

Questions:

1.) passive: transformation refers to same points in phase space before and after

active: transformation refers to different points -1-

2.) $g = L_2$

3.) i: Proton has charge:

$$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \text{ and } \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \text{ with } \begin{cases} \vec{A}(\vec{x}, t) \rightarrow \vec{A}(\vec{x}, t) + \vec{\nabla} \lambda(\vec{x}, t) \\ V(\vec{x}, t) \rightarrow V(\vec{x}, t) - \frac{\partial \lambda(\vec{x}, t)}{\partial t} \end{cases}$$

ii: Neutron does not have charge:

$$\text{Gauge: } \vec{E} = 0 = \vec{B}$$

1) Propagator of the Harmonic Oscillator

$$L = \frac{1}{2} m (\dot{x}^2 - \omega^2 x^2)$$

1) Show: $S_{cl}(x, x'; t) = \frac{m\omega}{2\sin(\omega t)} [\cos(\omega t)(x^2 + x'^2) - 2xx']$

$S_{cl} = \int_{t_0}^t dt' L(x_{cl}, \dot{x}_{cl}, t') = \frac{m}{2} \int_{t_0}^t dt' (\dot{x}_{cl}^2(t') - \omega^2 x_{cl}^2(t'))$ with $t_0 = 0$

$$\begin{cases} x_{cl}(t') = a \cos(\omega t') + b \sin(\omega t') & \text{with } x_{cl}(0) = x \text{ and } x_{cl}(t) = x' \\ \Rightarrow x_{cl}(0) = a = x & \text{and } x_{cl}(t) = a \cos(\omega t) + b \sin(\omega t) = x' \Rightarrow b = \frac{x' - a \cos(\omega t)}{\sin(\omega t)} = x' \cot(\omega t) - a \cot(\omega t) \\ \Rightarrow x_{cl}(t') = x \cos(\omega t') + x' \cot(\omega t) \sin(\omega t') - x \cot(\omega t) \sin(\omega t') = x \cos(\omega t') + (x' - x \cos(\omega t)) \frac{\sin(\omega t')}{\sin(\omega t)} \\ \Rightarrow \dot{x}_{cl}(t') = -x\omega \sin(\omega t') + \omega (x' - x \cos(\omega t)) \frac{\cos(\omega t')}{\sin(\omega t)} \end{cases}$$

$$\begin{aligned} \Rightarrow S_{cl} &= \frac{m}{2} \int_0^t dt' \left\{ \dot{x}_{cl}^2 \sin^2(\omega t') + \omega^2 (x' - x \cos(\omega t))^2 \frac{\cos^2(\omega t')}{\sin^2(\omega t)} - \omega^2 x (x' - x \cos(\omega t)) \frac{\sin(2\omega t')}{\sin(\omega t)} - \omega^2 [x^2 \cos^2(\omega t') + (x' - x \cos(\omega t))^2 \frac{\sin^2(\omega t')}{\sin^2(\omega t)} + x (x' - x \cos(\omega t)) \frac{\sin(2\omega t')}{\sin(\omega t)}] \right\} \\ &= \frac{m\omega^2}{2} \int_0^t dt' \left\{ -x^2 \cos(2\omega t') + (x' - x \cos(\omega t))^2 \frac{\cos(2\omega t')}{\sin^2(\omega t)} - 2x(x' - x \cos(\omega t)) \frac{\sin(2\omega t')}{\sin(\omega t)} \right\} \\ &= \frac{m\omega^2}{2} \int_0^t dt' \left\{ [(x' - x \cos(\omega t))^2 \frac{1}{\sin^2(\omega t)} - x^2] \cos(2\omega t') - 2x(x' - x \cos(\omega t)) \frac{\sin(2\omega t')}{\sin(\omega t)} \right\} \\ &= \frac{m\omega}{4} \left\{ [(x' - x \cos(\omega t))^2 \frac{1}{\sin^2(\omega t)} - x^2] \sin(2\omega t) + [2x(x' - x \cos(\omega t)) \frac{1}{\sin(\omega t)}] (\cos(2\omega t) - 1) \right\} \quad | \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x) \\ &= \frac{m\omega}{4} \left\{ (x'^2 + x^2 \cos^2(\omega t) - 2xx' \cos(\omega t)) \frac{2\cos(\omega t)}{\sin(\omega t)} - x^2 \sin(2\omega t) + [2x(x' - x \cos(\omega t)) \frac{1}{\sin(\omega t)}] (-2\sin^2(\omega t)) \right\} \\ &= \frac{m\omega}{2} \left\{ (x'^2 + x^2 \cos^2(\omega t) - 2xx' \cos(\omega t)) \frac{\cos(\omega t)}{\sin(\omega t)} - x^2 \sin(\omega t) \cos(\omega t) - (2x(x' - x \cos(\omega t)) \sin(\omega t)) \right\} \\ &= \frac{m\omega}{2\sin(\omega t)} \left\{ (x'^2 + x^2 \cos^2(\omega t)) \cos(\omega t) - 2xx' - x^2 \sin^2(\omega t) \cos(\omega t) + 2x^2 \cos(\omega t) \sin^2(\omega t) \right\} \\ &= \frac{m\omega}{2\sin(\omega t)} \left\{ (x'^2 + x^2 \cos^2(\omega t) + x^2 \sin^2(\omega t)) \cos(\omega t) - 2xx' \right\} \\ &= \frac{m\omega}{2\sin(\omega t)} [\cos(\omega t)(x^2 + x'^2) - 2xx'] \quad \square \end{aligned}$$

