AGT: Sheet 3; Marc Hauer, Angelo Brade; 22.10.2024

Exercise 1:

1.:

From Lagrange we know:

$$\vec{P} = \frac{\partial L}{\partial \dot{x}}$$

$$= \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m (\dot{\vec{x}})^2 - q(V - \dot{\vec{x}}\cdot\vec{A})\right)$$

$$= m\dot{\vec{x}} + q\vec{A}$$

So for the linear momenta $\vec{p} = m\dot{\vec{x}} \Rightarrow \vec{P} = \vec{p} + q\vec{A}$.

2.:

With the Legendre-Transformation of Lagrangean L we get:

$$H(q,p) = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$= \dot{\vec{x}}\cdot m\dot{\vec{x}} + \vec{A}\cdot\dot{\vec{x}} - \frac{1}{2}m\dot{\vec{x}}^2 + q(V - \dot{\vec{x}}\cdot\vec{A})$$

$$= \frac{1}{2}m\dot{\vec{x}}^2 + qV \quad \Big| \ \vec{p}^{\,2} = m^2\dot{\vec{x}}^2$$

$$= \frac{\vec{p}^{\,2}}{2m} + qV \quad \Big| \ \vec{P} = \vec{p} + q\cdot\vec{A} \Leftrightarrow \vec{p}^{\,2} = (\vec{P} - q\cdot\vec{A})^2$$

$$= \frac{1}{2m}(\vec{P} - q\cdot\vec{A})^2 + qV$$

Yes, this is equal to the total energie:

$T = \frac{\vec{p}^{\,2}}{2m}$ and $U = q\cdot V$, thus $E_{tot} = T + U = \frac{\vec{p}^{\,2}}{2m} + q\cdot V$.

3.:

$$\dot{\vec{p}} = \{\vec{p}, H\}$$

$$= \frac{\partial \vec{P}}{\partial \vec{x}}\frac{\partial H}{\partial \vec{p}} - \frac{\partial \vec{P}}{\partial \vec{p}}\frac{\partial H}{\partial \vec{x}}$$

$$= q\cdot\vec{\nabla}\cdot\vec{A}\cdot\frac{\vec{p}}{m} - q\vec{\nabla}\cdot V$$

$$= q\,(\vec{\nabla}\cdot\vec{A})\cdot\dot{\vec{x}} - q\vec{\nabla}\cdot V$$

$$= q\Big[\vec{\nabla}(\vec{A}\cdot\dot{\vec{x}}) - \vec{A}\cdot\underbrace{(\vec{\nabla}\cdot\dot{\vec{x}})}_{0} - \vec{\nabla}\cdot V\Big] \quad |(I)$$

$$= q\Big[-\vec{\nabla}\cdot V - \frac{\partial A}{\partial t} + \dot{\vec{x}}\times\vec{\nabla}\times\vec{A}\Big] \quad \square$$

This e.o.m is the Lorenz force.

NR:

$$\dot{\vec{A}} = \frac{d}{dt}\vec{A}(t,x) \quad \Big| \ \vec{A} = \vec{A}(t,\vec{x})$$

$$= \frac{\partial \vec{A}}{\partial t} + \frac{\partial A_i}{\partial x_i}\frac{d\vec{x}}{dt}$$

$$= \frac{\partial \vec{A}}{\partial t} + (\vec{\nabla}\cdot\vec{A})\cdot\dot{\vec{x}} \longrightarrow \nabla_i A_j x_i = x_i \nabla_i A_j = (\dot{\vec{x}}\cdot\vec{\nabla})\vec{A}$$

$$= \ \vdots \ ?$$

$$= \frac{\partial \vec{A}}{\partial t} + (\dot{\vec{x}}\cdot\vec{\nabla})\cdot\vec{A}$$

$$\dot{\vec{A}} = 0 \Rightarrow (\dot{\vec{x}}\cdot\vec{\nabla})\cdot\vec{A} = -\frac{\partial A}{\partial t}$$

