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Exircise 1:

1.: With $\langle \psi_1 | \hat{Q} \psi_2 \rangle = \int dx\, \psi_1^*(x) \hat{Q} \psi_2(x)$ and $\langle \psi_1 \hat{Q}^\dagger | \psi_2 \rangle = \int dx\, \psi_1^*(x) \hat{Q}^\dagger \psi_2(x)$

$\hat{Q} | \psi_n \rangle = q_n | \psi_n \rangle \wedge \hat{Q}^\dagger = \hat{Q}$

$\Rightarrow \langle \psi_1 | \hat{Q} | \psi_1 \rangle = q_1 \langle \psi_1 | \psi_1 \rangle = \langle \psi_1 \hat{Q} | \psi_1 \rangle = \bar{q}_1 \langle \psi_1 | \psi_1 \rangle$

$\Rightarrow q_1 = \bar{q}_1 \Rightarrow \operatorname{Im}(q_1) = 0 \Rightarrow q_1 \in \mathbb{R}$ $\square$

2.:

$q_n \langle \psi_{n'} | \psi_n \rangle = \langle \psi_{n'} | \hat{Q} | \psi_n \rangle = q_{n'} \langle \psi_{n'} | \psi_n \rangle$

$\Rightarrow \frac{q_n}{q_{n'}} \langle \psi_{n'} | \psi_n \rangle = \langle \psi_{n'} | \psi_n \rangle \quad | \; q_{n'} \neq q_n \Rightarrow \frac{q_n}{q_{n'}} \neq 1$

If $\langle \psi_{n'} | \psi_n \rangle \neq 0$:

I: $\Rightarrow \frac{q_n}{q_{n'}} = 1$ $\lightning$

$\Rightarrow \langle \psi_{n'} | \psi_n \rangle = 0$ $\square$

Proof does not work for $q_n = q_{n'}$ since then $\frac{q_n}{q_{n'}} = 1$, so there is no contradiction with (I).

3.: Be $(\phi_n)_n : n \in \mathbb{N}$ a basis in L^2 and $\{|i\rangle, |j\rangle\} \subseteq (\phi_n)_n$.

$\Rightarrow (\hat{Q}^\dagger)_{ij} = \langle i | \hat{Q}^\dagger | j \rangle = \langle i \hat{Q} | j \rangle = \langle j | \hat{Q} i \rangle^* = \langle j | \hat{Q} | i \rangle^* = (\hat{Q})_{ji}^*$ $\square$

Exercise 2:

1.:

$$\int dx\, \psi_n^*(x)\, \psi(x,t) = \int dx\, \psi_n^*(x) \sum_{n'} a_{n'}(t)\, \psi_{n'}(x) = \sum_{n'} a_{n'}(t) \int dx\, \psi_n^*(x)\, \psi_{n'}(x)$$

$$= \sum_{n'} a_{n'}(t)\, \delta_{nn'}$$

$$= a_n(t)$$

$\square$

2.:

$$1 = \int dx\, |\psi(x,t)|^2 = \int dx\, \psi^*(x,t)\, \psi(x,t) = \int dx \sum_{n'} a_{n'}^*(t)\, \psi_{n'}(x) \sum_n a_n(t)\, \psi_n(x)$$

$$= \sum_n \sum_{n'} a_{n'}^*(t)\, a_n(t) \int dx\, \psi_{n'}^*(x)\, \psi_n(x)$$

$$= \sum_{n,n'} a_{n'}^*(t)\, a_n(t)\, \delta_{nn'}$$

$$= \sum_n |a_n(t)|^2 = 1$$

$\square$

3.:

$$\langle \hat{Q} \rangle = \int dx\, \psi^*(x,t)\, \hat{Q}\, \psi(x,t)$$

$$= \int dx \sum_{n'} a_{n'}^*(t)\, \psi_{n'}^*(x) \sum_n q_n\, a_n(t)\, \psi_n(x)$$

$$= \sum_{n,n'} q_n \int dx\, a_{n'}^*(t)\, \psi_{n'}^*(x)\, a_n(t)\, \psi_n(x) \quad | \text{ This was already shown}$$

$$= \sum q_n |a_n(t)|^2$$

$\square$

Exercise 3:

1.:

$$\hat{L}_z \psi_m(\phi) = -i\hbar \frac{\partial}{\partial \phi} \psi_m(\phi) = \lambda_m \psi_m(\phi)$$

$$\Rightarrow \int d(\psi_m(\phi)) \frac{1}{\psi_m(\phi)} = \int d\phi \frac{i\lambda}{\hbar}$$

$$\Rightarrow \psi_m(\phi) = A e^{\frac{i}{\hbar}\lambda\phi} \quad \Big| \; m := \frac{\lambda_m}{\hbar}, 1 = \|\psi_m(\phi)\|^2 = \int_0^{2\pi} d\phi \psi_m^*(\phi) \psi_m(\phi) = \int_0^{2\pi} d\phi |A|^2 = 2\pi |A|^2$$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}} e^{i\xi} \quad \Big| \text{ Phase } e^{i\xi}\text{: wähle } \xi = 0$$

