AGT; Sheet 12; Marc Hanus, Angelo Brade;   10.01.2025

Exercise 1.

1.:

Lorentz trafo. $(SO(3,1))$: $(\Lambda : \Lambda^T g \Lambda = g)$ with $g = \mathrm{diag}(1,-1,-1,-1)$

- Associativity: is trivial since associativity is provided by matrix mult..
- Identity element: $\mathbb{1} = \mathrm{diag}(1,1,1,1) \Rightarrow A \cdot \mathbb{1} = A$ with $A \in \mathbb{R}^4$

  (for Λ := Lorentz boost: $\Lambda(u=0) = \mathbb{1}$ since $\gamma(u=0) = 1$ and $\beta(u=0) = 0$.)
- Inverse element: $g = \Lambda g \Lambda^T = (\Lambda g)\Lambda^T = (\Lambda(\Lambda g)^T)^T$

  $\Rightarrow$ for $\Lambda \circ g \to \Lambda \cdot g$ the inverse is

  $\Lambda^{-1} \circ g \to (\Lambda \cdot g^T)^T = g \cdot \Lambda^T$, since

  $\Lambda \circ (\Lambda^{-1} \circ g) \to (\Lambda(\Lambda g)^T)^T = \Lambda g \Lambda^T = g$

Rotation $(SO(3,1))$: $O_z(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- Associativity: is trivial since associativity is provided by matrix mult..
- Identity element: $O_z(\theta = 2\pi n) = \mathbb{1}$ with $n \in \mathbb{N}$

  $\mathbb{1} \cdot A = A$ with $A \in \mathbb{R}^4$
- Inverse element: $O_z^{-1}(\theta) = O_z(-\theta) \Rightarrow O_z(-\theta) \cdot O_z(\theta) = \mathbb{1}$

$$\Rightarrow \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\theta) & \sin(-\theta) & 0 \\ 0 & -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Bigg| \begin{array}{l} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & \cos^2\theta - (-)\sin^2\theta & -(-)\sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Bigg| \begin{matrix} -(-)1 = 1 \\ \sin^2\theta + \cos^2\theta = 1 \end{matrix}$$

$$= \mathrm{diag}(1, 1, 1, 1)$$

$$= \mathbb{1}$$

$\square$

For $O_x(\theta)$ and $O_y(\theta)$ the prove is analogously.
Thus with Λ, $O_z(\theta)$, $O_x(\theta)$ and $O_y(\theta)$ we can form a group: $SO(3,1)$

2.: As shown in part 1, $SO(3,1)$ contains the rotations around the x, y and z axis. Thus this set of all rotations is a subgroup: $SO(3) \in SO(3,1)$ with $SO(3) = (O_i(\theta))_i$ and $SO(3,1) = (O_i(\theta))_i \wedge \Lambda)$
That the set $SO(3)$ forms indeed a group was also shown.

3.: In part 1 we allredy showed that the rotation at $O_z(\theta)$ around the (x,y)-plane forms a group. This is analogously for $O_x(\theta)$ around the (y,z)-plane. So since $O_x(\theta) \in SO(3)$, $O_x(\theta)$ is a subgroup of $SO(3)$. Here the set of all rotations for one axis means: $\{O_i(\theta) : \theta \in \mathbb{R}\}$.
Sucessive rotations α_1 an α_2 in one rotation: $O_x(\alpha_1) \cdot O_x(\alpha_2) = O_x(\alpha_1 + \alpha_2)$.
For that relation the trig. identity $\sin(\alpha_1 + \alpha_2) = \sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2$ and $\cos(\alpha_1 + \alpha_2) = \cos\alpha_1 \sin\alpha_2 - \sin\alpha_1 \cos\alpha_2$ is crucial. The remainding part of the prove is matrix multiplication.

4.: We allredy showed in part 1, that the any rank two tensor Λ satisfying $\Lambda^T g \Lambda = g$ and $\Lambda = \mathbb{1}$ forms a group and together with $SO(3)$ form $SO(3,1)$.

