Ha3_Richter-Brade

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Theo 1 Übungsblatt Nr. 3 20. 4. 2023

H1 Differentialoperatoren II

Zeige mit $x \equiv |\vec{x}|$ und $|\vec{x}| = (x_i x_i)^{\frac{1}{2}}$ die folgenden Gleichungen:

a) Gradient:

$x = |\vec{x}| = \sqrt{x^2+y^2+z^2}$

$$\vec{\nabla}\frac{1}{x} = -\frac{\vec{x}}{x^3}$$

$$\vec{\nabla}\frac{1}{x} = \begin{pmatrix} \frac{\partial \frac{1}{|\vec{x}|}}{\partial x} \\ \frac{\partial \frac{1}{|\vec{x}|}}{\partial y} \\ \frac{\partial \frac{1}{|\vec{x}|}}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial \frac{1}{\sqrt{x^2+y^2+z^2}}}{\partial x} \\ \frac{\partial \frac{1}{\sqrt{x^2+y^2+z^2}}}{\partial y} \\ \frac{\partial \frac{1}{\sqrt{x^2+y^2+z^2}}}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\sqrt{x^3}}\cdot 2x \\ -\frac{1}{2\sqrt{x^3}}\cdot 2y \\ -\frac{1}{2\sqrt{x^3}}\cdot 2z \end{pmatrix} = \begin{pmatrix} -\frac{2x}{2|x|^3} \\ -\frac{2y}{2|x|^3} \\ -\frac{2z}{2|x|^3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{x}{|x|^3} \\ -\frac{y}{|x|^3} \\ -\frac{z}{|x|^3} \end{pmatrix} = -\frac{\vec{x}}{x^3} \qquad \sqrt{x} = |\vec{x}|$$

$$\vec{\nabla}(\ln x) = \begin{pmatrix} \frac{\partial \ln(\sqrt{x^2+y^2+z^2})}{\partial x} \\ \frac{\partial \ln(\sqrt{x^2+y^2+z^2})}{\partial y} \\ \frac{\partial \ln(\sqrt{x^2+y^2+z^2})}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{|x|}\cdot\frac{1}{2}\frac{1}{|x|}\cdot 2x \\ \frac{1}{|x|}\cdot\frac{1}{2}\frac{1}{|x|}\cdot 2y \\ \frac{1}{|x|}\cdot\frac{1}{2}\frac{1}{|x|}\cdot 2z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{|x|^2}\cdot x \\ \frac{1}{|x|^2}\cdot y \\ \frac{1}{|x|^2}\cdot z \end{pmatrix} = \frac{\vec{x}}{x^2}$$

b) Divergenz:

$\vec{\nabla}\cdot(x\vec{v}) = \vec{v}\,\frac{\vec{x}}{x}$, $\vec{v} =$ konst.

$$\vec{\nabla}\cdot(x\vec{v}) = \vec{\nabla}\cdot\begin{pmatrix} |\vec{x}|\cdot v_1 \\ |\vec{x}|\cdot v_2 \\ |\vec{x}|\cdot v_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial|\vec{x}|}{\partial x}\cdot v_1 \\ \frac{\partial|\vec{x}|}{\partial y}\cdot v_2 \\ \frac{\partial|\vec{x}|}{\partial z}\cdot v_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{1}{|\vec{x}|}\cdot 2x\cdot v_1 \\ \frac{1}{2}\frac{1}{|\vec{x}|}\cdot 2y\cdot v_2 \\ \frac{1}{2}\frac{1}{|\vec{x}|}\cdot 2z\cdot v_3 \end{pmatrix} = \begin{pmatrix} \frac{\vec{x}}{x} v_1 \\ \frac{\vec{x}}{x} v_2 \\ \frac{\vec{x}}{x} v_3 \end{pmatrix}$$

$$= \vec{v}\,\frac{\vec{x}}{x}$$

$$\vec{\nabla}(x^n\vec{x}) = (3+n)x^n$$

$$\vec{\nabla}\begin{pmatrix} |\vec{x}|^n\cdot x \\ |\vec{x}|^n\cdot y \\ |\vec{x}|^n\cdot z \end{pmatrix} = \begin{pmatrix} \frac{\partial|\vec{x}|^n}{\partial x}\cdot x \\ \frac{\partial|\vec{x}|^n}{\partial y}\cdot y \\ \frac{\partial|\vec{x}|^n}{\partial z}\cdot z \end{pmatrix} = \begin{pmatrix} \frac{\partial(x_ix_i)^{\frac{n}{2}}}{\partial x}\cdot x \\ \frac{\partial(x_ix_i)^{\frac{n}{2}}}{\partial y}\cdot y \\ \frac{\partial(x_ix_i)^{\frac{n}{2}}}{\partial z}\cdot z \end{pmatrix}$$