2) Show: $U(x, x', t) = \int_{x'}^x D(x(t')) \exp\left[-\frac{i}{\hbar} \int_0^t dt' L(x(t'), \dot{x}(t'))\right] = F(t) \cdot \exp\left[\frac{i}{\hbar} S_{cl}\right]$

• $U(x, x', t) = \int_{x'}^x D(x(t')) \exp\left[-\frac{i}{\hbar} \int_0^t dt' L(x(t'), \dot{x}(t'))\right] \quad | \quad x(t') = x_{cl}(t') + y(t') \quad \text{with} \quad y(0) = y(t) = 0$
 $= \int_{x'}^x D(x_{cl}(t') + y(t')) \exp\left[-\frac{i}{\hbar} \int_0^t dt' L(x_{cl}(t') + y(t'), \dot{x}_{cl}(t') + \dot{y}(t'))\right]$

• $S[x(t)] = \int_0^t dt' L(x_{cl} + y, \dot{x}_{cl} + \dot{y})$
 $= \frac{m}{2} \int_0^t dt' [(\dot{x}_{cl} + \dot{y})^2 - \omega^2 (x_{cl} + y)^2]$
 $= \frac{m}{2} \int_0^t dt' [(\dot{x}_{cl}^2 - \omega^2 x_{cl}^2) + (\dot{y}^2 - \omega^2 y^2) + 2(\dot{x}_{cl}\dot{y} - \omega^2 x_{cl}y)]$

(...)

2) Ordinary Path Integral from Phase Space Path Integral

Show: $U(x, x', t) = \frac{1}{B} \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int \frac{dx_k}{B} \exp\left[-\frac{i}{\hbar} \sum_{k=1}^N \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right] \quad \text{where} \quad B = \sqrt{\frac{2\pi\hbar i \varepsilon}{m}}$

