Theo3 Blatt 4 Angelo Brade

Aufgabe 1: Alle Operatoren $\hat{A}, \hat{B}$ und $\hat{C}$ können auch als Matrizen aufgefasst werden.

1) Aus der Distributivität für die Transponierung und komplexe Konjugation mit $\hat{A}^+ = (A^T)^*$ folgt $(\hat{A}+\hat{B})^+ = \hat{A}^+ + \hat{B}^+$ □

2) Aus der Assoziativität für die Transponierung und komplexe Konjugation folgt $(\hat{A}\cdot\hat{B})^+ = \hat{B}^+ \cdot \hat{A}^+$ □

3) Aus der Kommutativität der Invertierung und Adjungierung folgt direkt $(\hat{A}^{-1})^+ = (\hat{A}^+)^{-1}$ □

4) Aus der Distributivität und Assoziativität folgt $[\hat{A},\hat{B}]^+ = (\hat{A}\hat{B})^+ - (\hat{B}\hat{A})^+ = A^+B^+ - B^+A^+ = [\hat{A}^+, \hat{B}^+]$ □

5) $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -(\hat{B}\hat{A} - \hat{A}\hat{B}) = -[\hat{B},\hat{A}]$

6) $[\alpha\hat{A}+\beta\hat{B}, \hat{C}] = (\alpha\hat{A}+\beta\hat{B})\hat{C} - \hat{C}(\alpha\hat{A}+\beta\hat{B}) = \alpha\hat{A}\hat{C} - \alpha\hat{C}\hat{A} + \beta\hat{B}\hat{C} - \beta\hat{C}\hat{B} = \alpha[\hat{A},\hat{C}] + \beta[\hat{B},\hat{C}]$

7) $[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]]$

$= [A, \hat{B}\hat{C} - \hat{C}\hat{B}] + [\hat{B}, \hat{C}\hat{A} - \hat{A}\hat{C}] + [\hat{C}, \hat{A}\hat{B} - \hat{B}\hat{A}]$

$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) - (\hat{B}\hat{C} - \hat{C}\hat{B})\hat{A} + \hat{B}(\hat{C}\hat{A} - \hat{A}\hat{C}) - (\hat{C}\hat{A} - \hat{A}\hat{C})\hat{B} + \hat{C}(\hat{A}\hat{B} - \hat{B}\hat{A}) - (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C}$

$= \cancel{\hat{A}\hat{B}\hat{C}} - \cancel{\hat{A}\hat{C}\hat{B}} - \cancel{\hat{B}\hat{C}\hat{A}} + \cancel{\hat{C}\hat{B}\hat{A}} + \cancel{\hat{B}\hat{C}\hat{A}} - \cancel{\hat{B}\hat{A}\hat{C}} - \cancel{\hat{C}\hat{A}\hat{B}} + \cancel{\hat{A}\hat{C}\hat{B}} + \cancel{\hat{C}\hat{A}\hat{B}} - \cancel{\hat{C}\hat{B}\hat{A}} - \cancel{\hat{A}\hat{B}\hat{C}} + \cancel{\hat{B}\hat{A}\hat{C}}$

$= 0$

8) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B}$

$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B}$

$= \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

9)

$[\hat{A}^n, \hat{B}] = [\hat{A}\hat{A}^{n-1}, \hat{B}]$ (8)

$= \hat{A}^0[\hat{A},\hat{B}]\hat{A}^{n-1} + \hat{A}^1[\hat{A}^{n-1}, \hat{B}]\hat{A}^0$

$= \hat{A}^0[\hat{A},\hat{B}]\hat{A}^{n-1} + \hat{A}^1[\hat{A}\hat{A}^{n-2}, \hat{B}]\hat{A}^0$

$= \hat{A}^0[\hat{A},\hat{B}]\hat{A}^{n-1} + \hat{A}^1[\hat{A},\hat{B}]\hat{A}^{n-2} + \hat{A}^2[\hat{A}^{n-2}, \hat{B}]\hat{A}^0$

$= \hat{A}^0[\hat{A},\hat{B}]\hat{A}^{n-1} + \ldots + \hat{A}^{n-1}[\hat{A},\hat{B}]\hat{A}^0$

$= \sum_{k=0}^{n-1} \hat{A}^k[\hat{A},\hat{B}]\hat{A}^{n-k-1}$

Ich habe 2h an dieser Teilaufgabe gesessen :: Ich will jetzt 4 Punkt(e) dafür ! :)

Oder:

$\sum_{k=0}^{n-1} \hat{A}^k[\hat{A},\hat{B}]\hat{A}^{n-k-1} = \sum_{k=0}^{n-1} \hat{A}^k(\hat{A}\hat{B} - \hat{B}\hat{A})\hat{A}^{n-k-1}$