With $\dot{\vec{x}}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\dot{\vec{x}}\times\vec{A}) - (\dot{\vec{x}}\cdot\vec{\nabla})\cdot\vec{A}$:

$$I: \Rightarrow \vec{\nabla}(\dot{\vec{x}}\cdot\vec{A}) = \dot{\vec{x}}\times\vec{\nabla}\times\vec{A} - \frac{\partial A}{\partial t}$$

4.: $\hat{\vec{P}} = -i\hbar\vec{\nabla}$ follows from translation symmetries, ~~so a trafo. leave H invariant:~~
Sheet 2

~~$q = \hat{x} \Rightarrow \{\hat{x}, H\} = 0$~~

actually no! we don't have translational symmetry for nonzero external $\vec{A}$ field!
That is why $\vec{p}$ isn't a good operator but $\vec{P} = p + qA$ is.
You can check $\{P_i, x_j\} = \delta_{ij}$ and $\{p_i, x_j\} \neq \delta_{ij}$

Exercise 2:

2.) $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \quad | \rho(\vec{x}, t) = q|\psi(\vec{x}, t)|^2,$

$$\vec{j}(\vec{x}, t) = \frac{q}{2m}[\psi^*(\vec{x}, t)(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x}, t))\psi(\vec{x}, t) + h.c.]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(q|\psi(\vec{x}, t)|^2)$$

$$= q\left(\frac{\partial \psi^*(\vec{x}, t)}{\partial t}\psi(\vec{x}, t) + \psi^*(\vec{x}, t)\frac{\partial \psi(\vec{x}, t)}{\partial t}\right)$$

With $i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = \hat{H}\psi(\vec{x}, t) = \left(\frac{1}{2m}(\vec{P} - q\vec{A})^2 + qV\right)\psi(\vec{x}, t)$ follows:

$$\frac{\partial \rho}{\partial t}$$

$$= q\left[\left(\left(\frac{1}{2m}(\vec{P} - q\vec{A})^2 + qV\right)\frac{1}{i\hbar}\right)^* \psi^*(\vec{x}, t)\psi(\vec{x}, t) + \psi^*(\vec{x}, t)\left(\frac{1}{2m}(\vec{P} - q\vec{A})^2 + qV\right)\frac{1}{i\hbar}\psi(\vec{x}, t)\right]$$

$$= q\left[\psi^*(\vec{x}, t)\left(\frac{1}{2m}(\vec{P} - q\vec{A})^2 + qV\right)\frac{1}{i\hbar}\psi(\vec{x}, t) + h.c.\right]$$

$$= q\left[\psi^*(\vec{x}, t)\frac{1}{2m}(\vec{P} - q\vec{A})^2\frac{1}{i\hbar}\psi(\vec{x}, t) + h.c. + \underbrace{\psi^*(\vec{x}, t)qV\frac{1}{i\hbar}\psi(\vec{x}, t) + h.c.}_{[V, \psi(\vec{x}, t)] = 0 \Rightarrow \to 0}\right]$$

$$= q\left[\psi^*(\vec{x}, t)\left(\frac{1}{2m}(\vec{P} - q\vec{A})^2 + qV\right)\frac{1}{i\hbar}\psi(\vec{x}, t) + h.c.\right]$$

$\vec{\nabla} \cdot \vec{j} = \dots$

3.) $\rho(\vec{x}, t) = q|\psi(\vec{x}, t)|^2 \quad | \psi(\vec{x}, t) \to e^{\frac{i}{\hbar}q\lambda(\vec{x}, t)}\psi(\vec{x}, t)$

$$= q\underbrace{|e^{\frac{i}{\hbar}q\lambda(\vec{x}, t)}|^2}_{\to 1}|\psi(\vec{x}, t)|^2$$

