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1) Canonical transformation in Classical Trajectories

$$q_i \to \bar{q}_i = q_i + \delta q_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \quad ; \quad p_i \to \bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \qquad \text{and} \quad g = g(q_i, p_i)$$

1.1) Show: $(q_i(t), p_i(t)) \wedge \{g, H\} = 0 \Rightarrow (\bar{q}_i(t), \bar{p}_i(t))$

· $\dot{\bar{q}}_i = \frac{\partial H}{\partial \bar{p}_i} = \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \bar{p}_i} + \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \bar{p}_i}$ | $p_i = \bar{p}_i + \varepsilon \frac{\partial g}{\partial q_i} \Rightarrow \frac{\partial p_i}{\partial \bar{p}_i} = 1 + \varepsilon \frac{\partial}{\partial \bar{p}_i}\frac{\partial g}{\partial q_i}$, $q_i = \bar{q}_i - \varepsilon \frac{\partial g}{\partial p_i} \Rightarrow \frac{\partial q_i}{\partial \bar{p}_i} = -\varepsilon \frac{\partial}{\partial \bar{p}_i}\frac{\partial g}{\partial p_i}$

$\Rightarrow \dot{\bar{q}}_i = \frac{\partial H}{\partial p_i}\left(1 + \varepsilon \frac{\partial}{\partial \bar{p}_i}\frac{\partial g}{\partial q_i}\right) + \frac{\partial H}{\partial q_i}\left(-\varepsilon \frac{\partial}{\partial \bar{p}_i}\frac{\partial g}{\partial p_i}\right)$

$= \frac{\partial H}{\partial p_i} + \varepsilon \frac{\partial}{\partial \bar{p}_i}\left(\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$ | $\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i} = \{g, H\} = 0$

$= \frac{\partial H}{\partial p_i} = \dot{q}_i$

· $\dot{\bar{p}}_i = -\frac{\partial H}{\partial \bar{q}_i} = -\frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \bar{q}_i} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \bar{q}_i}$ | $\frac{\partial p_i}{\partial \bar{q}_i} = \varepsilon \frac{\partial}{\partial \bar{q}_i}\frac{\partial g}{\partial q_i}$, $\frac{\partial q_i}{\partial \bar{q}_i} = 1 - \varepsilon \frac{\partial}{\partial \bar{q}_i}\frac{\partial g}{\partial p_i}$

$\Rightarrow \dot{\bar{p}}_i = -\frac{\partial H}{\partial q_i}\left(1 - \varepsilon \frac{\partial}{\partial \bar{q}_i}\frac{\partial g}{\partial p_i}\right) - \frac{\partial H}{\partial p_i}\left(\varepsilon \frac{\partial}{\partial \bar{q}_i}\frac{\partial g}{\partial q_i}\right)$

$= -\frac{\partial H}{\partial q_i} + \varepsilon \frac{\partial}{\partial \bar{q}_i}\left(\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$ | $\frac{\partial H}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial g}{\partial q_i} = \{H, g\} = -\{g, H\} = 0$

$= -\frac{\partial H}{\partial q_i} = \dot{p}_i$

↳ if $(q_i(t), p_i(t))$ satisfies the e.o.m, then also $(\bar{q}(t), \bar{p}(t))$ satisfies the e.o.m. □
(⇔ valid trajectory)

1.2) $x_k \to \bar{x}_k = x_k + \delta$, $\delta \in \mathbb{R}$

$\Rightarrow g = p_k$, because $\bar{x}_k = x_k + \delta \frac{\partial p_k}{\partial p_k} = x_k + \delta$

· Hamilton function is invariant $\Leftrightarrow H(\bar{x}_i, p_i) = H(x_i, p_i)$ (*)

$\Rightarrow \dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}_i} \overset{*}{=} \frac{\partial H}{\partial p_i} = \dot{q}_i$ $\Rightarrow$ transformation generates valid trajectory $(q_i(t), p_i(t))$ □

$\Rightarrow \dot{\bar{p}} = -\frac{\partial H}{\partial \bar{q}_i} \overset{*}{=} -\frac{\partial H}{\partial q_i} = \dot{p}_i$

2) Canonical Transformation in Quantum Mechanics

$$\Psi(q_i, t) \to \bar{\Psi}(q_i, t) = \hat{U}_g(\xi)\Psi(q_i, t) \quad , \quad \hat{U}_g(\xi) = \exp\left(-\frac{i\xi}{\hbar} g\right)$$

2.1) $\bar{\Psi}(q_i, t) = \hat{U}_g(\xi)\Psi(q_i, t)$

$\Rightarrow \sum_n \bar{c}_n(t)\Psi_n(q_i) = \hat{U}_g(\xi)\sum_n c_n(t)\Psi_n(q_i)$

