Generalized Graph k-Coloring Games

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Abstract. We investigate pure Nash equilibria in *generalized graph k-coloring games* where we are given an edge-weighted undirected graph together with a set of k colors. Nodes represent players and edges capture their mutual interests. The strategy set of each player consists of k colors. The utility of a player v in a given state or coloring is given by the sum of the weights of edges $\{v, u\}$ incident to v such that the color chosen by v is different than the one chosen by u, plus the profit gained by using the chosen color. Such games form some of the basic payoff structures in game theory, model lots of real-world scenarios with selfish players and extend or are related to several fundamental class of games.

We first show that generalized graph k-coloring games are potential games. In particular, they are convergent and thus Nash Equilibria always exist. We then evaluate their performance by means of the widely used notions of price of anarchy and price of stability and provide tight bounds for two natural and widely used social welfare, i.e., utilitarian and egalitarian social welfare.

1 Introduction

We consider *generalized graph k-coloring games*. These are played on edge-weighted undirected graphs where nodes correspond to players and edges identify social connections or relations between players. The strategy set of each player is a set of k available colors (we assume that the colors are the same for each player). When players select a color they induce a k-coloring or simply a coloring. Each player has a *profit function* that expresses how much a player likes a color. Given a coloring, the *utility* (or *payoff*) of a player v colored i is the sum of the weights of edges $\{v, u\}$ incident to v, such that the color chosen by v is different than the one chosen by u, plus the profit deriving from choosing color i. This class of games forms some of the basic payoff structures in game theory, and can model lots of real-life scenarios. Consider, for example, a set of companies that have to decide which product to produce in order to maximize their revenue. Each company has its own competitors (for example the ones that are in its same region), and it is reasonable to assume that each company wants to minimize the number of competitors that produce the same product. However, this is not their only concern. Indeed, different products may guarantee different profits to a company according to many economic factors like the expected profit, the sponsorship revenue, and so on and so forth. Another possible scenario is the one with miners deciding which land to drill for resources. To a miner it is surely important to choose the land in which the number of rivals is minimized, but also the land itself is important: maybe there is a land with no miners that is very poor in resources, while another land that has been chosen by many miners may be very rich in resources. Thus, for a player is important to find a compromise between her neighbors' decisions and her own strategy choice. Other interesting applications can be found in [12,17,20].

Since players are assumed to be selfish, a well-known solution concept for this kind of setting is the Nash Equilibrium. Formally, a coloring is a (pure) Nash equilibrium if no player can improve her utility by unilaterally deviating from her actual strategy. We stress that in our setting it is not required that edges are properly colored, that is, in a Nash equilibrium, we can have edges whose two endpoints use the same color.

Nash equilibrium is one of the most important concepts in game theory and it provides a stable solution that is robust to deviations of single players. However, selfishness may cause loss of social welfare, that is, a stable solution is not always good with respect to the well-being of the society. We consider two natural and widely used notions of welfare. Given a coloring, the *utilitarian* social welfare is defined as the sum of the utilities of the players in the coloring, while the *egalitarian* social welfare is defined as the minimum utility among all the players in the coloring. Two used way of measuring the goodness of a Nash equilibrium with respect to a social welfare are the price of anarchy [19] and the price of stability [3,25]. We adopt such measures and study the quality of the worst (resp. best) Nash stable outcome and refer to the ratio of its social welfare to the one of the socially optimum one as to the price of anarchy (resp. stability). Roughly speaking, the price of anarchy says, in the worst case, how the efficiency of a system degrades due to selfish behavior of its players, while the price of stability has a natural meaning of stability, since it is the optimal solution among the ones which can be accepted by selfish players.

Our aim is to study the existence and the performance of Nash equilibria in generalized graph k-coloring games. We focus only on undirected graphs since for directed graphs even the problem of deciding whether an instance admits a Nash equilibria is an hard problem, and there exist instances for which a Nash equilibrium does not exist at all [20] (see Section 6 for further discussion about directed graphs).