$$= q|\psi(\vec{x}, t)|^2$$

$\Rightarrow \rho(\vec{x}, t)$ is invariant. ✓

$\square$

$$j(\vec{x},t) = \frac{q}{2m}\left[\psi^*(\vec{x},t)\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x},t)\right)\psi(\vec{x},t) + h.c.\right] \Bigg| \begin{array}{l} \vec{A}(\vec{x},t) \to \vec{A}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x},t), \\ \psi(\vec{x},t) \to e^{i\frac{q}{\hbar}\lambda(\vec{x},t)}\psi(\vec{x},t) \end{array}$$

$$= \frac{q}{2m}\left[\psi^*(\vec{x},t)\underbrace{e^{i\frac{q}{\hbar}\lambda(\vec{x},t)+h.c.}}_{\to 1}\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x},t) - \vec{\nabla}\lambda(\vec{x},t)\right)\psi(\vec{x},t) + h.c.\right]$$

$$= \frac{q}{2m}\left[\psi^*(\vec{x},t)\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x},t)\right)\psi(\vec{x},t) + h.c. - \left(\psi^*(\vec{x},t)\vec{\nabla}\lambda(\vec{x},t)\psi(\vec{x},t) + h.c.\right)\right]$$

$$= \frac{q}{2m}\left[\psi^*(\vec{x},t)\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x},t)\right)\psi(\vec{x},t) + h.c. - \left(\langle\psi|x,t\rangle\vec{\nabla}\lambda(\vec{x},t)\langle x,t|\psi\rangle + h.c.\right)\right]$$

$$= \frac{q}{2m}\Big[$$

I: $\psi^*(\vec{x},t)\vec{\nabla}\lambda(\vec{x},t)\psi(\vec{x},t) + h.c. = \langle\psi|x,t\rangle\vec{\nabla}\lambda(\vec{x},t)\langle x,t|\psi\rangle + h.c.$

$$= \langle\psi|x,t\rangle\vec{\nabla}\lambda(\vec{x},t)\langle x,t|\psi\rangle + \left(\langle\psi|x,t\rangle\vec{\nabla}\lambda(\vec{x},t)\langle x,t|\psi\rangle\right)^\dagger$$

$$= \langle\psi|x,t\rangle\vec{\nabla}\lambda(\vec{x},t)\langle x,t|\psi\rangle + ($$

3) Some Gaussian Integrals

1) Show: $I_1(a) = \int_0^\infty dx\, x e^{-ax^2} = \frac{1}{2a}$, $a \in \mathbb{C}: Re(a) \geq 0$

$\cdot\; I_1(a) = \int_0^\infty dx\, x e^{-ax^2} \quad | \; u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$= \frac{1}{2} \int_0^\infty du\, e^{-au}$

$= \frac{1}{2a} \left[-e^{-au} \right]_0^\infty \quad | \; Re(a) > 0$

$= \frac{1}{2a} \quad \square$

$\cdot$ If $Re(a) < 0 \Rightarrow Re(a') > 0$ with $a' = -a^*$:

$\Rightarrow \left[-e^{-au} \right]_0^\infty = \left[-e^{a'u} \right]_0^\infty \to -\infty$

2) $I_0(a) = \int_{-\infty}^\infty dx\, e^{-ax^2}$

$\Rightarrow [I_0(a)]^2 = \int_{-\infty}^\infty dx\, e^{-ax^2} \int_{-\infty}^\infty dy\, e^{-ay^2} = \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy\, e^{-a(x^2+y^2)} \quad |$ polar coordinates: $x = r\cos\phi$, $y = r\sin\phi \Rightarrow x^2 + y^2 = r^2$

$\Rightarrow [I_0(a)]^2 = \int_0^\infty dr \int_0^{2\pi} d\phi\, r e^{-ar^2}$

$= 2\pi \int_0^\infty dr\, r e^{-ar^2} \quad | \; \int_0^\infty dr\, r e^{-ra^2} = \frac{1}{2a} \quad$ (s.a)

$= \frac{\pi}{a}$

$\Rightarrow I_0(a) = \int_{-\infty}^\infty dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