Thus we need to show that Λ being the Lorentz boost:

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{satisfies} \quad \Lambda^T g \Lambda = g \quad \text{and} \quad \Lambda = \mathbb{1}.$$

At this point we should mention, that we use Λ very loosely and talk about it being a set. But Λ is an element of the set we talk about. More correct would be calling

$$\Lambda \in \{\Lambda' | (\Lambda'^T g \Lambda' = g : g = \mathrm{diag}(1,-1,-1,-1)) \wedge (\exists \Lambda : \Lambda = \mathbb{1})\}.$$

Back to the prove:

$$\Lambda^T g \Lambda = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} \cdot g \cdot \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & -\gamma & & 0 \\ & & -1 & \\ & 0 & & -1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & & & \\ & \beta^2\gamma^2 - \gamma^2 & & 0 \\ & & -1 & \\ & 0 & & -1 \end{pmatrix} \quad | \; \gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = \frac{1-\beta^2}{1-\beta^2} = 1$$

$$= g$$

As hinted in part 1: $\Lambda(u=0) = \mathbb{1}$, thus we found our identity element.

Successive boosts β_1 and β_2:

$$\Lambda(\beta_2) \cdot \Lambda(\beta_1) = \begin{pmatrix} \gamma_1 & \beta_1\gamma_1 & & \\ \beta_1\gamma_1 & \gamma_1 & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_2 & \beta_2\gamma_2 & & \\ \beta_2\gamma_2 & \gamma_2 & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\gamma_2 + \beta_1\gamma_1\beta_2\gamma_2 & \gamma_1\beta_1\gamma_2 + \gamma_1\beta_2\gamma_2 & 0 & 0 \\ \gamma_1\beta_1\gamma_2 + \gamma_1\beta_2\gamma_2 & \gamma_1\gamma_2 + \beta_1\gamma_1\beta_2\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow x_\mu = \Lambda^\alpha_\mu(\beta_1)\cdot\Lambda^\nu_\alpha(\beta_2)\cdot x_\nu$$

$$= \begin{pmatrix} \gamma_1\gamma_2+\beta_1\gamma_1\beta_2\gamma_2 & \gamma_1\beta_1\gamma_2+\gamma_1\beta_2\gamma_2 & 0 & 0 \\ \gamma_1\beta_1\gamma_2+\gamma_1\beta_2\gamma_2 & \gamma_1\gamma_2+\beta_1\gamma_1\beta_2\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} ct(\gamma_1\gamma_2+\beta_1\gamma_1\beta_2\gamma_2)+x(\gamma_1\beta_1\gamma_2+\gamma_1\beta_2\gamma_2) \\ ct(\gamma_1\beta_1\gamma_2+\gamma_1\beta_2\gamma_2)+x(\gamma_1\gamma_2+\beta_1\gamma_1\beta_2\gamma_2) \\ y \\ z \end{pmatrix}$$

Or for simply: $x_f = \gamma_2(x-v_2t)$ with $x=\gamma_1(x_i-v_1t)$

$$\Rightarrow x_f = \frac{\frac{x_i-v_1t}{\sqrt{1-\beta_1^2}}-v_2t}{\sqrt{1-\beta_2^2}} = \frac{\frac{x_i-\beta_1ct}{\sqrt{1-\beta_1^2}}-\beta_2ct}{\sqrt{1-\beta_2^2}}$$

5.:

Let β_x and β_y be two sucessive boosts in x and y direction. The order does not matter.

In 2-d space we have: $\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let $\vec{n} = (\cos\theta, \sin\theta)$.

If $\vec{n}_T \perp \vec{n} \Rightarrow \theta_T = \theta + \frac{\pi}{2} \Rightarrow \vec{n}_T = (-\sin\theta, \cos\theta)$.