Our results. We first show that generalized graph k-coloring games are potential games. In particular, they are convergent games and thus Nash Equilibria always exist. Moreover, if the graph is unweighted then it is possible to compute a Nash equilibrium in polynomial time. This is different from weighted undirected graph, for which the problem of computing a Nash equilibrium is PLS-complete even for $k = 2$ [24], since the max cut game is a special case of our game. We then evaluate the goodness of Nash equilibria by means of the widely used notions of price of anarchy and price of stability and show tight bounds for two natural and widely used social welfare, i.e., utilitarian and egalitarian social welfare. Moreover, we provide tight results for the egalitarian social welfare related to the case in which players have no personal preference on colors, that is, all color profits are set to 0. Our results are illustrated in Table 1 (our original results are marked with an $*$). Due to space constraints some proofs have been moved to the Appendix.

	Utilitarian SW		Egalitarian SW	
	PoA	PoS	PoA	PoS
Graph k-coloring without profits	$\frac{k}{k-1}$	1	$\frac{k}{k-1}$	$\frac{k}{k-1}$ *
Generalized graph k-coloring	2 *	$\frac{3}{2}$ *	2 *	2 *

Table 1: Overview of the results about the graph k-coloring games. New results are marked with a "*", while the other ones are obtained from [20].

Related work. The graph k-coloring games (also called Max k-Cut games and anticoordination games), that is the special case of the generalized graph k-coloring games where the colors profits are set to zero, have been studied in [17,20]. They consider the game applied to both undirected and directed graphs (in the last case, each player is interested only in her outgoing neighbors). They show that the graph k-coloring game is a potential game [21] in case of undirected graphs and therefore a Nash equilibrium always exists. They only consider the utilitarian social welfare and give a tight bound for the price of anarchy, which is $\frac{k}{k-1}$, and show that any optimum is a Nash equilibrium (i.e., the price of stability is 1). Conversely, they show that even deciding whether an unweighted directed graph admits a Nash equilibrium is NP-Hard, for any number of colors $k \geq 2$. As far as concerns graph k-coloring games in edge-weighted undirected graphs, computing a Nash equilibrium is PLS-Complete [24], while for unweighted undirected graphs the problem becomes polynomially solvable.

A more complex payoff function is considered in [18], where the utility of a player is equal to the sum of the distance of her color to the color of each of her neighbors, applying a non-negative, real-valued, concave function.

Apt et al. [4] consider a coordination game in which, given a graph, players are nodes and each player has to select a color so that the number of neighbors with her same color is maximized. Here, each player has her own set of colors. When the graph is undirected, the game converges to a Nash equilibria in polynomial time. Instead, for directed graphs computing a Nash equilibria is NP-Complete [5]. Feldman et al. [14] study the strong price of anarchy [2] of graph k-coloring games, that is, the ratio of the social optimum to the worst strong equilibrium [6], which is a Nash equilibrium resilient to deviations by group of players.

To the best of our knowledge there is no paper that considers the price of anarchy and stability for the graph k-coloring games under the egalitarian social welfare. However, we stress that the egalitarian social welfare has been studied in many other settings, like e.g., congestion games [13], hedonic games [7], and fair division problems [22].

The graph k-coloring games are strictly related to many fundamental games in the scientific literature. For instance, they are strictly related to the unfriendly partition problem [1,9]. Morevoer, they can be seen as a particular hedonic game (see [8] for an introduction to the topic), in which nodes with the same color belong to the same coalition, and the utility of each player is equal to her degree minus the number of neighbors that are in her own coalition. Nash equilibria in hedonic games have been largely investigated [11,15,16] (just to cite a few).

Finally, there are other related games on graphs that involve coloring. In [23] the authors study a game in which players are nodes of a graph. Each player has to choose a color among k available ones, and her utility is defined as follows: if no neighbor has chosen her same color then her utility is equal to the number of players (not in her neighborhood) that have chosen her same color, otherwise it is zero. They prove that this is a potential game and a Nash equilibria can be found in polynomial time. Moreover, they show that any pure equilibrium is a proper coloring.

