

# 1 State Parametrization

IMU State at Time k  $(16 \times 1)$ 

$$\mathbf{X}_{I_k} = \begin{bmatrix} \mathbf{q}_{I_kG} \\ \mathbf{p}_k^G \\ \mathbf{v}_k^G \\ \end{bmatrix} \leftarrow \text{Global to IMU rotation unit quaterion} \\ \leftarrow \text{IMU position in global frame} \\ \leftarrow \text{IMU velocity in global frame}$$

$$\mathbf{b}_{g,k} \\ \mathbf{b}_{a,k} \end{bmatrix} \leftarrow \text{Gyro bias}$$

$$\leftarrow \text{Accelerometer bias}$$

$$(1)$$

#### IMU Error State at Time k $(15 \times 1)$

$$\widetilde{\mathbf{X}}_{I_{k}} = \begin{bmatrix} \widetilde{\boldsymbol{\theta}}_{k}^{G} \\ \widetilde{\mathbf{p}}_{k}^{G} \\ \widetilde{\mathbf{b}}_{g,k} \\ \widetilde{\mathbf{b}}_{g,k} \\ \widetilde{\mathbf{b}}_{a,k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta}\boldsymbol{\theta}_{I} \\ \mathbf{p}_{k}^{G} - \hat{\mathbf{p}}_{k}^{G} \\ \mathbf{v}_{k}^{G} - \hat{\mathbf{v}}_{k}^{G} \\ \mathbf{v}_{k}^{G} - \hat{\mathbf{v}}_{k}^{G} \\ \mathbf{b}_{g,k} - \hat{\mathbf{b}}_{g,k} \\ \mathbf{b}_{a,k} - \hat{\mathbf{b}}_{a,k} \end{bmatrix}$$
(2)

where

$$\delta \mathbf{q} = (\hat{\mathbf{q}}_{IG})^{-1} \otimes \mathbf{q}_{IG} \simeq \begin{bmatrix} \frac{1}{2} \delta \boldsymbol{\theta} \\ 1 \end{bmatrix}$$
 (3)

#### MSCKF State at Time k $((16 + 7N) \times 1)$

$$\hat{\mathbf{X}}_{I_k} = \begin{bmatrix} \hat{\mathbf{X}}_{I_k} \\ \hat{\mathbf{q}}_{C_1G} \\ \hat{\mathbf{p}}_{C_1}^G \\ & \leftarrow \text{Camera pose 1 orientation estimate} \\ & \leftarrow \text{Camera pose 1 position estimate} \\ & \vdots \\ & \vdots \\ & \hat{\mathbf{q}}_{C_NG} \\ & \hat{\mathbf{p}}_{C_N}^G \end{bmatrix} \leftarrow \text{Camera pose N orientation estimate}$$

$$\leftarrow \text{Camera pose N orientation estimate}$$

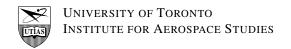
$$\leftarrow \text{Camera pose N position estimate}$$

$$\leftarrow \text{Camera pose N position estimate}$$

## **MSCKF Error State** $((15+6N)\times 1)$

$$\widetilde{\mathbf{X}}_{k} = \begin{bmatrix} \widetilde{\mathbf{X}}_{I_{k}} \\ \delta \boldsymbol{\theta}_{C_{1}} \\ \widetilde{\mathbf{p}}_{C_{1}}^{G} \\ \vdots \\ \delta \boldsymbol{\theta}_{C_{N}} \\ \widetilde{\mathbf{p}}_{C_{N}}^{G} \end{bmatrix}$$

$$(5)$$



# 2 IMU State Estimate Propagation

Integrate these with RK5 from  $t_{k-1}$  to  $t_{k-1} + T$  where T is the sampling period of the IMU:

$$\dot{\hat{\mathbf{q}}}_{IG} = \frac{1}{2} \mathbf{\Omega} \left( \boldsymbol{\omega}_m - \hat{\mathbf{b}}_a - \hat{\mathbf{C}}_{IG} \boldsymbol{\omega}_G \right) \hat{\mathbf{q}}_{IG} \tag{6}$$

$$\dot{\hat{\mathbf{b}}}_q = \mathbf{0}_{3 \times 1} \tag{7}$$

$$\dot{\hat{\mathbf{v}}}_{I}^{G} = \hat{\mathbf{C}}_{IG}^{T} \left( \mathbf{a}_{m} - \hat{\mathbf{b}}_{a} \right) - 2 \left( \boldsymbol{\omega}_{G}^{\times} \right) \hat{\mathbf{v}}_{I}^{G} - \left( \boldsymbol{\omega}_{G}^{\times} \right)^{2} \hat{\mathbf{p}}_{I}^{G} + \mathbf{g}^{G}$$

$$(8)$$

$$\dot{\hat{\mathbf{b}}}_a = \mathbf{0}_{3 \times 1} \tag{9}$$

$$\dot{\hat{\mathbf{p}}}_I^G = \hat{\mathbf{v}}_I^G \tag{10}$$

$$\dot{\widetilde{\mathbf{X}}}_I = \mathbf{F}\widetilde{\mathbf{X}}_I + \mathbf{G}\mathbf{n}_I \tag{11}$$

where

$$\mathbf{\Omega}\left(\boldsymbol{\omega}\right) = \begin{bmatrix} -\boldsymbol{\omega}^{\times} & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{T} & 0 \end{bmatrix} \text{ is } 4 \times 4, \tag{12}$$

