HA13

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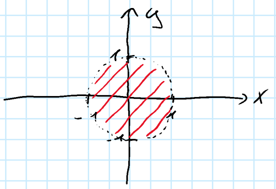
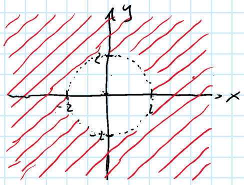
(2)

a) $A = \{(x,y): x^2+y^2<1\}$



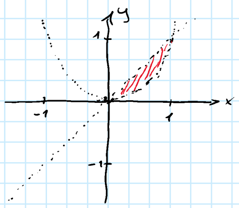
Offen, da der Rand nicht enthalten ist
$\Rightarrow$ nicht kompakt

b) $B = \{(x,y): x^2+y^2>2\}$



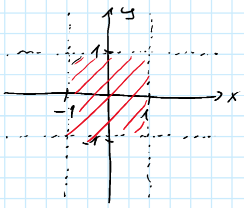
Offen, da der Rand nicht enthalten ist
$\Rightarrow$ nicht kompakt

c) $C = \{(x,y): x^2<y<x\}$



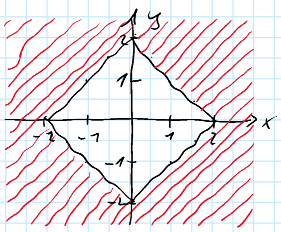
Offen, da der Rand nicht enthalten ist
$\Rightarrow$ nicht kompakt

d) $D = \{(x,y): \max(|x|,|y|)<1\}$



Offen, da der Rand nicht enthalten ist
$\Rightarrow$ nicht kompakt

e) $E = \{(x,y): |x|+|y|\geq 2\}$



Abgeschlossen, da das Komplement offen ist.
Ohne obere Schranke $\Rightarrow$ nicht kompakt

(3)

$f(x) = x^3 \exp(5x) + 7$

$f'(x) = 3x^2 \exp(5x) + x^3 \exp(5x) \cdot 5$

$g(x) = \exp(\exp(x))$

$g'(x) = \exp(\exp(x))$

$h(x) = \sin(\cos(x))$

$h'(x) = \cos(\cos(x)) \cdot \sin(x)$

$k(x) = \log(\log(x^2+2))$

$k'(x) = \log(x^2+2)^{-1} \cdot (x^2+2)^{-1} \cdot 2x$

(4)

$$f(t) := \frac{\exp(\sin(t)^2)}{2+\cos(t)}$$

$$f'(t) = \frac{(\exp(\sin(t)^2) \cdot 2\sin(t) \cdot \cos(t))(2+\cos(t)) - (\exp(\sin(t)^2))(-\sin(t))}{(2+\cos(t))^2}$$

$$f'(t) = \frac{\sin(t)(2\exp(\sin(t)^2) \cdot \cos(t)(2+\cos(t)) + \exp(\sin(t)^2)}{(2+\cos(t))^2}$$

(7)

$\lim_{x \to 0^+} \cos(x) = \cos(0) = 1,$

(7)

$$\lim_{x\to0^+}\frac{\overbrace{\cos(x)}^{\lim_{x\to0^+}\cos(x)=\cos(0)=1}}{x} = \lim_{x\to0^+}\frac{1}{x} = \lim_{x\to\infty} x = \infty$$

$$\lim_{x\to0^+}\frac{\exp(\sin(x))-1}{x} \overset{L'H}{=} \lim_{x\to0^+}\frac{\exp(\sin(x))\cdot\cos(x)}{1} = \frac{1}{1}=1$$

für $\lim_{x\to\infty}\frac{\log(1+\exp(x))}{\sqrt{1+x^2}}$ siehe Abbildung unten.