2 Preliminaries

We are given an undirected simple graph $G = (V, E, w)$, where $|V| = n$, $|E| = m$, and $w : E \to \mathbb{R}_{\geq 0}$ is the edge-weight function that associates a positive weight to each edge. When weights are omitted they are assumed to be 1. We denote by $\delta^v(G) = \sum_{u \in V:\{v,u\} \in E} w(\{v,u\})$ the sum of the weights of all the edges incident to v. The set of nodes with which a node v has an edge in common is called v's neighborhood. We will omit to specify (G) when clear from the context. An instance of the *generalized graph k-coloring game* is a tuple (G, K, P). $G = (V, E, w)$ is an undirected weighted graph without self loops, in which each node $v \in V$ is a selfish player (in the following we will use node and player interchangeably). K is a set of k available colors (we assume that $k \geq 2$). The strategy set of each player is given by the k available colors, that is, the players have the same set of actions. We denote with $P : V \times K \to \mathbb{R}_{\geq 0}$, the *color profit* function, that defines how much a player likes a color, that is, if player v chooses to use color i, then she gains $P_v(i)$. For each player v, we define P_v^M as the greatest profit that v can gain from a color, namely, $P_v^M = max_{i=1,\dots,k} P_v(i)$. When $P_v(i) = 0$ $\forall v \in V$ and $\forall i \in k$, that is the case without profits for the chooseen color, then the game is equivalent to the one analysed in [17,20], and we refer to it as *graph k-coloring game*. A state of the game $c = \{c_1, \dots, c_n\}$ is a *k-coloring*, or simply a *coloring*, where c_v is the color (i.e., a number from 1 to k) chosen by player v. In a certain coloring c, the payoff (or the utility) of a player v is the sum of the weights of edges $\{v, u\}$ incident to v, such that the color chosen by v is different than the one chosen by u, plus the profit gained by using the chosen color. Formally, for a coloring c, a player v's payoff $\mu_c(v) = \sum_{u \in V:\{v,u\} \in E \wedge c_v \neq c_u} w(\{v,u\}) + P_v(c_v)$. From now on, when an edge $\{v, u\}$ provides utility to its endpoints in a coloring c, that is, when $c_v \neq c_u$ we say that such edge is *proper*. We also say that an edge $\{v, u\}$ is *monochromatic* in a coloring c when $c_v = c_u$.

Let (c_{-v}, c'_v) denote the coloring obtained from c by changing the strategy of player v from c_v to c'_v. Given a coloring $c = \{c_1, \ldots, c_n\}$, an *improving move* of player v in the coloring c is a strategy c'_v such that $\mu_{(c_{-v}, c'_v)}(v) > \mu_c(v)$. A state of the game is a pure Nash or stable equilibrium if and only if no player can perform an improving move. Formally, $c = \{c_1, \ldots, c_n\}$ is a NE if $\mu_c(v) \geq \mu_{(c_{-v}, c'_v)}(v)$ for any possible color c'_v and for any player $v \in V$. An *improving dynamics* (shortly *dynamics*) is a sequence of improving moves. A game is said to be *convergent* if, given any initial state c, any sequence of improving moves leads to a Nash equilibrium.

Given a coloring c, we define the *utilitarian social welfare function* (denoted with $SW_{UT}(c)$) and the *egalitarian social welfare* (denoted with $SW_{EG}(c)$) as follows:

$$SW_{UT}(c) = \sum_{v \in V} \mu_c(v) = \sum_{v \in V} P_v(c_v) + \sum_{\{v,u\} \in E : c_v \neq c_u} 2w(\{v, u\}) \tag{1}$$

$$SW_{EG}(c) = \min_{v \in V} \mu_c(v) \tag{2}$$

Let us denote C the set of all the possible colorings, and let Q be the set of all the stable colorings. Given a social welfare function SW, we define the Price of Anarchy (PoA) of the generalized graph k-coloring game as the ratio of the maximum social welfare among all the possible colorings over the minimum social welfare among all the possible stable colorings. Formally, $PoA = \frac{max_{c \in C} SW(c)}{min_{c' \in Q} SW(c')}$. We further define the Price of Stability (PoS) of the generalized graph k-coloring game as the ratio of the maximum social welfare among all the possible colorings over the maximum social welfare among all the possible stable colorings. Formally, $PoS = \frac{max_{c \in C} SW(c)}{max_{c' \in Q} SW(c')}$. Intuitively, the PoA (resp. PoS) says us how much worse is the social welfare at a worst (resp. best) Nash equilibrium, relative to the social welfare of a centralized enforced optimum. We refer to the utilitarian price of anarchy and the egalitarian price of anarchy when we are dealing with utilitarian social welfare and egalitarian social welfare, respectively, and likewise the same is for the utilitarian price of stability and the egalitarian price of stability.

3 Existence of Nash equilibria

We first show that the generalized graph k-coloring game is convergent. It clearly implies that Nash equilibria always exist.

