AQT; Sheet 6; Marc Hauer, Angelo Bradei 12.11.2023

Quickies:

Q1: $\psi(x,t) = \int dx' U(x,x',t,t_0)\psi(x',t_0)$

Q2: $U(x_N, x_0, t_N, t_0) = A\int \mathcal{D}x(t) e^{\frac{i}{\hbar}S(x(t))}$

Q3: i) $\hat{U}(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'}$ is unitary.

ii) $U(x_N, x_0, t_N, t_0)$ is the time evolution operator for any path from x_0 to x_N. In the Schrödinger picture we do not look at multiple paths, thus the propagator (in S.P.: time evolution operator) is only time dependent: $\hat{U}(t,t_0)$.

Exercise 1:

1.: $e^{\hat{A}}\cdot e^{-\hat{A}} = \left(\sum_{n=0} \frac{\hat{A}^n}{n!}\right)\cdot\left(\sum_{n=0}\frac{(-\hat{A})^n}{n!}\right) \Bigg| \left(\sum_{i=0} a_i\right)\cdot\left(\sum_{j=0} b_j\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} a_l b_{k-l}$

$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\hat{A}^l}{l!}\frac{(-\hat{A})^{k-l}}{(k-l)!}$

$= \quad (-1)^{k-l}\cdot\hat{A}^k$

$$e^{\hat{A}}\cdot e^{-\hat{A}} = \left(\sum_{n=0}\frac{\hat{A}^n}{n!}\right)\cdot\left(\sum_{n=0}\frac{(-\hat{A})^n}{n!}\right)$$

$$= \left(1 + \sum_{n=1}\frac{\hat{A}^n}{n!}\right)\cdot\left(1+\sum_{n=1}\frac{(-\hat{A})^n}{n!}\right)$$

$$= 1 + \sum_{n=1}\frac{(-\hat{A})^n}{n!} + \sum_{n=1}\frac{\hat{A}^n}{n!} + \left(\sum_{n=1}\frac{\hat{A}^n}{n!}\right)\left(\sum_{n=1}\frac{(-\hat{A})^n}{n!}\right)$$

$$= 1 + \sum_{n=1}\frac{1}{n!}\left[(-\hat{A})^n + \hat{A}^n\right] + \sum_{n=0}^{\infty}\sum_{l=0}^{n+1}\frac{\hat{A}^{l+1}}{(l+1)!}\frac{(-\hat{A})^{n-l}}{(n-l)!}$$

$$= 1 + \sum_{n=1}\frac{\hat{A}^n}{n!}\left[1+(-1)^n\right] + \sum_{n=0}^{\infty}\sum_{l=0}^{n+1}\hat{A}^{n+1}\frac{(-1)^{n-l}}{(l+1)!(n-l)!}$$

$$= 1 + \sum_{n=0}\frac{\hat{A}^{n+1}}{(n+1)!}\left[1+(-1)^{n+1}\right] + \sum_{n=0}^{\infty}\hat{A}^{n+1}\sum_{l=0}^{n+1}\frac{(-1)^{n-l}}{(l+1)!(n-l)!}$$

$$\overset{!}{=} 1$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{\hat{A}^{n+1}}{(n+1)!}\left[1+(-1)^{n+1}\right] + \sum_{n=0}^{\infty}\hat{A}^{n+1}\sum_{l=0}^{n+1}\frac{(-1)^{n-l}}{(l+1)!(n-l)!} = 0$$

$$\Rightarrow \frac{1+(-1)^{n+1}}{(n+1)!} + \sum_{l=0}^{n+1}\frac{(-1)^{n-l}}{(l+1)!(n-l)!} = 0$$

2.:

Exercise 2:

1.: $c_f(t)$ is the coefficient, that „says" how much of $|f\rangle$, $|\psi(t)\rangle$ has:

I: $|\psi(t)\rangle = \sum_n c_n(t)|n\rangle = \sum_{n\neq f} c_n(t)|n\rangle + c_f(t)|f^{(0)}\rangle$

Starting from $|i^{(0)}\rangle = \sum_n c_n(t)|n^{(0)}\rangle = \sum_n \delta_{in}|n^{(0)}\rangle$

$|\psi(t_0)\rangle = |i^{(0)}\rangle \rightsquigarrow |\psi(t)\rangle = |f^{(0)}\rangle$

$\Rightarrow P_{fi}(t) = |\langle f^{(0)}|i^{(0)}\rangle|^2 \quad | \text{(I)}$

$= |c_f(t)|^2$

2.:

$$|\psi(t)\rangle = \sum_n d_n(t)\,|\psi_n^{(0)}(t)\rangle$$
$$= \sum_n d_n(t)\,\hat{U}(t,t_0)|\psi_n^{(0)}(t_0)\rangle$$
$$= \sum_n d_n(t)\,\hat{U}(t,t_0)|n^{(0)}\rangle$$

$|\psi(t)\rangle = \sum_n c_n(t)|n^{(0)}\rangle$, thus:

$$c_n(t) = d_n(t)\,\hat{U}(t,t_0)$$
$$= d_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$$

3.: $E_k^{(0)}|k\rangle = \hat{H}_0|k\rangle$

What is $\dot{d}_f$? $= \frac{d(d_f(t))}{dt}$?

$$\langle f^{(0)}|\hat{H}_1(t)|\psi(t)\rangle = \sum_n \langle f^{(0)}|\hat{H}_1(t)\cdot d_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}|n^{(0)}\rangle$$

$$\hat{H}(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

$$\Rightarrow \hat{H}(t)\,d_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}|n^{(0)}\rangle = i\hbar\frac{\partial}{\partial t}\left(d_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}|n^{(0)}\rangle\right) \quad \text{why?}$$

$$\Rightarrow (\hat{H}_1(t)+\cancel{\hat{H}_0})\,d_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}|n^{(0)}\rangle = i\hbar\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left(\dot{d}_n(t)|f^{(0)}\rangle + \frac{1}{i\hbar}\hat{H}_0\,d_n(t)\,|n^{(0)}\rangle\right)$$
$$= e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left(i\hbar\,\dot{d}_n(t)|f^{(0)}\rangle + \hat{H}_0\,d_n(t)\right)|n^{(0)}\rangle)$$

$$\Rightarrow \hat{H}_1(t)\,d_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}|n^{(0)}\rangle = i\hbar\,\dot{d}_n(t)\,e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}|f^{(0)}\rangle$$
$$\Rightarrow \hat{H}_1(t)\,d_n(t)\,e^{-\frac{i}{\hbar}E_n^{(0)}(t-t_0)}|n^{(0)}\rangle = i\hbar\,\dot{d}_n(t)\,e^{-\frac{i}{\hbar}E_f^{(0)}(t-t_0)}|f^{(0)}\rangle$$
$$\Rightarrow i\hbar\,\dot{d}_n = \sum_n \langle f^{(0)}|\hat{H}_1(t)|n^{(0)}\rangle\,d_n(t)\,e^{-i\omega_f(t-t_0)} \quad \text{with } \omega_f = \frac{E_f^{(0)}-E_n^{(0)}}{\hbar}$$

$\square$

4: