AQT, Shett 9; Marc Hauer, Angelo Brade; 08.12.2024

Exercise 1:

1.) zz: $f_{\vec{k}}^{dipole} = (1-e^{-i\vec{q}\cdot\vec{d}}) f_{\vec{k}}^{monopole}$ with $f_{\vec{k}}^{monopole} = -\frac{2\mu}{\hbar^2}\frac{z_1 z_2 e^2}{4\pi\varepsilon_0}\frac{1}{|q|^2}$

In the lecture we introduced the Yukawa potential $V(r) = g\,\frac{e^{-\mu r}}{r}$ and assumed $g = \frac{z_1 z_2 e^2}{4\pi\varepsilon_0}$ for the coulomb scattering.

particle at $\vec{x}' = \vec{0}$:

$$f_{\vec{k}}^{(1)} = -\frac{m}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3x'\, e^{-i(\vec{k}'-\vec{k})\vec{x}'} V(\vec{x}')$$

With potential $V(\vec{x}') = \frac{z_1 z_2}{4\pi\varepsilon_0} e^2 \frac{e^{-\mu r'}}{r'}$ and $\vec{q} = \vec{k}' - \vec{k}$ we get:

$$= -\frac{m}{2\pi\hbar^2}\frac{z_1 z_2}{4\pi\varepsilon_0} e^2 \int_0^{2\pi} d\varphi' \int_{-1}^{1} d(\cos\theta') \int_0^\infty dr'\, e^{-i|\vec{q}||\vec{x}'|\cos\theta'} \frac{1}{r'} r'^2\, e^{-\mu r'}$$

$$= -2i\frac{m}{\hbar^2}\frac{z_1 z_2}{4\pi\varepsilon_0} e^2 \int_0^\infty \frac{1}{-iq}\sin(|\vec{q}|r')\, e^{-\mu r'} dr'$$

$$= \frac{m}{\hbar^2}\frac{z_1 z_2 e^2}{4\pi\varepsilon_0 |\vec{q}| i}\int_0^\infty \left(e^{r'(\mu+|\vec{q}|i)} - e^{r'(\mu-|\vec{q}|i)}\right) dr'$$

$$= \frac{m}{\hbar^2}\frac{z_1 z_2 e^2}{4\pi\varepsilon_0 |\vec{q}| i}\left(\frac{1}{\mu+|\vec{q}|i} - \frac{1}{\mu-|\vec{q}|i}\right)$$

In the limit $\mu \to 0$ we get:

$$= -\frac{2m}{\hbar^2}\frac{z_1 z_2 e^2}{4\pi\varepsilon_0 |\vec{q}|^2}$$

Recalling the scattering amplitude in its original form

$$f_{\vec{k}}(\varphi,\theta) = -\frac{m}{2\pi\hbar^2}\int d^3x'\, e^{-i|\vec{k}|\hat{x}\cdot\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}} V(\vec{x}')\,\psi_{\vec{k}}(\vec{x}')$$

we get with the $\vec{x} \to \vec{d}$ transformation in the Born approximation:

$$f_{\vec{k}_d}^{(2)} = -\frac{m}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3x'\, e^{-i(\vec{k}'-\vec{k})(\vec{x}'+\vec{d})} V(\vec{x}')$$

$$= \underbrace{-\frac{m}{2\pi\hbar^2}\int_{\mathbb{R}^3} e^{-i(\vec{k}'-\vec{k})\vec{x}'} V(\vec{x}')\, d^3x'}_{f_{\vec{k}}^{(1)}}\; e^{-i\vec{q}\cdot\vec{d}}$$

$$= e^{-i\vec{q}\cdot\vec{d}} f_{\vec{k}}^{(1)}$$

Identefying $f_{\vec{k}}^{(1)}$ as $f_{\vec{k}}^{monopole}$ and $f_{\vec{k}}^{(1)}$ and $f_{\vec{k}}^{(2)}$ being in superposition with opposit charge,

we recieve:

$$f_{\vec{k}}^{dipole} = (1-e^{-i\vec{q}\cdot\vec{d}})\, f_{\vec{k}}^{monopole}$$

✓ Good!