Proposition 1. *For all k, any finite generalized graph k-coloring game (G, K, P) is convergent.*

We notice that, on the one hand, if the graph is unweighted the dynamics starting from the coloring in which each player v selects the color giving her the maximum possible profit, that is, the color i such that $P_v(i) = P_v^M$, converges to a Nash equilibrium in at most $|E|$ improving moves. On the other hand, if the graph is weighted, computing a Nash Equilibrium is PLS-complete. It follows from the fact that, when $k = 2$, our game is a generalization of the Cut Games that is one of the first problem proved to be PLS-complete [24].

4 Utilitarian Social Welfare

In this section we focus on the utilitarian social welfare. We show tight bounds both for utilitarian price of anarchy and stability.

4.1 Price of Anarchy

We recall that in the case with no color profits, the utilitarian price of anarchy is exactly $\frac{k}{k-1}$ [20]. Here we prove that for generalized graph k-coloring games the utilitarian price of anarchy is equal to 2, that is, it is independent from the number of colors. We start by showing that the utilitarian price of anarchy is at most 2.

Theorem 2. *The utilitarian price of anarchy of the generalized graph k-coloring games is at most* 2.

Proof. It is easy to see that, for any possible coloring, the utility of any player $v \in V$ cannot exceed $P_v^M + \delta^v$. Let c^* be the coloring that maximizes the social welfare, and c a stable coloring. We notice that, a player v, in any equilibrium, has utility at least $max\{\frac{k-1}{k}\delta^v, P_v^M\}$. In fact, on the one hand, by the Pigeonhole principle, there always exists a color i such that, $\sum_{u:c_u=i} w(\{v,u\}) \leq \frac{1}{k}\delta^v$. On the other hand, player v can always select the color that maximizes her profit function. We now consider the two following cases:
Case 1: $P_v^M \geq \delta^v$
In this case it holds that:

$$\frac{\mu_{c^*}(v)}{\mu_c(v)} \leq \frac{P_v^M + \delta^v}{P_v^M} \leq \frac{2P_v^M}{P_v^M} = 2 \tag{3}$$

Case 2: $P_v^M < \delta^v$
Let $P_v^M = \delta^v - x$, where $0 < x \leq \delta^v$. Let $Y_v(c)$ be the set of v's neighbors whose color in c is different from the one, say i, that maximizes the color profit of v, namely $u \in Y_v(c)$ if u is a neighbor of v and $c_u \neq i$, where $i \in K$ and $P_v(i) = P_v^M$. Let $y = \sum_{u \in Y_v(c)} w(\{v,u\})$. Thus, in any Nash equilibrium the v's payoff is at least $max\{P_v^M + y, \delta^v - y\}$. Therefore, both the following inequalities hold:

$$\frac{\mu_{c^*}(v)}{\mu_c(v)} \leq \frac{P_v^M + \delta^v}{\delta^v - x + y} = \frac{2\delta^v - x}{\delta^v - x + y} \tag{4}$$

$$\frac{\mu_{c^*}(v)}{\mu_c(v)} \leq \frac{P_v^M + \delta^v}{\delta^v - y} = \frac{2\delta^v - x}{\delta^v - y} \tag{5}$$

Inequality (4) is true because, if v chooses color i, then she earns at least P_v^M (that is equal to $\delta^v - x$) plus all the edges that are proper (and they are exactly y). Inequality (5) holds because if v chooses any other color than i, she earns at least $\delta^v - y$. Notice that the ratio is upper-bounded by the minimum between these two values, namely:

$$\frac{\mu_{c^*}(v)}{\mu_c(v)} \leq min \left\{ \frac{2\delta^v - x}{\delta^v - x + y}, \frac{2\delta^v - x}{\delta^v - y} \right\}, \tag{6}$$

and thus it is maximized when $\frac{2\delta^v - x}{\delta^v - x + y} = \frac{2\delta^v - x}{\delta^v - y}$, that is, when $x = 2y$.

By applying it to inequality (4), we obtain:

$$\frac{\mu_{c^*}(v)}{\mu_c(v)} \leq \frac{2\delta^v - 2y}{\delta^v - 2y + y} = \frac{2\left(\delta^v - y\right)}{\delta^v - y} = 2$$

Therefore, we have that for any player v, $\frac{\mu_{c^*}(v)}{\mu_c(v)} \leq 2$. By summing over all the players, the theorem follows. $\square$

We now show that the utilitarian price of anarchy is at least 2 even for the special case of unweighted star graphs.

