

1 State Parametrization

IMU State at Time k (16×1)

$$\mathbf{X}_{I_k} = \begin{bmatrix} \mathbf{q}_{I_k G} \\ \mathbf{b}_{g,k} \\ \mathbf{v}_k^G \\ \mathbf{b}_{a,k} \\ \mathbf{p}_k^G \end{bmatrix} \begin{array}{l} \leftarrow \text{Global to IMU rotation unit quaternion} \\ \leftarrow \text{Gyro bias} \\ \leftarrow \text{IMU velocity in global frame} \\ \leftarrow \text{Accelerometer bias} \\ \leftarrow \text{IMU position in global frame} \end{array} \quad (1)$$

IMU Error State at Time k (15×1)

$$\tilde{\mathbf{X}}_{I_k} = \begin{bmatrix} \tilde{\boldsymbol{\theta}}_k^G \\ \tilde{\mathbf{b}}_{g,k} \\ \tilde{\mathbf{v}}_k^G \\ \tilde{\mathbf{b}}_{a,k} \\ \tilde{\mathbf{p}}_k^G \end{bmatrix} = \begin{bmatrix} \delta \boldsymbol{\theta}_I \\ \mathbf{b}_{g,k} - \hat{\mathbf{b}}_{g,k} \\ \mathbf{v}_k^G - \hat{\mathbf{v}}_k^G \\ \mathbf{b}_{a,k} - \hat{\mathbf{b}}_{a,k} \\ \mathbf{p}_k^G - \hat{\mathbf{p}}_k^G \end{bmatrix} \quad (2)$$

where

$$\delta \mathbf{q} = (\hat{\mathbf{q}}_{IG})^{-1} \otimes \mathbf{q}_{IG} \simeq \begin{bmatrix} \frac{1}{2} \delta \boldsymbol{\theta} \\ 1 \end{bmatrix} \quad (3)$$

MSCKF State at Time k ($(16 + 7N) \times 1$)

$$\hat{\mathbf{X}}_k = \begin{bmatrix} \hat{\mathbf{X}}_{I_k} \\ \hat{\mathbf{q}}_{C_1 G} \\ \hat{\mathbf{p}}_{C_1}^G \\ \vdots \\ \hat{\mathbf{q}}_{C_N G} \\ \hat{\mathbf{p}}_{C_N}^G \end{bmatrix} \begin{array}{l} \leftarrow \text{IMU State estimate} \\ \leftarrow \text{Camera pose 1 orientation estimate} \\ \leftarrow \text{Camera pose 1 position estimate} \\ \vdots \\ \leftarrow \text{Camera pose N orientation estimate} \\ \leftarrow \text{Camera pose N position estimate} \end{array} \quad (4)$$

MSCKF Error State ($(15 + 6N) \times 1$)

$$\tilde{\mathbf{X}}_k = \begin{bmatrix} \tilde{\mathbf{X}}_{I_k} \\ \delta \boldsymbol{\theta}_{C_1} \\ \tilde{\mathbf{p}}_{C_1}^G \\ \vdots \\ \delta \boldsymbol{\theta}_{C_N} \\ \tilde{\mathbf{p}}_{C_N}^G \end{bmatrix} \quad (5)$$

2 IMU State Estimate Propagation

Integrate these with RK5 from t_{k-1} to $t_{k-1} + T$ where T is the sampling period of the IMU:

$$\dot{\mathbf{q}}_{IG} = \frac{1}{2} \boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}) \hat{\mathbf{q}}_{IG} \quad (6)$$

$$\dot{\mathbf{b}}_g = \mathbf{0}_{3 \times 1} \quad (7)$$

$$\dot{\mathbf{v}}_I^G = \hat{\mathbf{C}}_{IG}^T (\mathbf{a}_m - \hat{\mathbf{b}}_a) - 2(\boldsymbol{\omega}_G^\times) \hat{\mathbf{v}}_I^G - (\boldsymbol{\omega}_G^\times)^2 \hat{\mathbf{p}}_I^G + \mathbf{g}^G \quad (8)$$

$$\dot{\mathbf{b}}_a = \mathbf{0}_{3 \times 1} \quad (9)$$

$$\dot{\mathbf{p}}_I^G = \hat{\mathbf{v}}_I^G \quad (10)$$

where

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \hat{\mathbf{b}}_g - \hat{\mathbf{C}}_{IG} \boldsymbol{\omega}_G, \quad (11)$$