2.)

Since $\vec{k}' = k\begin{pmatrix}\sin\varphi\sin\theta\\ \cos\varphi\sin\theta\\ \cos\theta\end{pmatrix}$, $\vec{k} = k\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\vec{d} = \begin{pmatrix}0\\0\\d\end{pmatrix}$ the term $\vec{q}\cdot\vec{d}$ reduces to: $\vec{q}\cdot\vec{d} = d\cdot k\cdot(\cos\theta - 1)$ since $\vec{q} = \vec{k}' - \vec{k}$.

So $f_{\vec{k}}^{dipole} = (1-e^{-i\vec{q}\cdot\vec{d}})f_{\vec{k}}^{monopole} = \underline{(1-e^{-id\cdot k\cdot(\cos\theta-1)})f_{\vec{k}}^{monopole}}$ has no φ dependency.

As we approach $\theta\to 0$ we see $f_{\vec{k}}^{dipole}$ reduces to $f_{\vec{k}}^{dipole} = 0$

Thus the cross section $\frac{d\sigma}{d\Omega} = |f_{\vec{k}}|^2 = 0$ remains finite.

actually the zero in $(1-e^{-id\cdot k(\cos\theta-1)})$ cancels a pole in $f_{\vec{k}}^{monopole}$ and is $\neq 0$ but still finite!

3.)

Now with $\vec{d}' = \begin{pmatrix}0\\d\\0\end{pmatrix}$ we get: $\vec{q}\cdot\vec{d}' = \sin\theta\cos\varphi\, d\cdot k$. Thus

$f_{\vec{k}}^{dipole} = (1-e^{-id\cdot k\cdot\cos\varphi(\sin\theta - 1)})f_{\vec{k}}^{monopole}$ and $\frac{d\sigma}{d\Omega}$ is φ dependent.

Approaching $\theta\to 0$ we get: $f_{\vec{k}}^{dipole} = (1-e^{idk})f_{\vec{k}}^{monopole}$ wich is independent of φ and related to the distance $d = \frac{2\pi n}{k}$, for wich $f_{\vec{k}}^{dipole}$ and thus the cross section $\frac{d\sigma}{d\Omega}$ vanishes.

$f_{\vec{k}}^{monopole} \propto \frac{1}{|k|^2} \propto \frac{1}{(\cos\vartheta - 1)}$ (this is the pole I mentioned above)

so as $\vartheta\to 0$, $\frac{d\sigma}{d\Omega}\to\infty$!

4.)

$$\int\left(\frac{d\sigma}{d\Omega}\right)_{inc.} d\Omega \simeq \int\left(\frac{d\sigma}{d\Omega}\right) d\Omega \text{ for large } \Omega.$$

$$\int_\Omega |2\cdot f_{\vec{k}}^{monopole}|^2 d\Omega \simeq \int_\Omega \left|(1-e^{-i|\vec{q}|\cdot|\vec{d}|\cos\theta})f_{\vec{k}}^{monopole}\right|^2 d\Omega$$

Assume $\Omega = [-1+\varepsilon, 1-\varepsilon]\otimes[0,2\pi]$ with $\varepsilon \ll 1$ and $|\vec{q}| \gg \frac{1}{|\vec{d}|} \Leftrightarrow |\vec{q}||\vec{d}| \gg 1$ we can show that in 0-th order approximation:

$$\int_{-1}^{1}\int_0^{2\pi} |2\cdot f_{\vec{k}}^{monopole}|^2 d\Omega = \int_{-1}^{1}\int_0^{2\pi}\left|(1-e^{-i|\vec{q}|\cdot|\vec{d}|\cos\theta})f_{\vec{k}}^{monopole}\right|^2 d\Omega.$$

idk what to do with $|\vec{q}| \gg \frac{1}{|\vec{d}|}$:c

$$\int\left|\frac{d\sigma}{d\Omega}\right| d\Omega \propto \int d\Omega\left[\underbrace{|f_{\vec{k}}^{monopole}|^2 + |e^{-i\vec{q}\cdot\vec{d}} f_{\vec{k}}^{monopole}|^2}_{\text{incoherent sum}} + |(e^{i\vec{q}\cdot\vec{d}} + e^{-i\vec{q}\cdot\vec{d}}) f_{\vec{k}}^{monopole}|^2\right]$$

highly oscillatory and integral vanishes with large phase space

Quickies:

