1 State Parametrization

IMU State at Time k (16×1)

$$\mathbf{X}_{I_k} = \begin{bmatrix} {}^{I_k}_{G}\bar{q}^T \\ {}^{G}\mathbf{p}_k \\ {}^{G}\mathbf{v}_k \\ \mathbf{b}_{g_k} \\ \mathbf{b}_{a_k} \end{bmatrix} \begin{array}{l} \leftarrow \text{Global to IMU rotation unit quaterion} \\ \leftarrow \text{IMU position in global frame} \\ \leftarrow \text{IMU velocity in global frame} \\ \leftarrow \text{Gyro bias} \\ \leftarrow \text{Accelerometer bias} \end{array} \tag{1}$$

IMU Error State (15×1)

$$\widetilde{\mathbf{X}} = \begin{bmatrix} {}^{G}\widetilde{\boldsymbol{\theta}} \\ {}^{G}\widetilde{\mathbf{p}} \\ {}^{G}\widetilde{\mathbf{v}} \\ {}^{G}\widetilde{\mathbf{b}}_g \\ {}^{G}\widetilde{\mathbf{b}}_a \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta\theta}_I \\ {}^{G}\mathbf{p} -^{G}\hat{\mathbf{p}} \\ {}^{G}\mathbf{p} -^{G}\hat{\mathbf{v}} \\ \mathbf{b}_g - \hat{\mathbf{b}}_g \\ \mathbf{b}_a - \hat{\mathbf{b}}_a \end{bmatrix} \tag{2}$$

where

$$\delta\bar{q} =^{I}_{G}\hat{\bar{q}}^{-1} \otimes^{I}_{G}\bar{q} \simeq \begin{bmatrix} \frac{1}{2}\boldsymbol{\delta\theta} \\ 1 \end{bmatrix} \tag{3}$$

MSCKF State at Time k $((16 + 7N) \times 1)$

$$\hat{\mathbf{X}}_k = \begin{bmatrix} \hat{\mathbf{X}}_{I_k} \\ {}^{C_1}_{G}\hat{\bar{q}} \\ {}^{G}\hat{\mathbf{p}}_{C_1} \\ \vdots \\ {}^{C_N}_{G}\hat{\bar{q}} \\ {}^{G}\hat{\mathbf{p}}_{C_N} \end{bmatrix} \begin{array}{l} \leftarrow \text{IMU State estimate} \\ \leftarrow \text{Camera pose 1 orientation estimate} \\ \leftarrow \text{Camera pose 1 position estimate} \\ \vdots \\ \leftarrow \text{Camera pose N orientation estimate} \\ \leftarrow \text{Camera pose N position estimate} \end{array} \tag{4}$$

MSCKF Error State $((15 + 6N) \times 1)$

$$\widetilde{\mathbf{X}}_k = \begin{bmatrix} \widetilde{\mathbf{X}}_{I_k} \\ \boldsymbol{\delta\theta}_{C_1} \\ {}^{G}\widetilde{\mathbf{p}}_{C_1} \\ \vdots \\ \boldsymbol{\delta\theta}_{C_N} \\ {}^{G}\widetilde{\mathbf{p}}_{C_N} \end{bmatrix} \tag{5}$$

2 IMU State Estimate Propagation

Integrate these with RK5 from t_k to $t_k + T$ where T is the sampling period of the IMU:

$$ {}^{I}_{G}\dot{\hat{\bar{q}}} = \frac{1}{2}\boldsymbol{\Omega}\left(\boldsymbol{\omega}_m - \hat{\mathbf{b}}_a - \mathbf{C}\left({}^{I}_{G}\hat{\bar{q}}\right)\boldsymbol{\omega}_G\right){}^{I}_{G}\hat{\bar{q}} \tag{6} $$

$$ \dot{\hat{\mathbf{b}}}_g = \mathbf{0}_{3\times 1} \tag{7} $$

$$ {}^{G}\dot{\hat{\mathbf{v}}}_I = \mathbf{C}\left({}^{I}_{G}\hat{\bar{q}}\right)^T\left(\mathbf{a}_m - \hat{\mathbf{b}}_a\right) - 2\left(\boldsymbol{\omega}_G^{\times}\right){}^{G}\hat{\mathbf{v}}_I - \left(\boldsymbol{\omega}_G^{\times}\right)^2\,{}^{G}\hat{\mathbf{p}}_I + {}^{G}\mathbf{g} \tag{8} $$

$$ \dot{\hat{\mathbf{b}}}_a = \mathbf{0}_{3\times 1} \tag{9} $$

$$ {}^{G}\dot{\hat{\mathbf{p}}}_I = {}^{G}\hat{\mathbf{v}}_I \tag{10} $$

$$ \dot{\widetilde{\mathbf{X}}}_I = \mathbf{F}\widetilde{\mathbf{X}}_I + \mathbf{G}\mathbf{n}_I \tag{11} $$

where

$$ \boldsymbol{\Omega}\left(\boldsymbol{\omega}\right) = \begin{bmatrix} -\boldsymbol{\omega}^{\times} & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \text{ is } 4\times 4, \tag{12} $$