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\boldsymbol{\omega}^\times & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \text{ is } 4 \times 4, \quad (12)$$

$\hat{\mathbf{C}}_{IG} := \mathbf{C}(\hat{\mathbf{q}}_{IG})$ is the rotation matrix form of the quaternion $\hat{\mathbf{q}}_{IG}$,

$$\mathbf{F} = \begin{bmatrix} -\hat{\boldsymbol{\omega}}^\times & -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -\hat{\mathbf{C}}_{IG}^T \hat{\mathbf{a}}^\times & \mathbf{0}_{3 \times 3} & -2\boldsymbol{\omega}_G^\times & -\hat{\mathbf{C}}_{IG}^T & -(\boldsymbol{\omega}_G^\times)^2 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 15, \quad (13)$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & -\hat{\mathbf{C}}_{IG}^T & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 12, \quad (14)$$

and

$$\mathbf{n}_I = \begin{bmatrix} \mathbf{n}_g \\ \mathbf{n}_{wg} \\ \mathbf{n}_a \\ \mathbf{n}_{wa} \end{bmatrix} \begin{array}{l} \leftarrow \text{Gyro noise} \\ \leftarrow \text{Gyro-bias rate-of-change noise} \\ \leftarrow \text{Accelerometer noise} \\ \leftarrow \text{Accelerometer-bias rate-of-change noise} \end{array} \quad (15)$$

is the system noise whose covariance matrix \mathbf{Q}_I is computed offline during calibration. Note, $\boldsymbol{\omega}_G$ is the angular velocity of the spinning Earth in our global frame. The magnitude (i.e. rotational speed) is equal to $7.292 \times 10^5 \frac{\text{rad}}{\text{s}}$.

For the covariance,

$$\hat{\mathbf{P}}_k^- = \begin{bmatrix} \hat{\mathbf{P}}_{II,k}^- & \Phi(t_{k-1} + T, t_{k-1}) \hat{\mathbf{P}}_{IC,k-1} \\ \hat{\mathbf{P}}_{IC,k-1}^T \Phi(t_{k-1} + T, t_{k-1})^T & \hat{\mathbf{P}}_{CC,k-1} \end{bmatrix} \text{ is } (15 + 6N) \times (15 + 6N) \quad (16)$$

where:

- $\hat{\mathbf{P}}_{CC,k-1}$ is the $6N \times 6N$ covariance matrix of the camera pose estimates (see section on State Augmentation),
- $\hat{\mathbf{P}}_{IC,k-1}$ is the correlation between the errors in the IMU state and the camera pose estimates,
- the state transition matrix $\Phi(t_{k-1} + T, t_{k-1})$ is computed by integrating (with RK5)

$$\dot{\Phi}(t_{k-1} + \tau, t_{k-1}) = \mathbf{F}\Phi(t_{k-1} + \tau, t_{k-1}), \quad \tau \in [0, T] \quad (17)$$

with initial condition $\Phi(t_{k-1}, t_{k-1}) = \mathbf{1}_{15}$, and

- $\hat{\mathbf{P}}_{II,k}^-$ is obtained by integrating (with RK5)

$$\dot{\hat{\mathbf{P}}}_{II}(t_{k-1}, t_{k-1} + \tau) = \mathbf{F}\hat{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau) + \hat{\mathbf{P}}_{II}(t_{k-1}, t_{k-1} + \tau) \mathbf{F}^T + \mathbf{G}\mathbf{Q}_I\mathbf{G}^T, \quad \tau \in [0, T] \quad (18)$$

with initial condition $\mathbf{P}_{II}(t_{k-1}, t_{k-1}) = \hat{\mathbf{P}}_{II,k-1}$.

