The Complexity Class NP and NP-Completeness

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Nondeterminism

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Remark
The *nondeterministic* definition of the Turing machine mainly differs in the state transitioning function $\delta\colon Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{ L, N, R \})$.

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Definition
In analogy to the class TIME, the class *NTIME* of $f : \mathbb{N} \to \mathbb{R}^{+}$ is defined as:

$$\mathrm{NTIME}(f(n)) := \{ L \mid \text{There is a nondeterministic TM that decides } L \text{ in } \mathcal{O}(f(n)) \text{ time for an input of size } n \}$$

Verifiers

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An algorithm V is called a *verifier* for a language L, if:

$$L = \{ \ w \mid \exists \ c \in \Sigma^* \colon V \text{ accepts } \langle w, c \rangle \ \}$$

V is called a *polynomial verifier*, if it runs in polynomial time in terms of $|w|$. L is called *polynomially verifiable*, if such a polynomial verifier exists. c is called *certificate*.

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Example: Path for the PATH problem

Machines and Verifiers

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Machines

Verifiers

Machines and Verifiers

$$\text{Machines} \xrightarrow{\text{Guess certificate}} \text{Verifiers}$$

Machines and Verifiers

$$\begin{tikzcd}
\text{Machines} \arrow[r, bend left, "\text{Guess certificate}"] & \text{Verifiers} \arrow[l, bend left, "\text{Use certificate as step description}"]
\end{tikzcd}$$

NP and Equivalence of Concepts

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The class *NP* is the class of languages computable by nondeterministic turing machines in polynomial time.

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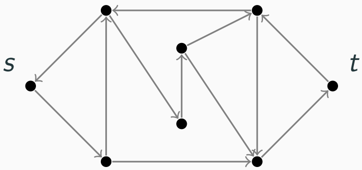
$$\mathrm{NP} := \bigcup_{k \in \mathbb{N}} \mathrm{NTIME}(n^k)$$

Corollary
$\mathrm{NP} = \{\ L \mid L \text{ has a polynomial time verifier }\}$

The HAMPATH Problem

$$\text{HAMPATH} := \{\ \langle G, s, t \rangle \mid \text{There is a hamiltonian path between } s \text{ and } t\ \}$$



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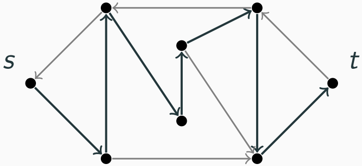

HAMPATH $\in$ NP

Theorem
HAMPATH $\in$ NP

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Proof.

Input: $\langle G, s, t \rangle$, where $G = (V, E)$ is a directed graph with nodes s, t.

Function:

1: Index the nodes from 1 to $m := |V|$ (number of nodes). Nondeterministically write a list of m numbers $p_1, ..., p_m$, where $p_i \in \{1, ..., m\}$.
2: Check for repetitions. If any are found, reject.
3: Check if p_1, p_m correspond to the node indices of s, t. If not, reject.
4: **for** each $1 \le i \le m - 1$ **do**
5: Check if $(p_i, p_{i+1}) \in E$. If not, reject.
6: Accept.

$\blacksquare$

The SUBSET-SUM Problem

$$\text{SUBSET-SUM} := \{\ \langle S, t\rangle \mid S = \{x_1, ..., x_n\} \wedge \\ \exists\ \{y_1, ..., y_k\} \subseteq S \colon \sum y_i = t\ \}$$

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Proof.
Construct a polynomial time verifier.

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SUBSET-SUM $\in$ NP

Proof.
Construct a polynomial time verifier.

Input: $\langle\langle S, t\rangle, C\rangle$ with S, C finite set of numbers and c number.

Function:

1: Test if $\sum c_i = t$ with $C = \{c_1, ..., c_k\}$.
2: Test if $C \subseteq S$.
3: Accept if both pass, otherwise reject at one stage.

