Sheet 2 Marc Maier, Angelo Brade 14.10.2024

Aufgabe 3:

1) $\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t) \quad | \; \vec{A}(\vec{x},t) \to \vec{A}(\vec{x},t)+\vec{\nabla}\lambda(\vec{x},t)$

$$= \vec{\nabla}\times\vec{A}(\vec{x},t) + \vec{\nabla}\times(\vec{\nabla}\cdot\lambda(\vec{x},t) \quad | \; \vec{\nabla}\times(\vec{\nabla}\cdot f) = 0$$

$$= \vec{\nabla}\times\vec{A}(\vec{x},t)$$

$\square$

$$\vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t} \quad \Big| \; \vec{U}(\vec{x},t) \to \vec{U}(\vec{x},t) - \frac{\partial\lambda(\vec{x},t)}{\partial t}, \; \vec{A}(\vec{x},t)\to\vec{A}(\vec{x},t)+\vec{\nabla}\lambda(\vec{x},t)$$

$$= -\vec{\nabla}\left(\vec{U}(\vec{x},t) - \cancel{\frac{\partial\lambda(\vec{x},t)}{\partial t}}\right) - \frac{\partial}{\partial t}\left(\vec{A}(\vec{x},t)+\cancel{\vec{\nabla}\lambda(\vec{x},t)}\right)$$

$$= -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

2)

$$0 = \vec{\nabla}\cdot\vec{B}(\vec{x},t) \quad | \; \vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t)$$

$$= \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}(\vec{x},t)) \quad | \; \vec{\nabla}\cdot(\vec{\nabla}\times\vec{f}) = 0$$

$$= 0$$

$\square$

$$\vec{\nabla}\times\vec{E}(\vec{x},t) = -\frac{\partial\vec{B}(\vec{x},t)}{\partial t} \quad | \; \vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t), \; \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

$$\vec{\nabla}\times\left(\cancel{-\vec{\nabla}U(\vec{x},t)} - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}\right) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}(\vec{x},t) \quad | \; \vec{\nabla}\times(\vec{\nabla}\cdot f) = 0, \; \vec{\nabla}\times\frac{\partial}{\partial t}f = \frac{\partial}{\partial t}\vec{\nabla}\times f$$

$$-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}(\vec{x},t) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}(\vec{x},t) \quad \square$$

3) $\vec{\nabla}\cdot\vec{E}(\vec{x},t) = \rho(\vec{x},t)/\varepsilon_0 \quad | \; \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}$

$$-\vec{\nabla}\cdot\vec{\nabla}U(\vec{x},t) - \vec{\nabla}\cdot\frac{\partial\vec{A}(\vec{x},t)}{\partial t} = \rho(\vec{x},t)/\varepsilon_0$$

$$\vec{\nabla}\times\vec{B}(\vec{x},t) = \mu_0\vec{j}(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}(\vec{x},t)}{\partial t} \quad | \; \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

$$= \mu_0\vec{j}(\vec{x},t) - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial}{\partial t}U(\vec{x},t) - \mu_0\varepsilon_0\frac{\partial^2\vec{A}(\vec{x},t)}{\partial t^2} \quad | \; -\mu_0\varepsilon_0\frac{\partial}{\partial t}U(\vec{x},t) = \vec{\nabla}\cdot\vec{A}(\vec{x},t)$$

$$\vec{\nabla}\times\vec{A}(\vec{x},t) = \mu_0\vec{j}(\vec{x},t) - \Delta\vec{A}(\vec{x},t) - \mu_0\varepsilon_0\frac{\partial^2\vec{A}(\vec{x},t)}{\partial t^2}$$

With rot $(\vec{\nabla}\times\vec{f})$ we get $\vec{B}(\vec{x},t)$.

$$\vec{\nabla}\cdot\vec{E}(\vec{x},t) = \rho(\vec{x},t)/\varepsilon_0 \qquad \Big|\ \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial \vec{A}(\vec{x},t)}{\partial t}$$

$$-\Delta U(\vec{x},t) - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}(\vec{x},t) = \rho(\vec{x},t)/\varepsilon_0 \qquad \Big|\ \vec{\nabla}\cdot\vec{A}(\vec{x},t) = -\mu_0\varepsilon_0\frac{\partial U(\vec{x},t)}{\partial t}$$

$$-\Delta U(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial^2 U(\vec{x},t)}{\partial t^2} = \rho(\vec{x},t)/\varepsilon_0$$

With grad $(\vec{\nabla}\cdot f)$ we get $\vec{E}(\vec{x},t)$.

4) $\frac{\partial\rho(\vec{x},t)}{\partial t} + \vec{\nabla}\cdot\vec{j}(\vec{x},t) = 0$

$$\frac{\varepsilon_0}{2}\vec{E}^2 - \frac{1}{2\mu_0}\vec{B}^2 - \rho U + \vec{j}\cdot\vec{A}$$

$$= \frac{\varepsilon_0}{2}\left(-\vec{\nabla}U - \frac{\partial\vec{A}}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\vec{\nabla}\times\vec{A})^2 - \rho U + \vec{j}\cdot\vec{A} \quad \Big|\ \text{gauge Trafo.}$$

$$= \frac{\varepsilon_0}{2}\Big(\underbrace{-\vec{\nabla}U + \cancel{\vec{\nabla}\frac{\partial\lambda}{\partial t}} - \frac{\partial\vec{A}}{\partial t}}_{\vec{E}} - \cancel{\frac{\partial}{\partial t}\vec{\nabla}\lambda}\Big)^2 - \frac{1}{2\mu_0}\Big(\underbrace{\vec{\nabla}\times\vec{A}}_{\vec{B}} + \underbrace{\cancel{\vec{\nabla}\times\vec{\nabla}\lambda}}_{=0}\Big)^2 - \rho U + \rho\frac{\partial}{\partial t}\lambda + \vec{j}\vec{A} + \vec{j}\vec{\nabla}\lambda$$

