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Aufgabe 1:

$$1 = \|\psi\|^2$$
$$= \int_{\mathbb{R}} A^2 e^{-ax^2} dx$$
$$= A^2 \sqrt{\frac{\pi}{a}}$$
$$\Rightarrow \boxed{A = \left(\frac{a}{\pi}\right)^{\frac{1}{4}}}$$

I)

$$ax^2 - bx + c^2 \overset{!}{=} (\sqrt{a}x - c)^2$$
$$= ax^2 - 2\sqrt{a}xc + c^2$$
$$\Rightarrow bx = 2\sqrt{a}xc$$
$$\Rightarrow c = \frac{b}{2\sqrt{a}}$$
$$\Rightarrow ax^2 - bx = \left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a}$$

Aufgabe 2:

$$\tilde{\psi} = \langle p | \psi \rangle$$
$$= \int_{\mathbb{R}} dx \, \langle p | x \rangle \langle x | \psi \rangle$$
$$= \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \psi(x,t)$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dx \, e^{-\frac{ipx}{\hbar}} \left(\frac{a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2} \quad | \mathrm{II}$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{2\pi}{a}} e^{\frac{-p^2}{2a\hbar^2}}$$
$$= \frac{1}{\sqrt{a\hbar}} \left(\frac{a}{\pi}\right)^{1/4} e^{\frac{-p^2}{2a\hbar^2}}$$
$$= \boxed{\frac{1}{\sqrt[4]{\hbar^2 a\pi}} e^{\frac{-p^2}{2a\hbar^2}}}$$

II)

$$\int_{\mathbb{R}} e^{-ax^2} \cdot e^{bx} dx \overset{\mathrm{I}}{=} \int_{\mathbb{R}} e^{-\left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2 + \frac{b^2}{4a}} dx \quad \Big| \sqrt{a}x - \frac{b}{2\sqrt{a}} = u \Rightarrow \frac{du}{dx} = \sqrt{a} \Rightarrow dx = \frac{du}{\sqrt{a}}$$
$$= \frac{e^{\frac{b^2}{4a}}}{\sqrt{a}} \sqrt{\pi}$$
$$= \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

Aufgabe 3:

$$\tilde{\psi}(p,t) = \hat{U}(t,t_0) \tilde{\psi}(p,t_0)$$
$$= \exp\left[-\frac{i}{\hbar} \hat{H}(t-t_0)\right] \tilde{\psi}(p,t_0)$$
$$= \exp\left[-\frac{i}{\hbar} \frac{\hat{p}^2}{2m}(t-t_0)\right] \tilde{\psi}(p,t_0)$$
$$= \exp\left[-\frac{i}{\hbar} \frac{p^2}{2m}(t-t_0)\right] \frac{e^{\frac{-p^2}{2a\hbar^2}}}{\sqrt[4]{a\pi\hbar^2}}$$
$$= \frac{1}{\sqrt[4]{a\pi\hbar^2}} \exp\left[-\frac{p^2}{2\hbar^2}\left(\frac{1}{a} + \frac{2i\hbar(t-t_0)}{m}\right)\right] \quad | t_0 = 0$$
$$= \boxed{\frac{1}{\sqrt[4]{a\pi\hbar^2}} \exp\left[-\frac{p^2}{2\hbar^2}\left(\frac{1}{a} + \frac{2i\hbar t}{m}\right)\right]}$$

Aufgabe 4:

$$\psi(x,t) = \langle x | \tilde{\psi}, p, t \rangle$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} \frac{1}{\sqrt[4]{a\pi\hbar^2}} \exp\left[-\frac{p^2}{\hbar}\left(\frac{1}{2a} + \frac{2i\hbar t}{\hbar m}\right) + \frac{ipx}{\hbar}\right] dp \quad | \mathrm{II}$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt[4]{a\pi\hbar^2}} \sqrt{\frac{\pi\hbar^2}{\left(\frac{i\hbar t}{2m} + \frac{1}{2a}\right)}} e^{-\frac{x^2 \hbar^2}{2\hbar^2\left(\frac{i\hbar t}{m} + \frac{1}{a}\right)}}$$
$$= \frac{1}{\sqrt[4]{a\pi\left(\frac{i\hbar t}{m} + \frac{1}{a}\right)^2}} \exp\left(-\frac{x^2}{2\left(\frac{i\hbar t}{2m} + \frac{1}{a}\right)}\right)$$
$$\boxed{= \alpha e^{-\beta x^2}} \text{ mit } \boxed{\alpha = \frac{1}{\sqrt[4]{a\pi\left(\frac{i\hbar t}{m} + \frac{1}{a}\right)^2}}}$$
$$\text{und } \boxed{\beta = \frac{1}{2\left(\frac{i\hbar t}{m} + \frac{1}{a}\right)}}$$

