



1 State Parametrization

IMU State at Time k (16×1)

$$\mathbf{X}_{I_k} = \begin{bmatrix} {}^G \tilde{q}^T \\ {}^G \mathbf{p}_k \\ {}^G \mathbf{v}_k \\ \mathbf{b}_{g_k} \\ \mathbf{b}_{a_k} \end{bmatrix} \begin{array}{l} \leftarrow \text{Global to IMU rotation unit quaternion} \\ \leftarrow \text{IMU position in global frame} \\ \leftarrow \text{IMU velocity in global frame} \\ \leftarrow \text{Gyro bias} \\ \leftarrow \text{Accelerometer bias} \end{array} \quad (1)$$

IMU Error State (15×1)

$$\tilde{\mathbf{X}} = \begin{bmatrix} {}^G \tilde{\boldsymbol{\theta}} \\ {}^G \tilde{\mathbf{p}} \\ {}^G \tilde{\mathbf{v}} \\ {}^G \tilde{\mathbf{b}}_g \\ {}^G \tilde{\mathbf{b}}_a \end{bmatrix} = \begin{bmatrix} \delta \boldsymbol{\theta}_I \\ {}^G \mathbf{p} - {}^G \hat{\mathbf{p}} \\ {}^G \mathbf{p} - {}^G \hat{\mathbf{v}} \\ \mathbf{b}_g - \hat{\mathbf{b}}_g \\ \mathbf{b}_a - \hat{\mathbf{b}}_a \end{bmatrix} \quad (2)$$

where

$$\delta \tilde{q} = {}^I_G \hat{q}^{-1} \otimes {}^I_G \tilde{q} \simeq \begin{bmatrix} \frac{1}{2} \delta \boldsymbol{\theta} \\ 1 \end{bmatrix} \quad (3)$$

MSCKF State at Time k ($(16 + 7N) \times 1$)

$$\hat{\mathbf{X}}_k = \begin{bmatrix} \hat{\mathbf{X}}_{I_k} \\ {}^{C_1}_G \hat{q} \\ {}^G \hat{\mathbf{p}}_{C_1} \\ \vdots \\ {}^{C_N}_G \hat{q} \\ {}^G \hat{\mathbf{p}}_{C_N} \end{bmatrix} \begin{array}{l} \leftarrow \text{IMU State estimate} \\ \leftarrow \text{Camera pose 1 orientation estimate} \\ \leftarrow \text{Camera pose 1 position estimate} \\ \vdots \\ \leftarrow \text{Camera pose N orientation estimate} \\ \leftarrow \text{Camera pose N position estimate} \end{array} \quad (4)$$

MSCKF Error State ($(15 + 6N) \times 1$)

$$\tilde{\mathbf{X}}_k = \begin{bmatrix} \tilde{\mathbf{X}}_{I_k} \\ \delta \boldsymbol{\theta}_{C_1} \\ {}^G \tilde{\mathbf{p}}_{C_1} \\ \vdots \\ \delta \boldsymbol{\theta}_{C_N} \\ {}^G \tilde{\mathbf{p}}_{C_N} \end{bmatrix} \quad (5)$$



2 IMU State Estimate Propagation

Integrate these with RK5 from t_k to $t_k + T$ where T is the sampling period of the IMU:

$${}^I_G \dot{\hat{q}} = \frac{1}{2} \Omega \left(\omega_m - \hat{\mathbf{b}}_a - \mathbf{C} \left({}^I_G \hat{q} \right) \omega_G \right) {}^I_G \hat{q} \quad (6)$$

$$\dot{\hat{\mathbf{b}}}_g = \mathbf{0}_{3 \times 1} \quad (7)$$

$${}^G \dot{\hat{\mathbf{v}}}_I = \mathbf{C} \left({}^I_G \hat{q} \right)^T \left(\mathbf{a}_m - \hat{\mathbf{b}}_a \right) - 2 \left(\omega_G^\times \right)^G \hat{\mathbf{v}}_I - \left(\omega_G^\times \right)^2 {}^G \hat{\mathbf{p}}_I + {}^G \mathbf{g} \quad (8)$$

$$\dot{\hat{\mathbf{b}}}_a = \mathbf{0}_{3 \times 1} \quad (9)$$

$${}^G \dot{\hat{\mathbf{p}}}_I = {}^G \hat{\mathbf{v}}_I \quad (10)$$

$$\dot{\tilde{\mathbf{X}}}_I = \mathbf{F} \tilde{\mathbf{X}}_I + \mathbf{G} \mathbf{n}_I \quad (11)$$

where

$$\Omega(\omega) = \begin{bmatrix} -\omega^\times & \omega \\ -\omega^T & 0 \end{bmatrix} \text{ is } 4 \times 4, \quad (12)$$

