Theo 3 Blatt 8 Angelo Brade, Florian Biolago, Pit Düster 17.06.2024

1.) $\hat{H} = \frac{\hat{p}^2}{2M} + V(x)$ mit $V(x) = \begin{cases} 0, & |x| = r < R \\ \infty, & |x| = r \geq R \end{cases}$ und $\hat{p} = -i\hbar\vec{\nabla}$

$$\phi(\vec{x}) = R_\ell(r) Y_{m_\ell}^\ell(\varphi, \theta)$$

$$\hat{H}\phi(\vec{x}) = E\phi(\vec{x})$$

$$-\left[\frac{\hbar^2}{2M}\Delta + V(x)\right]\left[R_\ell(r) Y_{m_\ell}^\ell(\varphi,\theta)\right] = E\left[R_\ell(r) Y_{m_\ell}^\ell(\varphi,\theta)\right] \quad \Big|\ \Delta = \Delta_r + \frac{\Delta_{\varphi\theta}}{r^2}$$

$$-\left[\frac{\hbar^2}{2M}\left(Y_{m_\ell}^\ell(\varphi,\theta)\Delta_r R_\ell(r) + \frac{R_\ell(r)}{r^2}\Delta_{\varphi\theta} Y_{m_\ell}^\ell(\varphi,\theta)\right] + \left[V(x) R_\ell(r) Y_{m_\ell}^\ell(\varphi,\theta)\right] = E\left(R_\ell(r) Y_{m_\ell}^\ell(\varphi,\theta)\right) \quad \Big|\ \Delta \quad \begin{cases} \Delta_r R_\ell(r) = \frac{\ell(\ell+1)}{r} R_\ell(r), \\ \Delta_{\varphi\theta} Y_{m_\ell}^\ell(\varphi,\theta) = -\ell(\ell+1) Y_{m_\ell}^\ell(\varphi,\theta) \end{cases}$$

$$-\left[\frac{\hbar^2}{2M}\left(Y_{m_\ell}^\ell(\varphi,\theta) \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R_\ell(r)\right) \quad - \frac{R_\ell(r)}{r^2}\ell(\ell+1)Y_{m_\ell}^\ell(\varphi,\theta)\right] + \left[V(x) R_\ell(r) Y_{m_\ell}^\ell(\varphi,\theta) = E\left(R_\ell(r) Y_{m_\ell}^\ell(\varphi,\theta)\right) \quad \Big|\ V(x < R) = 0$$

$$Y_{m_\ell}^\ell(\varphi,\theta)\left[-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R_\ell(r)\right) - \left(E - \frac{\hbar^2\ell(\ell+1)}{2Mr^2}\right)R_\ell(r)\right] = 0 \quad \Big|\ Y_{m_\ell}^\ell(\varphi,\theta) \neq 0$$

$$-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R_\ell(r)\right) - \left(E - \frac{\hbar^2\ell(\ell+1)}{2Mr^2}\right)R_\ell(r) = 0$$

2.)

$$-\frac{\hbar^2}{2M}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R_\ell(r)\right) - \left(E - \frac{\hbar^2\ell(\ell+1)}{2Mr^2}\right)R_\ell(r) = 0$$

$$\Rightarrow 2r R_\ell'(r) + r^2 R_\ell''(r) + \frac{2Mr^2}{\hbar^2}\left(E - \frac{\hbar^2\ell(\ell+1)}{2Mr^2}\right)R_\ell(r) = 0 \quad \Big|$$

$$\Rightarrow 2r R_\ell'(r) + r^2 R_\ell''(r) + \left(\frac{2ME}{\hbar^2}r^2 - \ell(\ell+1)\right)R_\ell(r) = 0$$

$$\begin{cases} R_\ell(r) = R_\ell(kr) \\ R_\ell'(r) = \frac{d}{dr}R_\ell(kr) = \frac{dR_\ell(kr)}{d(kr)}\frac{\partial(kr)}{\partial r} = R_\ell'(kr)\,k \\ R_\ell''(r) = R_\ell''(kr)\,k^2 \end{cases} \Rightarrow R_\ell^{(n)}(r) = R_\ell^{(n)}(kr)\,k^2 = R_\ell^{(n)}(kr)\,k^n$$

$$\Rightarrow 2rk\, R_\ell'(kr) + r^2k^2 R_\ell''(kr) + \left(\frac{2ME}{\hbar^2}r^2 - \ell(\ell+1)\right)R_\ell(kr) = 0 \quad \Big|\ \frac{2ME}{\hbar^2} = k^2$$

$$\Rightarrow 2z R_\ell'(z) + z^2 R_\ell''(z) + \left(z^2 - \ell(\ell+1)\right)R_\ell(z) = 0$$

Besselfunktion $\Rightarrow R_\ell(z) = (-z)^\ell\left(\frac{1}{z}\frac{d}{dz}\right)^\ell\frac{\sin z}{z}$

$$= j_\ell(z)$$

$$\Rightarrow R_\ell^{(n)}(r) = R_\ell^{(n)}(kr)\,k^n$$
$$= R_\ell^{(n)}(z)\,k^n$$
$$= j_\ell(z)\,k^n$$
$$= N\,j_\ell(z) \quad \text{mit } N = k^n$$

$\square$