AQT; Sheet 8; Marc Hauer, Angelo Brade; 01.12.2024

## Quickies:

Q.1: The probability is also like time dependent. thats a guess

its time dependent in general!

we take the $t \to \infty$ limit where we assume $\frac{d}{dt}$(probability) becomes constant

Q.2: 1: $\delta$ is dependent of the integrand.

2: $\delta$ can be rewritten as a Taylor-Approximation

3: $\delta$ can be rewritten as discret kronecka-deltas $\delta_{ij}$

even if we rewrite the $\delta$ function we need to get rid of it, typically by integrating over something, e.g. phase space of final state particles.

Q.3: If the source is far away.

## Exercise 1:

1.)

$$\psi(\vec{x},t) = \int d^3k \, \frac{a(\vec{k})}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x} - \omega(|\vec{k}|)t)}$$

Phase: $\phi(t) = \vec{k}\cdot\vec{x} - \omega(|\vec{k}|)t$

$\Rightarrow \dot{\phi}(t) = \vec{k}\cdot\dot{\vec{x}} - \omega(|\vec{k}|) \overset{!}{=} 0$ (const. phase)

$\Rightarrow |\vec{v}_{ph}| \equiv |\dot{\vec{x}}| = \frac{\omega(|\vec{k}|)}{|\vec{k}|}$

Phase velocity is perpendicular with wave ($\vec{k}$).

$\Rightarrow \vec{v}_{ph} = \vec{e}_k \frac{\omega(|\vec{k}|)}{|\vec{k}|}$ ✓

$\square$

2.)

$$v_{gr} \equiv \frac{\partial \omega}{\partial k}$$

$$\vec{k} = \vec{k}_0 + \vec{\delta} \quad \text{with} \quad |\vec{\delta}| \ll |\vec{k}_0|$$

$$\Rightarrow \quad v_{gr} = \left.\frac{\partial \omega(|\vec{k}|)}{\partial |\vec{k}|}\right|_{\vec{k}=\vec{k}_0} \quad \text{since} \quad \vec{k} = \vec{k}_0 + \mathcal{O}(\delta).$$

Group velocity perpendicular to wave ($\vec{k}$):

$$\Rightarrow \quad \vec{v}_{gr} = \vec{e}_{\vec{k}_0} \left.\frac{\partial \omega(|\vec{k}|)}{\partial |\vec{k}|}\right|_{\vec{k}=\vec{k}_0}$$

$\square$

$$\psi(\vec{x},t) = \int d^3k \, \frac{a(\vec{k})}{(2\pi)^3} e^{i(\vec{k}\cdot\vec{x} - \omega(|\vec{k}|)t)}$$

With $\vec{k} = \vec{k}_0 + \vec{\delta}$ and $|\vec{\delta}| \ll |\vec{k}_0|$ follows:

I: $|\vec{k}| = |\vec{k}_0 + \vec{\delta}| = |\vec{k}_0| + |\vec{\delta}| + \mathcal{O}(\vec{\delta}^2) \Rightarrow \frac{dk}{d|\delta|} \simeq 1 \Rightarrow d^3k \simeq d^3\delta$

II: $a(k) = a(k_0 + \delta)$

III: $\omega(|\vec{k}|) = \omega(|\vec{k}_0 + \vec{\delta}|) = \omega(|\vec{k}_0|) + \mathcal{O}(\delta) \simeq \omega(|\vec{k}_0|)$

Thus $\psi(\vec{x},t)$ approximates to:

$$\psi(\vec{x},t) \simeq \int d^3\delta \, \frac{a(\vec{k}_0 + \vec{\delta})}{(2\pi)^3} e^{i((\vec{k}_0 + \vec{\delta})\vec{x} - \omega(|\vec{k}_0|)\cdot t)}$$

