The Class PSPACE and Game Theory

valentinpi

5. Januar 2021 (neueste Version)

Proseminar Theoretische Informatik WiSe 2020-21
Institut für Informatik
Freie Universität Berlin

Space Complexity

Space Complexity

Definition
Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. The complexity classes *SPACE* and *NSPACE* of f are:

Definition
Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. The complexity classes *SPACE* and *NSPACE* of f are:

$$\begin{aligned}\mathrm{SPACE}(f(n)) := \{\ L \mid\ & L \text{ is decided by a deterministic turing machine} \\ & \text{using } \mathcal{O}(f(n)) \text{ space } \}\end{aligned}$$

Definition
Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. The complexity classes *SPACE* and *NSPACE* of f are:

$$\text{SPACE}(f(n)) := \{\ L \mid L \text{ is decided by a deterministic turing machine using } \mathcal{O}(f(n)) \text{ space}\ \}$$

$$\text{NSPACE}(f(n)) := \{\ L \mid L \text{ is decided by a nondeterministic turing machine using } \mathcal{O}(f(n)) \text{ space}\ \}$$

PSPACE and NPSPACE

PSPACE and NPSPACE

Definition
In analogy to the classes P and NP, *PSPACE* and *NPSPACE* are defined as:

$$\mathrm{PSPACE} := \bigcup_{k \in \mathbb{N}} \mathrm{SPACE}(n^k) \qquad \mathrm{NPSPACE} := \bigcup_{k \in \mathbb{N}} \mathrm{NSPACE}(n^k)$$

Another inequality?

Relations between PSPACE and NPSPACE?

Another inequality?

Relations between PSPACE and NPSPACE?

Surprising result (Savitch):

Fact (Corollary from Savitchs Theorem)
PSPACE = NPSPACE

PSPACE-Completeness

PSPACE-Completeness

Definition
A language B is *PSPACE-complete*, if:

1. $B \in \text{PSPACE}$
2. $A \leq_p B$ for all $A \in \text{PSPACE}$, also called *PSPACE-hard*

Effects on the Landscape

Fact
The following holds:

Effects on the Landscape

Fact
The following holds:

$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXPTIME} := \bigcup_{k \in \mathbb{N}} \mathsf{TIME}\left(2^{n^k}\right)$$

$$\mathsf{P} \neq \mathsf{EXPTIME}$$

Quantified Boolean Formulas

Quantified Boolean Formulas

Generalization of SAT.

Prerequisites:

Quantified Boolean Formulas

Generalization of SAT.

Prerequisites:

- *Prenex Normal Form*: $Q_1 x_1 Q_2 x_2 ... Q_n x_n : \phi$ with $Q_1, ..., Q_n \in \{\forall, \exists\}$

Quantified Boolean Formulas

Generalization of SAT.

Prerequisites:

- *Prenex Normal Form*: $Q_1 x_1 Q_2 x_2 ... Q_n x_n : \phi$ with $Q_1, ..., Q_n \in \{\forall, \exists\}$
- *Fully Quantified Boolean Formulas (Statements)*

The TQBF Problem

$$\text{TQBF} := \{ \langle \phi \rangle \mid \phi \text{ is a true fully quantified boolean formula} \}$$

TQBF is PSPACE-Complete

Theorem
TQBF is PSPACE-complete.

Proof.

TQBF is PSPACE-Complete

Theorem
TQBF is PSPACE-complete.

Proof.

TQBF $\in$ PSPACE:

Input: $\langle \phi \rangle$, where ϕ is a fully quantified boolean formula.

Function:

```
if φ = ψ, no quantifiers then
    Evaluate ψ, accept or reject depending on result.
else if φ = ∃x: ψ then
    Recursive call on ψ with x = true, x = false.
    If either accepts, accept. Otherwise, reject.
else φ = ∀x: ψ
    Recursive call on ψ with x = true, x = false.
    If both accept, accept. Otherwise, reject.
```

TQBF is PSPACE-Complete

Theorem
TQBF is PSPACE-complete.

Proof.

TQBF is PSPACE-Complete

Theorem
TQBF is PSPACE-complete.

Proof.

TQBF is PSPACE-hard. (Proof idea)

Use Cook-Levin?

$$\text{Machine} \longmapsto \text{Fully Quantified Boolean Formula}$$

TQBF is PSPACE-Complete

Theorem
TQBF is PSPACE-complete.

Proof.

TQBF is PSPACE-hard. (Proof idea)

Use Cook-Levin?

$$\text{Machine} \longmapsto \text{Fully Quantified Boolean Formula}$$

Machine may run in exponential time $\Rightarrow$ Exponential sized result

TQBF is PSPACE-Complete

Theorem
TQBF is PSPACE-complete.