 $\hat{\mathbf{C}}_{IG} := \mathbf{C}(\hat{\mathbf{q}}_{IG})$  is the rotation matrix form of the quaternion  $\hat{\mathbf{q}}_{IG}$ 

$$\mathbf{F} = \begin{bmatrix} -\hat{\boldsymbol{\omega}}^{\times} & -\mathbf{1}_{3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -\hat{\mathbf{C}}_{IG}^{T} \hat{\mathbf{a}}^{\times} & \mathbf{0}_{3\times3} & -2\boldsymbol{\omega}_{G}^{\times} & -\hat{\mathbf{C}}_{IG}^{T} & -(\boldsymbol{\omega}_{G}^{\times})^{2} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{1}_{3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \text{ is } 15 \times 15,$$

$$(13)$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_{3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{1}_{3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\hat{\mathbf{C}}_{IG}^{T} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{1}_{3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix} \text{ is } 15 \times 12,$$

$$(14)$$

and

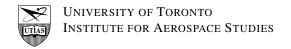
$$\mathbf{n}_{I} = \begin{bmatrix} \mathbf{n}_{g} \\ \mathbf{n}_{wg} \\ \mathbf{n}_{a} \\ \mathbf{n}_{wa} \end{bmatrix} \leftarrow \text{Gyro noise}$$

$$\leftarrow \text{Gyro-bias rate-of-change noise}$$

$$\leftarrow \text{Accelerometer noise}$$

$$\leftarrow \text{Accelerometer-bias rate-of-change noise}$$
(15)

is the system noise whose covariance matrix  $\mathbf{Q}_I$  is computed offline during calibration. Note,  $\boldsymbol{\omega}_G$  is the angular velocity of the spinning Earth in our global frame. The magnitude (i.e. rotational speed) is equal to  $7.292 \times 10^5 \frac{\mathrm{rad}}{\mathrm{s}}$ .



For the covariance,

$$\hat{\mathbf{P}}_{k}^{-} = \begin{bmatrix} \hat{\mathbf{P}}_{II,k}^{-} & \Phi(t_{k-1} + T, t_{k-1}) \hat{\mathbf{P}}_{IC,k-1} \\ \hat{\mathbf{P}}_{IC,k-1}^{T} \Phi(t_{k-1} + T, t_{k-1})^{T} & \hat{\mathbf{P}}_{CC,k-1} \end{bmatrix} \text{ is } (15 + 6N) \times (15 + 6N)$$
(16)

where:

- $\hat{\mathbf{P}}_{CC,k-1}$  is the  $6N \times 6N$  covariance matrix of the camera pose estimates (see section on State Augmentation),
- $\hat{\mathbf{P}}_{IC,k-1}$  is the correlation between the errors in the IMU state and the camera pose estimates,
- the state transition matrix  $\Phi(t_{k-1} + T, t_{k-1})$  is computed by integrating (with RK5)

$$\dot{\mathbf{\Phi}}(t_{k-1} + \tau, t_{k-1}) = \mathbf{F}\mathbf{\Phi}(t_{k-1} + \tau, t_{k-1}), \qquad \tau \in [0, T]$$
(17)

with initial condition  $\Phi(t_{k-1}, t_{k-1}) = \mathbf{1}_{15}$ , and

•  $\hat{\mathbf{P}}_{II\,k}^{-}$  is obtained by integrating (with RK5)

$$\dot{\hat{\mathbf{P}}}_{II}(t_{k-1}, t_{k-1} + \tau) = \mathbf{F}\hat{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau) + \hat{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau)\mathbf{F}^T + \mathbf{G}\mathbf{Q}_I\mathbf{G}^T, \qquad \tau \in [0, T]$$
(18)

with initial condition  $\mathbf{P}_{II}(t_{k-1}, t_{k-1}) = \hat{\mathbf{P}}_{II,k-1}$ .

