AQT; Sheet 04; Marc Hauer, Angelo Brade; 05.11.2024

Quickies:

1.) passive: transformation refers to same points in phase space before and after

active: transformation refers to different points —"—

2.) $g = L_z$

3.) i: Proton has charge:

$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad \text{with} \quad \begin{cases} \vec{A}(\vec{x},t) \to \vec{A}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x},t) \\ V(\vec{x},t) \to V(\vec{x},t) - \frac{\partial \lambda(\vec{x},t)}{\partial t} \end{cases}$$

ii: Neutron does not have charge:

Gauge: $\vec{E} = 0 = \vec{B}$

1) Propagator of the Harmonic Oszillator

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 - \omega^2 x^2)$$

1) Show: $S_{cl}(x,x';t) = \frac{m\omega}{2\sin(\omega t)}\left[\cos(\omega t)(x^2+x'^2) - 2xx'\right]$

- $S_{cl} = \int_{t_0}^{t} dt' \, \mathcal{L}(x_{cl}, \dot{x}_{cl}, t') = \frac{m}{2}\int_0^t dt' \, (\dot{x}_{cl}^2(t') - \omega^2 x_{cl}^2(t'))$ with $t_0 = 0$

$x_{cl}(t') = a\cos(\omega t') + b\sin(\omega t')$ with $x_{cl}(0) = x \wedge x_{cl}(t) = x'$

$\Rightarrow x_{cl}(0) = a = x \quad \wedge \quad x_{cl}(t) = a\cos(\omega t) + b\sin(\omega t) = x' \quad \Rightarrow b = \frac{x' - a\cos(\omega t)}{\sin(\omega t)} = x'\csc(\omega t) - a\cot(\omega t)$

$\Rightarrow x_{cl}(t') = x\cos(\omega t') + x'\csc(\omega t)\sin(\omega t') - x\cot(\omega t)\sin(\omega t') = x\cos(\omega t') + (x' - x\cos(\omega t))\frac{\sin(\omega t')}{\sin(\omega t)}$

$\Rightarrow \dot{x}_{cl}(t') = -x\omega\sin(\omega t') + \omega(x' - x\cos(\omega t))\frac{\cos(\omega t')}{\sin(\omega t)}$

$$\Rightarrow S_{cl} = \frac{m}{2}\int_0^t dt' \left\{ x^2\omega^2\sin^2(\omega t') + \omega^2(x'-x\cos(\omega t))^2\frac{\cos^2(\omega t')}{\sin^2(\omega t)} - \omega^2 x(x'-x\cos(\omega t))\frac{\sin(2\omega t')}{\sin(\omega t)} - \omega^2\left[x^2\cos^2(\omega t') + (x'-x\cos(\omega t))^2\frac{\sin^2(\omega t')}{\sin^2(\omega t)} + x(x'-x\cos(\omega t))\frac{\sin(2\omega t')}{\sin(\omega t)}\right]\right\}$$

$$= \frac{m\omega^2}{2}\int_0^t dt' \left\{ -x^2\cos(2\omega t') + (x'-x\cos(\omega t))^2\frac{\cos(2\omega t')}{\sin^2(\omega t)} - 2x(x'-x\cos(\omega t))\frac{\sin(2\omega t')}{\sin(\omega t)} \right\}$$

$$= \frac{m\omega^2}{2}\int_0^t dt' \left\{ \left[(x'-x\cos(\omega t))^2\frac{1}{\sin^2(\omega t)} - x^2\right]\cos(2\omega t') - 2x(x'-x\cos(\omega t))\frac{\sin(2\omega t')}{\sin(\omega t)} \right\}$$

