AGT; Sheet 8; Marc Hauer, Angelo Brade; 01.12.2024

Quickies:

Q.1: The probability is also life time dependent. thats a guess

Q.2: 1: δ is dependent of the integrand.

2: δ can be rewritten as a Taylor-Approximation

3: δ can be rewritten as discret kronecker-deltas δ_{ij}

Q.3: If the source is far away.

Exercise 1:

1.)

$$\psi(\vec{x},t)=\int d^3k \, \frac{a(\vec{k})}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x}-\omega(|\vec{k}|)t)}$$

Phase: $\phi(t)=\vec{k}\cdot\vec{x}-\omega(|\vec{k}|)t$

$$\Rightarrow \dot{\phi}(t)=\vec{k}\cdot\dot{\vec{x}}-\omega(|\vec{k}|) \overset{!}{=} 0 \quad \text{(const. phase)}$$

$$\Rightarrow |\vec{v}_{ph}| \equiv |\dot{\vec{x}}| = \frac{\omega(|\vec{k}|)}{|\vec{k}|}$$

Phase velocity is perpendicular with wave ($\vec{k}$).

$$\Rightarrow \vec{v}_{ph} = \vec{e}_k \frac{\omega(|\vec{k}|)}{|\vec{k}|}$$

$\square$

2.)

$$v_{gr} \equiv \frac{\partial \omega}{\partial k}$$

$$\vec{k} = \vec{k}_0 + \vec{\delta} \quad \text{with} \quad |\vec{\delta}| \ll |\vec{k}_0|$$

$$\Rightarrow v_{gr} = \left.\frac{\partial \omega(|\vec{k}|)}{\partial |\vec{k}|}\right|_{\vec{k}=\vec{k}_0} \quad \text{since} \quad \vec{k} = \vec{k}_0 + \mathcal{O}(\delta).$$

Group velocity perpendicular to wave ($\vec{k}$):

$$\Rightarrow \vec{v}_{gr} = \vec{e}_{\vec{k}_0} \left.\frac{\partial \omega(|\vec{k}|)}{\partial |\vec{k}|}\right|_{\vec{k}=\vec{k}_0}$$

$\square$

$$\psi(\vec{x},t) = \int d^3k \, \frac{a(\vec{k})}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x} - \omega(|\vec{k}|)t)}$$

With $\vec{k} = \vec{k}_0 + \vec{\delta}$ and $|\vec{\delta}| \ll |\vec{k}_0|$ follows:

I: $|\vec{k}| = |\vec{k}_0 + \vec{\delta}| = |\vec{k}_0| + |\vec{\delta}| + \mathcal{O}(\vec{\delta}^2) \Rightarrow \frac{d|k|}{d|\delta|} \simeq 1 \Rightarrow d^3k \simeq d^3\delta$

II: $a(k) = a(k_0 + \delta)$

III: $\omega(|\vec{k}|) = \omega(|\vec{k}_0 + \vec{\delta}|) = \omega(|\vec{k}_0|) + \mathcal{O}(\delta) \simeq \omega(|\vec{k}_0|)$

thus $\psi(\vec{x},t)$ approximates to:

$$\psi(\vec{x},t) \simeq \int d^3\delta \, \frac{a(\vec{k}_0 + \vec{\delta})}{(2\pi)^3} e^{i((\vec{k}_0 + \vec{\delta})\vec{x} - \omega(|\vec{k}_0|)\cdot t)}$$

Assume: $\vec{k}\cdot\vec{v}_{ph}(\vec{k}) = \overbrace{\vec{k}\cdot\vec{e}_{\vec{k}}}^{\vec{k}=\vec{e}_{\vec{k}}\cdot|\vec{k}|} \frac{\omega(|\vec{k}|)}{|\vec{k}|} = \omega(|\vec{k}|) \overset{(III)}{\simeq} \omega(|\vec{k}_0|) = \vec{k}_0\cdot\vec{v}_{ph}(\vec{k}_0)$