Q1: (i) $f_{\vec{k}} = -\frac{m}{2\pi\hbar^2}\int_{\mathbb{R}^3} d^3x'\, e^{-i(\vec{k}'-\vec{k})\vec{x}'}\, V(\vec{x}')$

(ii) $\vec{x} \to \tilde{\vec{x}} = \vec{x} + \vec{\varepsilon}$ with $|\vec{\varepsilon}| \ll \frac{1}{|\vec{q}|} \Leftrightarrow |\vec{\varepsilon}||\vec{q}| \ll 1$

As $|\vec{\varepsilon}||\vec{q}| \ll 1$ we can approximate $e^{-i\vec{q}\tilde{\vec{x}}} \simeq e^{-\vec{q}\vec{x}}$, thus having no effect on $f_{\vec{k}}$. The fast oscillating term is supressed by the fourier-transform.

Q2: Assuming a particle, thats not point like, we get for low energies only coulomb interaction, rather than an interaction with the expanded charge distribution, since the scattered particle wont get close. For high energies we get close to the core and observe a scattering cross section of a non point like particle, like a ball.

If the particle is however point like, we observe even for high energies no change in the cross section.

Q3:

(i) $|a_l(|\vec{k}|)| \leq \frac{1}{|\vec{k}|}$ with $a_l(|\vec{k}|) = \frac{e^{2i\delta_l(|\vec{k}|)} - 1}{2i|\vec{k}|}$

(ii) $\sigma_{tot}(|\vec{k}|) = \frac{4\pi}{|\vec{k}|}\, \mathrm{Im}\, f_{\vec{k}}(\Theta)$

2) Two-Particle Wave Function

$$\psi(x_1,x_2) = N\exp\left[-\frac{(x_1-x_2)^2}{\sigma^2}\right]\exp\left[-\frac{(x_1+x_2)^2}{\Sigma^2}\right]$$

1) $\int_{\mathbb{R}}dx_1\int_{\mathbb{R}}dx_2\, |\psi(x_1,x_2)|^2 \stackrel{!}{=} 1$

$$\begin{aligned}\Rightarrow 1 &= N^2\int_{\mathbb{R}}dx_1\int_{\mathbb{R}}dx_2\, \exp\left[-\frac{2(x_1-x_2)^2}{\sigma^2}\right]\exp\left[-\frac{2(x_1+x_2)^2}{\Sigma^2}\right] \\ &= \frac{N^2}{2}\int_{\mathbb{R}}du\int_{\mathbb{R}}dv\, \exp\left[-\frac{2v^2}{\sigma^2}\right]\exp\left[-\frac{2u^2}{\Sigma^2}\right] \\ &= \frac{N^2}{2}\sqrt{\frac{\sigma^2\pi}{2}}\sqrt{\frac{\Sigma^2\pi}{2}} \\ &= \frac{\pi\sigma\Sigma}{4}\cdot N^2 \\ \Rightarrow N &= \frac{2}{\sqrt{\pi\sigma\Sigma}}\end{aligned}$$

$u = x_1 + x_2 \quad \wedge \quad v = x_1 - x_2 \quad \Rightarrow x_1 = \frac{u+v}{2} \quad \wedge \quad x_2 = \frac{u-v}{2}$

$\Rightarrow |\det(J)| = \begin{vmatrix} \partial x_1/\partial u & \partial x_1/\partial v \\ \partial x_2/\partial u & \partial x_2/\partial v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \left|-\frac{1}{4}-\frac{1}{4}\right| = \frac{1}{2}$

