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1) Free Particle Propagator 1: Short-Time Evolution

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, U(x,x',t)\, \psi(x',0) \quad ; \quad U(x,x',t) = \underbrace{\sqrt{\frac{m}{2i\pi\hbar t}}}_{:=A} \exp\left[\frac{i(x-x')^2 m}{2\hbar t}\right] = A\cdot \exp\left[-\frac{m(x-x')^2}{2\hbar i t}\right]$$

$$\cdot\ \psi(x',0) = \psi(x,0) + (x'-x)\left.\frac{\partial\psi(x',0)}{\partial x'}\right|_{x'=x} + \frac{1}{2}(x'-x)^2 \left.\frac{\partial^2\psi(x',0)}{\partial x'^2}\right|_{x'=x} + \mathcal{O} \quad \rightarrow \text{Taylor expansion}$$

$$\Rightarrow \psi(x,t) = A \int_{-\infty}^{\infty} dx' \exp\left[-\frac{m(x-x')^2}{2\hbar i t}\right]\cdot\left\{\psi(x,0) + (x'-x)\left.\frac{\partial\psi(x',0)}{\partial x'}\right|_{x'=x} + \frac{1}{2}(x'-x)^2\left.\frac{\partial^2\psi(x',0)}{\partial x'^2}\right|_{x'=x} + \mathcal{O}\right\}$$

$$= A\cdot\left\{\sqrt{\frac{2\pi\hbar i t}{m}}\,\psi(x,0) + \frac{\hbar i t}{m}\left.\frac{\partial\psi(x',0)}{\partial x'}\right|_{x'=x} + \frac{1}{4}\sqrt{\frac{(2\hbar i t)^3\pi}{m^3}}\left.\frac{\partial^2\psi(x',0)}{\partial x'^2}\right|_{x'=x} + \mathcal{O}\right\}$$

$$= \psi(x,0) + \sqrt{\frac{\hbar i t}{2\pi m}}\left.\frac{\partial\psi(x',0)}{\partial x'}\right|_{x'=x} + \frac{\hbar i t}{2m}\left.\frac{\partial^2\psi(x',0)}{\partial x'^2}\right|_{x'=x} + \mathcal{O}$$

Gaussian Integrals:

$$\cdot \int_{-\infty}^{\infty} dx\, e^{-a(x+b)^2} = \sqrt{\frac{\pi}{a}}$$

$$\cdot \int_{-\infty}^{\infty} dx\, x\, e^{-ax^2} = \frac{1}{2a}$$

$$\cdot \int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\Rightarrow \lim_{t\to 0}\psi(x,t) = \lim_{t\to 0}\int_{-\infty}^{\infty} dx'\, U(x,x',t)\,\psi(x',0) = \psi(x,0) + \cancelto{0}{\sqrt{\frac{\hbar i t}{2\pi m}}}\left.\frac{\partial\psi(x',0)}{\partial x'}\right|_{x'=x} + \cancelto{0}{\frac{\hbar i t}{2m}}\left.\frac{\partial^2\psi(x',0)}{\partial x'^2}\right|_{x'=x} = \psi(x,0)$$

$$\Rightarrow U(x,x',t) \xrightarrow{t\to 0} \delta(x-x') \quad \text{, because} \quad \lim_{t\to 0}\int_{-\infty}^{\infty} dx'\, U(x',x,t)\,\psi(x',0) = \psi(x,0)$$