• $U(x, x', t) = \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int dx_k \prod_{k=1}^N \int \frac{dp_k}{2\pi\hbar} \exp\left[-\frac{i}{\hbar} \sum_{k=1}^N \left(\frac{\varepsilon p_k^2}{2m} - p_k(x_k - x_{k-1}) + \varepsilon V(x_{k-1})\right)\right]$
 $= \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int dx_k \prod_{k=1}^N \left\{ \int \frac{dp_k}{2\pi\hbar} \exp\left[-\frac{i}{\hbar} \left(\frac{\varepsilon p_k^2}{2m} - p_k(x_k - x_{k-1}) + \varepsilon V(x_{k-1})\right)\right] \right\} \quad | \quad \int dx e^{-\omega^2 x^2 + bx + c} = e^{\frac{b^2}{4\omega^2} + c} \sqrt{\frac{\pi}{\omega^2}}$
 $= \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int dx_k \prod_{k=1}^N \left\{ \frac{1}{2\pi\hbar} \exp\left[-\frac{(x_k - x_{k-1})^2 m}{2\hbar i \varepsilon} - \frac{i\varepsilon}{\hbar} V(x_{k-1})\right] \sqrt{\frac{2\pi\hbar m}{i\varepsilon}} \right\}$
 $= \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int dx_k \left(\frac{m}{2\pi\hbar i \varepsilon}\right)^{\frac{N}{2}} \exp\left[-\frac{i}{\hbar} \sum_{k=1}^N \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right]$
 $= \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int dx_k \left(\frac{1}{B}\right)^N \exp\left[-\frac{i}{\hbar} \sum_{k=1}^N \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right]$
 $= \frac{1}{B} \lim_{N \rightarrow \infty} \prod_{k=1}^{N-1} \int \frac{dx_k}{B} \exp\left[-\frac{i}{\hbar} \sum_{k=1}^N \left(\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\varepsilon} - \varepsilon V(x_{k-1})\right)\right] \quad \text{where} \quad B = \sqrt{\frac{2\pi\hbar i \varepsilon}{m}} \quad \square$

Exercise 3: $\hat{H} = -\frac{\hbar^2}{2m} \psi''(x,0) + V \psi(x,0)$

$$\begin{aligned} \psi(x+\eta, 0) &= \psi(x, 0) + \left(\frac{d}{d\eta} \psi(x+\eta, 0) \right) \Big|_{\eta=0} \cdot \eta + \left(\frac{1}{2} \frac{d^2}{d\eta^2} \psi(x+\eta, 0) \right) \Big|_{\eta=0} \eta^2 + O(\eta^3) \\ &= \psi(x, 0) + \cancel{\psi'(x+\eta, 0) \Big|_{\eta=0}} \cdot \eta + \frac{1}{2} \psi''(x+\eta, 0) \Big|_{\eta=0} \eta^2 + \cancel{O(\eta^3)} \\ &\approx \psi(x, 0) + \frac{1}{2} \psi''(x, 0) \eta^2 \end{aligned}$$

$$\begin{aligned} A(x+\alpha\eta, 0) &= A(x, 0) + \frac{d}{d\eta} A(x+\alpha\eta, 0) \Big|_{\eta=0} \cdot \eta + \frac{1}{2} \left(\frac{d^2}{d\eta^2} A(x+\alpha\eta, 0) \right) \Big|_{\eta=0} \eta^2 + O(\eta^3) \\ &= A(x, 0) + A'(x+\alpha\eta, 0) \Big|_{\eta=0} \alpha\eta + \frac{1}{2} A''(x+\alpha\eta, 0) \Big|_{\eta=0} \alpha^2 \eta^2 + \cancel{O(\eta^3)} \\ &\stackrel{I:}{=} A(x, 0) + A'(x, 0) \alpha\eta + \frac{1}{2} A''(x, 0) \alpha^2 \eta^2 \end{aligned}$$

$$\Rightarrow \varepsilon A(x+\alpha\eta, 0) \approx \varepsilon A(x, 0) + \alpha\eta \varepsilon A'(x, 0) + \frac{1}{2} A''(x, 0) \alpha^2 \eta^2 \varepsilon$$

$$= \varepsilon A(x, 0) + O(\eta\varepsilon, \eta^2) \quad \text{I think I made here a mistake. A should be } A' = \frac{\partial}{\partial x} A \text{ since } B = \vec{\nabla} \times \vec{A}. \text{ Also we need } \alpha. \text{ But since } O(\varepsilon\eta, \eta, \varepsilon^2) \text{ should be ignored, I don't find the mistake.}$$