2) $\psi(x_1,x_2) = N\exp\left[-\frac{(x_1-x_2)^2}{\sigma^2} - \frac{(x_1+x_2)^2}{\Sigma^2}\right] = N\underbrace{\exp\left[-\left(\frac{1}{\sigma^2}+\frac{1}{\Sigma^2}\right)x_1^2\right]}_{f(x_1)}\underbrace{\exp\left[-\left(\frac{1}{\sigma^2}+\frac{1}{\Sigma^2}\right)x_2^2\right]}_{g(x_2)}\underbrace{\exp\left[2\left(\frac{1}{\sigma^2}-\frac{1}{\Sigma^2}\right)x_1x_2\right]}_{\hookrightarrow \text{ impossible to write } e^{ab} \text{ as } f(a)\cdot f(b)} \neq f(x_1)\cdot g(x_2)$

3) $\psi(x_1,x_2) = \frac{1}{2\pi}\int_{\mathbb{R}}dk_1\int_{\mathbb{R}}dk_2\, \phi(k_1,k_2)\, e^{-ik_1x_1}e^{-ik_2x_2}$

$$\begin{aligned}\Rightarrow \phi(k_1,k_2) &= \int_{\mathbb{R}}dx_1\int_{\mathbb{R}}dx_2\, \psi(x_1,x_2)\, e^{ik_1x_1}e^{ik_2x_2} \\ &= N\cdot\int_{\mathbb{R}}dx_1\int_{\mathbb{R}}dx_2\, \exp\left[-\frac{(x_1-x_2)^2}{\sigma^2} + ik_1x_1\right]\exp\left[-\frac{(x_1+x_2)^2}{\Sigma^2} + ik_2x_2\right] \\ &= \frac{N}{2}\cdot\int_{\mathbb{R}}du\int_{\mathbb{R}}dv\, \exp\left[-\frac{v^2}{\sigma^2} + \frac{ik_1}{2}(u+v)\right]\exp\left[-\frac{u^2}{\Sigma^2} + \frac{ik_2}{2}(u-v)\right] \\ &= \frac{N}{2}\cdot\int_{\mathbb{R}}du\int_{\mathbb{R}}dv\, \exp\left[-\frac{v^2}{\sigma^2} + \frac{i(k_1-k_2)}{2}v\right]\exp\left[-\frac{u^2}{\Sigma^2} + \frac{i(k_1+k_2)}{2}u\right] \\ &= \frac{N}{2}\sqrt{\sigma^2\pi}\,\exp\left[-\frac{(k_1-k_2)^2}{16\sigma^2}\right]\sqrt{\Sigma^2\pi}\,\exp\left[-\frac{(k_1+k_2)^2}{16\Sigma^2}\right] \\ &= \sqrt{\pi\sigma\Sigma}\,\exp\left[-\left(\frac{k_1-k_2}{4\sigma}\right)^2 - \left(\frac{k_1+k_2}{4\Sigma}\right)^2\right]\end{aligned}$$

$\Big|\ \int_{\mathbb{R}}dx\, \exp[-ax^2+bx] = \sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2}{4a}\right]$

where $N = \frac{2}{\sqrt{\pi\sigma\Sigma}}$ (s.a.)

4) $\psi(x_1,x_2) = N\exp\left[-\frac{(x_1-x_2)^2}{\sigma^2}\right]\exp\left[-\frac{(x_1+x_2)^2}{\Sigma^2}\right] = N\exp\left[-\frac{(x_2-x_1)^2}{\sigma^2}\right]\exp\left[-\frac{(x_2+x_1)^2}{\Sigma^2}\right] = \psi(x_2,x_1) \quad \rightarrow$ symmetric wave function

i) $\psi(x_1,x_2) = \psi$ can be used to describe two identical bosons, because ψ is symmetrical under interchange of x_1 and x_2.

ii) $\psi(x_1,x_2) = \psi$ can't be used to describe two identical fermions, because then ψ must be antisymmetrical under interchange of x_1 and x_2.