$\partial(x_ix_i)^{\frac{n}{2}}\, x + \partial(x_ix_i)^{\frac{n}{2}}\, y + \partial(x_ix_i)^{\frac{n}{2}}\cdot z + 3\cdot(x_ix_i)^{\frac{n}{2}}$

$$\left(\frac{\partial |x|^n}{\partial z}\cdot z\right) \quad \left(\frac{\partial (x_i x_i)^{\frac{n}{2}}}{z}\cdot z\right)$$

$$= \frac{\partial (x_i x_i)^{\frac{n}{2}}}{\partial x}\cdot x + \frac{\partial (x_i x_i)^{\frac{n}{2}}}{\partial y}\cdot y + \frac{\partial (x_i x_i)^{\frac{n}{2}}}{\partial z}\cdot z + 3\cdot (x_i x_i)^{\frac{n}{2}}$$

$$= 3(x^2+y^2+z^2)^{\frac{n}{2}} + \frac{\partial (x^2+y^2+z^2)^{\frac{n}{2}}}{\partial x}\cdot x + \frac{\partial (x^2+y^2+z^2)^{\frac{n}{2}}}{\partial y}\cdot y + \frac{\partial (x^2+y^2+z^2)^{\frac{n}{2}}}{\partial z} z$$

$$= 3(x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2x\cdot x + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2y\cdot y + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2z\cdot z$$

$$= 3(x^2+y^2+z^2)^{\frac{n}{2}} + n(x^2+y^2+z^2)^{\frac{n}{2}-1} x^2 + n(x^2+y^2+z^2)^{\frac{n}{2}-1} y^2 + n(x^2+y^2+z^2)^{\frac{n}{2}-1} z^2$$

$$= 3(x^2+y^2+z^2)^{\frac{n}{2}} + n(x^2+y^2+z^2)^{\frac{n}{2}} \left(\frac{x^2+y^2+z^2}{x^2+y^2+z^2}\right)$$

$$= (3+n)(x^2+y^2+z^2)^n$$

$$= (3+n)x^n$$

Theo 1 Zettel 3

H1 Rotation:

$$\vec{\nabla} \times \vec{x} = \vec{0}$$

$$\begin{pmatrix} \frac{\partial f(x)}{\partial x} \\ \frac{\partial f(y)}{\partial y} \\ \frac{f(z)}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{array}{ll} \cancel{\partial f(x)} & \cancel{x} \\ \partial f(y) & y \\ \partial f(z) & z \\ \partial f(x) & x \\ \partial f(y) & y \\ \cancel{\partial f(z)} & \cancel{z} \end{array}$$

$$\left.\begin{array}{l} \partial f(y) \cdot z - y \cdot \partial f(z) \\ \partial f(z) \cdot x - z \cdot \partial f(x) \\ \partial f(x) \cdot y - x \cdot \partial f(y) \end{array}\right\} 0$$

$$\vec{\nabla} \times \left(\frac{1}{2} \vec{B} \times \vec{x}\right) = \vec{B} \quad , \vec{B} = \text{const}$$