$$\stackrel{II:}{=} \varepsilon A(x, 0)$$

$$\begin{aligned} \psi(x, \varepsilon) &= \frac{1}{B} \int_{-\infty}^{\infty} d\eta \psi(x+\eta, 0) \exp \left[\frac{i}{\hbar} \left(\frac{m\eta^2}{2\varepsilon} - q\varepsilon \frac{\eta}{2} A(x+\alpha\eta, 0) \right) \right] \Big|_{\frac{\eta}{\varepsilon} = -x, \text{ (I) \& (II)}} \\ &= \frac{1}{B} \int_{-\infty}^{\infty} d\eta \left(\psi(x, 0) + \frac{1}{2} \psi''(x, 0) \eta^2 \right) \exp \left[\frac{i}{\hbar} \left(\frac{m\eta^2}{2\varepsilon} - q x \varepsilon A(x, 0) \right) \right] \end{aligned}$$

With $\exp \left[\frac{i}{\hbar} q x \varepsilon A(x, 0) \right] \approx 1 - q x A(x, 0) \varepsilon + O(\varepsilon^2)$ follows:

$$= \frac{1}{B} \int_{-\infty}^{\infty} d\eta \left(\psi(x, 0) + \frac{1}{2} \psi''(x, 0) \eta^2 \right) \exp \left[\frac{i}{\hbar} \frac{m\eta^2}{2\varepsilon} \right] (1 - q x A(x, 0) \varepsilon)$$

$$= \frac{1}{B} \int_{-\infty}^{\infty} d\eta \left(\psi(x, 0) - \psi(x, 0) q x A(x, 0) \varepsilon + \frac{1}{2} \psi''(x, 0) \eta^2 - \frac{1}{2} \psi''(x, 0) \eta^2 q x A(x, 0) \varepsilon \right) \cdot \exp \left[\frac{i}{\hbar} \frac{m\eta^2}{2\varepsilon} \right]$$

$\rightarrow O(\eta^2 \varepsilon) = 0$

$$= \frac{1}{B} \int_{-\infty}^{\infty} d\eta \left(\psi(x, 0) - \psi(x, 0) q x A(x, 0) \varepsilon + \frac{1}{2} \psi''(x, 0) \eta^2 \right) \cdot \exp \left[\frac{i}{\hbar} \frac{m\eta^2}{2\varepsilon} \right]$$

$$= \frac{1}{B} \left[\left(\psi(x, 0) - \psi(x, 0) q x A(x, 0) \varepsilon \right) \exp \left[\frac{i}{\hbar} \frac{m}{2\varepsilon} \eta^2 \right] d\eta \right.$$

$$B = \sqrt{\frac{2\pi\hbar i \varepsilon}{m}}$$

$$+ \psi''(x, 0) \frac{1}{2} \int_{-\infty}^{\infty} \eta^2 \cdot \exp \left[\frac{i}{\hbar} \frac{m}{2\varepsilon} \eta^2 \right] d\eta \Big]$$

$$= \frac{1}{B} \left[\left(\psi(x, 0) - \psi(x, 0) q x A(x, 0) \varepsilon \right) \sqrt{\frac{2\pi\hbar i \varepsilon}{m}} + \frac{1}{4} \psi''(x, 0) \sqrt{\frac{2\pi\hbar i \varepsilon}{m}} \int_{-\infty}^{\infty} \eta^2 e^{-\eta^2} d\eta \right]$$

$\int_{-\infty}^{\infty} \eta^2 e^{-\eta^2} d\eta = \sqrt{\frac{\pi}{2}}$
 $\int_{-\infty}^{\infty} \eta^2 e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$

$$= \psi(x, 0) - \psi(x, 0) q x A(x, 0) \varepsilon + \frac{\partial^2}{\partial x^2} \psi(x, 0) \frac{\hbar \varepsilon}{2m} i$$

$$= \psi(x, 0) - i \frac{\varepsilon}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + q x A(x, 0) \varepsilon \right) \psi(x, 0)$$

$$= \psi(x, 0) - i \frac{\varepsilon}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, 0) \right) \psi(x, 0)$$

$$= \psi(x, 0) - i \frac{\varepsilon}{\hbar} \hat{H} \psi(x, 0)$$

□