Hamiltonian Formulation of Gauge Theories and its Use for Quantum Simulation

Uwe-Jens Wiese

Albert Einstein Center for Fundamental Physics
Institute for Theoretical Physics, Bern University

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Outline

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Different descriptions of dynamical Abelian gauge fields:

Maxwell's classical electromagnetic gauge fields

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \rho(\vec{x}, t), \quad \vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0, \quad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t)$$

Quantum Electrodynamics (QED) for perturbative treatment

$$E_i = -i\frac{\partial}{\partial A_i}, \quad [E_i(\vec{x}), A_j(\vec{x}\,')] = i\delta_{ij}\delta(\vec{x}-\vec{x}\,'), \quad \left[\vec{\nabla} \cdot \vec{E} - \rho\right] |\Psi[A]\rangle = 0$$

Wilson's $U(1)$ lattice gauge theory for classical simulation

$$U_{xy} = \exp\left(ie \int_x^y d\vec{l} \cdot \vec{A}\right) = \exp(i\varphi_{xy}) \in U(1), \quad E_{xy} = -i\frac{\partial}{\partial \varphi_{xy}}$$

$$[E_{xy}, U_{xy}] = U_{xy}, \quad \left[\sum_i (E_{x,x+\hat{i}} - E_{x-\hat{i},x}) - \rho\right] |\Psi[U]\rangle = 0$$

$U(1)$ quantum link models for quantum simulation

$$U_{xy} = S^+_{xy}, \quad U^\dagger_{xy} = S^-_{xy}, \quad E_{xy} = S^3_{xy},$$

$$[E_{xy}, U_{xy}] = U_{xy}, \quad [E_{xy}, U^\dagger_{xy}] = -U^\dagger_{xy}, \quad [U_{xy}, U^\dagger_{xy}] = 2E_{xy}$$

Canonical Quantization of the Electromagnetic Field

The homogeneous Maxwell equations

$$\vec{\nabla} \times \vec{E}(\vec{x}, t) + \frac{1}{c}\partial_t \vec{B}(\vec{x}, t) = 0 \ , \quad \vec{\nabla} \cdot \vec{B}(\vec{x}) = 0$$

are satisfied when we introduce scalar and vector potentials

$$\vec{E}(\vec{x}, t) = -\vec{\nabla}\phi(\vec{x}, t) - \frac{1}{c}\partial_t \vec{A}(\vec{x}, t) \ , \quad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t)$$

Under a gauge transformation

$$\phi'(\vec{x}, t) = \phi(\vec{x}, t) - \frac{1}{c}\partial_t \alpha(\vec{x}, t) \ , \quad \vec{A}'(\vec{x}, t) = \vec{A}(\vec{x}, t) - \vec{\nabla}\alpha(\vec{x}, t)$$

the electromagnetic fields are invariant

$$\begin{aligned}
\vec{E}'(\vec{x}, t) &= -\vec{\nabla}\phi'(\vec{x}, t) - \frac{1}{c}\partial_t \vec{A}'(\vec{x}, t) = -\vec{\nabla}\phi'(\vec{x}, t) - \frac{1}{c}\vec{\nabla}\partial_t \alpha(\vec{x}, t) \\
&- \frac{1}{c}\partial_t \vec{A}(\vec{x}, t) + \frac{1}{c}\partial_t \vec{\nabla}\alpha(\vec{x}, t) = \vec{E}(\vec{x}, t) \\
\vec{B}'(\vec{x}, t) &= \vec{\nabla} \times \vec{A}'(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t) - \vec{\nabla} \times \vec{\nabla}\alpha(\vec{x}, t) = \vec{B}(\vec{x}, t)
\end{aligned}$$

Relativistic Formulation of Electrodynamics with 4-Vectors

$$x^0 = ct\ ,\ x^\mu = (x^0, \vec{x})\ ,\ \partial_\mu = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right)\ ,\ A^\mu(x) = (\phi(\vec{x},t), \vec{A}(\vec{x},t))$$

The field strength tensor

$$\begin{aligned} F^{\mu\nu}(x) &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \\ &= \begin{pmatrix} 0 & -E_x(\vec{x},t) & -E_y(\vec{x},t) & -E_z(\vec{x},t) \\ E_x(\vec{x},t) & 0 & -B_z(\vec{x},t) & B_y(\vec{x},t) \\ E_y(\vec{x},t) & B_z(\vec{x},t) & 0 & -B_x(\vec{x},t) \\ E_z(\vec{x},t) & -B_y(\vec{x},t) & B_z(\vec{x},t) & 0 \end{pmatrix} \end{aligned}$$

is invariant under gauge transformations

$$\begin{aligned} A'^\mu(x) &= A^\mu(x) - \partial^\mu \alpha(x)\ , \\ F'^{\mu\nu}(x) &= \partial^\mu A'^\nu(x) - \partial^\nu A'^\mu(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) - \partial^\mu\partial^\nu\alpha(x) + \partial^\nu\partial^\mu\alpha(x) = F^{\mu\nu}(x) \end{aligned}$$