$\mathbf{C}(\bar{q})$ is the rotation matrix form of the quaternion \bar{q} ,

$$\mathbf{F} = \begin{bmatrix} -\hat{\omega}^\times & -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -\mathbf{C} \left({}^I_G \hat{q} \right)^T \hat{\mathbf{a}}^\times & \mathbf{0}_{3 \times 3} & -2\omega_G^\times & -\mathbf{C} \left({}^I_G \hat{q} \right)^T & -\left(\omega_G^\times \right)^2 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 15, \quad (13)$$

$$\mathbf{G} = \begin{bmatrix} -\mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{1}_3 & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & -\mathbf{C} \left({}^I_G \hat{q} \right) & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{1}_3 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \text{ is } 15 \times 12, \quad (14)$$

and

$$\mathbf{n}_I = \begin{bmatrix} \mathbf{n}_g \\ \mathbf{n}_{wg} \\ \mathbf{n}_a \\ \mathbf{n}_{wa} \end{bmatrix} \quad (15)$$

is the system noise whose covariance matrix \mathbf{Q}_I is computed offline during calibration. Note, ω_G is the angular velocity of the spinning Earth in our global frame. The magnitude (i.e. rotational speed) is equal to $7.292 \times 10^5 \frac{\text{rad}}{\text{s}}$.



For the covariance,

$$\mathbf{P}_{k+1|k} = \begin{bmatrix} \mathbf{P}_{II_{k+1|k}} & \Phi(t_k + T, t_k) \mathbf{P}_{IC_{k|k}} \\ \mathbf{P}_{IC_{k|k}}^T \Phi(t_k + T, t_k)^T & \mathbf{P}_{CC_{k|k}} \end{bmatrix} \text{ is } (15 + 6N) \times (15 + 6N) \quad (16)$$

where:

- $\mathbf{P}_{CC_{k|k}}$ is the $6N \times 6N$ covariance matrix of the camera pose estimates (see section on State Augmentation),
- $\mathbf{P}_{IC_{k|k}}$ is the correlation between the errors in the IMU state and the camera pose estimates,
- the state transition matrix $\Phi(t_k + T, t_k)$ is computed by integrating (with RK5)

$$\dot{\Phi}(t_k + \tau, t_k) = \mathbf{F}\Phi(t_k + \tau, t_k), \quad \tau \in [0, T] \quad (17)$$

with initial condition $\Phi(t_k, t_k) = \mathbf{1}_{15}$, and

- $\mathbf{P}_{II_{k+1|k}}$ is obtained by integrating (with RK5)

$$\dot{\mathbf{P}}_{II} = \mathbf{F}\mathbf{P}_{II} + \mathbf{P}_{II}\mathbf{F}^T + \mathbf{G}\mathbf{Q}_I\mathbf{G}^T \quad (18)$$

over the interval $(t_k, t_k + T)$.

3 State Augmentation

3.1 Camera Poses

For the $(N + 1)^{\text{th}}$ image, the camera pose is estimated as

$${}^C_G \hat{q} = {}^C_I \bar{q} \otimes {}^I_G \hat{q} \quad (19)$$

$${}^G \hat{\mathbf{p}}_C = {}^G \hat{\mathbf{p}}_I + \mathbf{C} ({}^I_G \hat{q})^T {}^I \mathbf{p}_C \quad (20)$$

and the EKF covariance matrix is augmented according to

$$\mathbf{P}_{k|k} \leftarrow \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix} \mathbf{P}_{k|k} \begin{bmatrix} \mathbf{1}_{15+6N} \\ \mathbf{J} \end{bmatrix}^T \quad (21)$$

where the Jacobian

$$\mathbf{J} = \begin{bmatrix} \mathbf{C} ({}^C_I \hat{q}) & \mathbf{0}_{3 \times 9} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 6N} \\ \left(\mathbf{C} ({}^I_G \hat{q})^T {}^I \mathbf{p}_C \right)^\times & \mathbf{0}_{3 \times 9} & \mathbf{I}_3 & \mathbf{0}_{3 \times 6N} \end{bmatrix} \text{ is } 6 \times (15 + 6N) \quad (22)$$