Proof.

TQBF is PSPACE-hard. (Proof idea)

Use Cook-Levin?

$$\text{Machine} \longmapsto \text{Fully Quantified Boolean Formula}$$

Machine may run in exponential time $\Rightarrow$ Exponential sized result

$\Rightarrow$ Shorten Formula using $\forall$ (Savitchs Theorem) ■

Proving more PSPACE-Completeness

From NP-completeness:

Proving more PSPACE-Completeness

From NP-completeness:

Corollary
If B is PSPACE-complete and $B \leq_p C$ for $C \in$ PSPACE, then C is PSPACE-complete.

Games

$$\underbrace{\text{Player I} \rightarrow \text{Goals} \leftarrow \text{Player II}}_{\text{Rules}}$$

Games

$$\underbrace{\text{Player I} \rightarrow \text{Goals} \leftarrow \text{Player II}}_{\text{Rules}}$$

Game: *Formula game* between players A and E.

Games

$$\underbrace{\text{Player I} \to \text{Goals} \leftarrow \text{Player II}}_{\text{Rules}}$$

Game: *Formula game* between players A and E.

Want: *Winning strategy*

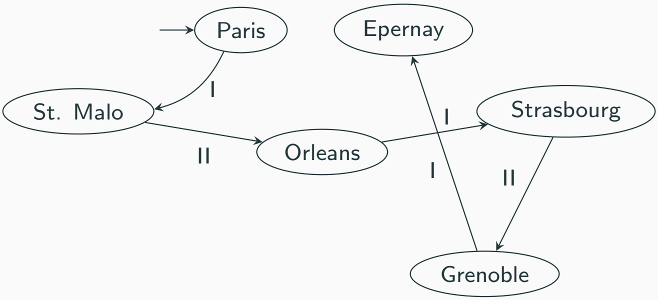
Equality of Problems

Equality of Problems

Corollary
FORMULA-GAME = TQBF and FORMULA-GAME is PSPACE-complete.

Geography and the World

A Game of Geography on France



Generalized Geography

Generalize for any directed graph:

$$\text{GG} := \{\ \langle G, s \rangle \mid \text{Player I has a winning strategy for the GG game played on } G\text{, starting from } s\ \}$$

GG is PSPACE-Complete

Theorem
GG is PSPACE-complete.

Proof.
FORMULA-GAME $\leq_p$ GG

GG $\in$ PSPACE

Input: $\langle G, s \rangle$, where G is a directed graph and s a node from G.

Function:

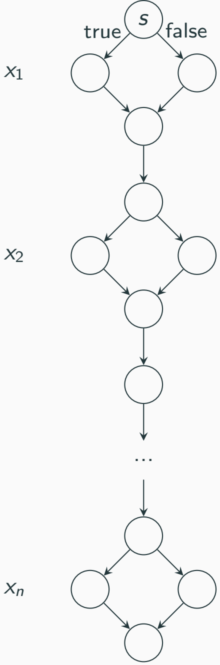
1: **if** $\deg s = 0$ **then**
2: Reject, since Player I instantly looses.
3: Remove s and its edges from G for a new graph G'.
4: For the outgoing neighbors of s, $s_1, ..., s_k$, do a recursive call on $\langle G', s_i \rangle$, $i \in \{1, ..., k\}$.
5: If all accept, reject. Otherwise, accept.

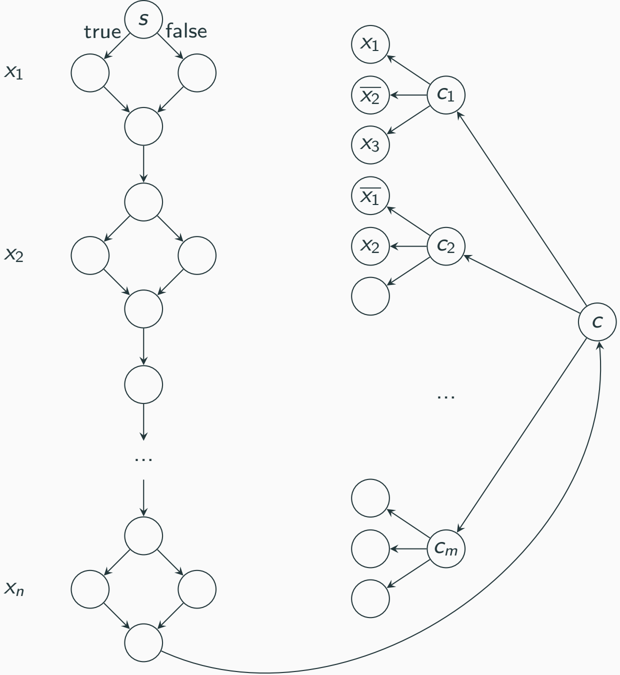
GG is PSPACE-Complete

Theorem
GG is PSPACE-complete.