With $\vec{\delta}\cdot\vec{e}_{\vec{k}_0} = |\vec{\delta}||\vec{e}_{\vec{k}_0}|\overbrace{\cos\theta}^{\substack{\theta \ll 1 \\ \cos\theta \simeq 1 + \mathcal{O}(\theta)}} \simeq |\vec{\delta}|\underbrace{|\vec{e}_{\vec{k}_0}|}_{1} = |\vec{\delta}|$ and $\overbrace{\frac{\partial\omega}{\partial|\vec{\delta}|} \simeq \frac{\omega}{|\vec{\delta}|} + \mathcal{O}}^{\text{this, i actually don't know why}}$ we also get:

$$\vec{\delta}\cdot\vec{v}_{gr}(\vec{k}) = \vec{\delta}\cdot\vec{e}_{\vec{k}_0} \left.\frac{d\omega}{d|\vec{k}|}\right|_{\vec{k}=\vec{k}_0} = \underbrace{\vec{\delta}\,\vec{e}_{\vec{k}_0}}_{\simeq|\vec{\delta}|} \underbrace{\frac{\partial\omega}{\partial|\vec{\delta}|}}_{\simeq\frac{\omega}{|\vec{\delta}|}} \underbrace{\frac{d|\vec{\delta}|}{d|\vec{k}|}}_{\simeq 1} \simeq \omega(|\vec{k}_0|)$$

$$\text{Thus } \psi(\vec{x},t) \simeq \int d^3\delta \, \frac{a(\vec{k}_0+\vec{\delta})}{(2\pi)^3} \underbrace{e^{i\vec{k}_0[\vec{x} - \vec{v}_{ph}(|\vec{k}_0|)t]}}_{\text{independent of } \delta} e^{i\vec{\delta}[\vec{x} - \vec{v}_{gr}\cdot t]}$$

$$\simeq e^{i\vec{k}_0[\vec{x} - \vec{v}_{ph}(|\vec{k}_0|)t]} \int d^3\delta \, \frac{a(\vec{k}_0+\vec{\delta})}{(2\pi)^3} e^{i\vec{\delta}[\vec{x} - \vec{v}_{gr}\cdot t]}.$$

$\square$

3.)

If $\vec{\tilde{x}} = \vec{x} - \vec{v}_{gr}\cdot t$:

$$\psi(\vec x,t) = e^{i\vec k_0\cdot[\vec x - \vec v_{ph}(|\vec k_0|)t]} \int d^3\delta \, \frac{a(\vec k_0+\vec\delta)}{(2\pi)^3}\, e^{i\vec\delta\cdot[\vec x - \vec v_{gr}\cdot t]} \quad \Big| \; \vec\delta = \vec k - \vec k_0 , \; \frac{d|\vec k|}{d|\vec\delta|} \simeq 1$$

$$\simeq e^{i\vec k_0\cdot[\cancel{\vec x} - \vec v_{ph}(|\vec k_0|)t]}\, e^{-i\vec k_0[\cancel{\vec x} - \vec v_{gr} t]} \underbrace{\int d^3k \, \frac{a(\vec k_0+\vec\delta)}{(2\pi)^3}\, e^{i\vec k[\vec x - \vec v_{gr}\cdot t]}}_{=\psi(\vec x - \vec v_{gr}\cdot t,\; t=0)}$$

$$= \psi(\vec x - \vec v_{gr}\cdot t, 0)\; e^{it\vec k_0[\vec v_{gr} - \vec v_{ph}(|\vec k_0|)]}$$

$\square$

4.)

$$\vec v_{ph} = \vec e_{\vec k}\,\frac{\omega(|\vec k|)}{|\vec k|}, \quad \vec v_{gr} = \vec e_{\vec k_0}\,\frac{\partial\omega(|\vec k|)}{\partial|\vec k|}\Big|_{\vec k=\vec k_0}, \quad \omega(|\vec k|) = E(|\vec k|)/\hbar$$

i) $E(\vec k) = \hbar c|\vec k|$

$\Rightarrow \vec v_{ph} = \vec e_{\vec k}\, c, \quad \vec v_{gr} = \vec e_{\vec k_0}\, c$

ii) $E(\vec k) = \hbar^2|\vec k|^2/(2\mu)$

$\Rightarrow \vec v_{ph} = \vec e_{\vec k}\cdot\frac{\hbar|\vec k|}{2\mu}, \quad \vec v_{gr} = \vec e_{\vec k_0}\cdot\frac{\hbar|\vec k|}{\mu}$