$\mathbf{C}(\bar{q})$ is the rotation matrix form of the quaternion $\bar{q}$,

$$ \mathbf{F} = \begin{bmatrix} -\hat{\boldsymbol{\omega}}^{\times} & -\mathbf{1}_3 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ -\mathbf{C}\left({}^{I}_{G}\hat{\bar{q}}\right)^T\hat{\mathbf{a}}^{\times} & \mathbf{0}_{3\times 3} & -2\boldsymbol{\omega}_G^{\times} & -\mathbf{C}\left({}^{I}_{G}\hat{\bar{q}}\right)^T & -\left(\boldsymbol{\omega}_G^{\times}\right)^2 \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{1}_3 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \text{ is } 15\times 15, \tag{13} $$

$$ \mathbf{G} = \begin{bmatrix} -\mathbf{1}_3 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{1}_3 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & -\mathbf{C}\left({}^{I}_{G}\hat{\bar{q}}\right) & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{1}_3 \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \text{ is } 15\times 12, \tag{14} $$

and

$$ \mathbf{n}_I = \begin{bmatrix} \mathbf{n}_g \\ \mathbf{n}_{wg} \\ \mathbf{n}_a \\ \mathbf{n}_{wa} \end{bmatrix} \tag{15} $$

is the system noise whose covariance matrix $\mathbf{Q}_I$ is computed offline during calibration.

For the covariance,

$$\mathbf{P}_{k+1|k} = \begin{bmatrix} \mathbf{P}_{II_{k+1|k}} & \mathbf{\Phi}\left(t_k+T, t_k\right)\mathbf{P}_{IC_{k|k}} \\ \mathbf{P}_{IC_{k|k}}^T \mathbf{\Phi}\left(t_k+T, t_k\right)^T & \mathbf{P}_{CC_{k|k}} \end{bmatrix} \text{ is } (15+6N)\times(15+6N) \tag{16}$$

where:

- $\mathbf{P}_{CC_{k|k}}$ is the $6N \times 6N$ covariance matrix of the camera pose estimates (see section on State Augmentation),
- $\mathbf{P}_{IC_{k|k}}$ is the correlation between the errors in the IMU state and the camera pose estimates,
- the state transition matrix $\mathbf{\Phi}\left(t_k+T, t_k\right)$ is computed by integrating (with RK5)

  $$\dot{\mathbf{\Phi}}\left(t_k+\tau, t_k\right) = \mathbf{F}\mathbf{\Phi}\left(t_k+\tau, t_k\right), \qquad \tau \in [0, T] \tag{17}$$

  with initial condition $\mathbf{\Phi}\left(t_k, t_k\right) = \mathbf{1}_{15}$, and
- $\mathbf{P}_{II_{k+1|k}}$ is obtained by integrating (with RK5)

  $$\dot{\mathbf{P}}_{II} = \mathbf{F}\mathbf{P}_{II} + \mathbf{P}_{II}\mathbf{F}^T + \mathbf{G}\mathbf{Q}_I\mathbf{G}^T \tag{18}$$

  over the interval (t_k, t_k+T).

3 State Augmentation

3.1 Camera Poses

For the $(N+1)^{\text{th}}$ image, the camera pose is estimated as

$${}^C_G\hat{\bar{q}} = {}^C_I\bar{q} \otimes {}^I_G\hat{\bar{q}} \tag{19}$$

$${}^G\hat{\mathbf{p}}_C = {}^G\hat{\mathbf{p}}_I + \mathbf{C}\left({}^I_G\hat{\bar{q}}\right)^T \, {}^I\mathbf{p}_C \tag{20}$$

and the EKF covariance matrix is augmented according to

$$\mathbf{P}_{k|k} \leftarrow \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix} \mathbf{P}_{k|k} \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix}^T \tag{21}$$

where the Jacobian

$$\mathbf{J} = \begin{bmatrix} \mathbf{C}\left({}^C_I\hat{\bar{q}}\right) & \mathbf{0}_{3\times9} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times6N} \\ \left(\mathbf{C}\left({}^I_G\hat{\bar{q}}\right)^T \, {}^I\mathbf{p}_C\right)^\times & \mathbf{0}_{3\times9} & \mathbf{I}_3 & \mathbf{0}_{3\times6N} \end{bmatrix} \text{ is } 6\times(15+6N) \tag{22}$$