∎

Additional NP Problems

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$$\text{COMPOSITES} = \{ \langle n \rangle \mid n = pq \text{ with } p, q > 1 \text{ natural numbers} \}$$

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$$\text{CLIQUE} = \{ \langle G, k \rangle \mid G \text{ is an undirected graph with a } k\text{-clique} \}$$

Polynomial Time Reducibility

Definition
A language A is *polynomial time reducible* to a language B, written $A \leq_p B$, if there is a polynomial time computable function $f : \Sigma^* \to \Sigma^*$, where for every $w \in \Sigma^*$:

$$w \in A \leftrightarrow f(w) \in B$$

SAT, 3SAT

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$$\text{3SAT} = \{ \langle \phi \rangle \mid \phi \text{ is a satisfiable 3cnf-formula } \}$$

3SAT $\leq_p$ CLIQUE

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3SAT $\leq_p$ CLIQUE

Proof.
Present reduction as function f: Generate an undirected graph G out from the input 3cnf-formula ϕ given. Let ϕ have k clausels.

- Organize clausels in triples $t_1, ..., t_k$
- Connect nodes if not from the same triple
- Or if not contradicting (e.g. x and $\neg x$ are not connected)

∎

Example for the Construction

$$\phi = (x_1 \vee \overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_2}) \wedge (\overline{x_2} \vee x_1 \vee \overline{x_2})$$

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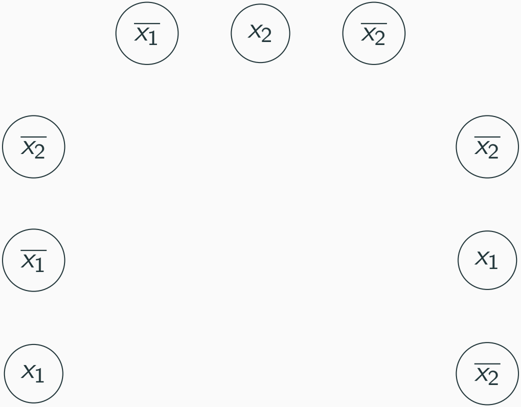

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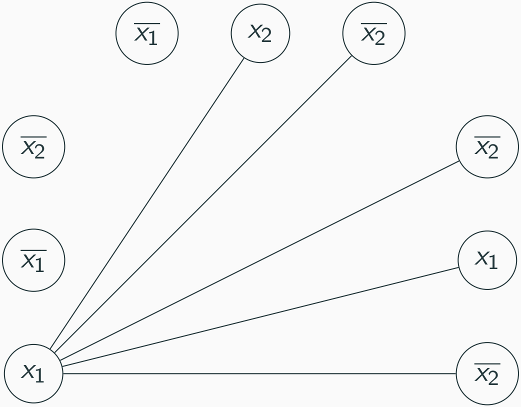

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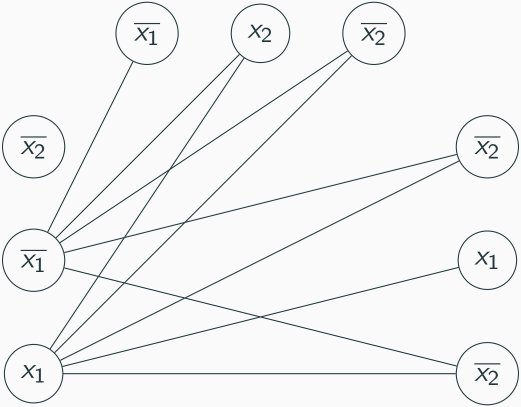