From the Lagrangian to the Hamilton Density

$$\mathcal{L}(\partial^\mu A^\nu) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = \frac{1}{2}\left(\vec{E}(x)^2 - \vec{B}(x)^2\right)$$

Temporal gauge fixing

$$A^0(x) = \phi(\vec{x}, t) = 0$$

Canonically conjugate momenta

$$E_i(x) = -\partial^0 A^i(x)\ ,\ \Pi_i(x) = \frac{\delta \mathcal{L}}{\delta \partial^0 A^i(x)} = \partial^0 A^i(x) = -E_i(x)$$

Classical Hamilton density

$$\begin{aligned}\mathcal{H}(A^i, \Pi_i) &= \Pi_i(x)\partial^0 A^i(x) - \mathcal{L} = \frac{1}{2}\left(\Pi_i(x)\Pi_i(x) + B_i(x)B_i(x)\right) \\ &= \frac{1}{2}\left(E_i(x)E_i(x) + B_i(x)B_i(x)\right)\end{aligned}$$

Classical Hamilton function

$$H = \int d^3x\ \mathcal{H} = \int d^3x\ \frac{1}{2}\left[\Pi_i(\vec{x})\Pi_i(\vec{x}) + \epsilon_{ijk}\partial_j A^k(\vec{x})\epsilon_{ilm}\partial_l A^m(\vec{x})\right]$$

From classical to quantum electrodynamics

Canonical commutation relations

$$[\hat{A}^i(\vec{x}), \hat{\Pi}_j(\vec{y})] = \mathrm{i}\ \delta_{ij}\delta(\vec{x}-\vec{y})\ ,\ [\hat{A}^i(\vec{x}), \hat{A}^j(\vec{y})] = [\hat{\Pi}_i(\vec{x}), \hat{\Pi}_j(\vec{y})] = 0$$

Conjugate momentum operator

$$\hat{\Pi}_i(\vec{x}) = -\mathrm{i}\ \frac{\delta}{\delta A^i(\vec{x})}$$

Hamilton operator of the electromagnetic field

$$\hat{H} = \int d^3x\ \frac{1}{2}\left[\hat{\Pi}_i(\vec{x})\hat{\Pi}_i(\vec{x}) + \epsilon_{ijk}\partial_j\hat{A}^k(\vec{x})\epsilon_{ilm}\partial_l\hat{A}^m(\vec{x})\right]$$

Fourier transform

$$\hat{A}^i(\vec{p}) = \int d^3x\ \hat{A}^i(\vec{x})\exp(-\mathrm{i}\ \vec{p}\cdot\vec{x})\ ,\ \hat{A}^i(\vec{p})^\dagger = \hat{A}^i(-\vec{p})$$

$$\hat{\Pi}_i(\vec{p}) = \int d^3x\ \hat{\Pi}_i(\vec{x})\exp(-\mathrm{i}\ \vec{p}\cdot\vec{x})\ ,\ \hat{\Pi}_i(\vec{p})^\dagger = \hat{\Pi}_i(-\vec{p})$$

$$[\hat{A}^i(\vec{p}), \hat{\Pi}_i(\vec{q})] = \mathrm{i}\ (2\pi)^3\delta_{ij}\delta(\vec{p}+\vec{q})$$

$$[\hat{A}^i(\vec{p}), \hat{A}^j(\vec{q})] = [\hat{\Pi}_i(\vec{p}), \hat{\Pi}_i(\vec{q})] = 0$$

Diagonalization of the Hamiltonian

$$\hat{H} = \frac{1}{(2\pi)^3} \int d^3p \; \frac{1}{2} \left[\hat{\Pi}_i(\vec{p})^\dagger \hat{\Pi}_i(\vec{p}) + \epsilon_{ijk} p_j \hat{A}^k(\vec{p})^\dagger \epsilon_{ilm} p_l \hat{A}^m(\vec{p}) \right]$$

Gauss law

$$\vec{\nabla} \cdot \hat{\vec{E}}(\vec{x}) \,|\Psi\rangle = 0 \;\Rightarrow\; p_i \hat{\Pi}_i(\vec{p}) \,|\Psi\rangle = 0$$