$$\begin{array}{ll} \cancel{\frac{1}{2}\vec{B}_1} & \cancel{x} \\ \frac{1}{2}\vec{B}_2 & y \\ \frac{1}{2}\vec{B}_3 & z \\ \frac{1}{2}\vec{B}_1 & x \\ \frac{1}{2}\vec{B}_2 & y \\ \cancel{\frac{1}{2}\vec{B}_3} & \cancel{z} \end{array} \qquad \vec{\nabla} \times \begin{pmatrix} \frac{1}{2}\vec{B}_2 \cdot z - y \frac{1}{2}\vec{B}_3 \\ \frac{1}{2}\vec{B}_3 \cdot x - z \frac{1}{2}\vec{B}_1 \\ \frac{1}{2}\vec{B}_1 y - x \frac{1}{2}\vec{B}_2 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{\frac{1}{2}\vec{B}_2 z - y \frac{1}{2}\vec{B}_3 \quad \partial f(x)} \\ (\frac{1}{2}\vec{B}_3 x - z \frac{1}{2}\vec{B}_1) \quad \partial f(y) \\ (\frac{1}{2}\vec{B}_1 y - x \frac{1}{2}\vec{B}_2) \quad \partial f(z) \\ (\frac{1}{2}\vec{B}_2 z - y \frac{1}{2}\vec{B}_3) \quad \partial f(x) \\ (\frac{1}{2}\vec{B}_3 x - z \frac{1}{2}\vec{B}_1) \quad \partial f(y) \\ \cancel{\frac{1}{2}\vec{B}_1 y - x \frac{1}{2}\vec{B}_2 \quad \partial f(z)} \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}\vec{B}_3 x - z \frac{1}{2}\vec{B}_1) \cdot \partial f(z) - \partial f(y) \cdot (\frac{1}{2}\vec{B}_1 y - x \frac{1}{2}\vec{B}_2) \\ \frac{1}{2}\vec{B}_1 y - x \frac{1}{2}\vec{B}_2 \cdot \partial f(x) - \partial f(z) \cdot (\frac{1}{2}\vec{B}_2 z - y \frac{1}{2}\vec{B}_3) \\ (\frac{1}{2}\vec{B}_2 z - y \frac{1}{2}\vec{B}_3) \cdot \partial f(y) - f(x) \cdot (\frac{1}{2}\vec{B}_3 x - z \frac{1}{2}\vec{B}_1) \end{pmatrix} = \vec{B}$$

$\vec{o} = \vec{\nabla} \times (f(x)\vec{x})$

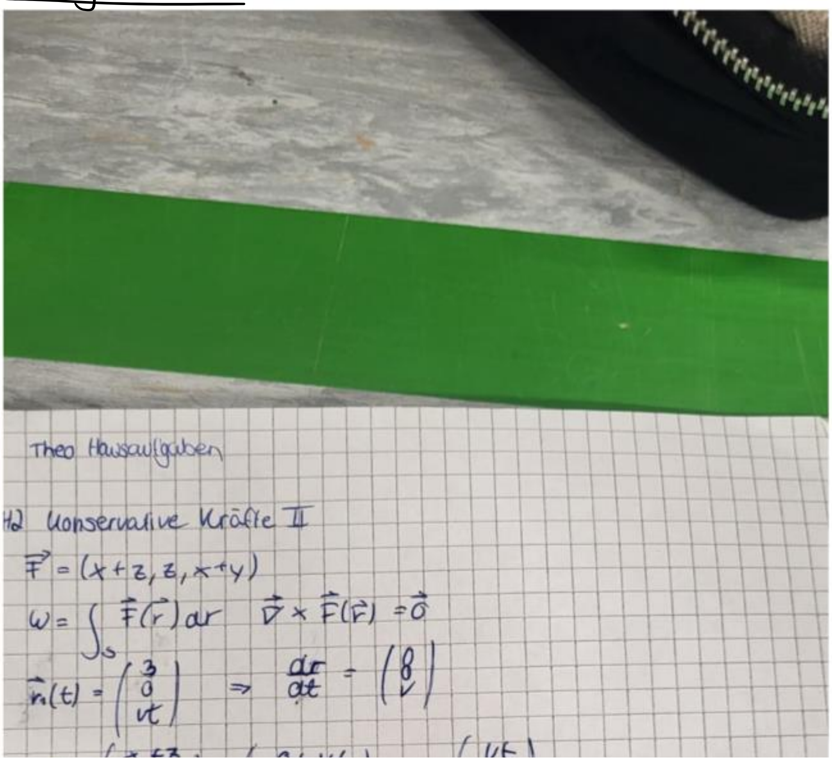
$$\vec{o} = \begin{pmatrix} \partial_{x_2} f(x) x_3 - \partial_{x_3} f(x) x_2 \\ \partial_{x_3} f(x) x_1 - \partial_{x_1} f(x) x_3 \\ \partial_{x_1} f(x) x_2 - \partial_{x_2} f(x) x_1 \end{pmatrix}$$

$$\vec{o} = \begin{pmatrix} \frac{\partial f(x)}{\partial x} \frac{\partial x}{\partial x_2} x_3 - \frac{\partial f(x)}{\partial x} \frac{\partial x}{\partial x_3} x_2 \\ \vdots \end{pmatrix}$$

$$\vec{o} = \begin{pmatrix} \frac{\partial f(x)}{\partial x} x^{-1} x_2 x_3 - \frac{\partial f(x)}{\partial x} x^{-1} x_3 x_2 \\ \vdots \end{pmatrix}$$