$$= \frac{m\omega}{4}\left\{ \left[(x'-x\cos(\omega t))^2\frac{1}{\sin^2(\omega t)} - x^2\right]\sin(2\omega t) + \left[2x(x'-x\cos(\omega t))\frac{1}{\sin(\omega t)}\right](\cos(2\omega t) - 1) \right\} \quad \Big| \; \sin(2x) = 2\sin(x)\cos(x), \; \cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= \frac{m\omega}{4}\left\{ (x'^2 + x^2\cos^2(\omega t) - 2xx'\cos(\omega t))\frac{2\cos(\omega t)}{\sin(\omega t)} - x^2\sin(2\omega t) + \left[2x(x'-x\cos(\omega t))\frac{1}{\sin(\omega t)}\right](-2\sin^2(\omega t)) \right\}$$

$$= \frac{m\omega}{2}\left\{ (x'^2 + x^2\cos^2(\omega t) - 2xx'\cos(\omega t))\frac{\cos(\omega t)}{\sin(\omega t)} - x^2\sin(\omega t)\cos(\omega t) - (2x(x'-x\cos(\omega t))\sin(\omega t)) \right\}$$

$$= \frac{m\omega}{2\sin(\omega t)}\left\{ (x'^2 + x^2\cos^2(\omega t))\cos(\omega t) - 2xx' - x^2\sin^2(\omega t)\cos(\omega t) + 2x^2\cos(\omega t)\sin^2(\omega t) \right\}$$

$$= \frac{m\omega}{2\sin(\omega t)}\left\{ (x'^2 + x^2\cos^2(\omega t) + x^2\sin^2(\omega t))\cos(\omega t) - 2xx' \right\}$$

$$= \frac{m\omega}{2\sin(\omega t)}\left[\cos(\omega t)(x^2 + x'^2) - 2xx'\right] \quad \square$$

2) Show: $\mathcal{U}(x,x',t) = \int_{x'}^{x} Dx(t') \exp\left[\frac{i}{\hbar}\int_0^t dt' \,\mathcal{L}(x(t'),\dot{x}(t'))\right] = F(t)\cdot \exp\left[\frac{i}{\hbar} S_{cl}\right]$

$\cdot\ \mathcal{U}(x,x',t) = \int_{x'}^{x} Dx(t') \exp\left[\frac{i}{\hbar}\int_0^t dt'\, \mathcal{L}(x(t'),\dot{x}(t'))\right] \quad \Big|\ x(t') = x_{cl}(t') + y(t') \quad \text{with} \quad y(0) = y(t) = 0$

$= \int_{x'}^{x} D(x_{cl}(t')+y(t')) \exp\left[\frac{i}{\hbar}\int_0^t dt'\, \mathcal{L}(x_{cl}(t')+y(t'), \dot{x}_{cl}(t')+\dot{y}(t'))\right]$

$\cdot\ S[x(t')] = \int_0^t dt'\, \mathcal{L}(x_{cl}+y, \dot{x}_{cl}+\dot{y})$

$= \frac{m}{2}\int_0^t dt' \left[(\dot{x}_{cl}+\dot{y})^2 - \omega^2(x_{cl}+y)^2\right]$

$= \frac{m}{2}\int_0^t dt' \left[(\dot{x}_{cl}^2 - \omega^2 x_{cl}^2) + (\dot{y}^2 - \omega^2 y^2) + 2(\dot{x}_{cl}\dot{y} - \omega^2 x_{cl} y)\right]$

(...)

2) Ordinary Path Integral from Phase Space Path Integral

Show: $\mathcal{U}(x,x',t) = \frac{1}{B}\lim_{N\to\infty}\prod_{k=1}^{N-1}\int \frac{dx_k}{B} \exp\left[\frac{i}{\hbar}\sum_{k=1}^{N}\left(\frac{m}{2}\frac{(x_k-x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right]$ where $B = \sqrt{\frac{2\pi\hbar i\varepsilon}{m}}$

$\cdot\ \mathcal{U}(x,x',t) = \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dx_k \prod_{k=1}^{N}\int\frac{dp_k}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\sum_{k=1}^{N}\left(\frac{\varepsilon p_k^2}{2m} - p_k(x_k-x_{k-1}) + \varepsilon V(x_{k-1})\right)\right]$