Quadratic form of the magnetic field term

$$\epsilon_{ijk} p_j \hat{A}^k(\vec{p})^\dagger \epsilon_{ilm} p_l \hat{A}^m(\vec{p}) =$$

$$(\hat{A}^1(\vec{p})^\dagger, \hat{A}^2(\vec{p})^\dagger, \hat{A}^3(\vec{p})^\dagger) \begin{pmatrix} \vec{p}^{\,2} - p_1^2 & -p_1 p_2 & -p_1 p_3 \\ -p_2 p_1 & \vec{p}^{\,2} - p_2^2 & -p_2 p_3 \\ -p_3 p_1 & -p_3 p_2 & \vec{p}^{\,2} - p_3^2 \end{pmatrix} \begin{pmatrix} \hat{A}^1(\vec{p}) \\ \hat{A}^2(\vec{p}) \\ \hat{A}^3(\vec{p}) \end{pmatrix}$$

Symmetric matrix

$$\mathcal{M}(\vec{p})_{ij} = \vec{p}^{\,2} \left(\delta_{ij} - e_{pi} e_{pj} \right) \;,\; \vec{e}_p = \vec{p}/|\vec{p}|$$

$$\vec{e}_1 \cdot \vec{e}_p = \vec{e}_2 \cdot \vec{e}_p = 0 \;,\; \vec{e}_1 \cdot \vec{e}_2 = 0 \;,\; \vec{e}_1 \times \vec{e}_2 = \vec{e}_p \;,\; \vec{e}_\pm = \frac{1}{\sqrt{2}} (\vec{e}_1 \pm \mathrm{i} \vec{e}_2)$$

$$\vec{e}_\pm^{\,*} \cdot \vec{e}_\pm = 1 \;,\; \vec{e}_\pm \cdot \vec{e}_p = 0 \;,\; \vec{e}_-^{\,*} \cdot \vec{e}_+ = 0 \;,\; \vec{e}_- \times \vec{e}_+ = \mathrm{i}\, \vec{e}_p \qquad (1)$$

Unitary transformation diagonalizes $\mathcal{M}(\vec{p})$

$$U(\vec{p}) = \begin{pmatrix} e_{+1} & e_{+2} & e_{+3} \\ e_{p1} & e_{p2} & e_{p3} \\ e_{-1} & e_{-2} & e_{-3} \end{pmatrix} , \quad U(\vec{p})\mathcal{M}(\vec{p})U(\vec{p})^\dagger = \vec{p}^{\,2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \hat{A}_+(\vec{p}) \\ \hat{A}_p(\vec{p}) \\ \hat{A}_-(\vec{p}) \end{pmatrix} = U(\vec{p}) \begin{pmatrix} \hat{A}^1(\vec{p}) \\ \hat{A}^2(\vec{p}) \\ \hat{A}^3(\vec{p}) \end{pmatrix} , \quad \begin{pmatrix} \hat{\Pi}_+(\vec{p}) \\ \hat{\Pi}_p(\vec{p}) \\ \hat{\Pi}_-(\vec{p}) \end{pmatrix} = U(\vec{p}) \begin{pmatrix} \hat{\Pi}_1(\vec{p}) \\ \hat{\Pi}_2(\vec{p}) \\ \hat{\Pi}_3(\vec{p}) \end{pmatrix}$$

Diagonalized Hamilton operator

$$\begin{aligned} \hat{H} &= \frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2} \Big[\hat{\Pi}_+(\vec{p})^\dagger \hat{\Pi}_+(\vec{p}) + \vec{p}^{\,2} \hat{A}_+(\vec{p})^\dagger \hat{A}_+(\vec{p}) \\ &+ \hat{\Pi}_-(\vec{p})^\dagger \hat{\Pi}_-(\vec{p}) + \vec{p}^{\,2} \hat{A}_-(\vec{p})^\dagger \hat{A}_-(\vec{p}) + \hat{\Pi}_p(\vec{p})^\dagger \hat{\Pi}_p(\vec{p}) \Big] \end{aligned}$$

Hamilton operator commutes with Gauss law constraint

$$[\hat{H}, \hat{\Pi}_p(\vec{p})] = 0 \ , \ \hat{\Pi}_p(\vec{p}) = U(\vec{p})_{pi} \hat{\Pi}_i(\vec{p}) = -e_{pi} \hat{E}_i(\vec{p}) = -\frac{\vec{p}}{|\vec{p}|} \cdot \hat{\vec{E}}(\vec{p})$$