$\vec{o} = \vec{o}$

Aufgabe 2:



Theo Hausaufgaben

H2 Konservative Kräfte II

$\vec{F} = (x+z, z, x+y)$

$W = \int_S \vec{F}(\vec{r})\, dr \quad \vec{\nabla} \times \vec{F}(\vec{r}) = \vec{0}$

$\vec{r}_1(t) = \begin{pmatrix} 3 \\ 0 \\ vt \end{pmatrix} \Rightarrow \frac{dr}{dt} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

$$\vec{r}_1(t) = \begin{pmatrix} 3 \\ 0 \\ vt \end{pmatrix} \Rightarrow \frac{d\vec{r}}{dt} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} x+z \\ z \\ x+y \end{pmatrix} \quad \begin{pmatrix} 0+vt \\ vt \\ 0+0 \end{pmatrix} \Rightarrow \begin{pmatrix} vt \\ vt \\ 0 \end{pmatrix}$$

$$\int_{t_0}^{t_1} \begin{pmatrix} 3+vt \\ vt \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} dt = 0$$

$$\begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} x+z \\ z \\ x+y \end{pmatrix} = \begin{pmatrix} \partial y(x+y) - \partial z(z) \\ \partial z(x+z) - \partial y(x+y) \\ \partial x(z) - \partial y(x+z) \end{pmatrix} = \vec{0}$$

Weil das Kraftfeld konservativ ist, muss auch die Schraubenwegparametrisierung die gesamt Arbeit 18 verichten

$$\int_{t_0}^{t_1} (3+vt) \cdot 0 + vt \cdot 0 + 3 \cdot v \, dt$$

$$\left[3vt \right]_{t_0}^{t_1} = 3vt_1 - 3vt_0 = 3v(t_1 - t_0)$$

$$3 \cdot 6 \cdot (1-0) = 18$$

Aufgabe 3:

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$$f(t) = 1 - \frac{2t}{T}$$

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\omega t)$$

Die Periode der Funktion beträgt π.
Die Funktion ist ganzwertig und $f(0) = 0$

$$a_0 = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(1 - \frac{2t}{T}\right) dt = \frac{2}{T} \left[1t - \frac{t^2}{T} \right]_{-\frac{T}{2}}^{\frac{T}{2}} = 0$$

$$a_n = \frac{2}{\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(n\omega t) \cdot f(t) \, dt \qquad \omega = \frac{2\pi}{T} \qquad \omega = 2\pi f$$

$$= -\frac{\cos(n\omega t)}{n\omega} \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{2t\cos n\omega t}{n\omega t} \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{2\sin(n\omega t)}{n\omega} \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi n}$$

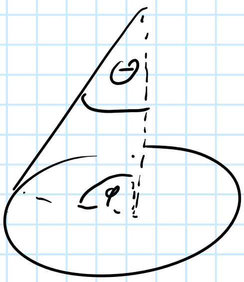
$b_n = 0$ weil $f(0) = 0$

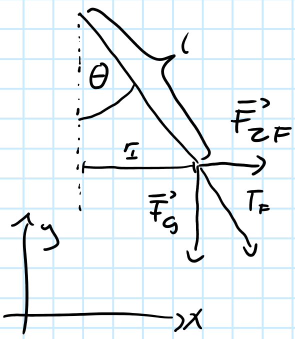
$$\hookrightarrow f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t) + b_n (n \sin \omega t)$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\omega t)$$

Aufgabe 6:

a)





$\sin\theta \, T_F = F_{ZF}$

$\cos\Theta \, T_F = F_g$

$$T_F = \frac{F_{ZF}}{\sin\Theta}$$

$$T_F = \frac{F_g}{\cos\Theta}$$

b) $$\frac{F_{ZF}}{\sin\Theta} = \frac{F_g}{\cos\Theta}$$

$$\Leftrightarrow \quad \frac{m r_1 \omega^2}{\sin\Theta} = \frac{m g}{\cos\Theta} \quad \Big| \; r_1 = \sin\Theta \, l$$

$$\Leftrightarrow \quad m \omega^2 l = \frac{m g}{\cos\Theta}$$

$$\Leftrightarrow \quad \omega = \sqrt{\frac{g}{\cos\Theta \cdot l}}$$

Aufgabe 5:

a)

