



1 State Parametrization

IMU State at Time k (16×1)

$$\mathbf{X}_{I_k} = \begin{bmatrix} \bar{q}_{I_k G} \\ \mathbf{p}_k^G \\ \mathbf{v}_k^G \\ \mathbf{b}_{g,k} \\ \mathbf{b}_{a,k} \end{bmatrix} \begin{array}{l} \leftarrow \text{Global to IMU rotation unit quaternion} \\ \leftarrow \text{IMU position in global frame} \\ \leftarrow \text{IMU velocity in global frame} \\ \leftarrow \text{Gyro bias} \\ \leftarrow \text{Accelerometer bias} \end{array} \quad (1)$$

IMU Error State at Time k (15×1)

$$\tilde{\mathbf{X}}_{I_k} = \begin{bmatrix} \tilde{\boldsymbol{\theta}}_k^G \\ \tilde{\mathbf{p}}_k^G \\ \tilde{\mathbf{v}}_k^G \\ \tilde{\mathbf{b}}_{g,k} \\ \tilde{\mathbf{b}}_{a,k} \end{bmatrix} = \begin{bmatrix} \delta \boldsymbol{\theta}_I \\ \mathbf{p}_k^G - \hat{\mathbf{p}}_k^G \\ \mathbf{v}_k^G - \hat{\mathbf{v}}_k^G \\ \mathbf{b}_{g,k} - \hat{\mathbf{b}}_{g,k} \\ \mathbf{b}_{a,k} - \hat{\mathbf{b}}_{a,k} \end{bmatrix} \quad (2)$$

where

$$\delta \bar{q} = (\hat{q}_{IG})^{-1} \otimes \bar{q}_{IG} \simeq \begin{bmatrix} \frac{1}{2} \delta \boldsymbol{\theta} \\ 1 \end{bmatrix} \quad (3)$$

MSCKF State at Time k ($(16 + 7N) \times 1$)

$$\hat{\mathbf{X}}_k = \begin{bmatrix} \hat{\mathbf{X}}_{I_k} \\ \hat{q}_{C_1 G} \\ \hat{\mathbf{p}}_{C_1}^G \\ \vdots \\ \hat{q}_{C_N G} \\ \hat{\mathbf{p}}_{C_N}^G \end{bmatrix} \begin{array}{l} \leftarrow \text{IMU State estimate} \\ \leftarrow \text{Camera pose 1 orientation estimate} \\ \leftarrow \text{Camera pose 1 position estimate} \\ \vdots \\ \leftarrow \text{Camera pose N orientation estimate} \\ \leftarrow \text{Camera pose N position estimate} \end{array} \quad (4)$$

MSCKF Error State ($(15 + 6N) \times 1$)

$$\tilde{\mathbf{X}}_k = \begin{bmatrix} \tilde{\mathbf{X}}_{I_k} \\ \delta \boldsymbol{\theta}_{C_1} \\ \tilde{\mathbf{p}}_{C_1}^G \\ \vdots \\ \delta \boldsymbol{\theta}_{C_N} \\ \tilde{\mathbf{p}}_{C_N}^G \end{bmatrix} \quad (5)$$



2 IMU State Estimate Propagation

Integrate these with RK5 from t_{k-1} to $t_{k-1} + T$ where T is the sampling period of the IMU:

$$\dot{\hat{q}}_{IG} = \frac{1}{2} \Omega(\omega_m - \hat{\mathbf{b}}_a - \hat{\mathbf{C}}_{IG} \omega_G) \hat{q}_{IG} \quad (6)$$

$$\dot{\hat{\mathbf{b}}}_g = \mathbf{0}_{3 \times 1} \quad (7)$$

$$\dot{\hat{\mathbf{v}}}_I^G = \hat{\mathbf{C}}_{IG}^T (\mathbf{a}_m - \hat{\mathbf{b}}_a) - 2(\omega_G^\times) \hat{\mathbf{v}}_I^G - (\omega_G^\times)^2 \hat{\mathbf{p}}_I^G + \mathbf{g}^G \quad (8)$$

$$\dot{\hat{\mathbf{b}}}_a = \mathbf{0}_{3 \times 1} \quad (9)$$

$$\dot{\hat{\mathbf{p}}}_I^G = \hat{\mathbf{v}}_I^G \quad (10)$$

$$\dot{\tilde{\mathbf{X}}}_I = \mathbf{F} \tilde{\mathbf{X}}_I + \mathbf{G} \mathbf{n}_I \quad (11)$$

where

$$\Omega(\omega) = \begin{bmatrix} -\omega^\times & \omega \\ -\omega^T & 0 \end{bmatrix} \text{ is } 4 \times 4, \quad (12)$$