Aufgabe 5:

$$\langle \hat{x} \rangle_t = \langle \psi, x, t | \hat{x} | \psi, x, t \rangle$$
$$= \int_{\mathbb{R}} \psi^*(x,t) \, x \, \psi(x,t) dx$$
$$= \alpha^* \alpha \int_{\mathbb{R}} \underbrace{\underbrace{e^{-x^2(\beta+\beta^*)}}_{\text{gerade}} \underbrace{x}_{\text{ungerade}}}_{\Rightarrow \text{ ungerade}} dx \quad \Big| \int_{a}^{-a} f(x) dx = 0 \text{ mit } f(x) \hat{=} \text{ ungerade Funktion}$$
$$\boxed{= 0}$$

$$(\Delta \hat{x})_t^2 = \langle \psi, x, t | \hat{x}^2 | \psi, x, t \rangle - \langle \hat{x} \rangle_t^2$$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = \frac{dx}{2x}$$

$$= \alpha^* \alpha \int_{\mathbb{R}} x^2 e^{-(\beta+\beta^*)x^2} dx \quad \Big| \text{part. Int. mit } f' = x e^{-\gamma x^2} = \frac{1}{2} e^{-\gamma u}$$

$$\Rightarrow f = -\frac{1}{2\gamma} e^{-\gamma u} = -\frac{1}{2\gamma} e^{-\gamma x^2}$$

$$= \alpha^* \alpha \left(\underbrace{\left[-x \cdot \frac{1}{2\gamma} e^{-\gamma x^2}\right]_{\mathbb{R}}}_{\to 0} + \int_{\mathbb{R}} \frac{1}{2\gamma} e^{-\gamma x^2}\right)$$
$$= \alpha^* \alpha \frac{1}{2\gamma} \sqrt{\frac{\pi}{\gamma}}$$
$$= \alpha^* \alpha \frac{\sqrt{\pi}}{2(\beta+\beta^*)^{\frac{3}{2}}}$$
$$= \frac{1}{\sqrt{a\pi} \sqrt{\left(\frac{1}{a} - \frac{i\hbar t}{m}\right)\left(\frac{1}{a} + \frac{i\hbar t}{m}\right)}} \cdot \frac{\sqrt{\pi}}{2\left(\frac{1}{2\left(\frac{1}{a} + \frac{i\hbar t}{m}\right)} + \frac{1}{2\left(\frac{1}{a} - \frac{i\hbar t}{m}\right)}\right)^{3/2}}$$
$$= \frac{1}{\sqrt{a} \sqrt{\frac{1}{a^2} + \frac{\hbar^2 t^2}{m^2}}} \frac{1}{2\left(\frac{\frac{2}{a}}{2\left(\frac{1}{a^2} + \frac{\hbar^2 t^2}{m^2}\right)}\right)^{3/2}}$$
$$= \frac{a^{3/2}}{2 a^{1/2}} \left(\frac{1}{a^2} + \frac{\hbar^2 t^2}{m^2}\right)$$
$$\boxed{= \frac{1}{2}\left(\frac{1}{a} + \frac{a\hbar^2 t^2}{m^2}\right)}$$