Assume: $\vec{k}\cdot\vec{v}_{ph}(\vec{k}) = \overbrace{\vec{k}}^{\vec{k}=\vec{e}_k\cdot|\vec{k}|}\cdot\vec{e}_k \frac{\omega(|\vec{k}|)}{|\vec{k}|} = \omega(|\vec{k}|) \overset{(III)}{\simeq} \omega(|\vec{k}_0|) = \vec{k}_0\cdot\vec{v}_{ph}(\vec{k}_0)$

hmm... I dont think this is right but in the end its okay

this, i actually don't know why

With $\vec{\delta}\cdot\vec{e}_{\vec{k}_0} = |\vec{\delta}||\vec{e}_{\vec{k}_0}| \overbrace{\cos\theta}^{\cos\theta \simeq 1+\mathcal{O}(\theta),\ \theta\ll 1} \simeq |\vec{\delta}|\underbrace{|\vec{e}_{\vec{k}_0}|}_{1} = |\vec{\delta}|$ and $\frac{\partial \omega}{\partial |\vec{\delta}|} \simeq \frac{\omega}{|\vec{\delta}|} + \mathcal{O}$ we also get:

$$\boxed{\vec{\delta}\cdot\vec{v}_{gr}(\vec{k}) = \vec{\delta}\cdot\vec{e}_{\vec{k}_0} \left.\frac{d\omega}{d|\vec{k}|}\right|_{\vec{k}=\vec{k}_0}} = \underbrace{\vec{\delta}\,\vec{e}_{\vec{k}_0}}_{\simeq|\vec{\delta}|} \underbrace{\frac{\partial \omega}{\partial |\vec{\delta}|}}_{\simeq\frac{\omega}{|\vec{\delta}|}} \underbrace{\frac{d|\vec{\delta}|}{d|\vec{k}|}}_{\simeq 1} \simeq \omega(|\vec{k}_0|)$$

$$\text{Thus} \quad \psi(\vec{x},t) \simeq \int d^3\delta \, \frac{a(\vec{k}_0 + \vec{\delta})}{(2\pi)^3} \underbrace{e^{i\vec{k}\cdot[\vec{x} - \vec{v}_{ph}(|\vec{k}_0|)t]}}_{\text{independent of } \delta} e^{i\vec{\delta}[\vec{x} - \vec{v}_{gr}\cdot t]}$$

$$\simeq e^{i\vec{k}_0\cdot[\vec{x} - \vec{v}_{ph}(|\vec{k}_0|)t]} \int d^3\delta \, \frac{a(\vec{k}_0 + \vec{\delta})}{(2\pi)^3} e^{i\vec{\delta}[\vec{x} - \vec{v}_{gr}\cdot t]}.$$

$\square$

3.)

If $\vec{x}' = \vec{x} - \vec{v}_{gr} \cdot t$ :

$$\psi(\vec{x},t) = e^{i\vec{k}_0 \cdot [\vec{x} - \vec{v}_{ph}(|\vec{k}_0|)t]} \int d^3\delta \, \frac{a(\vec{k}_0+\vec{\delta})}{(2\pi)^3} \, e^{i\vec{\delta}\cdot[\vec{x} - \vec{v}_{gr}\cdot t]} \; \Big| \; \vec{\delta} = \vec{k} - \vec{k}_0 , \; \frac{d|\vec{k}|}{d|\vec{\delta}|} \simeq 1$$

$$\simeq e^{i\vec{k}_0\cdot[\cancel{\vec{x}} - \vec{v}_{ph}(|\vec{k}_0|)t]} \, e^{-i\vec{k}_0[\cancel{\vec{x}} - \vec{v}_{gr} t]} \underbrace{\int d^3k \, \frac{a(\vec{k}_0+\vec{\delta})}{(2\pi)^3} \, e^{i\vec{k}[\vec{x} - \vec{v}_{gr}\cdot t]}}_{= \psi(\vec{x} - \vec{v}_{gr}\cdot t,\, t=0)}$$

$$= \psi(\vec{x} - \vec{v}_{gr}\cdot t, 0) \, e^{it\vec{k}_0[\vec{v}_{gr} - \vec{v}_{ph}(|\vec{k}_0|)]}$$

$\square$

4.)

$$\vec{v}_{ph} = \vec{e}_{\vec{k}} \frac{\omega(|\vec{k}|)}{|\vec{k}|}, \quad \vec{v}_{gr} = \vec{e}_{\vec{k}_0} \frac{\partial \omega(|\vec{k}|)}{\partial |\vec{k}|}\Big|_{\vec{k}=\vec{k}_0}, \quad \omega(|\vec{k}|) = E(|\vec{k}|)/\hbar$$

i) $E(\vec{k}) = \hbar c |\vec{k}|$

$\Rightarrow \vec{v}_{ph} = \vec{e}_{\vec{k}} c, \quad \vec{v}_{gr} = \vec{e}_{\vec{k}_0} c$

ii) $E(\vec{k}) = \hbar^2 |\vec{k}|^2/(2\mu)$

$\Rightarrow \vec{v}_{ph} = \vec{e}_{\vec{k}} \cdot \frac{\hbar|\vec{k}|}{2\mu}, \quad \vec{v}_{gr} = \vec{e}_{\vec{k}_0} \cdot \frac{\hbar|\vec{k}|}{\mu}$

