Theo 4; Marc Hauer, Franka Weronski, Angelo Brade; 29.10.2024

Aufgabe 1:

a)

$$f(x) = x^\alpha \Rightarrow z \equiv f'(x) = \alpha x^{\alpha-1} \Rightarrow x = \left(\frac{z}{\alpha}\right)^{\frac{1}{\alpha-1}} \equiv h(z)$$

$$\mathcal{L}[f](x) = f(h(x)) - h(z)z \quad | \ f(x) = x^\alpha \text{ mit } \alpha \in \mathbb{R}_{>1}$$

$$= \left[\left(\frac{z}{\alpha}\right)^{\frac{1}{\alpha-1}}\right]^\alpha - \left(\frac{z}{\alpha}\right)^{\frac{1}{\alpha-1}} z$$

$$= \left(\frac{z}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{z}{\alpha} z^{\alpha-1}\right)^{\frac{1}{\alpha-1}}$$

$$= \left(\frac{z}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{z^\alpha}{\alpha}\right)^{\frac{1}{\alpha-1}} = \left(\frac{z}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{z^\alpha}{\alpha}\frac{\alpha^{\alpha-1}}{\alpha^{\alpha-1}}\right)^{\frac{1}{\alpha-1}} = \left(\frac{z}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{z}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\alpha = \left(\frac{z}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha)$$

b)

$$f(\omega) = \mathcal{L}[\mathcal{L}[f]](\omega)$$

$$= \mathcal{L}[f(h(z)) - h(z)z](\omega)$$

$$= f(h(z)) - h(z)z + g(\omega)\omega \quad | \ z = g(\omega), \ \omega := \mathcal{L}[f]'(z)$$

Warum „+"?

$$= f(h(g(\omega))) - h(g(\omega))g(\omega) + g(\omega)\omega \quad | \ h(g(\omega)) = \omega$$

$$= f(\omega) - \omega g(\omega) + g(\omega)\omega$$

$$= f(\omega)$$

c)

$$\mathcal{L}[f](0) = f(h(0)) - h(0)\cdot 0$$

$$= f(h(0)) \qquad | \ z = 0 \Rightarrow f'(x) = 0 \Rightarrow x\Big|_{f'(x)=0} = x_0 \Rightarrow h(z)\Big|_{z=0} = x\Big|_{x=x_0}$$

$$= f(x)\Big|_{x=x_0}$$

Aufgabe 2:

a)

$$U(S,V,N) = TS - pV + \mu N$$

$$F(T,V,N) = \mathcal{L}[U(S;V,N)](T;V,N)$$
$$= U(S;V,N) - T\left(\frac{\partial U}{\partial T}\right)_{V,N} \qquad \Big| \left(\frac{\partial U}{\partial T}\right)_{V,N} = S$$
$$= U(S;N,V) - TS$$
$$= \mu N - pV$$

$$dF = dU - TdS - SdT$$
$$= \mu dN - SdT + pdV$$

$$a_i = \frac{\partial F}{\partial x_i} \wedge \frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$$

$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_{T,N} = \left(\frac{\partial p}{\partial T}\right)_{N,V}, \quad \left(\frac{\partial S}{\partial N}\right)_{V,T} = -\left(\frac{\partial \mu}{\partial T}\right)_{V,N}, \quad \left(\frac{\partial p}{\partial N}\right)_{V,T} = -\left(\frac{\partial \mu}{\partial V}\right)_{N,T}$$

b)

$$\phi(T,V,\mu) = \mathcal{L}[U(S,V,N)](T,V,\mu)$$
$$= U(S,V,N) - T\left(\frac{\partial U}{\partial T}\right)_{V,N} - \mu\left(\frac{\partial U}{\partial \mu}\right)_{S,V}$$
$$= U - TS - \mu N$$
$$= -pV$$

$$\Rightarrow d\phi = dU - SdT - TdS - \mu dN - Nd\mu \qquad \Big| dU = TdS - pdV + \mu dN$$
$$= -SdT - pdV - Nd\mu$$

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$$

$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_{T,\mu} = \left(\frac{\partial p}{\partial T}\right)_{V,\mu}, \quad \left(\frac{\partial S}{\partial \mu}\right)_{T,V} = \left(\frac{\partial N}{\partial T}\right)_{\mu,T}, \quad \left(\frac{\partial p}{\partial \mu}\right)_{T,V} = \left(\frac{\partial N}{\partial V}\right)_{\mu,T}$$

c) $U = TS - pV + \mu_i N_i$

$$\Rightarrow \Lambda(\mu, p, T) = \mathcal{L}[U(S,V,N)](\mu,p,T)$$

$$= U(S,V,N) - T\left(\frac{\partial U}{\partial T}\right)_{V,N} - p\left(\frac{\partial U}{\partial p}\right)_{S,N} - \mu_i\left(\frac{\partial U}{\partial \mu_i}\right)_{S,V}$$

I: $= U - TS + pV - \mu_i N_i \qquad | \, U = TS - pV + \mu_i N_i$

$= 0$

$$\Rightarrow d\Lambda(\mu,p,T) = dU - SdT + Vdp - N_i d\mu_i - TdS + pdV - \mu_i dN_i \quad | \, dU = TdS - pdV + \mu_i dN_i$$

$$= \cancel{TdS} - \cancel{pdV} + \cancel{\mu_i dN_i} - SdT + Vdp - N_i d\mu_i - \cancel{TdS} + \cancel{pdV} - \cancel{\mu_i dN_i}$$

$$= Vdp - SdT - N_i d\mu_i \quad | \, \Lambda(\mu,p,T) = 0 \Rightarrow d\Lambda = 0$$

$$\Rightarrow N_i d\mu_i = Vdp - SdT$$

$\square$

<u>Aufgabe 3:</u>

a) $U(T,V)$

$$\Rightarrow dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

b)

$$\left(\frac{\partial U}{\partial V}\right)_T = -\frac{\alpha}{V}U + \frac{\alpha T}{V}\left(\frac{\partial U}{\partial T}\right)_V \quad | \, U(T,V) = \frac{1}{V^\alpha}\phi(TV^\alpha)$$

$$\Rightarrow \cancel{-\alpha V^{-\alpha-1}\phi(TV^\alpha)} + \frac{1}{V^\alpha}\left(\frac{\partial \phi(TV^\alpha)}{\partial V}\right)_T = \cancel{-\frac{\alpha}{V^{\alpha+1}}\phi(TV^\alpha)} + \frac{\alpha T}{V^{\alpha+1}}\left(\frac{\partial \phi(TV^\alpha)}{\partial T}\right)_V$$

$$\Rightarrow \frac{1}{V^\alpha}\left(\frac{\partial \phi(u)}{\partial u}\frac{\partial u}{\partial V}\right)_T = \frac{\alpha T}{V^{\alpha+1}}\left(\frac{\partial \phi(u)}{\partial u}\frac{\partial u}{\partial T}\right)_V$$

$$\Rightarrow \frac{1}{V^\alpha}\phi'(TV^\alpha)\,\alpha V^{\alpha-1}\cdot T = \frac{\alpha T}{V^{\alpha+1}}\phi'(TV^\alpha)\,V^\alpha$$

$$\Rightarrow \frac{\alpha T}{V}\phi'(TV^\alpha) = \frac{\alpha T}{V}\phi'(TV^\alpha)$$

$\square$

c)