Theo 4 Zettel 1; Marc Hacuer, Franka Wronek, Angela Brede; 21.10.2024

2.1. Exaktheit und Integrabilität von Differentialen

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$n = 2$

$\alpha = x_1 x_2 dx_1 + x_1^2 dx_2$

$\alpha = \sum_{i=1}^{2} a_i(x_1, x_2) dx_i = \underbrace{a_1(x_1,x_2)}_{x_1 \cdot x_2} dx_1 + \underbrace{a_2(x_1,x_2)}_{x_1^2} dx_2$

a) zz: α nicht exakt

Integrabilitätsbedingung von α: $\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$

$\frac{\partial a_1}{\partial x_2} = x_1 \neq \frac{\partial a_2}{\partial x_1} = 2x_1$

$\rightarrow$ notwendige Bedingung für Exaktheit von α nicht erfüllt

b) zz: $x_1^{-1} \alpha$ exakt

$\alpha' = \frac{\alpha}{x_1} = \frac{x_1 x_2}{x_1} \cdot dx_1 + \frac{x_1^2}{x_1} dx_2$

$= \underbrace{x_2}_{a_1'} dx_1 + \underbrace{x_1}_{a_2'} dx_2$

$\frac{\partial a_1'}{\partial x_2} = 1 = \frac{\partial a_2'}{\partial x_1} = 1 \quad \Rightarrow$ notwendige Bedingung erfüllt

Satz von Green $\oint_\gamma (P dx_1 + Q dx_2) = \iint_A \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dA$

$\oint_\gamma \alpha' = 0 \overset{!}{=} \oint_\gamma (x_2 dx_1 + x_1 dx_2)$

$= \iint \left(\frac{\partial x_1}{\partial x_1} - \frac{\partial x_2}{\partial x_2} \right) dA$

$= \iint_A 0 \, dA = 0$

c) $dF = \frac{\alpha}{x_1}$

$a_i = \frac{\partial F}{\partial x_i}$

$a_1' = x_2 = \frac{\partial F}{\partial x_1}$ $\qquad$ $a_2' = x_1 = \frac{\partial F}{\partial x_2}$

$\Rightarrow F = x_1 x_2 + c(x_2)$ $\qquad$ $\Rightarrow F = x_1 x_2 + c(x_1)$

$$c(x_1) \stackrel{!}{=} c(x_2) = c$$

$$\Rightarrow F(x_1, x_2) = x_1 x_2 + c$$

d) $g\alpha$ ist exakt $\qquad$ $g\alpha = g(x_1 x_2 \, dx_1 + x_1^2 \, dx_2)$

$= g x_1 x_2 \, dx_1 + g x_1^2 \, dx_2$

Integrabilitätsbedingungen

$$\frac{\partial a_1}{\partial x_2} \stackrel{!}{=} \frac{\partial a_2}{\partial x_1}$$

$$\frac{\partial (g(x_1,x_2) x_1 x_2)}{\partial x_2} = \frac{\partial (g(x_1,x_2) x_1^2)}{\partial x_1}$$

$$\Leftrightarrow \frac{\partial g(x_1,x_2)}{\partial x_2} \cdot x_1 \cdot x_2 + g(x_1,x_2) \cdot x_1 = \frac{\partial g(x_1,x_2)}{\partial x_1} \cdot x_1^2 + g(x_1,x_2) \cdot 2x_1$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} \cdot x_2 + g(x_1,x_2) = \frac{\partial g(x_1,x_2)}{\partial x_1} \cdot x_1 + g(x_1,x_2) \cdot 2$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} \cdot x_2 - \frac{\partial g(x_1,x_2)}{\partial x_1} x_1 = g(x_1,x_2) \cdot 2 \quad - g(x_1,x_2)$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} \cdot x_2 - \frac{\partial g(x_1,x_2)}{\partial x_1} x_1 = + g(x_1,x_2)$$

Ansatz: $b(x_1) h(x_2)$

$$x_2 \cdot \partial x_2 (b(x_1) h(x_2)) - x_1 \partial x_1 (b(x_1) h(x_2)) = + b(x_1) h(x_2)$$

$$x_2 \cdot \frac{\partial h(x_2)}{\partial x_2} \cdot b(x_1) - x_1 \cdot \frac{\partial b(x_1)}{\partial x_1} \cdot h(x_2) = + b(x_1) h(x_2)$$

$$\left(x_2 \cdot \frac{\partial h(x_2)}{\partial x_2} - h(x_2) \right) b(x_1) = x_1 \frac{\partial b(x_1)}{\partial x_1} h(x_2)$$

$$\frac{x_2 \cdot \frac{\partial h(x_2)}{\partial x_2} - h(x_2)}{h(x_2)} = x_1 \cdot \frac{\frac{\partial b(x_1)}{\partial x_1}}{b(x_1)}$$