Creation and annihilation operators for photons

$$\hat{a}_\pm(\vec{p}) = \frac{1}{\sqrt{2}}\left[\sqrt{|\vec{p}|}\hat{A}_\pm(\vec{p}) + \frac{\mathrm{i}}{\sqrt{|\vec{p}|}}\hat{\Pi}_\pm(\vec{p})\right]$$

$$\hat{a}_\pm(\vec{p})^\dagger = \frac{1}{\sqrt{2}}\left[\sqrt{|\vec{p}|}\hat{A}_\pm(\vec{p})^\dagger - \frac{\mathrm{i}}{\sqrt{|\vec{p}|}}\hat{\Pi}_\pm(\vec{p})^\dagger\right]$$

Commutation relations

$$\begin{aligned}
&[\hat{a}_\pm(\vec{p}), \hat{a}_\pm(\vec{q})^\dagger] = (2\pi)^3\delta(\vec{p}-\vec{q}) \\
&[\hat{a}_+(\vec{p}), \hat{a}_-(\vec{q})^\dagger] = [\hat{a}_-(\vec{p}), \hat{a}_+(\vec{q})^\dagger] = 0 \\
&[\hat{a}_\pm(\vec{p}), \hat{a}_\pm(\vec{q})] = [\hat{a}_\pm(\vec{p})^\dagger, \hat{a}_\pm(\vec{q})^\dagger] = 0
\end{aligned} \tag{2}$$

Hamilton operator in the physical sector

$$\hat{H} = \frac{1}{(2\pi)^3}\int d^3p\ |\vec{p}|\left(\hat{a}_+(\vec{p})^\dagger\hat{a}_+(\vec{p}) + \hat{a}_-(\vec{p})^\dagger\hat{a}_-(\vec{p}) + V\right) \tag{3}$$

Vacuum and photon states

$$\hat{a}_\pm(\vec{p})|0\rangle = 0 \ , \ \hat{H}|0\rangle = E_0|0\rangle$$

$$|\vec{p},\pm\rangle = \hat{a}_\pm(\vec{p})^\dagger|0\rangle \ , \ \hat{H}|\vec{p},\pm\rangle = E(\vec{p})|\vec{p},\pm\rangle \ , \ E(\vec{p}) - E_0 = |\vec{p}|$$

Momentum operator and momentum of photons

$$\hat{\vec{P}} = \int d^3x \, \frac{1}{2}\left(\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x}) - \hat{\vec{B}}(\vec{x}) \times \hat{\vec{E}}(\vec{x})\right)$$

$$= \frac{1}{(2\pi)^3}\int d^3p \ \vec{p}\left(\hat{a}_+(\vec{p})^\dagger \hat{a}_+(\vec{p}) + \hat{a}_-(\vec{p})^\dagger \hat{a}_-(\vec{p})\right)$$

$$\left[\hat{\vec{P}}, \hat{a}_\pm(\vec{p})^\dagger\right] = \vec{p} \ \hat{a}_\pm(\vec{p})^\dagger \ \Rightarrow \ \hat{\vec{P}}|\vec{p},\pm\rangle = \vec{p} \ |\vec{p},\pm\rangle \tag{4}$$

Angular momentum operator and helicity of photons

$$\hat{\vec{J}} = \int d^3x \ \vec{x} \times \frac{1}{2}\left(\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x}) - \hat{\vec{B}}(\vec{x}) \times \hat{\vec{E}}(\vec{x})\right)$$

$$[\hat{P}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{P}_k \ \Rightarrow \ \left[\hat{\vec{P}}, \hat{\vec{P}} \cdot \hat{\vec{J}}\right] = 0$$

$$\left[\hat{\vec{J}} \cdot \vec{e}_p, \hat{a}_\pm(\vec{p})^\dagger\right] = \pm\hat{a}_\pm(\vec{p}^\dagger) \ \Rightarrow \ \hat{\vec{J}} \cdot \vec{e}_p|\vec{p},\pm\rangle = \pm|\vec{p},\pm\rangle \tag{5}$$

Homework: Recapitulate Lecture 1a and verify eqs.(1-5).