$\vec{x} = a\vec{n} + b\vec{n}_T$

$\Rightarrow \vec{x}' = \Lambda_{\vec{n}_T}\Lambda_{\vec{n}}\vec{x}$

$= \gamma_x(a-v_xt)\vec{n} + \gamma_y(b-v_yt)\vec{n}_T$

$$\vec{x}'' = O_z\left(\theta' = \theta + \tan^{-1}\left(\frac{v_y}{v_x}\right)\right) \Lambda \, \vec{x}' \qquad \Big| \; v_i = c\beta_i, \quad v = \sqrt{v_x^2 + v_y^2} \Rightarrow \beta = \sqrt{\beta_x^2 + \beta_y^2}$$

$$= O_z\left(\theta' = \theta + \tan^{-1}\left(\frac{v_y}{v_x}\right)\right) \cdot \begin{pmatrix} \gamma(a - vt) \\ \gamma(b - vt) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\theta + \tan^{-1}\left(\frac{v_y}{v_x}\right)\right)\gamma(a - vt) + \sin\left(\theta + \tan^{-1}\left(\frac{v_y}{v_x}\right)\right)\gamma(b - vt) \\ -\sin\left(\theta + \tan^{-1}\left(\frac{v_y}{v_x}\right)\right)\gamma(a - vt) + \cos\left(\theta + \tan^{-1}\left(\frac{v_y}{v_x}\right)\right)\gamma(b - vt) \end{pmatrix}$$

$$= \ldots$$

Essencially we see, that $\gamma \neq \sqrt{\gamma_x^2 + \gamma_y^2}$ thus we cant just do a boost with $v = \left|\left(\begin{smallmatrix} v_x \\ v_y \end{smallmatrix}\right)\right|$. Tbh i dont get why thats not possible. But i dont know what other thing the exercise wants from me. I think there should be a boost that is equivalent to to general boosts, since one could allway just use the absolut value of the total boost velocity and just rotate to the new direction $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$.

Exercise 2:

1.: ~~zz~~: $\tilde{a}\cdot\tilde{b} = a\cdot b$ with $\tilde{a}^\mu = \Lambda^\mu_{\ \nu} a^\nu$ and $\tilde{b}^\mu = \Lambda^\mu_{\ \nu} a^\nu$

$$\tilde{a}\cdot\tilde{b} = \tilde{a}_\mu \cdot \tilde{b}^\mu$$
$$= \tilde{a}_\mu \cdot g_{\mu\nu} \cdot \tilde{b}_\nu$$

maybe

$$= \Lambda^\nu_{\ \mu} a_\nu \cdot g_{\mu\nu} \cdot \Lambda^\mu_{\ \nu} \cdot b_\mu \quad | \ \Lambda^\nu_{\ \mu} \cdot a_\nu = a^\mu \Lambda^\nu_{\ \mu}$$
$$= a_\nu \Lambda^\nu_{\ \mu}\, g_{\mu\nu}\, \Lambda^\mu_{\ \nu} \cdot b_\mu \quad | \ \Lambda^\nu_{\ \mu} = (\Lambda^\mu_{\ \nu})^T$$
$$= a_\nu (\Lambda^\mu_{\ \nu})^T g_{\mu\nu}\, \Lambda^\mu_{\ \nu} \cdot b_\mu$$
$$= a_\nu\, g_{\mu\nu}\, b_\mu$$
$$= a_\nu b^\nu$$
$$= a\cdot b$$

2.: $p^\mu/\hbar = (\gamma m \frac{c}{\hbar}, \gamma \frac{\vec{p}}{\hbar})$

$$= (\gamma k_0, \gamma \vec{k})$$
$$= k^\mu$$

3.: $$p^\mu = k_1^\mu + k_2^\mu$$

$M(v) v^\mu = m_1(u_1)\cdot u_1^\mu + m_2(u_2)\cdot u_2^\mu \ |$ frame of $p \Rightarrow v = 0,\ M(v) = M\cdot\gamma(v) = M\cdot 1$

$M\cdot(c, \vec{0}) = m_1 \gamma(u_1)\cdot u_1^\mu + m_2 \gamma(u_2)\cdot u_2^\mu$

$\Rightarrow M = m_1 \cdot \gamma(u_1) + m_2\, \gamma(u_2)$

Since $u_1 \geq 0 \wedge u_2 \geq 0 \Rightarrow \gamma(u_1) \geq 1 \wedge \gamma(u_2) \geq 1$:

$M \geq m_1 + m_2$

$\square$