Präsenzaufgabenblatt 8

1)(i) Wir zeigen, dass $\sum_{k=-\infty}^{+\infty} |\alpha_k| < +\infty$

(I)

(II) $\sum_{k=-\infty}^{+\infty} |\alpha_k| = \underbrace{\sum_{k=-\infty}^{-1} |\alpha_k|(1)^k + \sum_{k=0}^{+\infty} |\alpha_k|(1)^k}_{\text{Laurent-Reihe evaluiert in } 1.}$

(I)

Nach dem Satz 6.1 vom Skript: Beide Reihen, Haupt und Nebenteil, konvergieren absolut auf $A_{r_1,r_2}(0) = \{z \in \mathbb{C} \mid 0 < r_1 \leq |z| \leq r_2 < \infty\}$

=

(II) (siehe $\sum_{k=0}^{+\infty} \alpha_k z^k = \frac{1}{2\pi i}\int_{\partial B_{r_2}(0)} \frac{f(y)}{y-z}\,dy < +\infty \Rightarrow$ auch Absolut ~~in~~ 1

$\sum_{k=-\infty}^{-1} \alpha_k z^k = \frac{1}{2\pi i}\int_{\partial B_{r_1}(0)} \frac{f(y)}{z-y}\,dy < +\infty \Rightarrow$ auch Absolut in 1)

(ii) $\hat{g}_k = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-ikt} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-ikt}\,dt$

=

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k'=-\infty}^{+\infty} \alpha_{k'} e^{ik't} e^{-ikt} \, dt$$

Sei $f_N(t) := \sum_{k'=-N}^{N} \alpha_{k'} e^{ik't} e^{-ikt}$

$$|f_N(t)| \leq \sum_{k'=-N}^{N} |\alpha_{k'}| < \underbrace{\sum_{k'=-\infty}^{+\infty} |\alpha_{k'}|}$$

Integrierbar auf $[0, 2\pi]$

Dominierte Konvergenz

$$\Rightarrow \lim_{N \to +\infty} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k'=-N}^{N} \alpha_{k'} e^{ik't} e^{-ikt} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \lim_{N \to +\infty} \sum_{k'=-N}^{N} \alpha_{k'} e^{ik't} e^{-ikt} \, dt$$

$$\Leftrightarrow \lim_{N \to +\infty} \sum_{k'=-N}^{N} \frac{\alpha_{k'}}{2\pi} \int_0^{2\pi} e^{ik't} e^{-ikt} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k'=-\infty}^{+\infty} \alpha_{k'} e^{ik't} e^{-ikt} \, dt$$

$$\Leftrightarrow \sum_{k'=-\infty}^{+\infty} \alpha_{k'} \delta_{k,k'} = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-ikt}$$

$$\Leftrightarrow \alpha_k = \hat{g}_k$$

2)

wir merken, dass

(I) $\int_0^{2\pi} \cos(k'x)\sin(kx)\,dx = 0$

für $k \neq k'$ oder $k \neq -k'$. $(k, k' \neq 0)$

(II) $\int_0^{2\pi} \cos(k'x)\cos(kx)\,dx = 0$

(I) $$\int_0^{2\pi} \underset{\downarrow \atop -k'\sin(k'x)}{\cos(k'x)}\ \overset{-\frac{\cos(kx)}{k} \atop \uparrow}{\sin(kx)}\,dx = -\underbrace{\left[\cos(k'x)\frac{\cos(kx)}{k}\right]_0^{2\pi}}_{\frac{1}{k}-\frac{1}{k}=0} - \frac{k'}{k}\int_0^{2\pi} \underset{\downarrow \atop k'\cos(k'x)}{\sin(k'x)}\ \overset{\frac{\sin(kx)}{k} \atop \uparrow}{\cos(kx)}\,dx$$

$$= \frac{k'}{k}\underbrace{\left[\sin(k'x)\frac{\sin(kx)}{k}\right]_0^{2\pi}}_{=0} + \left(\frac{k'}{k}\right)^2 \int_0^{2\pi} \cos(k'x)\sin(kx)\,dx$$

$$\Rightarrow \left(\int_0^{2\pi} \cos(k'x)\sin(kx)\,dx\right)\left(1-\left(\frac{k'}{k}\right)^2\right) = 0 \quad \Rightarrow \int_0^{2\pi} \cos(k'x)\sin(kx)\,dx = 0$$

für $k \neq k'$ oder $k \neq -k'$, $k, k' \neq 0$

(II) $$\int_0^{2\pi} \underset{\downarrow \atop -k'\sin(k'x)}{\cos(k'x)}\ \overset{\frac{\sin(kx)}{k} \atop \uparrow}{\cos(kx)}\,dx = \underbrace{\left[\cos(k'x)\frac{\sin(kx)}{k}\right]_0^{2\pi}}_{=0} - \int_0^{2\pi} \frac{(-k')}{k}\ \underset{\downarrow \atop k'\cos(k'x)}{\sin(k'x)}\ \overset{-\frac{\cos(kx)}{k} \atop \uparrow}{\sin(kx)}\,dx$$