Theorem 3. *The utilitarian price of anarchy of the generalized graph k-coloring games is at least* 2*, even for the special case of unweighted star graphs.*

4.2 Price of Stability

We now turn our attention to the utilitarian price of stability. We recall that in the case with no color profits, the utilitarian price of stability is 1 [20]. Here we start by showing that the utilitarian price of stability for the generalized graph k-coloring games is at least $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$, even for the special case of unweighted star graphs.

Theorem 4. *The utilitarian price of stability of the generalized graph k-coloring games is at least $\frac{3}{2} - \epsilon$, for any $\epsilon > 0$, even for the special case of unweighted star graphs.*

We now show that the utilitarian price of stability for the generalized graph k-coloring games is at most $\frac{3}{2}$.

Theorem 5. *The utilitarian price of stability of the generalized graph k-coloring games is at most $\frac{3}{2}$.*

Proof. In order to prove the upper bound for the utilitarian price of stability we first need some preliminary definitions. Given a coloring c we define the following two variables: let $A(c) = \sum_{v \in V} P_v(c_v)$ be the sum of the color profits gained by the players in the coloring c, and let $B(c) = \sum_{\{v,u\} \in E: c_v \neq c_u} w(\{v,u\})$ be the sum of the weights of the properly colored edges in c. Thus, we can rewrite the equations (1) and 16 for the utilitarian social welfare and the potential function with respect to a given coloring c as $SW_{UT}(c) = A(c) + 2B(c)$ and $\Phi(c) = A(c) + B(c)$, respectively.

Let c^* be the coloring that maximizes the social welfare, and let N be the Nash equilibrium that is reached by a dynamics starting from c^*. By the potential function argument, since every improving move increases the potential value, it must hold that:

$$A(N) + B(N) > A(c^*) + B(c^*) \tag{7}$$

On the other hand, since c^* is the coloring that maximizes the social welfare, it holds that:

$$A(c^*) + 2B(c^*) \geq A(N) + 2B(N) \tag{8}$$

In order to prove that $PoS \leq \frac{3}{2}$ it is sufficient to show that the ratio between $SW_{UT}(c^*)$ and $SW_{UT}(N)$ is less or equal than $\frac{3}{2}$. In fact, if such inequality holds then, if N^* is the best Nash equilibrium, $SW_{UT}(N) \leq SW_{UT}(N^*)$ and, consequently, $\frac{SW_{UT}(c^*)}{SW_{UT}(N^*)} \leq \frac{SW_{UT}(c^*)}{SW_{UT}(N)} \leq \frac{3}{2}$.

We want to prove that:

$$\frac{SW_{UT}(c^*)}{SW_{UT}(N)} \leq \frac{3}{2} \tag{9}$$

and, by using the above defined $A(c)$ and $B(c)$, we get that this is true if and only if:

$$A(c^*) + 2B(c^*) \leq \frac{3}{2}A(N) + 3B(N) \tag{10}$$

For the remainder of the proof we are going to show that:

$$B(c^*) \leq \frac{1}{2}A(N) + 2B(N). \tag{11}$$

Indeed, if inequality (11) holds, then by summing it with inequality (7) we get that (10) holds, and this ends the proof.

Given N, for every player v, let $B_v(N) = \sum_{u \in V : N_v \neq N_u} w(\{v,u\})$ be the sum of the weights of properly colored edges incident to v. Thus, $B(N) = \frac{1}{2}\sum_{v \in V} B_v(N)$. Similarly, let $\bar{B}_v(N) = \sum_{u \in V : N_v = N_u} w(\{v,u\}) = \delta_v - B_v(N)$ be the sum of the weights of monochromatic edges incident to v, and let $\bar{B}(N) = \sum_{v \in V} \delta_v - B(N) = \frac{1}{2}\sum_{v \in V} \bar{B}_v(N)$ be the sum of the weights of all monochromatic edges. Since N is a Nash equilibrium, it must hold that

$$P_v(N(v)) + B_v(N) \geq \bar{B}_v(N) \quad \forall v \in V, \tag{12}$$

otherwise, player v could switch to any other color, thus improving her utility. We can calculate $A(N)$ as follows:

$$A(N) = \sum_{v \in V} P_v(N(v)) \geq \sum_{v \in V} \left(\bar{B}_v(N) - B_v(N)\right) = \tag{13}$$

$$= \sum_{v \in V} \left(\bar{B}_v(N) - B_v(N)\right) + 2B(N) - 2B(N) = \tag{14}$$

$$= 2B(N) + \sum_{v \in V} \bar{B}_v(N) - \sum_{v \in V} B_v(N) - 2B(N) = \tag{15}$$

$$= 2\sum_{v \in V} \delta_v - 2B(N) - 2B(N) = 2\sum_{v \in V} \delta_v - 4B(N)$$