$\hat{\mathbf{C}}_{IG} := \mathbf{C}(\hat{q}_{IG})$ is the rotation matrix form of the quaternion \hat{q}_{IG} ,

$$\mathbf{F} = \begin{bmatrix} -\hat{\omega}^\times & -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -\hat{\mathbf{C}}_{IG}^T \hat{\mathbf{a}}^\times & \mathbf{0}_{3 \times 3} & -2\omega_G^\times & -\hat{\mathbf{C}}_{IG}^T & -(\omega_G^\times)^2 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 15, \quad (13)$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & -\hat{\mathbf{C}}_{IG}^T & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 12, \quad (14)$$

and

$$\mathbf{n}_I = \begin{bmatrix} \mathbf{n}_g \\ \mathbf{n}_{wg} \\ \mathbf{n}_a \\ \mathbf{n}_{wa} \end{bmatrix} \begin{array}{l} \leftarrow \text{Gyro noise} \\ \leftarrow \text{Gyro-bias rate-of-change noise} \\ \leftarrow \text{Accelerometer noise} \\ \leftarrow \text{Accelerometer-bias rate-of-change noise} \end{array} \quad (15)$$

is the system noise whose covariance matrix \mathbf{Q}_I is computed offline during calibration. Note, ω_G is the angular velocity of the spinning Earth in our global frame. The magnitude (i.e. rotational speed) is equal to $7.292 \times 10^5 \frac{\text{rad}}{\text{s}}$.



For the covariance,

$$\hat{\mathbf{P}}_k^- = \begin{bmatrix} \hat{\mathbf{P}}_{II,k}^- & \Phi(t_{k-1} + T, t_{k-1}) \hat{\mathbf{P}}_{IC,k-1} \\ \hat{\mathbf{P}}_{IC,k-1}^T \Phi(t_{k-1} + T, t_{k-1})^T & \hat{\mathbf{P}}_{CC,k-1} \end{bmatrix} \text{ is } (15 + 6N) \times (15 + 6N) \quad (16)$$

where:

- $\hat{\mathbf{P}}_{CC,k-1}$ is the $6N \times 6N$ covariance matrix of the camera pose estimates (see section on State Augmentation),
- $\hat{\mathbf{P}}_{IC,k-1}$ is the correlation between the errors in the IMU state and the camera pose estimates,
- the state transition matrix $\Phi(t_{k-1} + T, t_{k-1})$ is computed by integrating (with RK5)

$$\dot{\Phi}(t_{k-1} + \tau, t_{k-1}) = \mathbf{F}\Phi(t_{k-1} + \tau, t_{k-1}), \quad \tau \in [0, T] \quad (17)$$

with initial condition $\Phi(t_{k-1}, t_{k-1}) = \mathbf{1}_{15}$, and

- $\hat{\mathbf{P}}_{II,k}^-$ is obtained by integrating (with RK5)

$$\dot{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau) = \mathbf{F}\hat{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau) + \hat{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau) \mathbf{F}^T + \mathbf{G}\mathbf{Q}_I \mathbf{G}^T, \quad \tau \in [0, T] \quad (18)$$

with initial condition $\mathbf{P}_{II}(t_{k-1}, t_{k-1}) = \hat{\mathbf{P}}_{II,k-1}$.