Aufgabe 6:

$$\tilde{\Psi}(p,t) = \frac{1}{\sqrt[4]{a\pi\hbar^2}} \exp\left[-\frac{p^2}{2\hbar^2}\left(\frac{1}{a} + \frac{2i\hbar t}{m}\right)\right]$$

$$\langle \hat{p} \rangle_t = \langle \tilde{\Psi}, p, t | \hat{p} | \tilde{\Psi}, p, t \rangle$$

$$= \int_{\mathbb{R}} \tilde{\Psi}^* p \tilde{\Psi} \, dp$$

$$= \frac{1}{\sqrt{a\pi\hbar^2}} \int_{\mathbb{R}} \exp\left[-\frac{p^2}{\hbar^2 a}\right] p \, dp \quad \Big| \text{ sym. Intervall} \wedge \text{ungerade Fnk.}$$

$$= 0$$

$$\langle \Delta\hat{p} \rangle_t^2 = \langle \tilde{\Psi}, p, t | \hat{p}^2 | \tilde{\Psi}, p, t \rangle - \langle \hat{p} \rangle_t^2$$

$$= \frac{1}{\sqrt{a\pi\hbar^2}} \int_{\mathbb{R}} \exp\left(-\frac{p^2}{\hbar^2 a}\right) p^2 \, dp - 0$$

$$u = p^2 \Rightarrow \frac{du}{dp} = 2p \Rightarrow dp = \frac{du}{2p} \qquad p^2 = u$$

$$= \frac{1}{\sqrt{a\pi\hbar^2}} \left(\left[p \int p \exp\left(-\frac{p^2}{\hbar^2 a}\right) dp \right]_{\mathbb{R}} - \int_{\mathbb{R}} p \int p \exp\left(-\frac{p^2}{\hbar^2 a}\right) dp \, dp \right) \Bigg| \int p \exp\left(\frac{-p^2}{\hbar^2 a}\right) dp = \frac{1}{2}\int \exp\left(\frac{-u}{\hbar^2 a}\right) du = -\frac{\hbar^2 a}{2} \exp\left(\frac{-u}{\hbar^2 a}\right) = -\frac{\hbar^2 a}{2} \exp\left(-\frac{p^2}{\hbar^2 a}\right)$$

$$= \frac{1}{\sqrt{a\pi\hbar^2}} \left(\underbrace{\left[-p \frac{\hbar^2 a}{2} \exp\left(-\frac{p^2}{\hbar^2 a}\right) \right]_{\mathbb{R}}}_{\to 0} + \underbrace{\int_{\mathbb{R}} \frac{\hbar^2 a}{2} \exp\left(-\frac{p^2}{\hbar^2 a}\right) dp}_{\to \frac{\hbar^2 a}{2}\sqrt{\hbar^2 a \pi}} \right)$$

$$= \frac{\hbar^2 a}{2}$$

Aufgabe 7:

$$(\Delta\hat{x})(\Delta\hat{p}) \geq \frac{\hbar}{2}$$

$$\sqrt{\frac{1}{2}\left(\frac{1}{a} + \frac{a\hbar^2 t^2}{m^2}\right)} \sqrt{\frac{\hbar^2 a}{2}} \quad \Big| \geq \frac{\hbar}{2}$$

$$\sqrt{\frac{\hbar^2}{4} + \frac{a^2\hbar^4 t^2}{m^2}} \geq \frac{\hbar}{2}$$

$$\frac{\hbar}{2}\sqrt{1 + \underbrace{\frac{a^2\hbar^4 t^2}{m^2}}_{\geq 0}} \geq \frac{\hbar}{2}$$

$\square$

Aufgabe 8:

$$P(x>0)_{t>0} = \|\Psi(x>0,t)\|^2$$

$$= \langle \Psi, x, t | \Theta(x) | \Psi, x, t \rangle$$

$$= \int_{\mathbb{R}} \Psi^*(x,t) \Theta(x) \Psi(x,t) \, dx$$

$$= \int_0^\infty \Psi^* \Psi \, dx \quad \Big| \ \Psi(x,t) = \alpha e^{-\beta x^2} \text{ ist gerade (symmetrisch um } y\text{-Achse)}$$

$$= \frac{1}{2} \int_{\mathbb{R}} \Psi^* \Psi \, dx$$

$$= \frac{1}{2} \|\Psi\|^2$$

$$= \frac{1}{2}$$