$= \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dx_k \prod_{k=1}^{N}\left\{\int\frac{dp_k}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\left(\frac{\varepsilon p_k^2}{2m} - p_k(x_k-x_{k-1}) + \varepsilon V(x_{k-1})\right)\right]\right\} \quad \Big|\ \int dx\, e^{-ax^2+bx+c} = e^{\frac{b^2}{4a}+c}\sqrt{\frac{\pi}{a}}$

$= \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dx_k \prod_{k=1}^{N}\left\{\frac{1}{2\pi\hbar}\exp\left[-\frac{(x_k-x_{k-1})^2 m}{2\hbar i\varepsilon} - \frac{i\varepsilon}{\hbar}V(x_{k-1})\right]\sqrt{\frac{2\pi\hbar m}{i\varepsilon}}\right\}$

$= \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dx_k \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{\frac{N}{2}}\exp\left[\frac{i}{\hbar}\sum_{k=1}^{N}\left(\frac{m}{2}\frac{(x_k-x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right]$

$= \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dx_k \left(\frac{1}{B}\right)^{N}\exp\left[\frac{i}{\hbar}\sum_{k=1}^{N}\left(\frac{m}{2}\frac{(x_k-x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right]$

$= \frac{1}{B}\lim_{N\to\infty}\prod_{k=1}^{N-1}\int\frac{dx_k}{B}\exp\left[\frac{i}{\hbar}\sum_{k=1}^{N}\left(\frac{m}{2}\frac{(x_k-x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right]$ where $B = \sqrt{\frac{2\pi\hbar i\varepsilon}{m}}$ $\square$

Exercise 3: $\hat{H} = -\frac{\hbar^2}{2m}\psi''(x,0) + \hat{V}\psi(x,0)$

$$\psi(x+\eta,0) = \psi(x,0) + \left(\frac{d}{d\eta}\psi(x+\eta,0)\right)\Big|_{\eta=0}\cdot\eta + \left(\frac{1}{2}\frac{d^2}{d\eta^2}\psi(x+\eta,0)\right)\Big|_{\eta=0}\eta^2 + O(\eta^3)$$

$$= \psi(x,0) + \cancel{\psi'(x+\eta,0)\big|_{\eta=0}\cdot\eta}^{\to 0} + \frac{1}{2}\psi''(x+\eta,0)\big|_{\eta=0}\eta^2 + \cancel{O(\eta^3)}^{\to 0}$$

$$\simeq \psi(x,0) + \frac{1}{2}\psi''(x,0)\,\eta^2$$

$$A(x+\alpha\eta,0) = A(x,0) + \frac{d}{d\eta}A(x+\alpha\eta,0)\Big|_{\eta=0}\cdot\eta + \frac{1}{2}\left(\frac{d^2}{d\eta^2}A(x+\alpha\eta,0)\right)\Big|_{\eta=0}\eta^2 + O(\eta^3)$$

$$= A(x,0) + A'(x+\alpha\eta,0)\big|_{\eta=0}\alpha\eta + \frac{1}{2}A''(x+\alpha\eta,0)\big|_{\eta=0}\alpha^2\eta^2 + \cancel{O(\eta^3)}^{\to 0}$$

I: $\simeq A(x,0) + A'(x,0)\,\alpha\eta + \frac{1}{2}A''(x,0)\,\alpha^2\eta^2$

$$\Rightarrow \varepsilon A(x+\alpha\eta,0) \simeq \varepsilon A(x,0) + \alpha\eta\varepsilon A'(x,0) + \frac{1}{2}A''(x,0)\alpha^2\eta^2\varepsilon$$

$$= \varepsilon A(x,0) + O(\eta\varepsilon,\eta^2)$$

II: $\simeq \varepsilon A(x,0)$

I think i made here a mistake. A should be $A' = \frac{\partial}{\partial x}A$ since $B = \vec{\nabla}\times\vec{A}$. Also we need α. But since $O(\varepsilon\eta,\eta,\varepsilon^2)$ should be ignored, i dont find the mistake.