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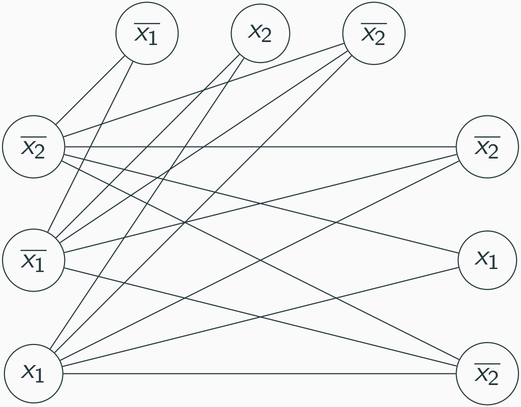
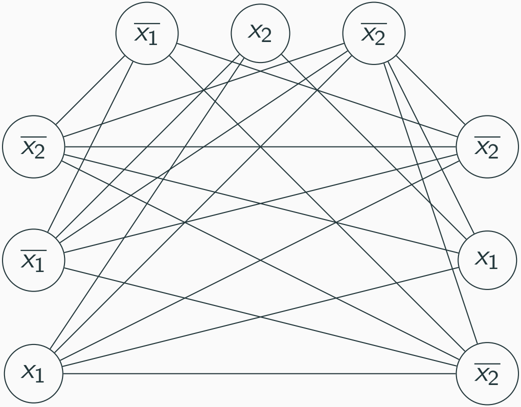


Example for the Construction

$$\phi = (x_1 \lor \overline{x_1} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor \overline{x_2}) \land (\overline{x_2} \lor x_1 \lor \overline{x_2})$$



NP-Completeness

Observation: There is some kind of link between the complexities of problems.

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Definition
A language B is called *NP-complete*, if it satisfies two conditions:

1. $B \in \mathrm{NP}$
2. $A \leq_p B$ for every language $A \in \mathrm{NP}$, also called *NP-hard*

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Fact (Cook-Levin Theorem)
SAT is NP-complete

Proof idea.

Colloraries

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Corollary
If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

Colloraries

Corollary
If B is NP-complete and $B \in$ P, then P $=$ NP.

Corollary
If B is NP-complete and $B \leq_p C$ for $C \in$ NP, then C is NP-complete.

Colloraries

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If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

Corollary
If B is NP-complete and $B \leq_p C$ for $C \in \mathrm{NP}$, then C is NP-complete.

Corollary
CLIQUE is NP-complete.

Resources

Michael Sipser.
Introduction to the Theory of Computation, Third Edition.
Cengage Learning, 2012.

The New Landscape

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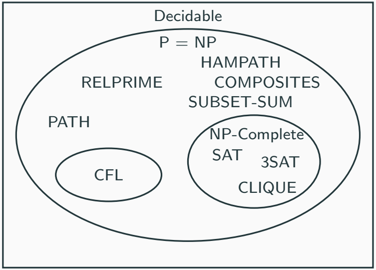

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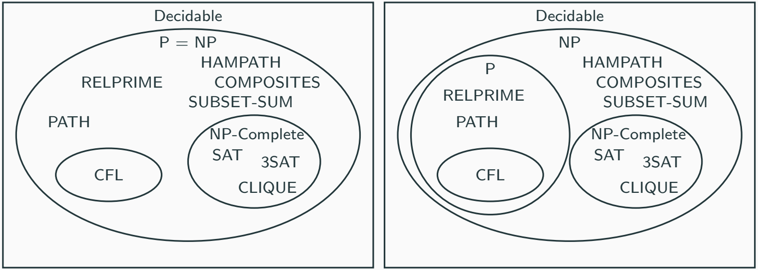

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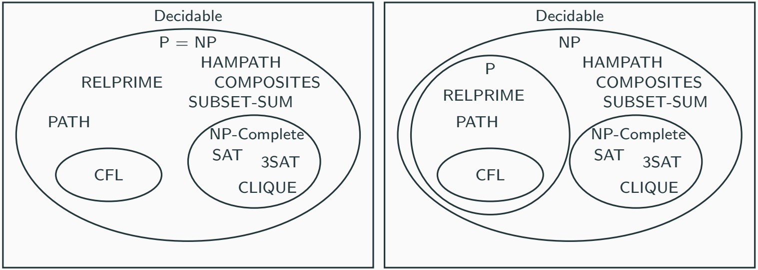

Questions?