$$x_2 \frac{\frac{\partial h(x_2)}{\partial x_2}}{h(x_2)} = x_1 \cdot \frac{\frac{\partial b(x_1)}{\partial x_1}}{b(x_1)} + 1 = c$$

$$x_2 \frac{\frac{\partial h(x_2)}{\partial x_2}}{h(x_2)} = c \quad \Rightarrow \quad c\, h(x_2) = x \cdot \frac{\partial h(x_2)}{\partial x_2}$$

$$\Rightarrow h(x_2) = C_1 x_2^c \quad , \quad \frac{\partial h(x_2)}{\partial x_2} = c \cdot C_1 \cdot x_2^{c-1}$$

$$x_1 \cdot \frac{\frac{\partial b(x_1)}{\partial x_1}}{b(x_1)} - 1 = c \quad \Rightarrow \quad (c-1)\, b(x_1) = x_1 \frac{\partial b(x_1)}{\partial x_1}$$

$$\frac{\partial b(x_1)}{\partial x_1} = \frac{(c-1)}{x_1} \cdot b(x_1)$$

$$b = C_2 \cdot x_1^{c-1}$$

$$\Rightarrow g(x_1, x_2) = C_1 \cdot x_2^c \cdot C_2 \cdot x_1^{c-1}$$

Für den Spezialfall, dass g nur von x_1 abhängt, $C_1 = C_2 = 1$ und $c = 0$ erhalten wir $g(x_1) = x_1^{-1}$, was dem Fall aus Aufgabe c)

Aufgabe 2:

a) $df = u(x,y)dx + v(x,y)dy$ exakt $\Rightarrow$ Integrabilitätsbedingung:

$$\frac{\partial a_1}{\partial x_2} = \frac{\partial a_2}{\partial x_1} \quad \Big| \; a_i = \frac{\partial f}{\partial x_i} \Rightarrow a_1 = u(x,y),\; a_2 = v(x,y)$$

$$\Rightarrow \frac{\partial u(x,y)}{\partial y} = \frac{\partial v(x,y)}{\partial x}$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_x = \left(\frac{\partial v}{\partial x}\right)_y$$

b)

$$F(x,y,z) = 0 \Rightarrow x = x(y,z) \wedge y = y(x,z) \wedge z = z(x,y)$$

$$\mathrm{I}: \Rightarrow \begin{cases} dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz \\ dy = \left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz \end{cases}$$

$$\Rightarrow df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy \quad \Big| \; (\mathrm{I})$$

$$= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z dy + \underbrace{\left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z}_{\left(\frac{\partial f}{\partial x}\right)_y} dx + \left[\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial z}\right)_x\right] dz$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_z = \left(\frac{\partial f}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z,$$

wobei ersichtlich wird, dass die Ableitungen nur von z unabhängig bleiben.

$\square$

c)

$$\mathrm{II}: \; dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$df = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z dx + \left[\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial z}\right)_x\right] dz$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_y \left\{\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial z}\right)_x\right\}\right]}_{\left(\frac{\partial f}{\partial x}\right)_y} dx$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z + \left(\frac{\partial z}{\cancel{\partial x}}\right)_y \left\{\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\cancel{\partial x}}{\partial z}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial z}\right)_x\right\}$$

$$= \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z + \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial z}\right)_x\}$$

d)

$dU(S, V, N) = T dS - p dV + \mu dN$ ist ein exaktes Differential:

$$dU(S, V, N) = \sum_i a_i(x_1, x_2, x_3)\, dx_i \quad \text{mit} \quad x_i = \{S, V, N\}$$

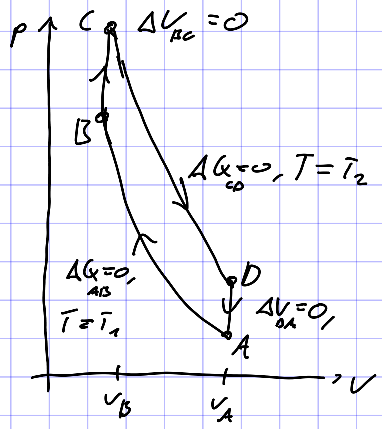
$$\Rightarrow \quad a_1 = \left(\frac{\partial U}{\partial S}\right)_{V,N} = T, \; a_2 = \left(\frac{\partial U}{\partial V}\right)_{S,N} = -p, \; a_3 = \left(\frac{\partial U}{\partial N}\right)_{S,V} = \mu$$

Int. bed.: $\dfrac{\partial a_i}{\partial x_j} = \dfrac{\partial a_j}{\partial x_i}$

$$\Rightarrow \begin{cases} \boxed{\left(\frac{\partial T}{\partial V}\right)_S = \left(\frac{-\partial p}{\partial S}\right)_V}, & \left(\frac{\partial T}{\partial N}\right)_S = \left(\frac{\partial \mu}{\partial S}\right)_N \\ \left(\frac{-\partial p}{\partial S}\right)_V = \left(\frac{\partial T}{\partial V}\right)_S, & \boxed{\left(\frac{\partial \mu}{\partial V}\right)_N = \left(\frac{-\partial p}{\partial N}\right)_V} \\ \boxed{\left(\frac{\partial \mu}{\partial S}\right)_N = \left(\frac{\partial T}{\partial N}\right)_S}, & \left(\frac{-\partial p}{\partial N}\right)_V = \left(\frac{\partial \mu}{\partial V}\right)_N \end{cases}$$