$= \sum_{k=0}^{n-1} \hat{A}^{k+1}\hat{B}\hat{A}^{n-k-1} - \sum_{k=0}^{n-1} \hat{A}^k\hat{B}\hat{A}^{n-k}$

$= \sum_{k=1}^{n} \hat{A}^k\hat{B}\hat{A}^{n-k} - \sum_{k=0}^{n-1} \hat{A}^k\hat{B}\hat{A}^{n-k}$

$= \sum_{k=0}^{n} \hat{A}^k\hat{B}\hat{A}^{n-k} - \hat{A}^0\hat{B}\hat{A}^n - \sum_{k=0}^{n} \hat{A}^k\hat{B}\hat{A}^{n-k} + \hat{A}^n\hat{B}\hat{A}^0$

$= \hat{A}^n\hat{B}\hat{A}^0 - \hat{A}^0\hat{B}\hat{A}^n$

$= [\hat{A}^n, \hat{B}]$

2) $\hat{U}^{-1} = \hat{U}^+$, $\hat{V}^{-1} = \hat{V}^+ \in \mathcal{H} \wedge \hat{A} = \hat{A}^+$

Z: $(a*b)*c = a*(b*c)$ (Assoziativität)

Sei $\hat{U}, \hat{V}, \hat{W} \in \mathcal{H}$ und unitär.

$(\hat{U}\cdot\hat{V})\cdot\hat{W} = \hat{U}\cdot(\hat{V}\cdot\hat{W})$

□

Z: $a*e = e*a = a$ (neutrales Element)

Sei $\hat{U} \in \mathcal{H}$ und unitär.

$\hat{U}\cdot\mathbb{1} = \hat{U}\hat{U}^{-1}\hat{U} \overset{\text{Assoziativ}}{=} \hat{U} = \hat{U}\cdot\mathbb{1}$ □

Z: $a*a^{-1} = e$ (inverses Element)

$\hat{U}\cdot\hat{U}^{-1} = \mathbb{1}$

Z: $\hat{U}\hat{A}\hat{U}^{-1} = (\hat{U}\hat{A}\hat{U}^{-1})^+$

$\hat{U}\hat{A}\hat{U}^{-1} = \hat{U}\hat{A}^+\hat{U}^+ = (U^+)^{-1}\hat{A}^+\hat{U}^+ = (U^+)^+\hat{A}^+\hat{U}^+ = (U^{-1}\hat{A}\,\hat{U})^+$ □

i dk

Aufgabe 2:

1)
Z: $[\hat{A}, e^{\hat{A}}] = 0$

$$[\hat{A}, e^{\hat{A}}] = A e^{\hat{A}} - e^{\hat{A}} \hat{A}$$
$$= A \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!} - \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!} \hat{A}$$
$$= \sum_{n=0}^{\infty} \frac{\hat{A}^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{\hat{A}^{n+1}}{n!}$$
$$= 0 \qquad \square$$

2) Z: $e^{\hat{B}\hat{A}\hat{B}^{-1}} = \hat{B} e^{\hat{A}} \hat{B}^{-1}$

$$e^{\hat{B}\hat{A}\hat{B}^{-1}} = \sum_{n=0}^{\infty} \frac{(\hat{B}\hat{A}\hat{B}^{-1})^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{\overbrace{(\hat{B}\hat{A}\hat{B}^{-1}) \cdot \ldots \cdot (\hat{B}\hat{A}\hat{B}^{-1})}^{n\text{-mal}}}{n!}$$
$$= \sum_{n=0} \frac{\hat{B}\hat{A}^n\hat{B}^{-1}}{n!}$$
$$= \hat{B} e^{\hat{A}} \hat{B}^{-1} \qquad \square$$

3) $f(t) = e^{t\hat{A}} e^{t\hat{B}}, \quad g(t) = e^{t(\hat{A}+\hat{B})}$

Z: $f(t) = g(t)$ für $\forall t \Rightarrow [\hat{A}, \hat{B}] = 0$

$$f'(t) = g'(t)$$
$$\Rightarrow \frac{d}{dt}\left(e^{t\hat{A}} e^{t\hat{B}}\right) = \frac{d}{dt} e^{t(\hat{A}+\hat{B})} \qquad \Big| \; \frac{\partial}{\partial t} e^{t\hat{A}} = \left(\sum_{n=0}^{\infty} \frac{(t\hat{A})^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{n\, t^{n-1} \hat{A}^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{n-1} \hat{A}^n}{(n-1)!} = \hat{A} \sum_{n=0}^{\infty} \frac{t^{n-1} \hat{A}^{n-1}}{(n-1)!} = \hat{A} e^{t\hat{A}}$$
$$\Rightarrow \left(\hat{A} e^{t\hat{A}} e^{t\hat{B}}\right) + \left(e^{t\hat{A}} \hat{B} e^{t\hat{B}}\right) = (\hat{A}+\hat{B}) e^{t(\hat{A}+\hat{B})}$$
$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