3 State Augmentation

3.1 Camera Poses

For the $(N + 1)^{\text{th}}$ image, the camera pose is estimated as

$$\hat{q}_{CG} = \bar{q}_{CI} \otimes \hat{q}_{IG} \quad (19)$$

$$\hat{\mathbf{p}}_C^G = \hat{\mathbf{p}}_I^G + \hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I \quad (20)$$

and the EKF covariance matrix is augmented according to

$$\hat{\mathbf{P}}_{k-1} \leftarrow \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix} \hat{\mathbf{P}}_{k-1} \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix}^T \quad (21)$$

where the Jacobian

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{C}}_{CI} & \mathbf{0}_{3 \times 9} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 6N} \\ \left(\hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I \right)^\times & \mathbf{0}_{3 \times 9} & \mathbf{1}_3 & \mathbf{0}_{3 \times 6N} \end{bmatrix} \text{ is } 6 \times (15 + 6N) \quad (22)$$



4 Correction Equations

Residuals (pretend everything has a k subscript...)

$$\mathbf{r}_n = \mathbf{T}_H \tilde{\mathbf{X}} + \mathbf{n}_n \quad (23)$$

where \mathbf{T}_H (an upper-triangular matrix) is obtained from the QR decomposition of \mathbf{H}_o (\mathbf{H}_x in the original tech report, but I think that's an error...)

$$\mathbf{H}_o = \begin{bmatrix} \mathbf{H}_o^{(1)} \\ \vdots \\ \mathbf{H}_o^{(L)} \end{bmatrix} = [\mathbf{Q}_1 \quad \mathbf{Q}_2] \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix}. \quad (24)$$

Each $\mathbf{H}_o^{(j)}$ is the projection of $\mathbf{H}_x^{(j)}$ onto the left nullspace of $\mathbf{H}_f^{(j)}$

\mathbf{T}_H and \mathbf{r}_n can be computed in $O(r^2d)$ time using Givens rotations without having to form \mathbf{Q}_1 explicitly. [FIGURE OUT HOW TO DO THIS]

Each $\mathbf{H}_x^{(j)}$ is a stack of Jacobians $\mathbf{H}_{x_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the state (only the entries corresponding to pose i are non-zero):

$$\mathbf{H}_{x_i}^{(j)} = \begin{bmatrix} \mathbf{0}_{2 \times 15} & \mathbf{0}_{2 \times 6} & \dots & \mathbf{J}_i^{(j)} \left(\hat{\mathbf{X}}_{f_j}^{C_i} \right)^\times & -\mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} & \dots & \mathbf{0}_{2 \times 6} \end{bmatrix} \text{ is } 2 \times (15 + 6N) \quad (25)$$

Each $\mathbf{H}_f^{(j)}$ is a stack of Jacobians $\mathbf{H}_{f_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the feature position:

$$\mathbf{H}_{f_i}^{(j)} = \mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} \text{ is } 2 \times 3 \quad (26)$$

In both equations,

$$\mathbf{J}_i^{(j)} = \frac{1}{\hat{Z}_j^{C_i}} \begin{bmatrix} 1 & 0 & -\frac{\hat{X}_j^{C_i}}{\hat{Z}_j^{C_i}} \\ 0 & 1 & -\frac{\hat{Y}_j^{C_i}}{\hat{Z}_j^{C_i}} \end{bmatrix} \quad (27)$$

where

$$\begin{bmatrix} \hat{X}_j^{C_i} \\ \hat{Y}_j^{C_i} \\ \hat{Z}_j^{C_i} \end{bmatrix} = \hat{\mathbf{C}}_{C_i G} \left(\hat{\mathbf{p}}_{f_j}^G - \hat{\mathbf{p}}_{C_i}^G \right) \quad (28)$$

are the 3D coordinates of feature j in the frame of image i .

\mathbf{n}_n is a noise vector with covariance matrix

$$\mathbf{R}_n = \sigma_{\text{im}}^2 \mathbf{1}_r \quad (29)$$

where r is the number of columns in \mathbf{Q}_1 .



Kalman Gain

$$\mathbf{K}_k = \hat{\mathbf{P}}_k^- \mathbf{T}_{H,k}^T \left(\mathbf{T}_{H,k} \hat{\mathbf{P}}_k^- \mathbf{T}_{H,k}^T + \mathbf{R}_{n,k} \right)^{-1} \quad (30)$$

State Vector Correction

$$\Delta \mathbf{X}_k = \mathbf{K}_k \mathbf{r}_{n,k} \quad (31)$$

State Covariance Correction

$$\hat{\mathbf{P}}_k = (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k}) \hat{\mathbf{P}}_k^- (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k})^T + \mathbf{K}_k \mathbf{R}_{n,k} \mathbf{K}_k^T \quad (32)$$