2) Free Particle Propagator: Late-Time Evolution

1) $\psi(x,0) = N\exp\left[-\frac{(x-x_0)^2}{4\sigma^2}\right]$

$$\cdot\ 1 \overset{!}{=} \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, |\psi(x,0)|^2 = N^2\int_{-\infty}^{\infty} dx \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] = N^2\cdot\sqrt{2\sigma^2\pi}$$

$$\Rightarrow N = (2\sigma^2\pi)^{-\frac{1}{4}}$$

2)
$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, U(x',x,t)\,\psi(x',0)$$
$$= N\cdot A\cdot\int dx' \exp\left[-\frac{m(x-x')^2}{2i\hbar t} - \frac{(x'-x_0)^2}{4\sigma^2}\right]$$
$$= N\cdot A\cdot\exp\left[\frac{mix^2}{2\hbar t} - \frac{x_0^2}{4\sigma^2}\right]\cdot\int dx' \exp\left[\left(\frac{mx}{i\hbar t} + \frac{x_0}{2\sigma^2}\right)x' + \left(\frac{mi}{2\hbar t} - \frac{1}{4\sigma^2}\right)x'^2\right] \quad \Big| \int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}$$
$$= N\cdot A\cdot\exp\left[\frac{mix^2}{2\hbar t} - \frac{x_0^2}{4\sigma^2}\right]\cdot\exp\left[\left(\frac{mx}{i\hbar t} + \frac{x_0}{2\sigma^2}\right)^2\cdot\frac{1}{4}\left(\frac{1}{4\sigma^2} - \frac{mi}{2\hbar t}\right)^{-1}\right]\cdot\left(\frac{\pi}{\frac{1}{4\sigma^2} - \frac{mi}{2\hbar t}}\right)^{\frac{1}{2}} \quad \Big| \left(\frac{1}{4\sigma^2} - \frac{mi}{2\hbar t}\right)^{-1} = \left(\frac{2\hbar t - 4\sigma^2 mi}{8\hbar\sigma^2 t}\right)^{-1} = \frac{4\hbar\sigma^2 t}{\hbar t - 2\sigma^2 mi}$$
$$= \left(\frac{1}{2\sigma^2\pi}\right)^{\frac{1}{4}}\cdot\left(\frac{m}{2i\pi\hbar t}\right)^{\frac{1}{2}}\cdot\left(\frac{4\pi\hbar\sigma^2 t}{\hbar t - 2\sigma^2 mi}\right)^{\frac{1}{2}}\cdot\exp\left[\frac{\hbar\sigma^2 t}{\hbar t - 2\sigma^2 mi}\left(\left(\frac{mx}{i\hbar t}\right)^2 + \left(\frac{x_0}{2\sigma^2}\right)^2 + \frac{mxx_0}{i\hbar t\sigma^2}\right) - \frac{mx^2}{2\hbar i t} - \frac{x_0^2}{4\sigma^2}\right]$$
$$= \left[\left(\frac{1}{2\sigma^2\pi}\right)^{\frac{1}{2}}\cdot\frac{2\sigma^2 m}{i\hbar t + 2\sigma^2 m}\right]^{\frac{1}{2}}\cdot\exp\left[\left(\frac{\hbar t}{\hbar t - 2\sigma^2 mi} - 1\right)\frac{x_0^2}{4\sigma^2} + \frac{mxx_0}{i\hbar t + 2\sigma^2 m} + \left(\frac{2m\sigma^2}{i\hbar t + 2\sigma^2 m} - 1\right)\frac{mx^2}{2\hbar i t}\right]$$
$$= \left[\frac{2\sigma^2 m^2}{\pi(i\hbar t + 2\sigma^2 m)^2}\right]^{\frac{1}{4}}\cdot\exp\left[\frac{1}{i\hbar t + 2\sigma^2 m}\left(-\frac{mx_0^2}{2} + mx_0 x - \frac{mx^2}{2}\right)\right]$$
$$= \left[\frac{2\sigma^2}{\pi\left(i\frac{\hbar t}{m} + 2\sigma^2\right)^2}\right]^{\frac{1}{4}}\cdot\exp\left[-\frac{(x-x_0)^2}{2\left(i\frac{\hbar t}{m} + 2\sigma^2\right)}\right] \quad \Big| \quad \frac{\hbar t}{m} \gg \sigma^2$$

$$\Rightarrow \psi(x,t) \approx (2\pi\sigma)^{-\frac{1}{4}}\cdot\exp\left[-\frac{(x-x_0)^2}{4\sigma^2}\right]$$

t independent ?