Blatt 4)

Aufgabe 1) $\displaystyle\sum_{k=0}^{+\infty} k z^k \sum_{k=0}^{+\infty} z^k \overset{\forall z,\ |z|<1 \text{ (Produkt von Cauchy)}}{=} \sum_{n=0}^{+\infty} \sum_{k=0}^{n} k z^k z^{n-k}$ 1

$$= \sum_{n=0}^{+\infty} z^n \underbrace{\sum_{k=0}^{n} \cancel{z}k}_{= \frac{n(n+1)}{2}}$$

$$= \sum_{n=0}^{+\infty} \frac{n(n+1)}{2} z^n$$

$\forall z_1, z_2 \in \mathbb{C},\quad e^{z_1} e^{z_2} \overset{\text{Def.}}{=} \left(\sum_{k=0}^{+\infty} \frac{z_1^k}{k!}\right)\left(\sum_{k=0}^{+\infty} \frac{z_2^k}{k!}\right)$

$$\overset{\text{Produkt von Cauchy}}{=} \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} \left(\times \frac{n!}{n!}\right)$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

$$\overset{\text{Binom}}{=} \sum_{n=0}^{+\infty} \frac{(z_1+z_2)^n}{n!} \overset{\text{Def.}}{=} e^{z_1+z_2}$$

Aufgabe 2)

(i) $f(z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!}$, Potenzreihen sind holomorphe Funktionen

$\Rightarrow f'(z) \overset{①}{=} \sum_{k=1}^{+\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{+\infty} \frac{z^k}{k!} = e^z$

und $f''(z) = \left(f'(z)\right)' = \left(\sum_{k=0}^{+\infty} \frac{z^k}{k!}\right)' \overset{①}{=} \sum_{k=0}^{+\infty} \frac{z^k}{k!}$

und gleich $f'''(z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!}$

(ii) $f(z) = \sum_{k=0}^{+\infty} 2^k z^k$ Potenzreihen sind holomorphe $\forall z, |z|<R$ $\Rightarrow \forall z, |z| < \frac{1}{2}$

$f'(z) = \sum_{k=1}^{+\infty} k 2^k z^{k-1}$

$f''(z) = \sum_{k=2}^{+\infty} k(k-1) 2^k z^{k-2}$

$f'''(z) = \sum_{k=3}^{+\infty} k(k-1)(k-2) 2^k z^{k-3}$

(iii) $\forall z, |z|<1$, $f(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k} \Rightarrow f'(z) = \sum_{k=1}^{+\infty} z^{k-1} = \frac{1}{1-z}$

und $f''(z) = \left(\sum_{k=0}^{+\infty} z^k\right)' = \sum_{k=1}^{+\infty} k z^{k-1}$ und $f'''(z) = \sum_{k=2}^{+\infty} k(k-1) z^{k-2}$

Aufgabe 3) $\sum_{k=0}^{+\infty} 2^{-k}(i+z)^k = \sum_{k=0}^{+\infty} \left(\frac{i+z}{2}\right)^k = \frac{1}{1-\left(\frac{i+z}{2}\right)}$ für $\left|\frac{i+z}{2}\right|<1$

$$= \frac{1}{1-\frac{i}{2}-\frac{z}{2}} = \frac{1}{\left(1-\frac{i}{2}\right)\left[1-\frac{z}{2\left(1-\frac{i}{2}\right)}\right]} \overset{\left|\frac{z}{2\left(1-\frac{i}{2}\right)}\right|<1}{=} \sum_{k=0}^{+\infty} \frac{1}{\left(1-\frac{i}{2}\right)}\left[\frac{1}{2\left(1-\frac{i}{2}\right)}\right]^k z^k$$

$$= \sum_{k=0}^{+\infty} \frac{1}{2^k}\left[\frac{1}{\left(1-\frac{i}{2}\right)}\right]^{k+1} z^k$$

Aufgabe 4)

(i) $z \in \mathbb{C}$ $e^{iz} \overset{\text{Def}}{=} \sum_{k=0}^{+\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{+\infty} \frac{i^{2k} z^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{i^{2k+1} z^{2k+1}}{(2k+1)!}$

$$= \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\overset{\text{Def}}{=} \cos(z) + i \sin(z)$$

(ii) $\begin{cases} e^{iz} \overset{i)}{=} \cos(z) + i\sin(z) \\ e^{-iz} \overset{i),\, z \to -z}{=} \cos(z) - i\sin(z) \end{cases} \Rightarrow \begin{cases} \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \end{cases}$

$$\sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) = \frac{(e^{iz_1} - e^{-iz_1})(e^{iz_2} + e^{-iz_2})}{4i} + \frac{(e^{iz_1} + e^{-iz_1})(e^{iz_2} - e^{-iz_2})}{4i}$$

$$= \Big[e^{i(z_1+z_2)} + \cancel{e^{i(z_1-z_2)}} - \cancel{e^{i(z_2-z_1)}} - e^{-i(z_1+z_2)} + e^{i(z_1+z_2)} - \cancel{e^{i(z_1-z_2)}}$$
$$+ \cancel{e^{i(z_2-z_1)}} - e^{-i(z_1+z_2)}\Big] \frac{1}{4i}$$

$$= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \sin(z_1 + z_2)$$

(iii) $\cos^2(z) + \sin^2(z) = \frac{(e^{iz} + e^{-iz})^2 - (e^{iz} - e^{-iz})^2}{4} = \frac{e^{2iz} + \cancel{e^{-2iz}} + 2 - e^{2iz} - \cancel{e^{-2iz}} + 2}{4} = 1$