



# 1 State Parametrization

**IMU State at Time k** ( $16 \times 1$ )

$$\mathbf{X}_{I_k} = \begin{bmatrix} \bar{q}_{I_k G} \\ \mathbf{p}_k^G \\ \mathbf{v}_k^G \\ \mathbf{b}_{g,k} \\ \mathbf{b}_{a,k} \end{bmatrix} \begin{array}{l} \leftarrow \text{Global to IMU rotation unit quaternion} \\ \leftarrow \text{IMU position in global frame} \\ \leftarrow \text{IMU velocity in global frame} \\ \leftarrow \text{Gyro bias} \\ \leftarrow \text{Accelerometer bias} \end{array} \quad (1)$$

**IMU Error State at Time k** ( $15 \times 1$ )

$$\tilde{\mathbf{X}}_{I_k} = \begin{bmatrix} \tilde{\boldsymbol{\theta}}_k^G \\ \tilde{\mathbf{p}}_k^G \\ \tilde{\mathbf{v}}_k^G \\ \tilde{\mathbf{b}}_{g,k} \\ \tilde{\mathbf{b}}_{a,k} \end{bmatrix} = \begin{bmatrix} \delta \boldsymbol{\theta}_I \\ \mathbf{p}_k^G - \hat{\mathbf{p}}_k^G \\ \mathbf{v}_k^G - \hat{\mathbf{v}}_k^G \\ \mathbf{b}_{g,k} - \hat{\mathbf{b}}_{g,k} \\ \mathbf{b}_{a,k} - \hat{\mathbf{b}}_{a,k} \end{bmatrix} \quad (2)$$

where

$$\delta \bar{q} = (\hat{q}_{IG})^{-1} \otimes \bar{q}_{IG} \simeq \begin{bmatrix} \frac{1}{2} \delta \boldsymbol{\theta} \\ 1 \end{bmatrix} \quad (3)$$

**MSCKF State at Time k** ( $(16 + 7N) \times 1$ )

$$\hat{\mathbf{X}}_k = \begin{bmatrix} \hat{\mathbf{X}}_{I_k} \\ \hat{q}_{C_1 G} \\ \hat{\mathbf{p}}_{C_1}^G \\ \vdots \\ \hat{q}_{C_N G} \\ \hat{\mathbf{p}}_{C_N}^G \end{bmatrix} \begin{array}{l} \leftarrow \text{IMU State estimate} \\ \leftarrow \text{Camera pose 1 orientation estimate} \\ \leftarrow \text{Camera pose 1 position estimate} \\ \vdots \\ \leftarrow \text{Camera pose N orientation estimate} \\ \leftarrow \text{Camera pose N position estimate} \end{array} \quad (4)$$

**MSCKF Error State** ( $(15 + 6N) \times 1$ )

$$\tilde{\mathbf{X}}_k = \begin{bmatrix} \tilde{\mathbf{X}}_{I_k} \\ \delta \boldsymbol{\theta}_{C_1} \\ \tilde{\mathbf{p}}_{C_1}^G \\ \vdots \\ \delta \boldsymbol{\theta}_{C_N} \\ \tilde{\mathbf{p}}_{C_N}^G \end{bmatrix} \quad (5)$$



## 2 IMU State Estimate Propagation

Integrate these with RK5 from  $t_{k-1}$  to  $t_{k-1} + T$  where  $T$  is the sampling period of the IMU:

$$\dot{\hat{q}}_{IG} = \frac{1}{2} \Omega(\omega_m - \hat{\mathbf{b}}_a - \hat{\mathbf{C}}_{IG} \omega_G) \hat{q}_{IG} \quad (6)$$

$$\dot{\hat{\mathbf{b}}}_g = \mathbf{0}_{3 \times 1} \quad (7)$$

$$\dot{\hat{\mathbf{v}}}_I^G = \hat{\mathbf{C}}_{IG}^T (\mathbf{a}_m - \hat{\mathbf{b}}_a) - 2(\omega_G^\times) \hat{\mathbf{v}}_I^G - (\omega_G^\times)^2 \hat{\mathbf{p}}_I^G + \mathbf{g}^G \quad (8)$$

$$\dot{\hat{\mathbf{b}}}_a = \mathbf{0}_{3 \times 1} \quad (9)$$

$$\dot{\hat{\mathbf{p}}}_I^G = \hat{\mathbf{v}}_I^G \quad (10)$$

$$\dot{\tilde{\mathbf{X}}}_I = \mathbf{F} \tilde{\mathbf{X}}_I + \mathbf{G} \mathbf{n}_I \quad (11)$$

where

$$\Omega(\omega) = \begin{bmatrix} -\omega^\times & \omega \\ -\omega^T & 0 \end{bmatrix} \text{ is } 4 \times 4, \quad (12)$$