$$= \frac{k'}{k}\underbrace{\left[\sin(k'x)\left(\frac{-\cos(kx)}{k}\right)\right]_0^{2\pi}}_{=0} + \left(\frac{k'}{k}\right)^2 \int_0^{2\pi} \cos(k'x)\cos(kx)\,dx$$

$$\Rightarrow \left[\int_0^{2\pi} \cos(k'x)\cos(kx)\,dx\right]\left(1-\left(\frac{k'}{k}\right)^2\right) = 0 \quad \Rightarrow \int_0^{2\pi} \cos(k'x)\cos(kx)\,dx = 0$$

für $k \neq k'$, $k \neq -k'$, $k, k' \neq 0$

und $(\text{III}) \displaystyle\int_0^{2\pi} \frac{\cos(kx)^2}{\pi}\,dx = \frac{1}{\pi}\int_0^{2\pi} \cos(kx)\cos(kx)\,dx$

$\uparrow \frac{\sin(kx)}{k}$, $\downarrow -k\sin(kx)$

$$= \frac{1}{\pi}\underbrace{\left[\frac{\sin(kx)}{k}\cos(kx)\right]_0^{2\pi}}_{=0} + \frac{1}{\pi}\int_0^{2\pi} \sin(kx)^2\,dx$$

$1 = \cos^2 + \sin^2$

$$\frac{1}{\pi}\int_0^{2\pi}\cos(kx)^2\,dx \overset{\curvearrowleft}{=} \frac{2\pi}{\pi} - \frac{1}{\pi}\int_0^{2\pi}\cos^2(kx)\,dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^{2\pi}\cos(kx)^2\,dx = 2 \Rightarrow \frac{1}{\pi}\int_0^{2\pi}\cos(kx)^2\,dx = 1$$

siehe $1 = \cos^2 + \sin^2$

und $\displaystyle\int_0^{2\pi}\frac{\sin(kx)^2}{\pi}\,dx \overset{!}{=} 1 \quad (\text{IV})$

$$\cdot\ \partial_{xx}u = -f_1(y)\cos(x) + f_{10}(y)\left(-10^2\cos(10x)\right) + g_2(y)\left(-2^2\sin(2x)\right)$$
$$- g_5(y)\cos(5x)\,5^2$$

$$\partial_{yy}u = f_1''(y)\cos(x) + f_{10}''(y)\cos(10x) + g_2''(y)\sin(2y) + g_5''(y)\cos(5x)$$

wenn $\partial_{xx}u + \partial_{yy}u = 0 \overset{(\text{I})(\text{II})(\text{III})(\text{IV})}{\Longrightarrow} \begin{cases} f_1(y) = f_1''(y) \\ f_{10}(y)\,10^2 = f_{10}''(y) \\ g_2(y)\,2^2 = g_2''(y) \\ g_5(y)\,5^2 = g_5''(y) \end{cases} \Rightarrow \begin{cases} f_1(y) = A_1e^{y} + B_1e^{-y} \\ f_{10}(y) = A_{10}e^{10y} + B_{10}e^{-10y} \\ g_2(y) = A_2e^{2y} + B_2e^{-2y} \\ g_5(y) = A_5e^{5y} + B_5e^{-5y} \end{cases}$

$u(x,0) = \cos(x) + \cos(10x) \overset{(\text{I})(\text{II})(\text{III})(\text{IV})}{\Longrightarrow} \begin{cases} f_1(0) = 1,\ g_2(0) = 0 \\ f_{10}(0) = 1,\ g_5(0) = 0 \end{cases}$

$\partial_y u(x,0) = \sin(2x) + \cos(5x) \overset{(\text{I})(\text{II})(\text{III})(\text{IV})}{\Longrightarrow} \begin{cases} g_2'(0) = 1,\ f_1'(0) = 0 \\ g_5'(0) = 1,\ f_{10}'(0) = 0 \end{cases}$

$$\begin{cases} f_1(0) = 1 \\ f_1'(0) = 0 \end{cases} \Rightarrow f_1(y) = \cosh(y)$$

$$\begin{cases} f_{10}(0) = 1 \\ f_{10}'(0) = 0 \end{cases} \Rightarrow f_{10}(y) = \cosh(10y)$$

$$\begin{cases} g_2(0) = 0 \\ g_2'(0) = 1 \end{cases} \Rightarrow g_2(y) = \frac{\sinh(2y)}{2}$$

$$\begin{cases} g_5(0) = 0 \\ g_5'(0) = 1 \end{cases} \Rightarrow g_5(y) = \frac{\sinh(5y)}{5}$$

$$\Rightarrow u(x,y) = \cosh(y)\cos(x) + \cosh(10y)\cos(10x) + \frac{\sinh(2y)}{2}\sin(2x) + \frac{\sinh(5y)}{5}\cos(5x)$$

(leicht zu prüfen dass es eine Lösung ist).