$\Rightarrow \sum_n \bar{c}_n(t)\Psi_n(q_i) = \sum_n c_n(t)\cdot\exp\left(-\frac{i\xi}{\hbar}\hat{g}\right)\Psi_n(q_i) = \sum_n c_n \cdot \exp\left(-\frac{i\xi}{\hbar}g_n\right)\Psi_n(q_i)$

$\Rightarrow \bar{c}_n(t) = \exp\left(-\frac{i\xi}{\hbar}g_n\right)c_n(t)$

2.2) $\hat{g} = \hat{L}_z \to \hat{U}_{L_z} = \exp\left(-\frac{i\xi}{\hbar}\hat{L}_z\right)$

$\Rightarrow \bar{\Psi}(q_i, t) = \hat{U}_{L_z}\Psi(q_i, t) = \exp\left(-\frac{i\xi}{\hbar}\hat{L}_z\right)\sum_m c_m(t)\Psi_m(q_i)$ | $\hat{L}_z\Psi_m(q_i) = \hbar m\Psi_m(q_i)$

$= \sum_m c_m(t)\exp(-i\xi m)\Psi_m(q_i)$ | eigenfunctions $\Psi_m = Y_l^m(\theta, \phi) \propto \exp(im\phi)$
↳ sperical harmonics

$= \sum_m c_m(t)\, Y_l^m(\theta, \phi - \xi)$

↳ L_z geresats rotation around z-axis $[\phi \to \phi - \xi]$

2.3) $\hat{g} = \hat{L}^2 \quad \rightarrow \quad \hat{U}_{L^2} = \exp\left(-\frac{i\xi}{\hbar}\hat{L}^2\right)$

$\Rightarrow \quad \bar{\Psi}(q_i,t) = \hat{U}_{L^2}\Psi(q_i,t) = \exp\left(-\frac{i\xi}{\hbar}\hat{L}^2\right)\Psi(q_i,t)$

i) $\Psi(q_i,t)$ is eigenfunction of $\hat{L}^2 \wedge \hat{L}_z$, with fixed l and m

$\hookrightarrow \quad \Psi(q_i,t) = \Psi_{lm}$

$\Rightarrow \quad \bar{\Psi}(q_i,t) = \exp(-i\hbar\xi\, l(l+1))\,\Psi_{lm} = \exp(-i\hbar\xi\, l(l+1))\,\Psi(q_i,t)$ → absolut phase has no significance → <u>not</u> a physical change !

ii) fixed l, different m

$\hookrightarrow \quad \Psi(q_i,t) = \sum_m \Psi_{lm}$

$\Rightarrow \quad \bar{\Psi}(q_i,t) = \exp\left(-\frac{i\xi}{\hbar}\hat{L}^2\right)\sum_m \Psi_{lm} = \sum_m \exp(-i\hbar\xi\, l(l+1))\,\Psi_{lm} = \exp(-i\hbar\xi\, l(l+1))\sum_m \Psi_{lm} = \exp(-i\hbar\xi\, l(l+1))\,\Psi(q_i,t)$ → <u>not</u> a physical change !

iii) different l and m

$\hookrightarrow \quad \Psi(q_i,t) = \sum_{l,m}\Psi_{lm}$

$\Rightarrow \quad \bar{\Psi}(q_i,t) = \exp\left(-\frac{i\hbar}{\hbar}\hat{L}^2\right)\sum_{lm}\Psi_{lm} = \sum_{l,m}\exp(-i\hbar\xi\, l(l+1))\,\Psi_{lm}$ → different relativ phases → real physical change !

3) Gauge Invariance in classical Electrodynamics

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t) \quad ; \quad \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t}$$

1) $\vec{A}'(\vec{x},t) = \vec{A}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x},t) \quad , \quad U'(\vec{x},t) = U(\vec{x},t) - \frac{\partial\lambda(\vec{x},t)}{\partial t}$

Show: $\vec{B}'(\vec{x},t) = \vec{B}(x,t) \wedge \vec{E}'(\vec{x},t) = \vec{E}(x,t)$ (⇔ gauge transf. leaves $\vec{B}$ & $\vec{E}$ unchanged)

- $\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(A+\vec{\nabla}\lambda) = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times\vec{\nabla}\lambda} \quad \Big| \ \vec{\nabla}\times\vec{\nabla}\lambda = 0$