# 3 State Augmentation

#### 3.1 Camera Poses

For the  $(N+1)^{th}$  image, the camera pose is estimated as

$$\hat{\mathbf{q}}_{CG} = \mathbf{q}_{CI} \otimes \hat{\mathbf{q}}_{IG} \tag{19}$$

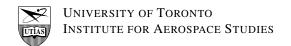
$$\hat{\mathbf{p}}_C^G = \hat{\mathbf{p}}_I^G + \hat{\mathbf{C}}_{IG}^T \, \mathbf{p}_C^I \tag{20}$$

and the EKF covariance matrix is augmented according to

$$\hat{\mathbf{P}}_{k-1} \leftarrow \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix} \hat{\mathbf{P}}_{k-1} \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix}^T \tag{21}$$

where the Jacobian

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{C}}_{CI} & \mathbf{0}_{3\times9} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times6N} \\ (\hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I)^{\times} & \mathbf{0}_{3\times9} & \mathbf{1}_3 & \mathbf{0}_{3\times6N} \end{bmatrix} \text{ is } 6 \times (15+6N)$$
(22)



# 4 Correction Equations

Residuals (pretend everything has a k subscript...)

$$\mathbf{r}_n = \mathbf{T}_H \widetilde{\mathbf{X}} + \mathbf{n}_n \tag{23}$$

where  $T_H$  (an upper-triangular matrix) is obtained from the QR decomposition of  $H_o$  ( $H_x$  in the original tech report, but I think that's an error...)

$$\mathbf{H}_{o} = \begin{bmatrix} \mathbf{H}_{o}^{(1)} \\ \vdots \\ \mathbf{H}_{o}^{(L)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{1} & \mathbf{Q}_{2} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{H} \\ \mathbf{0} \end{bmatrix}. \tag{24}$$

Each  $\mathbf{H}_o^{(j)}$  is the projection of  $\mathbf{H}_\mathbf{x}^{(j)}$  onto the left nullspace of  $\mathbf{H}_f^{(j)}$ 

 $T_H$  and  $\mathbf{r}_n$  can be computed in  $O(r^2d)$  time using Givens rotations without having to form  $\mathbf{Q}_1$  explicitly. [FIGURE OUT HOW TO DO THIS]

Each  $\mathbf{H}_{\mathbf{x}}^{(j)}$  is a stack of Jacobians  $\mathbf{H}_{\mathbf{x}_i}^{(j)}$  of the  $i^{\text{th}}$  measurement of feature j with respect to the state (only the entries corresponding to pose i are non-zero):

$$\mathbf{H}_{\mathbf{x}_{i}}^{(j)} = \begin{bmatrix} \mathbf{0}_{2\times15} & \mathbf{0}_{2\times6} & \dots & \mathbf{J}_{i}^{(j)} \left( \hat{\mathbf{X}}_{f_{j}}^{C_{i}} \right)^{\times} & -\mathbf{J}_{i}^{(j)} \hat{\mathbf{C}}_{C_{i}G} & \dots & \mathbf{0}_{2\times6} \end{bmatrix} \text{ is } 2 \times (15+6N)$$

$$(25)$$

Each  $\mathbf{H}_f^{(j)}$  is a stack of Jacobians  $\mathbf{H}_{f_i}^{(j)}$  of the  $i^{\text{th}}$  measurement of feature j with respect to the feature position:

$$\mathbf{H}_{f_i}^{(j)} = \mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} \text{ is } 2 \times 3 \tag{26}$$

In both equations,

$$\mathbf{J}_{i}^{(j)} = \frac{1}{\hat{Z}_{j}^{C_{i}}} \begin{bmatrix} 1 & 0 & -\frac{\hat{X}_{j}^{C_{i}}}{\hat{Z}_{i}^{C_{i}}} \\ 0 & 1 & -\frac{\hat{Y}_{j}^{C_{i}}}{\hat{Z}_{i}^{C_{i}}} \end{bmatrix}$$
(27)

where

$$\begin{bmatrix} \hat{X}_{j}^{C_{i}} \\ \hat{Y}_{j}^{C_{i}} \\ \hat{Z}_{j}^{C_{i}} \end{bmatrix} = \hat{\mathbf{C}}_{C_{i}G} \left( \hat{\mathbf{p}}_{f_{j}}^{G} - \hat{\mathbf{p}}_{C_{i}}^{G} \right)$$

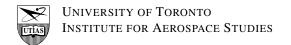
$$(28)$$

are the 3D coordinates of feature j in the frame of image i.

 $\mathbf{n}_n$  is a noise vector with covariance matrix

$$\mathbf{R}_n = \sigma_{\rm im}^2 \mathbf{1}_r \tag{29}$$

where r is the number of columns in  $\mathbf{Q}_1$ .



December 1, 2014

## Kalman Gain

$$\mathbf{K}_{k} = \hat{\mathbf{P}}_{k}^{-} \mathbf{T}_{H,k}^{T} \left( \mathbf{T}_{H,k} \hat{\mathbf{P}}_{k}^{-} \mathbf{T}_{H,k}^{T} + \mathbf{R}_{n,k} \right)^{-1}$$
(30)

# **State Vector Correction**

$$\Delta \mathbf{X}_k = \mathbf{K}_k \mathbf{r}_{n,k} \tag{31}$$

## **State Covariance Correction**

$$\hat{\mathbf{P}}_k = (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k}) \hat{\mathbf{P}}_k^- (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k})^T + \mathbf{K}_k \mathbf{R}_{n,k} \mathbf{K}_k^T$$
(32)