Y_l^m(\theta, \phi) \propto \exp(im\phi)$
↳ sperical harmonics

$= \sum_m c_m(t)\, Y_l^m(\theta, \phi - \xi)$

↳ L_z generats rotation around z-axis [$\phi \to \phi - \xi$]

2.3) $\hat{g} = L^2 \rightarrow \hat{U}_{L^2} = \exp\left(-\frac{i\xi}{\hbar} L^2\right)$

$\Rightarrow \bar{\Psi}(q_i,t) = \hat{U}_{L^2}\Psi(q_i,t) = \exp\left(-\frac{i\xi}{\hbar}L^2\right)\Psi(q_i,t)$

i) $\Psi(q_i,t)$ is eigenfunction of $L^2 \wedge L_z$, with fixed l and m

$\hookrightarrow \Psi(q_i,t) = \Psi_{lm}$

$\Rightarrow \bar{\Psi}(q_i,t) = \exp(-i\hbar\xi\, l(l+1))\,\Psi_{lm} = \exp(-i\hbar\xi\, l(l+1))\,\Psi(q_i,t)$ → absolut phase has no significance → <u>not</u> a physical change !

ii) fixed l, different m

$\hookrightarrow \Psi(q_i,t) = \sum_m \Psi_{lm}$

$\Rightarrow \bar{\Psi}(q_i,t) = \exp\left(-\frac{i\xi}{\hbar}L^2\right)\sum_m \Psi_{lm} = \sum_m \exp(-i\hbar\xi\, l(l+1))\,\Psi_{lm} = \exp(-i\hbar\xi\, l(l+1))\sum_m \Psi_{lm} = \exp(-i\hbar\xi\, l(l+1))\,\Psi(q_i,t)$ → <u>not</u> a physical change !

iii) different l and m

$\hookrightarrow \Psi(q_i,t) = \sum_{l,m}\Psi_{lm}$

$\Rightarrow \bar{\Psi}(q_i,t) = \exp\left(-\frac{i\hbar}{\hbar}L^2\right)\sum_{l,m}\Psi_{lm} = \sum_{l,m}\exp(-i\hbar\xi\, l(l+1))\,\Psi_{lm}$ → different relativ phases → real physical change ! ✓

3) Gauge Invariance in classical Electrodynamics

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t) \quad ; \quad \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

1) $\vec{A}'(\vec{x},t) = \vec{A}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x},t)$, $U'(\vec{x},t) = U(\vec{x},t) - \frac{\partial\lambda(\vec{x},t)}{\partial t}$

Show: $\vec{B}'(\vec{x},t) = \vec{B}(x,t) \wedge \vec{E}'(\vec{x},t) = \vec{E}(x,t)$ (⇔ gauge transf. leaves $\vec{B}$ & $\vec{E}$ unchanged)

· $\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(A+\vec{\nabla}\lambda) = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times\vec{\nabla}\lambda}$ $\quad | \ \vec{\nabla}\times\vec{\nabla}\lambda = 0$

$\Rightarrow \vec{B}' = \vec{\nabla}\times\vec{A} = \vec{B}$

· $\vec{E}' = -\vec{\nabla}U' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\left(U - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial(A+\vec{\nabla}\lambda)}{\partial t} = -\vec{\nabla}U + \vec{\nabla}\frac{\partial}{\partial t}\lambda - \frac{\partial}{\partial t}\vec{\nabla}\lambda - \frac{\partial A}{\partial t}$ $\quad | \ \vec{\nabla}\frac{\partial}{\partial t}\lambda = \frac{\partial}{\partial t}\vec{\nabla}\lambda$

$\Rightarrow \vec{E}' = -\vec{\nabla}U - \frac{\partial A}{\partial t} = \vec{E}$ ✓ □

2) Show: $\vec{\nabla}\cdot\vec{B}(\vec{x},t) = 0 \wedge \vec{\nabla}\times\vec{E}(\vec{x},t) = -\frac{\partial B(\vec{x},t)}{\partial t}$

· $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \partial_i \varepsilon_{ijk}\partial_j a_k = \varepsilon_{ijk}\partial_i\partial_j a_k = \varepsilon_{ijk}\partial_j\partial_i a_k = -\partial_j\varepsilon_{jik}\partial_i a_k = -\partial_i\varepsilon_{ijk}\partial_j a_k = 0$ ✓

· $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\cancel{\vec{\nabla}}U - \frac{\partial A}{\partial t}\right) = -\vec{\nabla}\times\frac{\partial A}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial B}{\partial t}$ □

3) $\vec{\nabla}\cdot\vec{A}(\vec{x},t) = -\mu_0\varepsilon_0\frac{\partial U(\vec{x},t)}{\partial t}$

· $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \vec{\nabla}\cdot\left(-\vec{\nabla}U - \frac{\partial\vec{A}}{\partial t}\right) = -\Delta U - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}$ to decouple $\vec{A}$ & U

· $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}U - \frac{\partial\vec{A}}{\partial t}\right)$

$= \mu_0\vec{j} + \vec{\nabla}\left(-\mu_0\varepsilon_0\frac{\partial U}{\partial t}\right) - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}$

$= \mu_0\vec{j} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}$ ✓

4) $L = \int d^3x \left[\frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} - \frac{1}{2\mu_0}\vec{B}\cdot\vec{B} - \rho U + \vec{j}\cdot\vec{A}\right]$

· $\left[\frac{\varepsilon_0}{2}\vec{E}'\cdot\vec{E}' - \frac{1}{2\mu_0}\vec{B}'\cdot\vec{B}' - \rho U' + \vec{j}\cdot\vec{A}'\right] = \frac{\varepsilon_0}{2}\left(-\vec{\nabla}U' - \frac{\partial A'}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\vec{\nabla}\times A')^2 - \rho U' + \vec{j}\cdot\vec{A}'$

$= \frac{\varepsilon_0}{2}\left(-\vec{\nabla}\left(U - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial(A+\vec{\nabla}\lambda)}{\partial t}\right)^2 - \frac{1}{2\mu_0}\left(\vec{\nabla}\times(\vec{A}+\vec{\nabla}\lambda)\right)^2 - \rho\left(U - \frac{\partial\lambda}{\partial t}\right) + \vec{j}\cdot(\vec{A}+\vec{\nabla}\lambda)$

$= \frac{\varepsilon_0}{2}\left(-\vec{\nabla}U - \frac{\partial A}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\vec{\nabla}\times\vec{A})^2 - \rho\left(U - \frac{\partial\lambda}{\partial t}\right) + \vec{j}\cdot(\vec{A}+\vec{\nabla}\lambda)$

$= \left[\frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B} - \rho U + \vec{j}A\right] + \underbrace{\rho\frac{\partial\lambda}{\partial t} + \vec{j}(\vec{\nabla}\lambda)}_{*}$ → Integrand not invariant !

*) $\rho\frac{\partial\lambda}{\partial t} + j(\vec{\nabla}\cdot\lambda) = \rho\frac{\partial\lambda}{\partial t} + \vec{\nabla}\cdot(\lambda\vec{j}) - \lambda(\vec{\nabla}\cdot\vec{j})$ $\quad | \ \vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

$= \rho\frac{\partial\lambda}{\partial t} + \lambda\frac{\partial\rho}{\partial t} + \vec{\nabla}(\lambda\vec{j}) = \frac{d}{dt}(\lambda\rho) + \vec{\nabla}(\lambda\vec{j})$

$\Rightarrow \int dx^3\left(\frac{d}{dt}(\lambda\rho) + \vec{\nabla}(\lambda\vec{j})\right) = 0 \Rightarrow L$ is invariant !

good!