$$= \frac{\varepsilon_0}{2}\vec{E}^2 - \frac{1}{2\mu_0}\vec{B}^2 - \rho U + \vec{j}\vec{A} + \rho\frac{\partial}{\partial t}\lambda + \vec{j}\vec{\nabla}\lambda \quad \Big|\ \text{chain rule}$$

$$= \frac{\varepsilon_0}{2}\vec{E}^2 - \frac{1}{2\mu_0}\vec{B}^2 - \rho U + \vec{j}\vec{A} + \underbrace{\left(\frac{\partial}{\partial t} + \vec{\nabla}\right)\lambda\cdot(\rho + \vec{j})}_{\text{surface term}} + \lambda\Big(\underbrace{\cancel{\frac{\partial}{\partial t}\rho + \vec{\nabla}\vec{j}}}_{=0}\Big)$$

why?

$\square$

Exercise 2:

1.) $\Psi(q_i,t) = \sum_n c_n(t)\,\psi_n(q_i)$

$\Rightarrow \bar{\Psi}(q_i,t) = \hat{U}_g(\xi)\,\Psi(q_i,t)$

$= \exp(-i\xi\hat{g}/\hbar)\cdot\sum_n c_n(t)\,\psi_n(q_i)$

$= \sum_n c_n(t)\exp(-i\xi g_n/\hbar)\,\psi_n(q_i) \quad | \quad \bar{c}_n = c_n(t)\exp(-i\xi g_n/\hbar)$

$= \sum_n \bar{c}_n(t)\,\psi_n(q_i)$

$\Rightarrow c_n(t) \to \bar{c}_n(t) = c_n(t)\exp(-i\xi g_n/\hbar)$

2.) $\hat{g} = \hat{L}_z$

$\hat{g}\,\psi_n(q_i) = g_n\,\psi_n(q_i)$

$\Rightarrow \hat{L}_z\,\psi_m(q_i) = m\hbar\,\psi_m(q_i)$

$\Rightarrow \psi_m(q_i) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$

$\Rightarrow \bar{\Psi}(q_i,t) = \sum_m \frac{c_n(t)}{\sqrt{2\pi}}\,e^{-i\xi m}\cdot e^{im\varphi}$

$= \sum_m \frac{c_n(t)}{\sqrt{2\pi}}\,e^{im(\varphi-\xi)}$

$\Rightarrow e^{im(\varphi-\xi)}$ is rotation in φ and $\varphi \perp z \Rightarrow$ rotation around z-axis.

3.) physical trafo. must be measurable $\Rightarrow \hat{L}^2 = \hat{L}^{2\dagger} \Rightarrow \hat{L}^2$ has eigenfunctions with different eigenvalues.

In cases i) and ii) the eigenvalues of $\hat{L}^2$ cannot change $\Rightarrow$ no physical trafo.

In case iii) the eigenvalues can change $\Rightarrow$ physical trafo.

Not sure about this logic.

Exercise 1:

1) E.o.m: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

(q_i, p_i) valid, iff satisfies e.o.m.

Z: $(\bar{q}_i, \bar{p}_i)$ satisfies e.o.m, iff $\{g, H\} = 0$.

$$\dot{\bar{q}}_i = \frac{d}{dt}\left(q_i + \varepsilon \frac{\partial g}{\partial p_i}\right)$$

$$= \frac{\partial H}{\partial p_i} + \frac{d}{dt}\left(\varepsilon \frac{\partial g}{\partial p_i}\right)$$

$$= \frac{\partial H}{\partial p_i} + \frac{\partial}{\partial p_i} \varepsilon \frac{\partial g}{\partial t}$$

$$= \frac{\partial H}{\partial p_i} + \frac{\partial}{\partial p_i} \varepsilon \left(\frac{\partial g}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial g}{\partial p_i}\frac{\partial p_i}{\partial t}\right) \Big| \text{ e.o.m.}$$

$$= \frac{\partial H}{\partial p_i} + \frac{\partial}{\partial p_i} \underbrace{\varepsilon}_{\neq 0} \underbrace{\left(\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i}\right)}_{\overset{!}{=} 0}$$

$\Rightarrow \{g, H\} = 0$

Analog for $\dot{p}_i$.

$\square$

2.)

$$\bar{x}_k = x_k + \delta \Rightarrow \delta = \varepsilon \underbrace{\frac{\partial g}{\partial p_i}}_{\overset{!}{=} \delta_{ik}} \Rightarrow \underline{g = p_k}$$

$$\{g, H\} = \{p_k, H\} = \underbrace{\frac{\partial p_k}{\partial q_i}}_{0}\frac{\partial H}{\partial p_i} - \underbrace{\frac{\partial p_k}{\partial p_i}}_{\delta_{ki}}\frac{\partial H}{\partial q_i}$$

$$= -\delta_{ki}\frac{\partial H}{\partial q_i}$$

$$= \delta_{ki}\dot{p}_i$$

$$= \underbrace{\dot{p}_k}_{\overset{!}{=} 0}$$

$$\Rightarrow \frac{\partial H}{\partial q_k} = 0 \Rightarrow H = H(q_{i \neq k}, p_i)$$

With $H = H(q_{i\neq k}, p_i)$ the e.o.m. are still satisfied $\Rightarrow$ valid trajectory.