Since $\vec k \simeq \vec k_0 \Rightarrow \vec k\cdot\vec k_0 \simeq |\vec k_0|^2 \simeq |\vec k|^2$

$\Rightarrow e^{it\vec k_0(\vec v_{gr} - \vec v_{ph})} = e^{it|\vec k_0|^2(\frac{\hbar}{\mu} - \frac{\hbar}{2\mu})} = e^{it\frac{E(|\vec k_0|)}{\hbar}}$

$\square$

5.) $E(|\vec k|) = \sqrt{\mu^2c^4 + \hbar^2\vec k^2c^2}$

$$\Rightarrow \vec v_{ph} = \vec e_{\vec k}\, c\sqrt{\frac{\mu^2c^2}{\hbar^2|\vec k|^2}+1}, \quad \vec v_{gr} = \vec e_{\vec k_0}\,\frac{\hbar|\vec k|c^2}{\sqrt{\mu^2c^4+\hbar^2\vec k^2c^2}} = \vec e_{\vec k_0}\, c\,\frac{1}{\sqrt{\left(\frac{\mu c}{\hbar|\vec k|}\right)^2+1}}$$

$\mu \to 0$: $\vec v_{ph} = \vec e_{\vec k}\, c, \quad \vec v_{gr} = \vec e_{\vec k_0}\, c$

As $|\vec k| \to 0$: $\vec v_{ph} = \vec e_{\vec k}\,\frac{\sqrt{\mu^2c^4+\hbar^2\vec k^2c^2}}{\hbar|\vec k|} \overset{\mu^2c^4 \ll \hbar^2|\vec k|^2c^2}{\simeq} \vec e_{\vec k}\,\frac{\sqrt{\hbar^2|\vec k|^2c^2}}{\hbar|\vec k|} = \vec e_{\vec k}\, c$

$\square$

2) Scattering on a constant potential

$$V(\vec{x}) = \begin{cases} V_0 & \text{for } |\vec{x}| < r_0 \\ 0 & \text{for } |\vec{x}| \geq r_0 \end{cases}$$

1) $f_{\vec{k}} = -\frac{2\mu}{|\vec{q}|\hbar^2} \int_0^\infty dr' \, V(r') \, r' \sin(|\vec{q}| r')$

$\Rightarrow f_{\vec{k}} = -\frac{2\mu}{|\vec{q}|\hbar^2} \int_0^{r_0} dr' \, V_0 \, r' \sin(|\vec{q}| r')$

$= -\frac{2\mu V_0}{|\vec{q}|\hbar^2} \left[-\frac{r'}{|\vec{q}|} \cos(|\vec{q}| r') + \frac{1}{|\vec{q}|} \int dr' \cos(|\vec{q}| r') \right]_0^{r_0}$

$= -\frac{2\mu V_0}{|\vec{q}|^3 \hbar^2} \left[-r' \cos(|\vec{q}| r') + \frac{1}{|\vec{q}|} \sin(|\vec{q}| r') \right]_0^{r_0}$

$= \underbrace{-\frac{2\mu V_0}{|\vec{q}|^3 \hbar^2}}_{=: A/|\vec{q}|^3} \left[\sin(|\vec{q}| \cdot r_0) - |\vec{q}| r_0 \cos(|\vec{q}| r_0) \right]$

- $\lim_{|\vec{q}| \to 0} f_{\vec{k}} = \lim_{|\vec{q}| \to 0} \left(\frac{A \sin(|\vec{q}| r_0) - A r_0 |\vec{q}| \cos(|\vec{q}| r_0)}{|\vec{q}|^3} \right)$ | L'Hospital

$= \lim_{|\vec{q}| \to 0} \left(\frac{A r_0 \cos(|\vec{q}| r_0) - A r_0 \cos(|\vec{q}| r_0) + A r_0^2 |\vec{q}| \sin(|\vec{q}| r_0)}{3|\vec{q}|^2} \right)$