In (13) we apply inequality (12), then in (14) we add and remove $2B(N)$. Since each edge is either proper or monochromatic, it appears in only one of the two summations in (15). Moreover, each edge is counted exactly twice in a summation (one for endpoint), that is, $\sum_{v \in V} \bar{B}_v(N) = 2\bar{B}(N)$ and $\sum_{v \in V} B_v(N) = 2B(N)$. By summing $2\bar{B}(N)$ and $2B(N)$, we get $2\sum_{v \in V} \delta_v$. Thus, $A(N) \geq 2\sum_{v \in V} \delta_v - 4B(N)$, that is, $A(N) + 4B(N) \geq 2\sum_{v \in V} \delta_v \geq 2B(c^*)$. Dividing by 2, we get that inequality (11) holds, and this concludes the proof. □

5 Egalitarian Social Welfare

In this section we focus on the egalitarian social welfare. We show tight bounds both for egalitarian price of anarchy and stability.

5.1 Price of Anarchy

For the graph k-coloring game (i.e., without color profits), a lower bound on the egalitarian price of anarchy is provided by the instance in [20] (Section 3, Proposition 2). In such instance, the optimal solution is such that each player v has utility equal to her degree δ^v, however there exists a stable coloring in which each player v has utility equal to $\frac{k-1}{k}\delta^v$, for any number of colors $k \geq 2$. This result together with the fact that by the pigeonhole principle, in any stable coloring each player v achieves utility at least $\frac{k-1}{k}\delta^v$, imply that the egalitarian price of anarchy is exactly $\frac{k}{k-1}$ for the graph k-coloring games.

Therefore, we now consider the generalized graph k-coloring games. We first notice that the instance defined in Theorem 3 gives us a lover bound of 2 to the egalitarian price of anarchy. In fact, in the optimal coloring the minimum utility is 2, and there exists a stable coloring in which the minimum utility is 1. Moreover, we point out that the proof of Theorem 2 basically shows that for any player i, her utility in any stable outcome is at least half the payoff that she has in the optimum. Therefore we easily get the following theorem which says that the egalitarian price of anarchy of the generalized graph k-coloring games is 2.

Theorem 6. *The egalitarian price of anarchy of the generalized graph k-coloring games is 2.*

5.2 Price of Stability

We now turn our attention to the egalitarian price of stability. We start by showing that for the graph k-coloring games (i.e., without color profits), the egalitarian price of stability is exactly $\frac{k}{k-1}$.

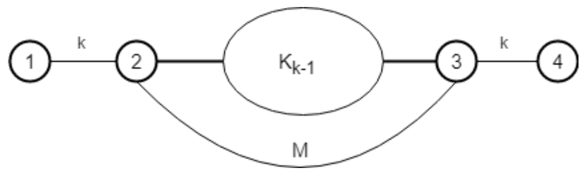

Fig. 1: Instance used in Theorem 7.

Theorem 7. *The egalitarian price of stability of the graph k-coloring games is* $\frac{k}{k-1}$.

Proof. Consider the weighted graph of Figure 1 with $k \geq 2$ colors, where K_{k-1} stands for a clique of size $k-1$, and the bold arrows indicate that there is complete incidence between the two adjacent subgraphs, i.e., there is an edge of weight 1 between any node of the clique K_{k-1} and node 2, and any node of the clique K_{k-1} and node 3. Moreover, there is an edge from 2 to 3 of weight M, where M is an integer greater or equal than $2k$. The only way to get a coloring that maximizes the egalitarian social welfare is the following: nodes 2 and 3 pick the same color, the nodes in the clique choose the remaining $k-1$ colors, one per vertex, and players 1 and 4 pick any color different from the one chosen by 2 and 3. By coloring the nodes in this way, the resulting egalitarian social welfare is equal to k. We notice that this is the unique possible optimal coloring, since the maximum possible utility that each node in the clique K_{k-1} can achieve is exactly k, and the only way to get it is by assigning the same color to nodes 2 and 3. However, this coloring is not stable, since player 2 or 3 can switch to a different available color in order to improve her utility, passing from $2k-1$ to an utility of at least M. Indeed, it is easy to see that in any Nash equilibrium nodes 2 and 3 must choose different colors, and by doing in this way there is at least a node in the clique, say i, that has the same color as 2 or 3, thus achieving an utility of at most $k-1$. Moreover, player i cannot increase her utility by changing color because she has exactly one neighbor colored h, for each color $h = 1, \dots, k$. Thus, we get that the egalitarian price of stability of the graph k-coloring games is at least $\frac{k}{k-1}$.