$\Rightarrow \vec{B}' = \vec{\nabla}\times\vec{A} = \vec{B}$

- $\vec{E}' = -\vec{\nabla}U' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\left(U - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial(A+\vec{\nabla}\lambda)}{\partial t} = -\vec{\nabla}U + \vec{\nabla}\frac{\partial}{\partial t}\lambda - \frac{\partial}{\partial t}\vec{\nabla}\lambda - \frac{\partial A}{\partial t} \quad \Big| \ \vec{\nabla}\frac{\partial}{\partial t}\lambda = \frac{\partial}{\partial t}\vec{\nabla}\lambda$

$\Rightarrow \vec{E}' = -\vec{\nabla}U - \frac{\partial A}{\partial t} = \vec{E} \quad \square$

2) Show: $\vec{\nabla}\cdot\vec{B}(\vec{x},t) = 0 \ \wedge \ \vec{\nabla}\times\vec{E}(\vec{x},t) = -\frac{\partial B(\vec{x},t)}{\partial t}$

- $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \partial_i\varepsilon_{ijk}\partial_j a_k = \varepsilon_{ijk}\partial_i\partial_j a_k = \varepsilon_{ijk}\partial_j\partial_i a_k = -\partial_j\varepsilon_{jik}\partial_i a_k = -\partial_i\varepsilon_{ijk}\partial_j a_k = 0$

- $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\cancel{\vec{\nabla}}U - \frac{\partial A}{\partial t}\right) = -\vec{\nabla}\times\frac{\partial A}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial B}{\partial t} \quad \square$

3) $\vec{\nabla}\cdot\vec{A}(\vec{x},t) = -\mu_0\varepsilon_0\frac{\partial U(\vec{x},t)}{\partial t}$

- $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \vec{\nabla}\cdot\left(-\vec{\nabla}U - \frac{\partial\vec{A}}{\partial t}\right) = -\Delta U - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}$

- $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}U - \frac{\partial\vec{A}}{\partial t}\right)$

$$= \mu_0\vec{j} + \vec{\nabla}\left(-\mu_0\varepsilon_0\frac{\partial U}{\partial t}\right) - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}$$

$$= \mu_0\vec{j} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}$$

4) $L = \int d^3x \left[\frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} - \frac{1}{2\mu_0}\vec{B}\cdot\vec{B} - \rho U + \vec{j}\cdot\vec{A}\right]$

- $\left[\frac{\varepsilon_0}{2}\vec{E}'\cdot\vec{E}' - \frac{1}{2\mu_0}\vec{B}'\cdot\vec{B}' - \rho U' + \vec{j}\cdot\vec{A}'\right] = \frac{\varepsilon_0}{2}\left(-\vec{\nabla}U' - \frac{\partial A'}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\vec{\nabla}\times A')^2 - \rho U' + \vec{j}\cdot\vec{A}'$

$$= \frac{\varepsilon_0}{2}\left(-\vec{\nabla}\left(U - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial(A+\vec{\nabla}\lambda)}{\partial t}\right)^2 - \frac{1}{2\mu_0}\left(\vec{\nabla}\times(\vec{A}+\vec{\nabla}\lambda)\right)^2 - \rho\left(U - \frac{\partial\lambda}{\partial t}\right) + \vec{j}\cdot(\vec{A}+\vec{\nabla}\lambda)$$

$$= \frac{\varepsilon_0}{2}\left(-\vec{\nabla}U - \frac{\partial A}{\partial t}\right)^2 - \frac{1}{2\mu_0}(\vec{\nabla}\times\vec{A})^2 - \rho\left(U - \frac{\partial\lambda}{\partial t}\right) + \vec{j}\cdot(\vec{A}+\vec{\nabla}\lambda)$$

$$= \left[\frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B} - \rho U + \vec{j}A\right] + \underbrace{\rho\frac{\partial\lambda}{\partial t} + \vec{j}(\vec{\nabla}\lambda)}_{*}$$ → Integrand not invariant !

*) $\rho\frac{\partial\lambda}{\partial t} + j(\vec{\nabla}\cdot\lambda) = \rho\frac{\partial\lambda}{\partial t} + \vec{\nabla}\cdot(\lambda\vec{j}) - \lambda(\vec{\nabla}\cdot\vec{j}) \quad \Big| \ \vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

$$= \rho\frac{\partial\lambda}{\partial t} + \lambda\frac{\partial\rho}{\partial t} + \vec{\nabla}(\lambda\vec{j}) = \frac{d}{dt}(\lambda\rho) + \vec{\nabla}(\lambda\vec{j})$$

$\Rightarrow \quad \int dx^3\left(\frac{d}{dt}(\lambda\rho) + \vec{\nabla}(\lambda\vec{j})\right) = 0 \quad \Rightarrow \quad L$ is invariant !