$$\lim_{x\to0}\frac{\overbrace{\sin(x)}^{\lim_{x\to0}\sin(x)=\lim_{x\to0}x}}{\exp(x)-1} = \overbrace{\lim_{x\to0}\frac{x}{\exp(x)-1}}^{\lim_{x\to0}\exp(x)=1 \Rightarrow \lim_{x\to0}x+1=1} = \lim_{x\to0}\frac{x}{x}=1$$

$$\lim_{x\to0^+}\frac{\log(x)}{x} \overset{L'H}{=} \lim_{x\to0^+}\frac{-1}{x} = -\infty$$

$$\lim_{x\to0^+}\frac{\log(\sin(x))}{\log(\sin(2x))} \overset{L'H}{=} \lim_{x\to0^+}\frac{\sin(2x)\cos(x)}{\sin(x)\cos(2x)} = \overbrace{\lim_{x\to0^+}\frac{\sin(2x)}{\sin(x)}\cdot\lim\frac{\cos(x)}{\cos(2x)}}^{\lim_{x\to0^+}\frac{\sin(2x)}{\sin(x)}=2 \,\land\, \lim_{x\to0^+}\frac{\cos(x)}{\cos(2x)}=\frac{1}{2}} = 1$$

(1)

$$A=\begin{pmatrix}1 & i\\ -i & 0\end{pmatrix}$$

$$\det(A) = 1\cdot0+(-i)\cdot i = 1$$

$$\det(A-\lambda E) = \det\begin{pmatrix}1-\lambda & i\\ -i & -\lambda\end{pmatrix} = (1-\lambda)(-\lambda)-(-i)\,i = -\lambda+\lambda^2-1 \quad |\text{p-q-Formel}$$

$$= \frac{1}{2}\pm\sqrt{\left(\frac{-1}{2}\right)^2+1}$$

$$\lambda_1 = \frac{1}{2}+\sqrt{\frac{5}{4}} \quad\land\quad \lambda_2=\frac{1}{2}-\sqrt{\frac{5}{4}}$$

$\Rightarrow$ zwei unterschiedliche Eigenwerte

$\Rightarrow$ zwei Eigenräume

$\Rightarrow$ <u>diagonalisierbar</u>

$$B=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$\det(B) = \det\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = -1-1=-2 \Rightarrow$ <u>diagonalisierbar</u>

$$C=\begin{pmatrix}2 & i\\ i & 0\end{pmatrix}$$

$$\det(C) = \det\begin{pmatrix}2 & i\\ i & 0\end{pmatrix} = 2\cdot0+ii=-1$$

$$\det(A-\lambda E) = \det\begin{pmatrix}2-\lambda & i\\ i & -\lambda\end{pmatrix} = (2-\lambda)(-\lambda)+i^2 = -2\lambda+\lambda^2-1 \quad |\text{pq-For.}$$

$$\Leftrightarrow\ = \frac{2}{2}\pm\sqrt{\left(\frac{-2}{2}\right)^2+1}$$

$$= 1\pm\sqrt{1+1}$$

$$\lambda_1 = 1+\sqrt{2} \quad\lor\quad \lambda_2 = 1-\sqrt{2}$$

$\Rightarrow$ <u>diagonalisierbar</u>

(5) $G(f) := \frac{d}{dt}(tf) - 2f \quad f \in Pol_n$

Vektoraddition:

$f_1, f_2 \in Pol_n$

$$
\begin{aligned}
G(f_1+f_2) = G(f_1)+G(f_2) \\
&= \left(\frac{d}{dt}(tf_1)-2f_1\right)+\left(\frac{d}{dt}(tf_2)-2f_2\right) \\
&= \frac{d}{dt}(tf_1)-2f_1+\frac{d}{dt}(tf_2)-2f_2 \qquad \frac{d}{dt} \text{ ist eine lin. Abb.} \\
&= \frac{d}{dt}(tf_1+tf_2)-2(f_1+f_2) \\
&= \frac{d}{dt}(t(f_1+f_2)) - 2(f_1+f_2)
\end{aligned}
$$

$\underline{G(f_1+f_2) = G(f_1+f_2)}$

Skalarmultiplikation:

$\lambda \in \mathbb{R},\ f \in Pol_n$

$$
\begin{aligned}
G(\lambda f) = \lambda G(f) \\
&= \lambda\left(\frac{d}{dt}(tf)-2f\right) \\
&= \lambda \frac{d}{dt}(tf) - 2\lambda f \\
&= \frac{d}{dt}(t\lambda f) - 2\lambda f \\
&= \frac{d}{dt}(t(\lambda f)) - 2(\lambda f)
\end{aligned}
$$

