Blatt 5) Aufgabe 1)

(i) $z^{a+b} \overset{\text{Def.}}{=} e^{(a+b)\ln(z)} = e^{a\ln(z)+b\ln(z)}$

$= e^{a\ln(z)} \cdot e^{b\ln(z)} \overset{\text{Def.}}{=} z^a z^b$

siehe ($e^{z+w} = e^z e^w \ \forall z,w \in \mathbb{C}$)

$\Rightarrow z^{a+b} = z^a z^b \quad \forall z, a, b \in \mathbb{C}.$

(ii) Falls $a \in \mathbb{Z} \cap \mathbb{N}$, $(z_1 z_2)^a = z_1^a z_2^a$

Falls $a \in \mathbb{Z}^-$ und $z_1, z_2 \neq 0$, $\left(\frac{1}{z_1}\frac{1}{z_2}\right)^{-a} = \left(\frac{1}{z_1}\right)^{-a}\left(\frac{1}{z_2}\right)^{-a} = z_1^a z_2^a$

(iii) Nein, wir wählen den Zweig $\arg(z) \in (0, 2\pi)$ und

$z_1 = e^{i\pi}$, $z_2 = e^{i\frac{3\pi}{2}}$ und $a = i$,

$(z_1 z_2)^i = e^{i\ln(z_1 z_2)} = e^{i\ln(e^{i\frac{5\pi}{2}})} = e^{i\cdot i\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$

$z_1^i z_2^i = e^{i\ln(z_1)} e^{i\ln(z_2)} = e^{i[\ln(z_1)+\ln(z_2)]}$

$= e^{i[i\pi + i\frac{3\pi}{2}]} = e^{-\frac{5\pi}{2}}$

Aufgabe 2) Wir wählen den Zweig $\arg(z) \in (0, 2\pi)$

(i) $\ln(1+i) = \ln\left(\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right) = \ln\left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)$

$= \ln(\sqrt{2}) + i\frac{\pi}{4}$

(ii) $\ln(-e^{10}) = \ln(e^{i\pi} e^{10}) = \ln(e^{10}) + i\pi = 10 + i\pi$

(iii) $(1+i)^{(1+i)} = e^{(1+i)\ln(1+i)} \overset{(i)}{=} e^{(1+i)[\ln(\sqrt{2})+i\frac{\pi}{4}]}$

$= e^{\ln(\sqrt{2})} e^{i\frac{\pi}{4}} e^{i\ln(\sqrt{2})} e^{-\frac{\pi}{4}}$

$= \sqrt{2} e^{-\pi/4} e^{i\left(\ln(\sqrt{2})+\frac{\pi}{4}\right)}$

Aufgabe 3)

(i) $\partial M_1 = \overline{M_1} \setminus \mathring{M}_1 = \overline{M_1} = M_1$

(ii) Wir haben $\overline{M_2} = \{z \in \mathbb{C} : |z| \in [1,2]\}$ und

$\mathring{M}_2 = M_2$

und somit $\partial M_2 = \overline{M_2} \setminus \mathring{M}_2 = \{z \in \mathbb{C} : |z| \in \{1,2\}\}$.

(iii) Wir haben $\overline{M_3} = \{z \in \mathbb{C} : Re(z) \geq 0\} \dot{\cup} \{\infty\}$ und

somit ist $\partial M_3 = \overline{M_3} \setminus \underset{M_3}{\underset{\parallel}{\mathring{M}_3}} = \{z \in \mathbb{C}, Re(z) = 0\} \cup \{\infty\}$

(iv) $z = x + iy$ und $Re(e^{x+iy}) = e^x \cos(y)$

$\Rightarrow Re(e^z) > 0 \Leftrightarrow \cos(y) > 0 \Leftrightarrow y \in \bigcup_{k \in \mathbb{Z}}]-\frac{\pi}{2}+2\pi k, \frac{\pi}{2}+2\pi k[$