The theorem follows from the fact that the egalitarian price of anarchy in the graph k-coloring games is at most $\frac{k}{k-1}$. □

We now consider the egalitarian price of stability of the generalized graph k-coloring games and show a tight bound of 2.

Theorem 8. *The egalitarian price of stability of the generalized graph k-coloring games is* 2.

6 Future work

A possible future research direction is the study of other types of equilibria for the generalized graph k-coloring game. For instance, Feldman et al. [14] prove that, when strong Nash equilbria exist, the strong price of anarchy in the graph k-coloring game without profits depends on the number of colors and its maximum value is $\frac{3}{2}$ when $k = 2$. It would be interesting to study how much worse the strong price of anarchy gets for the generalized graph k-coloring games.

We also believe that questions related to the computational complexity of (approximate) Nash equilibria for the generalized graph k-coloring games deserve investigation. Indeed, already for graph k-coloring games, if the graph is unweighted and directed the problem of deciding whether an instance admits a Nash equilibria is NP-hard, and there exist instances for which a Nash equilibrium does not exist at all [20]. Moreover, for the case of weighted undirected graphs, even if Nash equilibria always exist, computing them is PLS-complete [24] even for $k = 2$, since the max cut game is a special case. A γ-Nash equilibrium is a coloring such that no player can improve her payoff by a (multiplicative) factor of γ by changing color. To the best of our knowledge, there are few papers which deal with the problem of computing approximate Nash equilibria for graph k-coloring games. In [10], the authors show that it is possible to compute in polynomial time a $(3 + \epsilon)$-Nash equilibrium, for any $\epsilon > 0$, for max cut games, while in [12] the authors present a randomized polynomial time algorithm that computes a constant approximate Nash equilibrium for a large class of directed unweighted graphs.

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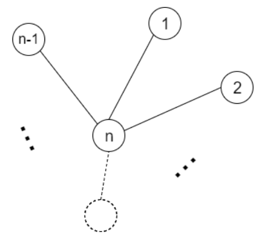

Fig. 2: Instance used in Theorems 3, 4 and 8.

A Appendix: Omitted proofs

A.1 Proof of Proposition 1

Given a coloring c, we define the potential function $\Phi(c)$ as the sum of the weights of proper colored edges plus the profits that each player v gains for using color c_v. Formally:

$$\Phi(c) = \sum_{\{v,u\}\in E: c_v \neq c_u} w(\{v,u\}) + \sum_{v \in V} P_v(c_v) \tag{16}$$

We first notice that this function is bounded. We now show that Φ is an exact potential function, namely the change that occurs to a player's utility when she performs an improving move is equal to the change in Φ. In a given coloring c, we call a player v *unhappy* if she can perform an improving move, by creating the new coloring $c' = (c_{-v}, c'_v)$. Then the change in Φ, resulting from an improving move of player v that induces the new coloring c', is:

$$\begin{aligned}
&\Phi(c') - \Phi(c) = \\
&= \sum_{\{u,z\}\in E: c'_u \neq c'_z} w(\{u,z\}) + \sum_{u\in V} P_u(c'_u) - \sum_{\{u,z\}\in E: c_u \neq c_z} w(\{u,z\}) - \sum_{u \in V} P_u(c_u) = \\
&= \sum_{u\in V: \{v,u\}\in E \wedge c'_v \neq c'_u} w(\{v,u\}) + P_v(c'_v) - \sum_{u\in V: \{v,u\}\in E \wedge c_v \neq c_u} w(\{v,u\}) - P_v(c_v) = \\
&= \mu_{c'}(v) - \mu_c(v).
\end{aligned}$$

Thus, each improving move increases the value of Φ in a proportional way. It follows that, after a finite number of improving moves, no player can be unhappy anymore, that is, the resulting coloring is Nash stable.