$\underline{G(\lambda f) = G(\lambda f)}$

$\boxed{a \frac{d}{dt} f(t) = \frac{d}{dt}(a f(t))}$

$G(f) := \frac{d}{dt}(tf) - 2f$

$v_B = \begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ \vdots \end{pmatrix} \qquad G(v_B) = \begin{pmatrix} -1 \\ 2t-2t \\ 3t^2-2t^2 \\ 4t^3-2t^3 \\ \vdots \end{pmatrix}$

Basis von Kern: Basis von Bild:

$\Rightarrow DM = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2 & & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \end{pmatrix} \Rightarrow v_{BK_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge v_{BB_1} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_{BB_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_{BB_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, v_{BB_n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ n-1 \end{pmatrix}$

(6) $G(f) := f(t) \mapsto (t \cdot f(t))'$

$v_B = \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \end{pmatrix} \qquad \begin{aligned} 1 &\mapsto 1 \\ t &\mapsto 2t \\ t^2 &\mapsto 3t^2 \\ t^3 &\mapsto 4t^3 \\ t^4 &\mapsto 5t^4 \\ t^5 &\mapsto 6t^5 \end{aligned} \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; v_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \dots; v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \end{pmatrix} \Rightarrow DM(G(f)) = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 2 & 0 & & & \\ & 0 & 3 & 0 & & \vdots \\ \vdots & & 0 & 4 & 0 & \\ & & & 0 & 5 & 0 \\ 0 & \cdots & & & 0 & 6 \end{pmatrix}; \det(DM(G(f))) = 6! \neq 0 \Rightarrow$ invertierbar

$\begin{matrix} 1 & 0 & \cdots & 0 \\ 6 & 2 & 0 & \ddots \end{matrix} \;\Big|\; \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots \end{matrix} \;\Big|\; \begin{matrix} :1 \\ :2 \end{matrix}$

$$\left(\begin{array}{cccccc|cccccc} 1 & 0 & \cdots & & & 0 & 1 & 0 & \cdots & & & 0 \\ 6 & 2 & 0 & & \iddots & & 0 & 1 & 0 & & \iddots & \\ & 0 & 3 & 0 & & \vdots & & 0 & 1 & 0 & & \vdots \\ \vdots & & 0 & 4 & 0 & & \vdots & & 0 & 1 & 0 & \\ & \iddots & & 0 & 5 & 0 & & \iddots & & 0 & 1 & 0 \\ 0 & & \cdots & & 0 & 6 & 0 & & \cdots & & 0 & 1 \end{array}\right) \begin{array}{l} :1 \\ :2 \\ :3 \\ :4 \\ :5 \\ :6 \end{array}$$

$$\left(\begin{array}{cccccc|cccccc} 1 & 0 & \cdots & & & 0 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & & \iddots & & 0 & \frac{1}{2} & 0 & & \iddots & \\ & 0 & 1 & 0 & & \vdots & & 0 & \frac{1}{3} & 0 & & \vdots \\ \vdots & & 0 & 1 & 0 & & \vdots & & 0 & \frac{1}{4} & 0 & \\ & \iddots & & 0 & 1 & 0 & & \iddots & & 0 & \frac{1}{5} & 0 \\ 0 & & \cdots & & 0 & 1 & 0 & & \cdots & & 0 & \frac{1}{6} \end{array}\right) = DM^{-1}(G(t)) = \underline{\underline{DM(G(t)^{-1})}}$$

Auf 7.

$$\lim_{x \to \infty} \frac{\log(1+\exp(x))}{\sqrt{1+x^2}}$$

l'hospital

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

$f(x) = \ln(1+\exp(x))$

$f'(x) = \frac{1}{1+e^x} \cdot e^x = 1$

$g(x) = \sqrt{1+x^2}$

$g'(x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$

$\rightarrow$ l'hospital

$$\lim_{x \to \infty} \frac{\log(1+\exp(x))}{\sqrt{1+x^2}} = \lim_{x \to \infty} \frac{1}{\frac{x}{\sqrt{1+x^2}}}$$

$$\lim_{x \to \infty} \frac{1}{\frac{x}{\sqrt{1+x^2}}} = \frac{1}{1} = 1$$

Der Grenzwert ist 1