$$\psi(x,\varepsilon) = \frac{1}{B}\int_{-\infty}^{\infty}d\eta\,\psi(x+\eta,0)\exp\left[\frac{i}{\hbar}\left(\frac{m\eta^2}{2\varepsilon} - q\varepsilon\frac{\eta}{\varepsilon}A(x+\alpha\eta,0)\right)\right] \quad \Big|\ \frac{\eta}{\varepsilon} = -\dot{x},\ \text{(I) \& (II)}$$

$$\simeq \frac{1}{B}\int_{-\infty}^{\infty}d\eta\left(\psi(x,0) + \frac{1}{2}\psi''(x,0)\eta^2\right)\exp\left[\frac{i}{\hbar}\left(\frac{m\eta^2}{2\varepsilon} - q\dot{x}\,\varepsilon A(x,0)\right)\right]$$

Mit $\exp\left[-\frac{i}{\hbar}q\dot{x}\varepsilon A(x,0)\right] \simeq 1 - q\dot{x}A(x,0)\varepsilon + O(\varepsilon^2)$ folgt:

$$= \frac{1}{B}\int_{-\infty}^{\infty}d\eta\left(\psi(x,0) + \frac{1}{2}\psi''(x,0)\eta^2\right)\exp\left[\frac{i}{\hbar}\frac{m\eta^2}{2\varepsilon}\right]\left(1 - q\dot{x}A(x,0)\varepsilon\right)$$

$$= \frac{1}{B}\int_{-\infty}^{\infty}d\eta\Big(\psi(x,0) - \psi(x,0)q\dot{x}A(x,0)\varepsilon + \frac{1}{2}\psi''(x,0)\eta^2 - \underbrace{\frac{1}{2}\psi''(x,0)\eta^2 q\dot{x}A(x,0)\varepsilon}_{\to O(\eta^2\varepsilon)=0}\Big)\cdot\exp\left[\frac{i}{\hbar}\frac{m\eta^2}{2\varepsilon}\right]$$

$$= \frac{1}{B}\int_{-\infty}^{\infty}d\eta\left(\psi(x,0) - \psi(x,0)q\dot{x}A(x,0)\varepsilon + \frac{1}{2}\psi''(x,0)\eta^2\right)\cdot\exp\left[\frac{i}{\hbar}\frac{m\eta^2}{2\varepsilon}\right]$$

$$= \frac{1}{B}\left[\int_{-\infty}^{\infty}\left(\psi(x,0) - \psi(x,0)q\dot{x}A(x,0)\varepsilon\right)\exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}\eta^2\right]d\eta + \psi''(x,0)\frac{1}{2}\int_{-\infty}^{\infty}\eta^2\cdot\exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}\eta^2\right]d\eta\right]$$

$B = \sqrt{\frac{2\pi\hbar i\varepsilon}{m}}$

$$= \frac{1}{B}\left[\left(\psi(x,0) - \psi(x,0)q\dot{x}A(x,0)\varepsilon\right)\sqrt{\frac{2i\hbar\varepsilon\pi}{m}} + \frac{1}{4}\psi''(x,0)\sqrt{\frac{\pi 8\hbar^3\varepsilon^3}{i m^3}}\right]$$

$\int_{\mathbb{R}}d\eta\, e^{-a\eta^2} = \sqrt{\frac{\pi}{a}}$

$\int_{\mathbb{R}}d\eta\,\eta^2 e^{-a\eta^2} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$

$$= \psi(x,0) - \psi(x,0)q\dot{x}A(x,0)\varepsilon i + \frac{\partial^2}{\partial x^2}\psi(x,0)\frac{\hbar\varepsilon}{2m}i$$

$$= \psi(x,0) - i\frac{\varepsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + q\dot{x}A(x,0)\varepsilon\right)\psi(x,0)$$

$$= \psi(x,0) - i\frac{\varepsilon}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,0)\right)\psi(x,0)$$

$$= \psi(x,0) - i\frac{\varepsilon}{\hbar}\hat{H}\,\psi(x,0)$$

$\square$