$$\Rightarrow \psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

$\square$

2.: $\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$

$$\psi_m(\phi) = \psi_m(\phi + 2\pi)$$

$$\Rightarrow e^{im\phi} = e^{im(\phi+2\pi)}$$

$$= e^{im\phi} e^{i2\pi m}$$

$$\Rightarrow 1 = e^{i2\pi m}$$

$$\Rightarrow m \in \mathbb{N}$$

$\square$

3.:

$$\int_0^{2\pi} d\phi \, \psi_l^*(\phi) \psi_m(\phi) = \int_0^{2\pi} d\phi \frac{1}{2\pi} e^{-il\phi} e^{im\phi}$$

$$= \int_0^{2\pi} d\phi \frac{1}{2\pi} e^{i\phi(m-l)} \quad \Big| \text{ Fourier-Trafo.}$$

$$= \delta_{ml}$$

$\square$

Exercise 4:

$$H(q,p,t) = \dot q_i p_i - L(q,\dot q,t), \quad p_i(q,\dot q_i t) = \frac{\partial L}{\partial \dot q_i}, \quad \frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$$

$$\dot{\bar q}_i = \frac{\partial K}{\partial \bar p_i} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \bar p_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \bar p_i} \qquad \frac{\partial \dot q_j}{\partial \dot{\bar p}_i} = \frac{\partial}{\partial \dot{\bar p}_i}\frac{\partial H}{\partial \dot p_j} =$$

$$= -\dot p_j \frac{\partial q_j}{\partial \bar p_i} + \dot q_j \frac{\partial p_j}{\partial \bar p_i}$$

$$= \{q, p\}_{t,\bar p} \qquad \underbrace{\dots}_{\neq 0} \underbrace{\{\bar q_i, \bar q_k\}}_{\Rightarrow\ =0} \overset{!}{=} 0$$

$$\frac{\partial \bar q_i}{\partial q_i}\frac{\partial \bar q_k}{\partial q_k} - \frac{\partial \bar q_i}{\partial q_k}\frac{\partial \bar q_k}{\partial q_i}$$

$$\text{I)}\ \dot{\bar p}_i = -\frac{\partial H}{\partial \bar q_i} = -\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \bar q_i} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \bar q_i}$$

$$\text{II)}\ \dot{\bar p}_i = -\frac{d}{dt}\frac{\partial L}{\partial q_i}$$

$$0 = \frac{\partial(\dot q_i p_i - L)}{\partial \bar q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar q}_i}$$

$$(\text{I})\,\&\,(\text{II}) \Rightarrow \quad 0 = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \bar q_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \bar q_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i}$$

$$= -\dot p_j \frac{\partial q_j}{\partial \bar q_i} + \dot q_j \frac{\partial p_j}{\partial \bar q_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i}$$

$$= -\frac{d}{dt}\Big(p_j \partial$$

$$\dot q$$

2.: $\bar q = \ln(q^{-1}\sin p), \quad \bar p = q\cot p$

$$\{\bar q_i, \bar p_k\} = \frac{-q^{-2}\sin p}{q^{-1}\sin p}\cdot\frac{-q}{\sin^2 p} - \frac{q^{-1}\cos p}{q^{-1}\sin p}\cot p$$

$$= \frac{1}{\sin^2 p} - \cot^2 p \qquad \Big|\ \sin^2 p + \cos^2 p = 1,\ \cot = \frac{\cos p}{\sin p}$$

$$= \frac{1}{\sin^2 p} - \frac{1-\sin^2 p}{\sin^2 p}$$

$$= 1$$

$$\{\bar q_i, \bar q_k\} = \{\bar q, \bar q\} = 0$$

$$\{\bar p_i, \bar p_k\} = \{\bar p, \bar p\} = 0$$

$\square$

3.:

$$\{A(\bar{q},\bar{p}),\ B(\bar{q},\bar{p})\}_{\bar{q},\bar{p}} = \frac{\partial A}{\partial \bar{q}}\frac{\partial B}{\partial \bar{p}} - \frac{\partial A}{\partial \bar{p}}\frac{\partial B}{\partial \bar{q}}$$

$$= \left(\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar{q}} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar{q}}\right)\left(\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar{p}} + \frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar{p}}\right) -$$

$$\left(\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar{q}} + \frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar{q}}\right)\left(\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar{p}} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar{p}}\right)$$

$$= \cancel{\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar{q}}\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar{p}}} + \frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar{q}}\frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar{p}} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar{q}}\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar{p}} + \cancel{\frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar{q}}\frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar{p}}}$$

$$- \cancel{\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar{q}}\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar{p}}} - \frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar{q}}\frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar{p}} - \frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar{q}}\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar{p}} - \cancel{\frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar{q}}\frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar{p}}}$$

$$= \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}\underbrace{\left(\frac{\partial q}{\partial \bar{q}}\frac{\partial p}{\partial \bar{p}} - \frac{\partial p}{\partial \bar{q}}\frac{\partial q}{\partial \bar{p}}\right)}_{\delta_{ij} = 1} + \frac{\partial B}{\partial q}\frac{\partial A}{\partial p}\underbrace{\left(\frac{\partial p}{\partial \bar{q}}\frac{\partial q}{\partial \bar{p}} - \frac{\partial q}{\partial \bar{q}}\frac{\partial p}{\partial \bar{p}}\right)}_{-\delta_{ij} = -1}$$

$$= \{A, B\}_{q,p}$$

$\square$