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Exircise 1:

1.: With $\langle \psi_1 | \hat{Q} \psi_2 \rangle = \int dx\, \psi_1^*(x) \hat{Q}\, \psi_2(x)$ and $\langle \psi_1 \hat{Q}^\dagger | \psi_2 \rangle = \int dx\, \psi_1^*(x) \hat{Q}^\dagger \psi_2(x)$

$\hat{Q} | \psi_n \rangle = q_n | \psi_n \rangle \wedge \hat{Q}^\dagger = \hat{Q}$

$\Rightarrow \langle \psi_1 | \hat{Q} | \psi_1 \rangle = q_1 \langle \psi_1 | \psi_1 \rangle = \langle \psi_1 \hat{Q} | \psi_1 \rangle = \bar{q}_1 \langle \psi_1 | \psi_1 \rangle$

$\Rightarrow q_1 = \bar{q}_1 \Rightarrow \mathrm{Im}(q_1) = 0 \Rightarrow q_1 \in \mathbb{R}$ $\square$

2.:

$q_n \langle \psi_{n'} | \psi_n \rangle = \langle \psi_{n'} | \hat{Q} | \psi_n \rangle = q_{n'} \langle \psi_{n'} | \psi_n \rangle$

$\Rightarrow \frac{q_n}{q_{n'}} \langle \psi_{n'} | \psi_n \rangle = \langle \psi_{n'} | \psi_n \rangle \quad | \; q_{n'} \neq q_n \Rightarrow \frac{q_n}{q_{n'}} \neq 1$

If $\langle \psi_{n'} | \psi_n \rangle \neq 0$:

I: $\Rightarrow \frac{q_n}{q_{n'}} = 1$ ↯

$\Rightarrow \langle \psi_{n'} | \psi_n \rangle = 0$

$\square$

Proof does not work for $q_n = q_{n'}$ since then $\frac{q_n}{q_{n'}} = 1$, so there is no contradiction with (I).

3.: Be $(\phi_n)_n : n \in \mathbb{N}$ a basis in L^2 and $\{ |i\rangle, |j\rangle \} \subseteq (\phi_n)_n$.

$\Rightarrow (\hat{Q}^\dagger)_{ij} = \langle i | \hat{Q}^\dagger | j \rangle = \langle i \hat{Q} | j \rangle = \langle j | \hat{Q} i \rangle^* = \langle j | \hat{Q} | i \rangle^* = (\hat{Q})_{ji}^*$ $\square$

Exercise 2:

1.:

$$\int dx\, \psi_n^*(x)\, \psi(x,t) = \int dx\, \psi_n^*(x) \sum_{n'} a_{n'}(t)\, \psi_{n'}(x) = \sum_{n'} a_{n'}(t) \int dx\, \psi_n^*(x)\, \psi_{n'}(x)$$

$$= \sum_{n'} a_{n'}(t)\, \delta_{nn'}$$

$$= a_n(t)$$

$\square$

2.:

$$1 = \int dx\, |\psi(x,t)|^2 = \int dx\, \psi^*(x,t)\, \psi(x,t) = \int dx \sum_{n'} a_{n'}^*(t)\, \psi_{n'}(x) \sum_n a_n(t)\, \psi_n(x)$$

$$= \sum_n \sum_{n'} a_{n'}^*(t)\, a_n(t) \int dx\, \psi_{n'}^*(x)\, \psi_n(x)$$

$$= \sum_{n,n'} a_{n'}^*(t)\, a_n(t)\, \delta_{nn'}$$

$$= \sum_n |a_n(t)|^2 = 1$$

$\square$

3.:

$$\langle \hat{Q} \rangle = \int dx\, \psi^*(x,t)\, \hat{Q}\, \psi(x,t)$$

$$= \int dx \sum_{n'} a_{n'}^*(t)\, \psi_{n'}^*(x) \sum_n q_n\, a_n(t)\, \psi_n(x)$$

$$= \sum_{n,n'} q_n \int dx\, a_{n'}^*(t)\, \psi_{n'}^*(x)\, a_n(t)\, \psi_n(x) \quad | \text{ This was already shown}$$

$$= \sum q_n |a_n(t)|^2$$

$\square$

Exercise 3:

1.:

$$\hat{L}_z \psi_m(\phi) = -i\hbar \frac{\partial}{\partial \phi} \psi_m(\phi) = \lambda_m \psi_m(\phi)$$

$$\Rightarrow \int d(\psi_m(\phi)) \frac{1}{\psi_m(\phi)} = \int d\phi \frac{i\lambda}{\hbar}$$

$$\Rightarrow \psi_m(\phi) = A e^{\frac{i}{\hbar}\lambda\phi} \quad \Big| \; m := \frac{\lambda_m}{\hbar}, 1 = \|\psi_m(\phi)\|^2 = \int_0^{2\pi} d\phi \, \psi_m^*(\phi) \psi_m(\phi) = \int_0^{2\pi} d\phi \, |A|^2 = 2\pi |A|^2$$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}} e^{i\xi} \quad \Big| \text{ Phase } e^{i\xi}\text{: wähle } \xi = 0$$

$$\Rightarrow \psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

$\square$

2.: $\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$

$$\psi_m(\phi) = \psi_m(\phi + 2\pi)$$

$$\Rightarrow e^{im\phi} = e^{im(\phi + 2\pi)}$$

$$= e^{im\phi} e^{i2\pi m}$$

$$\Rightarrow 1 = e^{i2\pi m}$$

$$\Rightarrow m \in \mathbb{N}$$

$\square$

3.:

$$\int_0^{2\pi} d\phi \, \psi_l^*(\phi) \psi_m(\phi) = \int_0^{2\pi} d\phi \, \frac{1}{2\pi} e^{-il\phi} e^{im\phi}$$

$$= \int_0^{2\pi} d\phi \, \frac{1}{2\pi} e^{i\phi(m-l)} \quad \Big| \text{ Fourier-Trafo.}$$

$$= \delta_{ml}$$

$\square$

Exercise 4:

$$H(q,p,t) = \dot{q}_i p_i - L(q,\dot{q},t), \quad p_i(q,\dot{q}_i t) = \frac{\partial L}{\partial \dot{q}_i}, \quad \frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

$$\dot{\bar{q}}_i = \frac{\partial K}{\partial \bar{p}_i} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \bar{p}_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \bar{p}_i} \qquad \frac{\partial \dot{q}_j}{\partial \bar{p}_i} = \frac{\partial}{\partial \bar{p}_i}\frac{\partial H}{\partial \dot{p}_j} =$$

want to show this

this is true.

$$= -\dot{p}_j \frac{\partial q_j}{\partial \bar{p}_i} + \dot{q}_j \frac{\partial p_j}{\partial \bar{p}_i}$$

start with $\dot{\bar{q}}_i = \frac{d\bar{q}_i}{dt} = \sum_j \left(\frac{\partial \bar{q}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{q}_i}{\partial p_j}\dot{p}_j \right)$

$$= \{q,p\}_{\cancel{t},\bar{p}} \qquad \underbrace{\dots}_{\neq 0} \underbrace{\{\bar{q}_i, \bar{q}_k\}}_{\Rightarrow\ =0} \overset{!}{=} 0$$

use exactly the same procedure you did to show part 3!

$$\frac{\partial \bar{q}_i}{\partial q_i}\frac{\partial \bar{q}_k}{\partial q_k} - \frac{\partial \bar{q}_i}{\partial q_k}\frac{\partial \bar{q}_k}{\partial q_i}$$

$$\text{I)}\ \dot{\bar{p}}_i = -\frac{\partial H}{\partial \bar{q}_i} = -\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \bar{q}_i} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \bar{q}_i}$$

$$\text{II)}\ \dot{\bar{p}}_i = -\frac{d}{dt}\frac{\partial L}{\partial q_i}$$

$$0 = \frac{\partial(\dot{q}_i p_i - L)}{\partial \bar{q}_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar{q}}_i}$$

$$(\text{I})\&(\text{II}) \Rightarrow \ 0 = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial \bar{q}_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \bar{q}_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i}$$

$$= -\dot{p}_j \frac{\partial q_j}{\partial \bar{q}_i} + \dot{q}_j \frac{\partial p_j}{\partial \bar{q}_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i}$$

$$\dot{q}$$

$$= -\frac{d}{dt}\left(p_j \partial\right.$$

2.: $\bar{q} = \ln(q^{-1}\sin p), \quad \bar{p} = q\cot p$

$$\{\bar{q}_i, \bar{p}_k\} = \frac{-q^{-2}\sin p}{q^{-1}\sin p}\cdot\frac{-q}{\sin^2 p} - \frac{q^{-1}\cos p}{q^{-1}\sin p}\cot p$$

$$= \frac{1}{\sin^2 p} - \cot^2 p \qquad \Big|\ \sin^2 p + \cos^2 p = 1, \ \cot = \frac{\cos p}{\sin p}$$

$$= \frac{1}{\sin^2 p} - \frac{1-\sin^2 p}{\sin^2 p}$$

$$= 1$$

$$\{\bar{q}_i, \bar{q}_k\} = \{\bar{q}, \bar{q}\} = 0$$

$$\{\bar{p}_i, \bar{p}_k\} = \{\bar{p}, \bar{p}\} = 0$$

3.:

$$\{A(\bar q,\bar p),\ B(\bar q,\bar p)\}_{\bar q,\bar p} = \frac{\partial A}{\partial \bar q}\frac{\partial B}{\partial \bar p} - \frac{\partial A}{\partial \bar p}\frac{\partial B}{\partial \bar q}$$

$$= \left(\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar q} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar q}\right)\left(\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar p} + \frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar p}\right) -$$

$$\left(\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar q} + \frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar q}\right)\left(\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar p} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar p}\right)$$

$$= \cancel{\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar q}\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar p}} + \frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar q}\frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar p} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar q}\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar p} + \cancel{\frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar q}\frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar p}}$$

$$- \cancel{\frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar q}\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar p}} - \frac{\partial B}{\partial q}\frac{\partial q}{\partial \bar q}\frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar p} - \frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar q}\frac{\partial A}{\partial q}\frac{\partial q}{\partial \bar p} - \cancel{\frac{\partial B}{\partial p}\frac{\partial p}{\partial \bar q}\frac{\partial A}{\partial p}\frac{\partial p}{\partial \bar p}}$$

$$= \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}\underbrace{\left(\frac{\partial q}{\partial \bar q}\frac{\partial p}{\partial \bar p} - \frac{\partial p}{\partial \bar q}\frac{\partial q}{\partial \bar p}\right)}_{\delta_{ij} = 1} + \frac{\partial B}{\partial q}\frac{\partial A}{\partial p}\underbrace{\left(\frac{\partial p}{\partial \bar q}\frac{\partial q}{\partial \bar p} - \frac{\partial q}{\partial \bar q}\frac{\partial p}{\partial \bar p}\right)}_{-\delta_{ij} = -1}$$

$$= \{A, B\}_{q,p}$$

good!