4 Correction Equations

Residuals

$$\mathbf{r}_n = \mathbf{T}_H \tilde{\mathbf{X}} + \mathbf{n}_n \quad (23)$$

where \mathbf{T}_H (an upper-triangular matrix) is obtained from the QR decomposition of \mathbf{H}_o (\mathbf{H}_x in the original tech report, but I think that's an error...)

$$\mathbf{H}_o = \begin{bmatrix} \mathbf{H}_o^{(1)} \\ \vdots \\ \mathbf{H}_o^{(L)} \end{bmatrix} = [\mathbf{Q}_1 \quad \mathbf{Q}_2] \begin{bmatrix} \mathbf{T}_H \\ \mathbf{0} \end{bmatrix}. \quad (24)$$

Each $\mathbf{H}_o^{(j)}$ is the projection of $\mathbf{H}_x^{(j)}$ onto the left nullspace of $\mathbf{H}_f^{(j)}$

\mathbf{T}_H and \mathbf{r}_n can be computed in $O(r^2 d)$ time using Givens rotations without having to form \mathbf{Q}_1 explicitly. [FIGURE OUT HOW TO DO THIS]

Each $\mathbf{H}_x^{(j)}$ is a stack of Jacobians $\mathbf{H}_{x_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the state (only the entries corresponding to pose i are non-zero):

$$\mathbf{H}_{x_i}^{(j)} = \begin{bmatrix} \mathbf{0}_{2 \times 15} & \mathbf{0}_{2 \times 6} & \dots & \mathbf{J}_i^{(j)} \left(C_i \hat{\mathbf{X}}_{f_j} \right)^\times & -\mathbf{J}_i^{(j)} \mathbf{C} \left(\begin{smallmatrix} C_i \\ G \end{smallmatrix} \hat{\mathbf{q}} \right) & \dots & \mathbf{0}_{2 \times 6} \end{bmatrix} \text{ is } 2 \times (15 + 6N) \quad (25)$$

Each $\mathbf{H}_f^{(j)}$ is a stack of Jacobians $\mathbf{H}_{f_i}^{(j)}$ of the i^{th} measurement of feature j with respect to the feature position:

$$\mathbf{H}_{f_i}^{(j)} = \mathbf{J}_i^{(j)} \mathbf{C} \left(\begin{smallmatrix} C_i \\ G \end{smallmatrix} \hat{\mathbf{q}} \right) \text{ is } 2 \times 3 \quad (26)$$

In both equations,

$$\mathbf{J}_i^{(j)} = \frac{1}{C_i \hat{Z}_j} \begin{bmatrix} 1 & 0 & -\frac{C_i \hat{X}_j}{C_i \hat{Z}_j} \\ 0 & 1 & -\frac{C_i \hat{Y}_j}{C_i \hat{Z}_j} \end{bmatrix} \quad (27)$$

where

$$\begin{bmatrix} C_i \hat{X}_j \\ C_i \hat{Y}_j \\ C_i \hat{Z}_j \end{bmatrix} = \mathbf{C} \left(\begin{smallmatrix} C_i \\ G \end{smallmatrix} \hat{\mathbf{q}} \right) ({}^G \hat{\mathbf{p}}_{f_j} - {}^G \hat{\mathbf{p}}_{C_i}) \quad (28)$$

are the 3D coordinates of feature j in the frame of image i .

\mathbf{n}_n is a noise vector with covariance matrix

$$\mathbf{R}_n = \sigma_{\text{im}}^2 \mathbf{1}_r \quad (29)$$

where r is the number of columns in \mathbf{Q}_1 .



Kalman Gain

$$\mathbf{K} = \mathbf{P}_{k+1|k} \mathbf{T}_H^T (\mathbf{T}_H \mathbf{P}_{k+1|k} \mathbf{T}_H^T + \mathbf{R}_n)^{-1} \quad (30)$$

State Vector Correction

$$\Delta \mathbf{X} = \mathbf{K} \mathbf{r}_n \quad (31)$$

State Covariance Correction

$$\mathbf{P}_{k+1|k+1} = (\mathbf{1}_{15+6N} - \mathbf{K} \mathbf{T}_H) \mathbf{P}_{k+1|k} (\mathbf{1}_{15+6N} - \mathbf{K} \mathbf{T}_H)^T + \mathbf{K} \mathbf{R}_n \mathbf{K}^T \quad (32)$$