Since $\vec{k} \simeq \vec{k}_0 \Rightarrow \vec{k}\cdot\vec{k}_0 \simeq |\vec{k}_0|^2 \simeq |\vec{k}|^2$

$\Rightarrow e^{it\vec{k}_0(\vec{v}_{gr} - \vec{v}_{ph})} = e^{it|\vec{k}_0|^2(\frac{\hbar}{\mu} - \frac{\hbar}{2\mu})} = e^{it\frac{E(|\vec{k}_0|)}{\hbar}}$

$\square$

5.) $E(|\vec{k}|) = \sqrt{\mu^2 c^4 + \hbar^2 \vec{k}^2 c^2}$

$$\Rightarrow \vec{v}_{ph} = \vec{e}_{\vec{k}} c \sqrt{\frac{\mu^2 c^2}{\hbar^2 |\vec{k}|^2} + 1}, \quad \vec{v}_{gr} = \vec{e}_{\vec{k}_0} \frac{\hbar|\vec{k}| c^2}{\sqrt{\mu^2 c^4 + \hbar^2 \vec{k}^2 c^2}} = \vec{e}_{\vec{k}_0} c \frac{1}{\sqrt{\left(\frac{\mu c}{\hbar |\vec{k}|}\right)^2 + 1}}$$

$\mu \to 0$: $\vec{v}_{ph} = \vec{e}_{\vec{k}} c, \quad \vec{v}_{gr} = \vec{e}_{\vec{k}_0} c$

As $|\vec{k}| \to 0$: $\vec{v}_{ph} = \vec{e}_{\vec{k}} \frac{\sqrt{\mu^2 c^4 + \hbar^2 \vec{k}^2 c^2}}{\hbar |\vec{k}|} \overset{\mu^2 c^4 \ll \hbar^2 |\vec{k}|^2 c^2}{\simeq} \vec{e}_{\vec{k}} \frac{\sqrt{\hbar^2 |\vec{k}|^2 c^2}}{\hbar |\vec{k}|} = \vec{e}_{\vec{k}} c$

$\square$

as $|\vec{k}| \to 0$ first term $\propto \frac{1}{|\vec{k}|} \to \infty$ dominates!

$|\vec{v}_{ph}| \to \infty$ in this limit!

2) Scattering on a constant potential

$$V(\vec{x}) = \begin{cases} V_0 & \text{for } |\vec{x}| < r_0 \\ 0 & \text{for } |\vec{x}| \geq r_0 \end{cases}$$

1) $f_{\vec{k}} = -\frac{2M}{|\vec{q}|\hbar^2} \int_0^\infty dr' \, V(r') r' \sin(|\vec{q}|r')$

$$\Rightarrow f_{\vec{k}} = -\frac{2M}{|\vec{q}|\hbar^2} \int_0^{r_0} dr' \, V_0 r' \sin(|\vec{q}|r')$$

$$= -\frac{2MV_0}{|\vec{q}|\hbar^2} \left[ -\frac{r'}{|\vec{q}|} \cos(|\vec{q}|r') + \frac{1}{|\vec{q}|} \int dr' \cos(|\vec{q}|r') \right]_0^{r_0}$$

$$= -\frac{2MV_0}{|\vec{q}|^3\hbar^2} \left[ -r' \cos(|\vec{q}|r') + \frac{1}{|\vec{q}|} \sin(|\vec{q}|r') \right]_0^{r_0}$$

$$= \underbrace{-\frac{2MV_0}{|\vec{q}|^3\hbar^2}}_{=: A/|\vec{q}|^3} \left[ \sin(|\vec{q}| \cdot r_0) - |\vec{q}| r_0 \cos(|\vec{q}| r_0) \right]$$

· $\lim_{|\vec{q}| \to 0} f_{\vec{k}} = \lim_{|\vec{q}| \to 0} \left( \frac{A \sin(|\vec{q}|r_0) - A r_0 |\vec{q}| \cos(|\vec{q}|r_0)}{|\vec{q}|^3} \right)$ | L'Hospital

$$= \lim_{|\vec{q}| \to 0} \left( \frac{A r_0 \cos(|\vec{q}|r_0) - A r_0 \cos(|\vec{q}|r_0) + A r_0^2 |\vec{q}| \sin(|\vec{q}|r_0)}{3|\vec{q}|^2} \right)$$