A.2 Proof of Theorem 3

In order to prove the lower bound of the utilitarian price of anarchy, consider the star $G = (V, E)$ depicted in Figure 2, where n is the central node, and all the other nodes are leaves, namely, $V = \{1, \ldots, n\}$, and $E = \{\{j, n\} : j = 1, \ldots, n-1\}$. There are k colors available. We define the following profit function:

for all the leaves $j = 1, \ldots, n-1$, $P_j(1) = 1$, and $P_j(h) = 0$, for $h = 2, \ldots, k$, while for the central node n, $P_n(h) = 0$, for $h = 1, \ldots, k-1$, and $P_n(k) = n-1$.

It is easy to see that the coloring c^* with maximum social welfare is the one in which the leaves are colored 1, and the central node is colored k. The resulting social welfare is $SW_{UT}(c^*) = 4(n-1)$. Even though c^* is stable, it is not the unique Nash equilibrium. In fact, the coloring c' in which the leaves are colored k and the central node is colored 1 is a Nash equilibrium as well, and its social welfare is $SW_{UT}(c') = 2(n-1)$. Therefore, the price of anarchy of the generalized graph k-coloring game is at least 2. We remark that such lower bound does not depend on the number of colors k.

A.3 Proof of Theorem 4

Consider an unweighted star graph G of Figure 2, and any k. We define the following profit function: for all the leaves $j = 1, \ldots, n-1$, $P_j(1) = 1 + \gamma$, where $\gamma > 0$ is a small value, and $P_j(h) = 0$, for $h = 2, \ldots, k$. Concerning the central node n, we have that $P_n(1) = n - 1 + \gamma$, and $P_n(h) = 0$, for $h = 2, \ldots, k$. It is easy to see that the optimal coloring c^* with maximum social welfare is the one in which the central node n is colored h, for some $h = 2, \ldots, k$, and the leaves are colored 1. Its social welfare is $SW_{UT}(c^*) = 2(n-1) + (n-1)(1+\gamma) = 3(n-1) + \gamma(n-1)$. However, c^* is not a Nash equilibrium, in fact, the central player n can perform an improving move by selecting the color 1, since in this case her payoff goes up from $n-1$ to $n-1+\gamma$. The resulting coloring c' is now a Nash equilibirum, since no player has incentive to deviate (even if all edges are now monochromatic). The resulting social welfare is $SW_{UT}(c') = (n-1)(1+\gamma) + n - 1 + \gamma = 2(n-1) + n\gamma$. Moreover, c' is the unique Nash equilibrium. In fact, the only other possible type of coloring is the one in which the central node n is colored 1, while each leaf is colored h, for some $h = 1, \ldots, k$. But here, any leaf that is not colored 1, can improve her utility by switching to color 1. Therefore we get that:

$$PoS(G) \geq \frac{3(n-1) + \gamma(n-1)}{2(n-1) + n\gamma} = \frac{3}{2} - \epsilon, \tag{17}$$

where $\epsilon = \dfrac{\gamma(n+2)}{2(2(n-1) + n\gamma)}$. The theorem follows.

A.4 Proof of Theorem 8

We first show a lower bound of 2. Consider the game induced by the unweighted star G of Figure 2 with the following profit function: for all the leaves $j = 1, \ldots, n-1$, $P_j(1) = 1$, and $P_j(h) = 0$, for $h = 2, \ldots, k$. Concerning the central node n, we have that $P_n(1) = n - 1 + \epsilon$ for some small $\epsilon > 0$, and $P_n(h) = 0$, for $h = 2, \ldots, k$. An optimal coloring c^* can be the one in which the central node is colored h, for some $h = 2, \ldots, n$, and the leaves are colored 1. Its egalitarian social welfare is $SW_{EG}(c^*) = 2$. We notice that in any coloring the leaves cannot have utility greater than 2. However c^* is not stable since player n can switch to color 1, and thus improving her utility from $n-1$ to $n-1+\epsilon$. Indeed, it is easy to see that in any Nash equilibrium the central node n chooses color 1 and in this way any leaf can achieve utility at most 1. Therefore, we get that the egalitarian price of stability of the generalized graph k-coloring games is at least 2.

The claim follows from the fact that the egalitarian price of anarchy of the generalized graph k-coloring games is 2 (Theorem 6).