3 State Augmentation

3.1 Camera Poses

For the $(N + 1)^{\text{th}}$ image, the camera pose is estimated as

$$\hat{\mathbf{q}}_{CG} = \mathbf{q}_{CI} \otimes \hat{\mathbf{q}}_{IG} \quad (19)$$

$$\hat{\mathbf{p}}_C^G = \hat{\mathbf{p}}_I^G + \hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I \quad (20)$$

and the EKF covariance matrix is augmented according to

$$\hat{\mathbf{P}}_{k-1} \leftarrow \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix} \hat{\mathbf{P}}_{k-1} \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix}^T \quad (21)$$

where the Jacobian

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{C}}_{CI} & \mathbf{0}_{3 \times 9} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 6N} \\ \left(\hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I \right)^\times & \mathbf{0}_{3 \times 9} & \mathbf{1}_3 & \mathbf{0}_{3 \times 6N} \end{bmatrix} \text{ is } 6 \times (15 + 6N) \quad (22)$$

4 Correction Equations

Residuals (pretend everything has a k subscript...)

$$\mathbf{r}_n = \mathbf{T}_H \tilde{\mathbf{X}} + \mathbf{n}_n \quad (23)$$

where \mathbf{T}_H (an upper-triangular matrix) is obtained from the QR decomposition of \mathbf{H}_o (\mathbf{H}_x in the original tech report, but I think that's an error...)

$$\mathbf{H}_o = \begin{bmatrix} \mathbf{H}_o^{(1)} \\ \vdots \\ \mathbf{H}_o^{(L)} \end{bmatrix} = [\mathbf{Q}_1 \quad \mathbf{Q}_2] \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix}. \quad (24)$$

Each $\mathbf{H}_o^{(j)}$ is the projection of $\mathbf{H}_x^{(j)}$ onto the left nullspace of $\mathbf{H}_f^{(j)}$

\mathbf{T}_H and \mathbf{r}_n can be computed in $O(r^2d)$ time using Givens rotations without having to form \mathbf{Q}_1 explicitly. [FIGURE OUT HOW TO DO THIS]

Each $\mathbf{H}_x^{(j)}$ is a stack of Jacobians $\mathbf{H}_{x_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the state (only the entries corresponding to pose i are non-zero):

$$\mathbf{H}_{x_i}^{(j)} = \begin{bmatrix} \mathbf{0}_{2 \times 15} & \mathbf{0}_{2 \times 6} & \dots & \mathbf{J}_i^{(j)} \left(\hat{\mathbf{x}}_{f_j}^{C_i} \right)^\times & -\mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} & \dots & \mathbf{0}_{2 \times 6} \end{bmatrix} \text{ is } 2 \times (15 + 6N) \quad (25)$$

Each $\mathbf{H}_f^{(j)}$ is a stack of Jacobians $\mathbf{H}_{f_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the feature position:

$$\mathbf{H}_{f_i}^{(j)} = \mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} \text{ is } 2 \times 3 \quad (26)$$

In both equations,

$$\mathbf{J}_i^{(j)} = \frac{1}{\hat{Z}_j^{C_i}} \begin{bmatrix} 1 & 0 & -\frac{\hat{X}_j^{C_i}}{\hat{Z}_j^{C_i}} \\ 0 & 1 & -\frac{\hat{Y}_j^{C_i}}{\hat{Z}_j^{C_i}} \end{bmatrix} \quad (27)$$

where

$$\begin{bmatrix} \hat{X}_j^{C_i} \\ \hat{Y}_j^{C_i} \\ \hat{Z}_j^{C_i} \end{bmatrix} = \hat{\mathbf{C}}_{C_i G} \left(\hat{\mathbf{p}}_{f_j}^G - \hat{\mathbf{p}}_{C_i}^G \right) \quad (28)$$

are the 3D coordinates of feature j in the frame of image i .