Aufgabe 4:

a)



$$dU = dQ - dW + \mu dN \quad | \; N = \text{const.} \Rightarrow dN = 0$$
$$= dQ - dW$$

$A \to B$: adiabat. $\Rightarrow dQ = 0$

$$\Rightarrow dW = -dU$$
$$= p\,dV \quad | \; pV = N k_B T$$
$$\Rightarrow \int_{V_A}^{V_B} dW = N k_B T_1 \int_{V_A}^{V_B} \frac{1}{V}\, dV$$
$$W_{AB} = N k_B T_1 \ln \frac{V_B}{V_A}$$

$B \to C$: isochor $\Rightarrow dV = 0 \Rightarrow dW = 0$

$$\Rightarrow dQ = dU \quad | \; U = \frac{f}{2} N k_B \cdot T$$
$$= \frac{f}{2} N k_B\, dT$$
$$\Rightarrow \int_B^C dQ = \int_{T_1}^{T_2} \frac{f}{2} N k_B\, dT$$
$$\Delta Q_{BC} = \frac{f}{2} N k_B (T_2 - T_1)$$

$C \to D$: analog zu $(A \to B)$

$$\Rightarrow W_{CD} = N k_B T_2 \ln \frac{V_A}{V_B}$$

$D \to A$: analog zu $(B \to C)$

$$\Rightarrow \Delta Q_{DA} = \frac{f}{2} N k_B (T_1 - T_2)$$

b)

$$\cancel{\eta = 1 - \frac{W_{AB}}{W_{CD}} = 1 + \frac{T_1}{T_2} = 1 + \left(\frac{V_B}{V_A}\right)^{\kappa - 1}}$$

$$\ln \frac{V_B}{V_A} = -\ln \frac{V_A}{V_B}$$

$$\cancel{\eta = \frac{|W_{tot}|}{W_{CD}} = \frac{T_2 \ln \frac{V_A}{V_B} + T_1 \ln \frac{V_B}{V_A}}{T_2 \ln \frac{V_A}{V_B}} = \frac{T_2 - T_1}{T_2} = 1 - \frac{T_1}{T_2} = 1 - \left(\frac{V_B}{V_A}\right)^{\kappa - 1}}$$

Vorlesung (2.59): Alle irreversiblen Prozesse haben Wirkungsgrad

$$\eta = 1 - \frac{T_C}{T_?}$$

Otto-Prozess ist irreversiebel.

$$\Rightarrow \eta = 1 - \frac{T_1}{T_2} \quad | \; T V^{\frac{2}{f}} = T V^{\kappa - 1} = \text{const. mit } \kappa = \frac{f+2}{f}$$

$$= 1 - \left(\frac{V_B}{V_A}\right)^{\kappa - 1} \quad \text{mit } \kappa = \frac{f+2}{f}$$

2.3) Adiabatische Zustandsänderung

$$U = \frac{f}{2} N k_B T \quad ; \quad pV = N k_B T \quad ; \quad dS = \frac{1}{T} dU + \frac{p}{T} dV + \frac{\mu}{T} dN$$

a) z: $T V^{\frac{2}{f}} = \text{konst.}$

$\cdot \quad dS = \frac{1}{T} dU + \frac{p}{T} dV - \frac{\mu}{T} dN \quad \Big| \; dS = 0 = dN$

$\Rightarrow \quad 0 = dU + p\, dV \quad \Big| \quad dU = \frac{f}{2} N k_B \, dT$

$\Rightarrow \quad 0 = \frac{f}{2} N k_B dT + N k_B T \frac{1}{V} dV$

$\Rightarrow \quad -\frac{f}{2} \int dT \frac{1}{T} = \int dV \frac{1}{V}$

$\Rightarrow \quad -\frac{f}{2} \ln(T) = \ln(V) + c$

$\Rightarrow \quad \frac{2}{f} \ln(V) + \ln(T) = -c'$

$\Rightarrow \quad \ln(T V^{\frac{2}{f}}) = -c'$

$\Rightarrow \quad T V^{\frac{2}{f}} = e^{-c'} = \text{konst.} \quad \square$

b) z: $p V^{\frac{f+2}{f}} = \text{konst.}$

$\cdot \quad T V^{\frac{2}{f}} = \text{konst.} \quad \Big| \; T = \frac{pV}{N k_B} \propto pV$

$\Rightarrow \quad pV \cdot V^{\frac{2}{f}} = \text{konst.}$

$\Rightarrow \quad p V^{\frac{f+2}{f}} = \text{konst.} \quad \square$