$\Rightarrow \partial M_4 = \overline{M_4} \setminus \mathring{M}_4 = \bigcup_{k \in \mathbb{Z}} \{z \in \mathbb{C}, Im(z) = \frac{\pi}{2} + k\pi\} \cup \{\infty\}$

Aufgabe 4)

$$\int_\gamma \bar{z}\,dz = \int_{[z_1, z_2]} \bar{z}\,dz = \int_0^1 \overline{((1-t)z_1 + t z_2)}\,(z_2 - z_1)\,dt$$

$$= \int_0^1 \left[(1-t)\overline{z_1} + t\overline{z_2}\right](z_2 - z_1)\,dt$$

$$= \underbrace{\left[\int_0^1 (1-t)\,dt\right]}_{=\frac{1}{2}} \overline{z_1}(z_2 - z_1) + \underbrace{\int_0^1 t\,dt}_{=\frac{1}{2}} \overline{z_2}(z_2 - z_1)$$

$$= \frac{1}{2}(z_2 - z_1)\left(\overline{z_1 + z_2}\right).$$

(i) $z_1 = 0,\ z_2 = i \Rightarrow \int_{[0,i]} \bar{z}\,dz = \frac{1}{2}\, i\,(-i) = \frac{1}{2}$

$z_1 = i,\ z_2 = 1+i \Rightarrow \int_{[i,1+i]} \bar{z}\,dz = \frac{1}{2}\,(1)(-i + 1 - i) = \frac{1}{2}(1 - 2i)$

$\Rightarrow \int_\gamma \bar{z}\,dz = 1 - i$

(ii) $z_1 = 0,\ z_2 = 1 \Rightarrow \int_{[0,1]} \bar{z}\,dz = \frac{1}{2}$

$z_1 = 1,\ z_2 = 1+i \Rightarrow \int_{[1,1+i]} \bar{z}\,dz = \frac{1}{2}\, i\,(1 + 1 - i) = \frac{1}{2}(2i + 1)$

$\Rightarrow \int_\gamma \bar{z}\,dz = 1 + i$

(iii) $$\int_{\gamma_R} z^n \bar{z}^m\,dz = \int_0^{2\pi} R^{n+m+1}\, e^{i(n-m)t}\, i e^{it}\,dt = R^{n+m+1} \int_0^{2\pi} e^{i(n-m+1)t}\, i\,dt$$

$\left(\int e^{it} = \int \cos(t) + i \int \sin t = -ie^{it}\right)$

$$= \begin{cases} 0 & \text{falls } n+1 \neq m \\ R^{\cancel{2n+1}}\, 2\pi & n+1 = m \end{cases}$$

(iv) $$\int_{\gamma_R} Re(z)\,dz = \int_0^{2\pi} R^2 \cos(t)\, i e^{it}\, dt$$

$$= R^2 \int_0^{2\pi} \left[i\cos(t)^2 - \sin(t)\cos(t) \right] dt$$

$$= R^2 i \int_0^{2\pi} \cos(t)^2\, dt - R^2 \int_0^{2\pi} \sin(t)\cos(t)\, dt$$

and $$\int_0^{2\pi} \cos(t)^2\, dt = \int_0^{2\pi} \frac{\cos(2t)+1}{2}\, dt = \int_0^{2\pi} \cos(2t)\, dt + \pi = \underbrace{\left[\frac{\sin(2t)}{2}\right]_0^{2\pi}}_{=0} + \pi$$

and $$\int_0^{2\pi} \sin(t)\cos(t)\, dt = \int_0^{\pi} \sin(t)\cos(t)\, dt + \int_{\pi}^{2\pi} \sin(t)\cos(t)\, dt$$

$$= \int_0^{\pi} \sin(t)\cos(t)\, dt + \int_0^{\pi} \underbrace{\sin(u+\pi)}_{=\sin(u)} \underbrace{\cos(u+\pi)}_{=-\cos(u)}\, du = 0$$

$$\Rightarrow \int_{\gamma_R} Re(z)\,dz = R^2 i\pi$$