\mathbf{n}_n is a noise vector with covariance matrix

$$\mathbf{R}_n = \sigma_{\text{im}}^2 \mathbf{1}_r \quad (29)$$

where r is the number of columns in \mathbf{Q}_1 .

Kalman Gain

$$\mathbf{K}_k = \hat{\mathbf{P}}_k^- \mathbf{T}_{H,k}^T \left(\mathbf{T}_{H,k} \hat{\mathbf{P}}_k^- \mathbf{T}_{H,k}^T + \mathbf{R}_{n,k} \right)^{-1} \quad (30)$$

State Vector Correction

$$\Delta \mathbf{X}_k = \mathbf{K}_k \mathbf{r}_{n,k} \quad (31)$$

State Covariance Correction

$$\hat{\mathbf{P}}_k = (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k}) \hat{\mathbf{P}}_k^- (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k})^T + \mathbf{K}_k \mathbf{R}_{n,k} \mathbf{K}_k^T \quad (32)$$

5 Feature Triangulation

We triangulate features using an inverse depth least squares Gauss-Newton optimization. The procedure takes as input N camera poses and N sets of ideal pixel measurements. We define *ideal* measurements as pixel measurements that have been corrected for the focal length and principal points of the camera:

$$u' = (u - c_u)/f_u \quad (33)$$

$$v' = (v - c_v)/f_v \quad (34)$$

$$(35)$$

We calculate the 3D position of the feature in the first camera frame ($\mathbf{p}_f^{C_1}$) using a linear least squares method with measurements from the first two camera poses:

$$\mathbf{p}_f^{C_1} = \gamma \hat{\mathbf{p}}_f^{C_1} \quad (36)$$

where

$$\gamma = \left[(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{p}_{C_2}^{C_1} \right]_1, \quad \hat{\mathbf{p}}_f^C := \frac{1}{\sqrt{u'^2 + v'^2 + 1}} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} \hat{\mathbf{p}}_f^{C_1} & -\hat{\mathbf{p}}_f^{C_2} \end{bmatrix} \quad (37)$$

Next, we define a vector of three parameters:

$$\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \\ \rho \end{bmatrix}, \quad \alpha = \frac{x_f^{C_1}}{z_f^{C_1}}, \quad \beta = \frac{y_f^{C_1}}{z_f^{C_1}}, \quad \rho = \frac{1}{z_f^{C_1}} \quad (38)$$

The feature position in the i^{th} camera frame can be expressed as:

$$\mathbf{p}_f^{C_i} = \mathbf{C}_{in} \mathbf{p}_f^{C_1} + \mathbf{p}_{C_1}^{C_i} \quad (39)$$

With our state parameters, we can use this equation to define three functions:

$$\begin{bmatrix} h_1(\alpha, \beta, \rho) \\ h_2(\alpha, \beta, \rho) \\ h_3(\alpha, \beta, \rho) \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \\ h_3(\mathbf{x}) \end{bmatrix} = \mathbf{C}_{i1} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} + \rho \mathbf{p}_{C_1}^{C_i} = \begin{bmatrix} C_{11}\alpha + C_{12}\beta + C_{13} + \rho x_{C_1}^{C_i} \\ C_{21}\alpha + C_{22}\beta + C_{23} + \rho y_{C_1}^{C_i} \\ C_{31}\alpha + C_{32}\beta + C_{33} + \rho z_{C_1}^{C_i} \end{bmatrix} \quad (40)$$

The camera measurement error can then be written as:

$$\mathbf{e}(\mathbf{x}) = \hat{\mathbf{z}} - \frac{1}{h_3(\mathbf{x})} \begin{bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \end{bmatrix} \quad (41)$$