$= \lim_{|\vec{q}| \to 0} \left( \frac{A r_0^2 |\vec{q}| \sin(|\vec{q}|r_0)}{3|\vec{q}|^2} \right)$ | L'Hospital

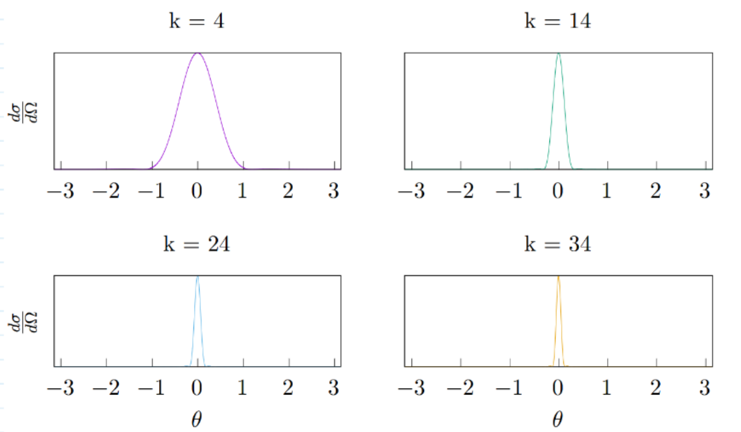
$= \lim_{|\vec{q}| \to 0} \left( \frac{A r_0^2 \sin(|\vec{q}|r_0) + A r_0^3 |\vec{q}| \cos(|\vec{q}|r_0)}{6|\vec{q}|} \right)$ | L'Hospital

$$= \lim_{|\vec{q}| \to 0} \left( \frac{A r_0^3 \underbrace{\cos(|\vec{q}|r_0)}_{\to 1} + A r_0^3 \underbrace{\cos(|\vec{q}|r_0)}_{\to 1} - \underbrace{A r_0^4 |\vec{q}| \sin(|\vec{q}|r_0)}_{\to 0}}{6} \right)$$

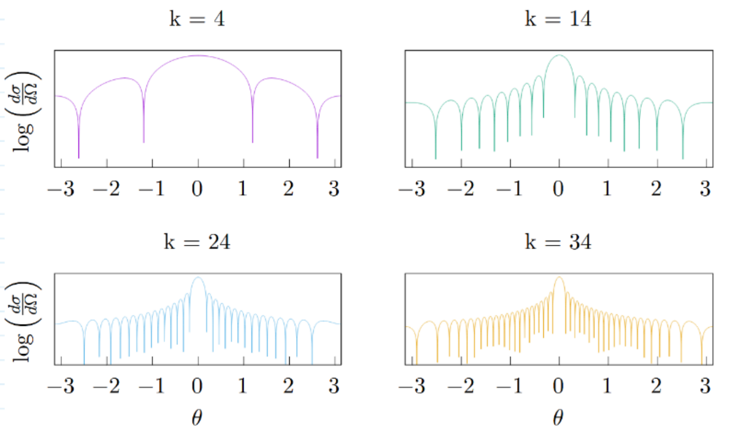
$$= \frac{A r_0^3}{3} \quad \propto r_0^3$$

2) $|\vec{q}| = 2k \sin\left(\frac{\theta}{2}\right)$ where $k = |\vec{k}|$

· Plot of the cross section $\frac{d\sigma}{d\Omega}$:




· Plot of $\log\left(\frac{d\sigma}{d\Omega}\right)$ to visualize the zeros:




• For larger k the number of zero points, and so the number of maxima and minima becomes large.

always infinitely many!

3) Show: $\frac{MV_0 r_0^2}{\hbar^2} \ll 1$

· $|f_{\vec{k}}(\theta, \varphi)| \ll r_0$

$$\Rightarrow \left| -\frac{2MV_0 r_0^3}{3\hbar^2} \right| \ll r_0$$

$$\Rightarrow \frac{2}{3} \frac{M|V_0|r_0^2}{\hbar^2} \ll 1$$

$$\Rightarrow \frac{M|V_0|r_0^2}{\hbar^2} \ll 1 \quad \square$$