$\hat{\mathbf{C}}_{IG} := \mathbf{C}(\hat{q}_{IG})$  is the rotation matrix form of the quaternion  $\hat{q}_{IG}$ ,

$$\mathbf{F} = \begin{bmatrix} -\hat{\omega}^\times & -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -\hat{\mathbf{C}}_{IG}^T \hat{\mathbf{a}}^\times & \mathbf{0}_{3 \times 3} & -2\omega_G^\times & -\hat{\mathbf{C}}_{IG}^T & -(\omega_G^\times)^2 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 15, \quad (13)$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & -\hat{\mathbf{C}}_{IG}^T & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 12, \quad (14)$$

and

$$\mathbf{n}_I = \begin{bmatrix} \mathbf{n}_g \\ \mathbf{n}_{wg} \\ \mathbf{n}_a \\ \mathbf{n}_{wa} \end{bmatrix} \begin{array}{l} \leftarrow \text{Gyro noise} \\ \leftarrow \text{Gyro-bias rate-of-change noise} \\ \leftarrow \text{Accelerometer noise} \\ \leftarrow \text{Accelerometer-bias rate-of-change noise} \end{array} \quad (15)$$

is the system noise whose covariance matrix  $\mathbf{Q}_I$  is computed offline during calibration. Note,  $\omega_G$  is the angular velocity of the spinning Earth in our global frame. The magnitude (i.e. rotational speed) is equal to  $7.292 \times 10^5 \frac{\text{rad}}{\text{s}}$ .



For the covariance,

$$\hat{\mathbf{P}}_k^- = \begin{bmatrix} \hat{\mathbf{P}}_{II,k}^- & \Phi(t_{k-1} + T, t_{k-1}) \hat{\mathbf{P}}_{IC,k-1} \\ \hat{\mathbf{P}}_{IC,k-1}^T \Phi(t_{k-1} + T, t_{k-1})^T & \hat{\mathbf{P}}_{CC,k-1} \end{bmatrix} \text{ is } (15 + 6N) \times (15 + 6N) \quad (16)$$

where:

- $\hat{\mathbf{P}}_{CC,k-1}$  is the  $6N \times 6N$  covariance matrix of the camera pose estimates (see section on State Augmentation),
- $\hat{\mathbf{P}}_{IC,k-1}$  is the correlation between the errors in the IMU state and the camera pose estimates,
- the state transition matrix  $\Phi(t_{k-1} + T, t_{k-1})$  is computed by integrating (with RK5)

$$\dot{\Phi}(t_{k-1} + \tau, t_{k-1}) = \mathbf{F}\Phi(t_{k-1} + \tau, t_{k-1}), \quad \tau \in [0, T] \quad (17)$$

with initial condition  $\Phi(t_{k-1}, t_{k-1}) = \mathbf{1}_{15}$ , and

- $\hat{\mathbf{P}}_{II,k}^-$  is obtained by integrating (with RK5)

$$\dot{\hat{\mathbf{P}}}_{II}^- = \mathbf{F}\hat{\mathbf{P}}_{II}^- + \hat{\mathbf{P}}_{II}^- \mathbf{F}^T + \mathbf{G}\mathbf{Q}_I\mathbf{G}^T \quad (18)$$

over the interval  $(t_{k-1}, t_{k-1} + T)$ .

## 3 State Augmentation

### 3.1 Camera Poses

For the  $(N + 1)^{\text{th}}$  image, the camera pose is estimated as

$$\hat{q}_{CG} = \bar{q}_{CI} \otimes \hat{q}_{IG} \quad (19)$$

$$\hat{\mathbf{p}}_C^G = \hat{\mathbf{p}}_I^G + \hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I \quad (20)$$

and the EKF covariance matrix is augmented according to

$$\hat{\mathbf{P}}_{k-1} \leftarrow \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix} \hat{\mathbf{P}}_{k-1} \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix}^T \quad (21)$$

where the Jacobian

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{C}}_{CI} & \mathbf{0}_{3 \times 9} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 6N} \\ \left( \hat{\mathbf{C}}_{IG}^T \mathbf{p}_C^I \right)^\times & \mathbf{0}_{3 \times 9} & \mathbf{1}_3 & \mathbf{0}_{3 \times 6N} \end{bmatrix} \text{ is } 6 \times (15 + 6N) \quad (22)$$