The least squares linear system then becomes:

$$(\mathbf{E}^T \mathbf{W}^{-1} \mathbf{E}) \delta \mathbf{x}^* = -\mathbf{E}^T \mathbf{W}^{-1} \mathbf{e}(\bar{\mathbf{x}}) \quad (42)$$

where

$$\bar{\mathbf{x}} := \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\rho} \end{bmatrix} = , \quad \mathbf{E} := \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \in \mathbb{R}^{2N \times 3}, \quad \mathbf{W} := \begin{bmatrix} \mathbf{R} & & \\ & \ddots & \\ & & \mathbf{R} \end{bmatrix} \in \mathbb{R}^{2N \times 2N} \quad (43)$$

The Jacobian elements are given by:

$$\frac{\partial \mathbf{e}}{\partial \alpha} = \begin{bmatrix} -\frac{1}{h_3} \frac{\partial h_1}{\partial \alpha} + \frac{1}{h_3^2} \frac{\partial h_3}{\partial \alpha} h_1 \\ -\frac{1}{h_3} \frac{\partial h_2}{\partial \alpha} + \frac{1}{h_3^2} \frac{\partial h_3}{\partial \alpha} h_2 \end{bmatrix}, \quad \frac{\partial \mathbf{e}}{\partial \beta} = \begin{bmatrix} -\frac{1}{h_3} \frac{\partial h_1}{\partial \beta} + \frac{1}{h_3^2} \frac{\partial h_3}{\partial \beta} h_1 \\ -\frac{1}{h_3} \frac{\partial h_2}{\partial \beta} + \frac{1}{h_3^2} \frac{\partial h_3}{\partial \beta} h_2 \end{bmatrix}, \quad \frac{\partial \mathbf{e}}{\partial \rho} = \begin{bmatrix} -\frac{1}{h_3} \frac{\partial h_1}{\partial \rho} + \frac{1}{h_3^2} \frac{\partial h_3}{\partial \rho} h_1 \\ -\frac{1}{h_3} \frac{\partial h_2}{\partial \rho} + \frac{1}{h_3^2} \frac{\partial h_3}{\partial \rho} h_2 \end{bmatrix} \quad (44)$$

with

$$\frac{\partial h_1}{\partial \alpha} = C_{11}, \quad \frac{\partial h_1}{\partial \beta} = C_{12}, \quad \frac{\partial h_1}{\partial \rho} = x_{C_1}^{C_i} \quad (45)$$

$$\frac{\partial h_2}{\partial \alpha} = C_{21}, \quad \frac{\partial h_2}{\partial \beta} = C_{22}, \quad \frac{\partial h_2}{\partial \rho} = y_{C_1}^{C_i}$$

$$\frac{\partial h_3}{\partial \alpha} = C_{31}, \quad \frac{\partial h_3}{\partial \beta} = C_{32}, \quad \frac{\partial h_3}{\partial \rho} = z_{C_1}^{C_i}$$

6 AER 1513 Mods

6.1 State Parametrization

IMU State at Time k (13×1)

$$\mathbf{X}_{I_k} = \begin{bmatrix} \mathbf{q}_{I_k G} \\ \mathbf{b}_{g,k} \\ \mathbf{b}_{v,k} \\ \mathbf{p}_k^G \end{bmatrix} \begin{array}{l} \leftarrow \text{Global to IMU rotation unit quaternion} \\ \leftarrow \text{Gyro bias} \\ \leftarrow \text{Velocity measurement bias} \\ \leftarrow \text{IMU position in global frame} \end{array} \quad (46)$$

IMU Error State at Time k (12×1)

$$\tilde{\mathbf{X}}_{I_k} = \begin{bmatrix} \tilde{\boldsymbol{\theta}}_k^G \\ \tilde{\mathbf{b}}_{g,k} \\ \tilde{\mathbf{b}}_{v,k} \\ \tilde{\mathbf{p}}_k^G \end{bmatrix} = \begin{bmatrix} \delta \boldsymbol{\theta}_I \\ \mathbf{b}_{g,k} - \hat{\mathbf{b}}_{g,k} \\ \mathbf{b}_{v,k} - \hat{\mathbf{b}}_{v,k} \\ \mathbf{p}_k^G - \hat{\mathbf{p}}_k^G \end{bmatrix} \quad (47)$$