$= \lim_{|\vec{q}| \to 0} \left(\frac{A r_0^2 |\vec{q}| \sin(|\vec{q}| r_0)}{3|\vec{q}|^2} \right)$ | L'Hospital

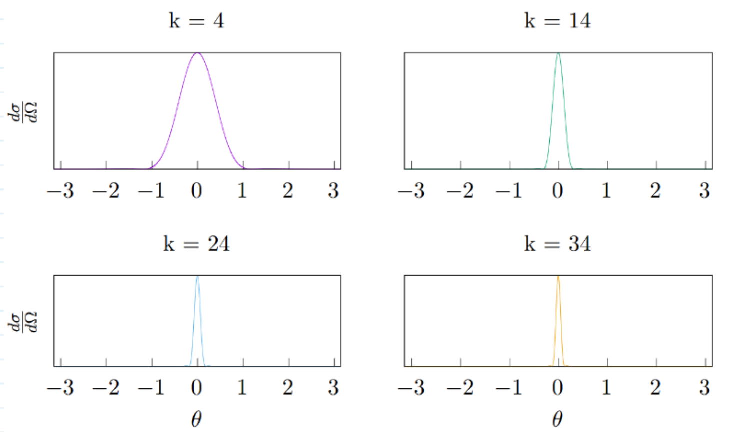
$= \lim_{|\vec{q}| \to 0} \left(\frac{A r_0^2 \sin(|\vec{q}| r_0) + A r_0^3 |\vec{q}| \cos(|\vec{q}| r_0)}{6|\vec{q}|} \right)$ | L'Hospital

$= \lim_{|\vec{q}| \to 0} \left(\frac{A r_0^3 \cancelto{1}{\cos(|\vec{q}| r_0)} + A r_0^3 \cancelto{1}{\cos(|\vec{q}| r_0)} - \cancelto{0}{A r_0^4 |\vec{q}| \sin(|\vec{q}| r_0)}}{6} \right)$

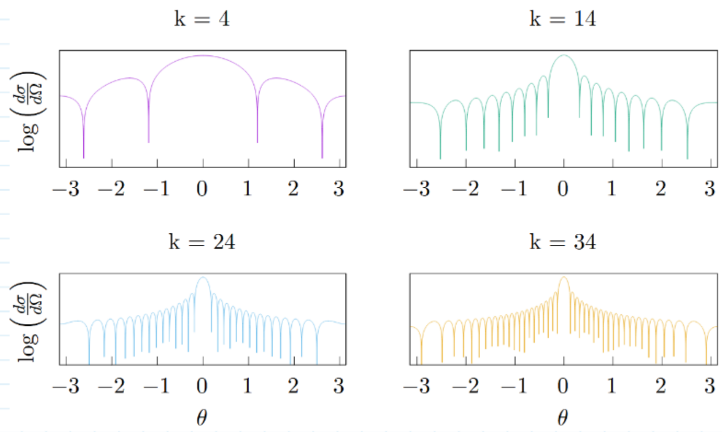
$= \frac{A r_0^3}{3} \quad \propto r_0^3$

2) $|\vec{q}| = 2k \sin\left(\frac{\theta}{2}\right)$ where $k = |\vec{k}|$

- Plot of the cross section $\frac{d\sigma}{d\Omega}$:



- Plot of $\log\left(\frac{d\sigma}{d\Omega}\right)$ to visualize the zeros:



- For larger k the number of zero points, and so the number of maxima and minima becomes large.

3) Show: $\frac{\mu V_0 r_0^2}{\hbar^2} \ll 1$

- $|f_{\vec{k}}(\theta, \varphi)| \ll r_0$

$\Rightarrow \left| -\frac{2\mu V_0 r_0^3}{3\hbar^2} \right| \ll r_0$

$\Rightarrow \frac{2}{3} \frac{\mu |V_0| r_0^2}{\hbar^2} \ll 1$

$\Rightarrow \frac{\mu |V_0| r_0^2}{\hbar^2} \ll 1 \quad \square$