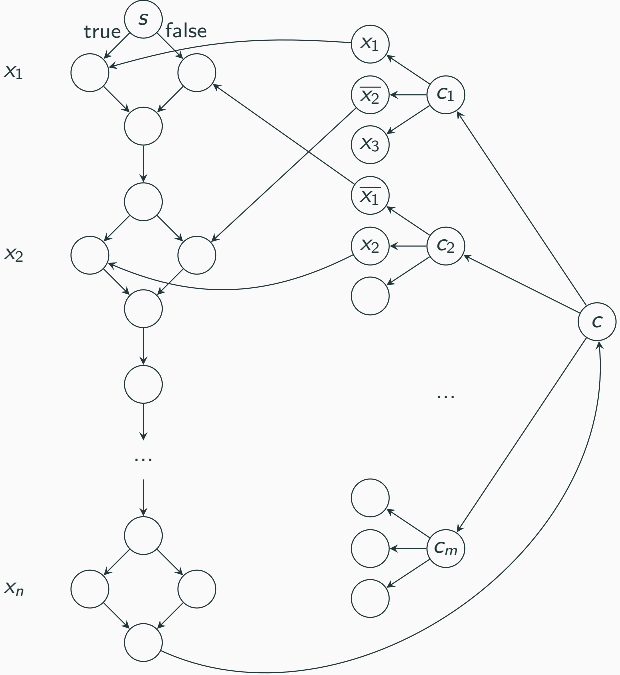
Proof.
FORMULA-GAME $\leq_p$ GG

$$\phi = \exists x_1 \forall x_2 \exists x_3 ... \exists x_n \colon \psi \mapsto (G, s)$$


s
true
false
x_1
x_2
...
x_n



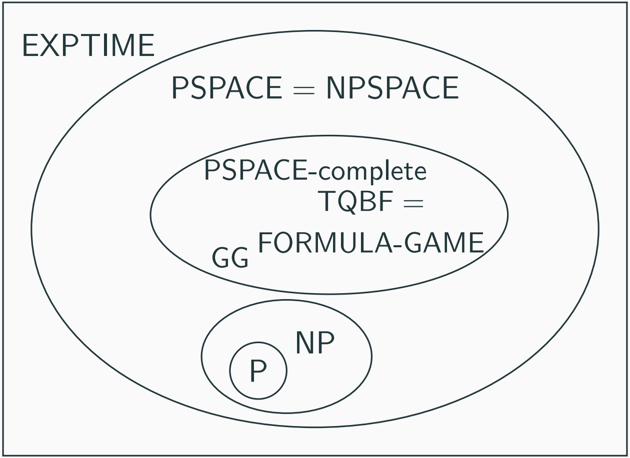
s
true
false
x_1
x_2
...
x_n
x_1
$\overline{x_2}$
x_3
c_1
$\overline{x_1}$
x_2
c_2
c
...
c_m


s
true
false
x_1
x_2
x_n
…
x_1
$\overline{x_2}$
x_3
c_1
$\overline{x_1}$
x_2
c_2
c
…
c_m

Resources

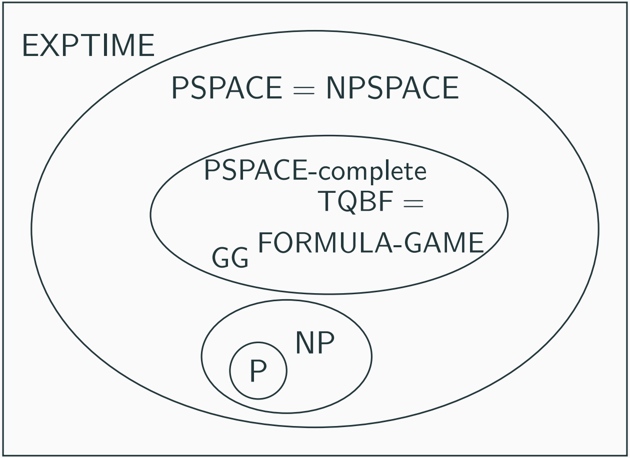
Another New Landscape

Another New Landscape



(Assuming relationships are proper)

Another New Landscape



(Assuming relationships are proper)

Questions?