6.2 IMU State Propagation

$$\dot{\hat{\mathbf{q}}}_{IG} = \frac{1}{2} \boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}) \hat{\mathbf{q}}_{IG} \quad (48)$$

$$\dot{\hat{\mathbf{b}}}_g = \mathbf{0}_{3 \times 1} \quad (49)$$

$$\dot{\hat{\mathbf{b}}}_v = \mathbf{0}_{3 \times 1} \quad (50)$$

$$\dot{\hat{\mathbf{p}}}_I^G = \hat{\mathbf{C}}_{IG}^T \hat{\mathbf{v}} \quad (51)$$

where

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \hat{\mathbf{b}}_g, \quad (52)$$

$$\hat{\mathbf{v}} = \mathbf{v}_m - \hat{\mathbf{b}}_v, \quad (53)$$

$$\mathbf{F} = \begin{bmatrix} -\hat{\boldsymbol{\omega}}^\times & -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -\hat{\mathbf{C}}_{IG}^T \hat{\mathbf{v}}^\times & \mathbf{0}_{3 \times 3} & -\hat{\mathbf{C}}_{IG}^T & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 12 \times 12, \quad (54)$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & -\hat{\mathbf{C}}_{IG}^T & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 12 \times 12, \quad (55)$$

and

$$\mathbf{n}_I = \begin{bmatrix} \mathbf{n}_g \\ \mathbf{n}_{wg} \\ \mathbf{n}_v \\ \mathbf{n}_{wv} \end{bmatrix} \begin{array}{l} \leftarrow \text{Gyro noise} \\ \leftarrow \text{Gyro-bias rate-of-change noise} \\ \leftarrow \text{Velocity measurement noise} \\ \leftarrow \text{Velocity bias rate-of-change noise} \end{array} \quad (56)$$

$\hat{\mathbf{P}}_k^-$ is $(12 + 6N) \times (12 + 6N)$.

6.3 State Augmentation

$$\hat{\mathbf{P}}_{k-1} \leftarrow \begin{bmatrix} \mathbf{1}_{12+6N} \\ \mathbf{J} \end{bmatrix} \hat{\mathbf{P}}_{k-1} \begin{bmatrix} \mathbf{1}_{12+6N} \\ \mathbf{J} \end{bmatrix}^T \quad (57)$$

where the Jacobian (not quite sure if I took out the right columns)...

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{C}}_{CI} & \mathbf{0}_{3 \times 6} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 6N} \\ \left(\hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^L \right)^\times & \mathbf{0}_{3 \times 6} & \mathbf{1}_3 & \mathbf{0}_{3 \times 6N} \end{bmatrix} \text{ is } 6 \times (12 + 6N) \quad (58)$$

6.4 Correction Equations

$$\mathbf{H}_{\mathbf{x}_i}^{(j)} = \begin{bmatrix} \mathbf{0}_{2 \times 12} & \mathbf{0}_{2 \times 6} & \dots & \mathbf{J}_i^{(j)} \left(\hat{\mathbf{x}}_{f_j}^{C_i} \right)^\times & -\mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} & \dots & \mathbf{0}_{2 \times 6} \end{bmatrix} \text{ is } 2 \times (12 + 6N) \quad (59)$$

$$\hat{\mathbf{P}}_k = (\mathbf{1}_{12+6N} - \mathbf{K}_k \mathbf{T}_{H,k}) \hat{\mathbf{P}}_k^- (\mathbf{1}_{12+6N} - \mathbf{K}_k \mathbf{T}_{H,k})^T + \mathbf{K}_k \mathbf{R}_{n,k} \mathbf{K}_k^T \quad (60)$$