Die Exponentialfunktion

1.

$$[\hat{A}, e^{\hat{A}}] = [\hat{A}, \sum \frac{\hat{A}^n}{n!}] = \sum \frac{[\hat{A}, \hat{A}^n]}{n!}$$

$$\rightsquigarrow \hat{A}\hat{A}^n - \hat{A}^n\hat{A} = \hat{A}^{n+1} - \hat{A}^{n+1} = 0$$

also $[\hat{A}, e^{\hat{A}}] = 0$

2.)

$$e^{BAB^{-1}} = \sum_n \frac{(BAB^{-1})^n}{n!}$$

für $(BAB^{-1})^n$ gilt

$$\underbrace{BAB^{-1}BAB^{-1} \dots B^{-1}BAB^{-1}}_{n \text{ mal}} = BA^nB^{-1}$$

sonit gilt $e^{BAB^{-1}} = \sum_n \frac{(BAB^{-1})^n}{n!} = \sum_n \frac{BA^nB^{-1}}{n!} = Be^AB^{-1}$

3.

- Sei $f(t) = e^{tA}e^{tB}$
  $g(t) = e^{t(A+B)}$

4

$$M = \begin{pmatrix} 1 & 2-i \\ 2+i & -3 \end{pmatrix}$$

$$\begin{aligned} \det(M - \lambda I) &= (1-\lambda)(-3-\lambda) - (2+i)(2-i) \\ &= -3 - \lambda + 3\lambda + \lambda^2 - (4 - 2i + 2i + 1) \\ &= \lambda^2 + 2\lambda - 3 - 5 \\ &= \lambda^2 + 2\lambda - 8 \end{aligned}$$

$$\det(M - \lambda I) \stackrel{!}{=} 0 = \lambda^2 + 2\lambda - 8 = (\lambda - 2)(\lambda + 4)$$

$$\lambda - 2 \,/\, \lambda^2 + 2\lambda - 8 = \lambda + 4$$
$$\lambda^2 - 2\lambda$$
$$4\lambda - 8$$

Eigenwerte $\lambda_1 = 2$
$\lambda_2 = -4$

Eigenvektoren

$\lambda = 2$

$$\left(\begin{array}{cc|c} -1 & 2-i & 0 \\ 2+i & -5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -1 & 2-i & 0 \\ 1 & -(2-i) & 0 \end{array}\right)$$

$(2+i)(2-i) = 5$ $\rightsquigarrow \left(\begin{array}{cc|c} 1 & -(2-i) & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x = (2-i)y \qquad y = y_0$

$$L = \{v \in \mathbb{C}^2 / v = r\begin{pmatrix} 2-i \\ 1 \end{pmatrix}, r \in \mathbb{C}\}$$

$\lambda = -4$

$$\left(\begin{array}{cc|c} 5 & 2-i & 0 \\ 2+i & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 5 & 2-i & 0 \\ 5 & 2-i & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 5 & 2-i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$5x = -(2-i)y$

$x = -\frac{2-i}{5}y$

$$L = \{v \in \mathbb{C}^2 / v = r\begin{pmatrix} -2+i \\ 5 \end{pmatrix}, r \in \mathbb{C}\}$$

wir haben also die Diagonalmatrix

$$D = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$$

mit den Transformationsmatrizen

$$U = \begin{pmatrix} -2+i & 2-i \\ 5 & 1 \end{pmatrix}$$

$$U^{-1} = \begin{pmatrix} -\frac{1}{15} - \frac{i}{30} & \frac{1}{6} \\ \frac{1}{3} + \frac{i}{6} & \frac{1}{6} \end{pmatrix}$$

somit $M = UDU^{-1}$

- Wir Wissen $e^M = e^{UDU^{-1}} = Ue^DU^{-1}$

Für $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n$ gilt $\begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

somit $e^D = \begin{pmatrix} e^{-4} & 1 \\ 1 & e^2 \end{pmatrix}$

Also

$$e^M = \begin{pmatrix} \frac{1}{6}(-6 + \frac{1}{e^4} + 5e^2) & \frac{(\frac{1}{3} - \frac{i}{6})(e^6 - 1)}{e^4} \\ \frac{(\frac{1}{15} + \frac{i}{30})(-5 + 24e^4 + 5e^6)}{e^4} & 1 + \frac{5}{6e^4} + \frac{e^2}{6} \end{pmatrix}$$