$\Theta_0 \neq 0$

$D = F \cdot r_\perp = I \cdot \dot{\omega}$

$F = mg, \; r_\perp = \sin\Theta l, \; \dot{\omega} = \ddot{\Theta}, \; I = ml^2$

$-mg \sin\Theta l = ml^2 \cdot \ddot{\Theta}$

$\Leftrightarrow \quad \ddot{\Theta} + \frac{g}{l}\sin\Theta = 0$

$\Leftrightarrow \quad \frac{d\Theta}{dt}\frac{d\dot{\Theta}}{d\Theta} = -\frac{g}{l}\sin\Theta$

$\Leftrightarrow \quad \dot{\Theta}\, d\dot{\Theta} = -\frac{g}{l}\sin\Theta\, d\Theta$

$\Leftrightarrow \quad \int_0^{\dot{\Theta}} \dot{\Theta}' d\dot{\Theta}' = -\frac{g}{l}\int_{\Theta_0}^{\Theta} \sin\Theta' d\Theta'$

$\Leftrightarrow \quad \frac{1}{2}\dot{\Theta}^2 = \frac{g}{l}(\cos\Theta - \cos\Theta_0)$

$$
\begin{aligned}
E_{tot} &= E_{rot} + E_{pot} \\
&= \frac{1}{2} I \dot{\Theta}^2 + mgh \quad \Big|\; h = l(1-\cos\Theta),\; I = ml^2,\; \tfrac{1}{2}\dot{\Theta}^2 = \tfrac{g}{l}(\cos\Theta - \cos\Theta_0) \\
&= \frac{g}{l}(\cos\Theta - \cos\Theta_0)\, ml^2 + mg\, l(1-\cos\Theta) \\
&= gml(\cos\Theta - \cos\Theta_0 + 1 - \cos\Theta) \\
&= gml(1-\cos\Theta_0)
\end{aligned}
$$

$\frac{d}{dt}E(t) = \frac{d}{dt} gml(1-\cos\Theta_0)$

$\Leftrightarrow \quad \underline{\underline{\dot{E}(t) = 0}}$

b) $I\ddot{\theta} = -mg\sin\Theta l \quad | \; \sin\Theta \approx \Theta$

$I\ddot{\theta} = -mg\Theta l$

$\ddot{\Theta} + \frac{mgl}{I}\theta = 0 \quad | \; I = ml^2$

$\Leftrightarrow \quad \ddot{\theta} + \frac{g}{l}\Theta = 0$

$\Rightarrow \quad \omega_0 = \sqrt{\frac{g}{l}}$

$$\Leftrightarrow \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\Rightarrow \quad \omega_0 = \sqrt{\frac{g}{l}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Das Ergebnis ist ähnlich zu dem aus Aufgabe 4.b, indem die Eigenfrequenz genauso von $\sqrt{\frac{g}{l}}$ abhängt. Bei der Schwingung in A. 4.b ist noch die Frequenz von der Auslenkung θ abhängig, da die Zentripetalkraft also die Winkelgeschwindigkeit höher sein muss, um eine stärker wirkende Gravitationskraft auszugleichen.

c)

$$E_{kin} = gml(1-\cos\theta_0)$$

$$\Leftrightarrow \quad -mgl\cos\theta_0 \overset{!}{=} \frac{1}{2}I\dot{\theta}^2 + mgh \quad / \; h = l(1-\cos\theta), \; I = ml^2$$

$$\Leftrightarrow \quad = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$\Leftrightarrow \quad -g\cos\theta_0 = \frac{1}{2}l\,\dot{\theta}^2 + g - \cos\theta g$$

$$\Leftrightarrow \quad \frac{d\theta}{dt} = \sqrt{2\frac{g}{l}(\cos\theta - \cos\theta_0 - 1)}$$

$$\Leftrightarrow \quad \frac{d\theta}{dt} = \omega_0\sqrt{2(\cos\theta - \cos\theta_0 - 1)}$$

$$\Leftrightarrow \quad d\theta = \omega_0\sqrt{2(\cos\theta - \cos\theta_0 - 1)}\,dt$$

$$\Leftrightarrow \quad \int_0^{\theta} d\theta' = \omega_0\sqrt{2}\int_{t'=2\pi\sqrt{\frac{l}{g}}}^{t'=t}\sqrt{(\cos\theta - \cos\theta_0 - 1)}\,dt'$$

$$= \omega_0\sqrt{2}\,\frac{2}{3}(\cos\theta) - \cos\theta_0 - 1)^{\frac{3}{2}}.$$