Outline

Wilson's concept of a parallel transporter

$$U_{xy} = \exp\left[ie \int_{x_k}^{x_k+a} dx_k \ A_k(x)\right] \in U(1)$$

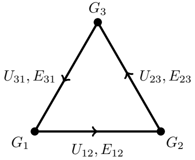
Behavior under gauge transformations

$$\begin{aligned}
A_k(x)' &= A_k(x) - \partial_k \alpha(x) \ \Rightarrow \\
U'_{xy} &= \exp\left[ie \int_{x_k}^{x_k+a} dx_k \ A'_k(x)\right] \\
&= \exp\left[ie \int_{x_k}^{x_k+a} dx_k \ \{A_k(x) - \partial_k \alpha(x)\}\right] \\
&= \exp\left[ie \left\{\int_{x_k}^{x_k+a} dx_k \ A_k(x) + \alpha(x) - \alpha(y)\right\}\right] \\
&= \Omega_x U_{xy} \Omega_y^\dagger \ , \ \Omega_x = \exp\left[i\alpha(x)\right] \in U(1)
\end{aligned}$$

Quantum mechanical analog "particle" on a circle $S^1 = U(1)$

$$U = \exp(i\varphi) \ , \ U^\dagger = \exp(-i\varphi) \ , \ E = -i\partial_\varphi$$

$$[E, U] = U \ , \ [E, U^\dagger] = -U^\dagger \ , \ [U, U^\dagger] = 0$$



Three analog "particles" on a plaquette

$$E_{12} = -i\partial_{\varphi_{12}} \ , \ E_{23} = -i\partial_{\varphi_{23}} \ , \ E_{31} = -i\partial_{\varphi_{31}}$$

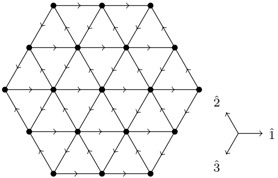
Three-"particle" Hamiltonian

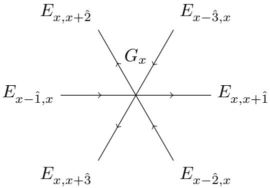
$$\begin{aligned} H &= T_{12} + T_{23} + T_{31} + V_{123} \\ &= \frac{E_{12}^2}{2I} + \frac{E_{23}^2}{2I} + \frac{E_{31}^2}{2I} - \frac{1}{e^2}\cos(\varphi_1 + \varphi_2 + \varphi_3) \\ &= \frac{E_{12}^2}{2I} + \frac{E_{23}^2}{2I} + \frac{E_{31}^2}{2I} - \frac{1}{2e^2}(U_{12}U_{23}U_{31} + U_{31}^\dagger U_{23}^\dagger U_{12}^\dagger) \end{aligned}$$

Invariance against relative rotations

$$\begin{aligned} &G_1 = E_{12} - E_{31} \ , \ G_2 = E_{23} - E_{12} \ , \ G_3 = E_{31} - E_{23} \\ &[H, G_1] = [H, G_2] = [H, G_3] = 0 \qquad (6) \end{aligned}$$

Many "particles" in S^1 forming a $U(1)$ lattice gauge theory





$$H = \frac{e^2}{2}\sum_{\langle xy\rangle} E_{xy}^2 - \frac{1}{2e^2}\sum_{\langle xyz\rangle}\left(U_{xy}U_{yz}U_{zx} + U_{zx}^\dagger U_{yz}^\dagger U_{xy}^\dagger\right) , \quad I = \frac{1}{e^2}$$

Link-based operator algebra

$$[E_l, E_{l'}] = 0 , \quad [E_l, U_{l'}] = i\delta_{ll'}U_l , \quad [E_l, U_{l'}^\dagger] = -i\delta_{ll'}U_l^\dagger$$
$$[U_l, U_{l'}] = [U_l^\dagger, U_{l'}^\dagger] = [U_l, U_{l'}^\dagger] = 0$$

Invariance against gauge transformations

$$G_x = \sum_k (E_{x,x+\hat{k}} - E_{x-\hat{k},x}) , \ [G_x, G_y] = 0 , \ [H, G_x] = 0 \qquad (7)$$

General gauge transformations and Gauss law

$$V = \prod_x \exp(i\alpha_x G_x)\ ,\ \ V U_{xy} V^\dagger = \Omega_x U_{xy} \Omega_y^\dagger \tag{8}$$

States with external charges $Q_x \in \mathbb{Z}$

$$G_x|\Psi, Q\rangle = Q_x|\Psi, Q\rangle\ ,\ \ Q = \{Q_x\}$$

Standard Gauss law

$$G_x|\Psi\rangle = 0$$

Canonical quantum statistical partition function

$$Z_Q = \mathrm{Tr}[\exp(-\beta H) P_Q]$$

Potential between external charges $Q_x = 1,\ Q_y = -1$

$$\frac{Z_Q}{Z} = \exp(-\beta V(x-y))\ ,\ \ V(x-y) \sim \sigma|x-y|$$

Homework: Recapitulate Lecture 1b and verify eqs.(6-8).

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