4 Correction Equations

Residuals

$$\mathbf{r}_n = \mathbf{T}_H \widetilde{\mathbf{X}} + \mathbf{n}_n \tag{23}$$

where $\mathbf{T}_H$ (an upper-triangular matrix) is obtained from the QR decomposition of $\mathbf{H}_o$ ($\mathbf{H}_{\mathbf{x}}$ in the original tech report, but I think that's an error...)

$$\mathbf{H}_o = \begin{bmatrix} \mathbf{H}_o^{(1)} \\ \vdots \\ \mathbf{H}_o^{(L)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix}. \tag{24}$$

Each $\mathbf{H}_o^{(j)}$ is the projection of $\mathbf{H}_{\mathbf{x}}^{(j)}$ onto the left nullspace of $\mathbf{H}_f^{(j)}$

T_H and $\mathbf{r}_n$ can be computed in $O(r^2 d)$ time using Givens rotations without having to form $\mathbf{Q}_1$ explicitly. [FIGURE OUT HOW TO DO THIS]

Each $\mathbf{H}_{\mathbf{x}}^{(j)}$ is a stack of Jacobians $\mathbf{H}_{\mathbf{x}_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the state (only the entries corresponding to pose i are non-zero):

$$\mathbf{H}_{\mathbf{x}_i}^{(j)} = \begin{bmatrix} \mathbf{0}_{2\times 15} & \mathbf{0}_{2\times 6} & \dots & \mathbf{J}_i^{(j)} \left({}^{C_i}\hat{\mathbf{X}}_{f_j} \right)^{\times} & -\mathbf{J}_i^{(j)} \mathbf{C}\left({}^{C_i}_{G}\hat{\bar{q}} \right) & \dots & \mathbf{0}_{2\times 6} \end{bmatrix} \text{ is } 2 \times (15 + 6N) \tag{25}$$

Each $\mathbf{H}_f^{(j)}$ is a stack of Jacobians $\mathbf{H}_{f_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the feature position:

$$\mathbf{H}_{f_i}^{(j)} = \mathbf{J}_i^{(j)} \mathbf{C}\left({}^{C_i}_{G}\hat{\bar{q}} \right) \text{ is } 2 \times 3 \tag{26}$$

In both equations,

$$\mathbf{J}_i^{(j)} = \frac{1}{{}^{C_i}\hat{Z}_j} \begin{bmatrix} 1 & 0 & -\frac{{}^{C_i}\hat{X}_j}{{}^{C_i}\hat{Z}_j} \\ 0 & 1 & -\frac{{}^{C_i}\hat{Y}_j}{{}^{C_i}\hat{Z}_j} \end{bmatrix} \tag{27}$$

where

$$\begin{bmatrix} {}^{C_i}\hat{X}_j \\ {}^{C_i}\hat{Y}_j \\ {}^{C_i}\hat{Z}_j \end{bmatrix} = \mathbf{C}\left({}^{C_i}_{G}\hat{\bar{q}} \right) \left({}^{G}\hat{\mathbf{p}}_{f_j} -{}^{G} \hat{\mathbf{p}}_{C_i} \right) \tag{28}$$

are the 3D coordinates of feature j in the frame of image i.

$\mathbf{n}_n$ is a noise vector with covariance matrix

$$\mathbf{R}_n = \sigma_{\text{im}}^2 \mathbf{1}_r \tag{29}$$

where r is the number of columns in $\mathbf{Q}_1$.

Kalman Gain

$$\mathbf{K} = \mathbf{P}_{k+1|k}\mathbf{T}_H^T \left(\mathbf{T}_H\mathbf{P}_{k+1|k}\mathbf{T}_H^T + \mathbf{R}_n\right)^{-1} \tag{30}$$

State Vector Correction

$$\Delta\mathbf{X} = \mathbf{K}\mathbf{r}_n \tag{31}$$

State Covariance Correction

$$\mathbf{P}_{k+1|k+1} = \left(\mathbf{1}_{15+6N} - \mathbf{K}\mathbf{T}_H\right)\mathbf{P}_{k+1|k}\left(\mathbf{1}_{15+6N} - \mathbf{K}\mathbf{T}_H\right)^T + \mathbf{K}\mathbf{R}_n\mathbf{K}^T \tag{32}$$