## 4 Correction Equations

**Residuals (pretend everything has a k subscript...)**

$$\mathbf{r}_n = \mathbf{T}_H \tilde{\mathbf{X}} + \mathbf{n}_n \quad (23)$$

where  $\mathbf{T}_H$  (an upper-triangular matrix) is obtained from the QR decomposition of  $\mathbf{H}_o$  ( $\mathbf{H}_x$  in the original tech report, but I think that's an error...)

$$\mathbf{H}_o = \begin{bmatrix} \mathbf{H}_o^{(1)} \\ \vdots \\ \mathbf{H}_o^{(L)} \end{bmatrix} = [\mathbf{Q}_1 \quad \mathbf{Q}_2] \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix}. \quad (24)$$

Each  $\mathbf{H}_o^{(j)}$  is the projection of  $\mathbf{H}_x^{(j)}$  onto the left nullspace of  $\mathbf{H}_f^{(j)}$

$\mathbf{T}_H$  and  $\mathbf{r}_n$  can be computed in  $O(r^2d)$  time using Givens rotations without having to form  $\mathbf{Q}_1$  explicitly. [FIGURE OUT HOW TO DO THIS]

Each  $\mathbf{H}_x^{(j)}$  is a stack of Jacobians  $\mathbf{H}_{x_i}^{(j)}$  of the  $i^{\text{th}}$  measurement of feature  $j$  with respect to the state (only the entries corresponding to pose  $i$  are non-zero):

$$\mathbf{H}_{x_i}^{(j)} = \begin{bmatrix} \mathbf{0}_{2 \times 15} & \mathbf{0}_{2 \times 6} & \dots & \mathbf{J}_i^{(j)} \left( \hat{\mathbf{X}}_{f_j}^{C_i} \right)^\times & -\mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} & \dots & \mathbf{0}_{2 \times 6} \end{bmatrix} \text{ is } 2 \times (15 + 6N) \quad (25)$$

Each  $\mathbf{H}_f^{(j)}$  is a stack of Jacobians  $\mathbf{H}_{f_i}^{(j)}$  of the  $i^{\text{th}}$  measurement of feature  $j$  with respect to the feature position:

$$\mathbf{H}_{f_i}^{(j)} = \mathbf{J}_i^{(j)} \hat{\mathbf{C}}_{C_i G} \text{ is } 2 \times 3 \quad (26)$$

In both equations,

$$\mathbf{J}_i^{(j)} = \frac{1}{\hat{Z}_j^{C_i}} \begin{bmatrix} 1 & 0 & -\frac{\hat{X}_j^{C_i}}{\hat{Z}_j^{C_i}} \\ 0 & 1 & -\frac{\hat{Y}_j^{C_i}}{\hat{Z}_j^{C_i}} \end{bmatrix} \quad (27)$$

where

$$\begin{bmatrix} \hat{X}_j^{C_i} \\ \hat{Y}_j^{C_i} \\ \hat{Z}_j^{C_i} \end{bmatrix} = \hat{\mathbf{C}}_{C_i G} \left( \hat{\mathbf{p}}_{f_j}^G - \hat{\mathbf{p}}_{C_i}^G \right) \quad (28)$$

are the 3D coordinates of feature  $j$  in the frame of image  $i$ .

$\mathbf{n}_n$  is a noise vector with covariance matrix

$$\mathbf{R}_n = \sigma_{\text{im}}^2 \mathbf{1}_r \quad (29)$$

where  $r$  is the number of columns in  $\mathbf{Q}_1$ .



### Kalman Gain

$$\mathbf{K}_k = \hat{\mathbf{P}}_k^- \mathbf{T}_{H,k}^T \left( \mathbf{T}_{H,k} \hat{\mathbf{P}}_k^- \mathbf{T}_{H,k}^T + \mathbf{R}_{n,k} \right)^{-1} \quad (30)$$

### State Vector Correction

$$\Delta \mathbf{X}_k = \mathbf{K}_k \mathbf{r}_{n,k} \quad (31)$$

### State Covariance Correction

$$\hat{\mathbf{P}}_k = (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k}) \hat{\mathbf{P}}_k^- (\mathbf{1}_{15+6N} - \mathbf{K}_k \mathbf{T}_{H,k})^T + \mathbf{K}_k \mathbf{R}_{n,k} \mathbf{K}_k^T \quad (32)$$