3) $I_2(a) = \int_{-\infty}^\infty dx\, x^2 e^{-ax^2} = \int_{-\infty}^\infty dx\, \frac{\partial}{\partial a}\left(-e^{-ax^2}\right) = \frac{\partial}{\partial a}\left(-\int_{-\infty}^\infty dx\, e^{-ax^2}\right) = \underline{\frac{\partial}{\partial a}\left(-\sqrt{\frac{\pi}{a}}\right)} = -\frac{1}{2}\sqrt{\frac{a}{\pi}}$ $\quad + \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$

4) Show: $I_0(a,b) = \int_{-\infty}^\infty dx\, e^{-ax^2+bx} = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}$

$\cdot\; I_0(a,b) = \int_{-\infty}^\infty dx\, e^{-ax^2+bx} = \int_{-\infty}^\infty dx\, e^{-(ax^2 - bx + \frac{b^2}{4a}) + \frac{b^2}{4a}} = e^{\frac{b^2}{4a}} \int_{-\infty}^\infty dx\, e^{-(\sqrt{a}x - \frac{b}{2\sqrt{a}})^2} = e^{\frac{b^2}{4a}} \int_{-\infty}^\infty dx\, e^{-a(x - \frac{b}{2a})^2} \quad | \; u = x - \frac{b}{2a} \Rightarrow \frac{du}{dx} = 1$

$\Rightarrow I_0(a,b) = e^{\frac{4b}{a}} \int_{-\infty}^\infty du\, e^{-au^2} = e^{\frac{b^2}{4a}} \cdot \sqrt{\frac{\pi}{a}} \quad \square$

2) Charge Conservation

$\cdot$ continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \qquad (*)$

2) Show: $\rho(\vec{x},t) = q\,|\Psi(\vec{x},t)|^2 \;\wedge\; \vec{j}(\vec{x},t) = \frac{q}{2m}\left[\Psi^*(\vec{x},t)\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{x},t)\right)\Psi(\vec{x},t) + h.c.\right]$ satisfy $*$.

$\cdot\; \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = \frac{\partial}{\partial t}\left[q\,|\Psi(\vec{x},t)|^2\right] + \vec{\nabla}\cdot\left\{\frac{q}{2m}\left[\Psi^*(-i\hbar\vec{\nabla} - q\vec{A})\Psi + h.c.\right]\right\}$

$= \frac{\partial}{\partial t}(\Psi^* \Psi) + \frac{1}{2m}\vec{\nabla}\cdot\left[\Psi^*(-i\hbar\vec{\nabla} - q\vec{A})\Psi + h.c.\right] \qquad | \; i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t) = \left(\frac{1}{2m}(-i\hbar\vec{\nabla} - q\vec{A})^2 + q\Phi\right)\Psi(\vec{x},t) \qquad (+ V\Psi(x,t))$?

$= \left[\Psi^* \frac{1}{i\hbar}\left(\frac{1}{2m}(-\hbar^2\vec{\nabla}^2 + q^2\vec{A}^2 + 2iq\hbar\,\vec{\nabla}\cdot\vec{A}) + q\Phi\right)\Psi + h.c.\right] + \frac{1}{2m}\vec{\nabla}\cdot\left[\Psi^*(-i\hbar\vec{\nabla} - q\vec{A})\Psi + h.c.\right]$

$= \left[\Psi^*\left(i\hbar\vec{\nabla}^2 + \underbrace{\frac{q^2\vec{A}^2}{i\hbar}} + 2q\,\vec{\nabla}\cdot\vec{A} + \underbrace{\frac{2m}{i\hbar}q\Phi}\right)\Psi + h.c.\right] - \left[\Psi^*(i\hbar\vec{\nabla}^2 + q\,\vec{\nabla}\cdot\vec{A})\Psi + h.c.\right]$

these two pieces are $\propto -i\,$(real constant)$\,|\Psi|^2$

so is purely imaginary. So adding h.c. these vanish.