Overview of reduced-rank regression with dense and sparse coefficients
Supplementary material

July 18, 2024

1 Simulation study: basic definitions and settings

The performance of the proposed method **CCSRRR** is assessed by conducting a simulation study and comparing it against several other similar estimators. The estimation, the prediction and the variable selection performances will be compared using the following metrics. The prediction performance is assessed by the mean squared prediction error (MSPE) calculated as follows

$$MSPE = ||\boldsymbol{Y}_t - \hat{\boldsymbol{Y}}_t||_F^2/n_t q \tag{1}$$

where $\hat{\boldsymbol{Y}}_t = \boldsymbol{X}_t\boldsymbol{C}$ is the prediction on the test sample, n_t is the size of the test sample and $||.||_F$ represents the Frobenius norm. The estimation accuracy of the coefficient matrix C is measured by

$$\Delta(\boldsymbol{C}) = ||\boldsymbol{C} - \hat{\boldsymbol{C}}||_F^2/pq. \tag{2}$$

The level of sparseness achieved by the methods is given by the ratio of nonzero elements in $\boldsymbol{C}$ to the number of all elements of $\boldsymbol{C}$: $p \times q$ and is denoted by Nnz. The performance of the variable selection is evaluated using true positive rate (TPR) and false positive rate (FPR). TPR is the ratio of truly important variables that the method selects as important while FPR is the ratio of unimportant predictors that the method selects as important. Using the notation in Table 1, TPR and FPR are defined as follows:

$$\begin{aligned} TPR &= \frac{TP}{TP+FN} \\ FPR &= \frac{FP}{FP+TN} \end{aligned}$$

TPR values close to one (many of the truly important variables are selected) and FPR values close to zero (only few non important variables are selected) indicate a better variable selection performance.

	Actual (generated)	
Predicted	Important variables (s)	Non-important variables ($p - s$)
Important (selected) ($\hat{s}$)	True Positives (TP)	False Positives (FP)
Non-important (not selected) ($p - \hat{s}$)	False Negatives (FN)	True Negatives (TN)

Table 1

In all simulation settings, to minimize the effect of parameter tuning, a large test set (with 10,000 observations) will be generated as it was proposed in a similar situation by Bunea et al. (2012). The simulation process is repeat 50 times and the results are reported in tables and visualized by boxplots.

CSRRR is compared to the following sparse estimation methods for multivariate linear regression.

1. **SRRR**, Chen and Huang (2012), row-sparse reduced rank regression using adaptive group lasso as implemented in the functions `cv.srrr()` and `srrr()` in the R package **rrpack**.

2. **SARRS**, Ma et al. (2020), subspace assisted regression with row-sparsity (SARRS) method. SARRS is carried out by Algorithm 1 in Ma et al. (2020).

3. **SiER**, Luo and Qi (2017), is a signal extraction approach for sparse multivariate response regression. This method exploits the reduced rank structure by assuming there exist matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}^\top$, and seeks such $\boldsymbol{A}$ and $\boldsymbol{B}$ that lead to the best rank r approximation of the signal matrix $\boldsymbol{X}\boldsymbol{C}$. The implementation in the R package **SiER** is used and the tuning parameters are selected using the `cv.SiER` function with 5-fold cross-validation.

4. **SeCURE**, Mishra et al. (2017), is an efficient computational procedure, called sequential factor extraction via co-sparse unit-rank estimation (SeCURE), that completely bypasses the orthogonality requirements. Each latent factor is a sparse linear combination of the predictors and may influence only a subset of responses. To choose the optimal solution along the path a hybrid information criterion is used. When $p < n$ this criteria is equivalent to the familiar BIC criterion, otherwise, when $p > n$ it corresponds to the generalized information criterion proposed by Fan and Tang (2012). The algorithm does not require the rank to be specified in advance. The implementation in the R package **secure** is used.

5. **MRCE**, Rothman et al. (2010), multivariate regression with covariance estimation (MRCE) algorithm which involves penalized likelihood with simultaneous estimation of the regression coefficients and the covariance structure. The implementation we use in this simulation is from the R package **MRCE**.

6. **RSSVD**, Chen et al. (2012), is a reduced-rank regression with sparse singular value decomposition using adaptive lasso (RSSVD). The implementation in the R package **rrpack** is used.

7. **remMap**, Peng et al. (2008), is a regularized multivariate regression for identifying master predictors which does not assume the reduced rank structure and solves a penalized least squares problem with both row-wise and element-wise sparsity imposed on the coefficient matrix. The implementation in the R package **remMap** is used.

8. **SPLS**, Chung et al. (2009), identifies sparse latent components by maximizing the covariance between them and the responses with sparsity inducing penalty imposed. The implementation in the R package **spls** is used The tuning parameters are selected with 5-fold cross-validation using the function `cv.spls()`.

9. **SOFAR**, Uematsu et al. (2017), sparse orthogonal factor regression (SOFAR) via the sparse singular value decomposition with orthogonality constrained optimization as implemented in the functions `cv.sofar()` and `sofar()` in the R package **rpack**.

For a fair comparison, the tuning parameter selection methods presented in the corresponding papers are used, as well as their implementations in the listed above software.

The comparison of the methods is done under different scenarios characterized by the covariance matrix of the predictors, as well as different rank values, and signal-to-noise ratios. In the definition of the scenarios we follow the simulation done in Bunea et al. (2012) and followed in Hilafu et al. (2020). The rows of the design matrix $\boldsymbol{X}$ are i.i.d. random vectors sampled from a multivariate Gaussian distribution with zero mean vector and covariance matrix $\boldsymbol{\Sigma}$ with $\Sigma_{ij} = \rho^{|i-j|}$. The coefficient matrix $\boldsymbol{C}$ has the form

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} b\boldsymbol{B}_0\boldsymbol{B}_1 \\ \boldsymbol{0} \end{bmatrix} \tag{3}$$

where $b > 0$, $\boldsymbol{B}_0$ is an $s \times r$ matrix with elements i.i.d. random numbers from $N(0, 1)$ and $\boldsymbol{B}_1$ is an $r \times q$ matrix also with all elements i.i.d. random numbers from $N(0, 1)$. Large values of b correspond to large signal-to-noise ratios. The response matrix $\boldsymbol{Y}$ is generated as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{C} + \boldsymbol{E} \tag{4}$$

where $\boldsymbol{E}$ is an $n \times q$ matrix with all entries i.i.d. random numbers from $N(0, 1)$. The following four cases are considered defined by the size of the model (n, p, q), its rank r, the number of important variables s and the parameters ρ and b:

1. $n > p = q$: $n = 100, p = 25, q = 25, s = 15, r = 5, b = 0.2, 0.4, \rho = 0.1, 0.5, 0.9$.
2. $q < n < p$: $n = 30, p = 100, q = 10, s = 15, r = 2, b = 0.5, 1, \rho = 0.1, 0.5, 0.9$.
3. $n < p = q$: $n = 30, p = 100, q = 100, s = 15, r = 2, b = 0.5, 1, \rho = 0.1, 0.5, 0.9$.
4. $n < p < q$: $n = 30, p = 100, q = 1000, s = 15, r = 5, b = 0.5, 1, \rho = 0.1, 0.5, 0.9$.

Additional simulations can be conducted where the error term matrix has entries from a non-Gaussian distribution. Concretely, the following two additional noise distributions can be considered: $E_{ij} \sim \sqrt{\frac{3}{5}} t_5$, and $E_{ij} \sim 3U[-1, 1]$, where "$3U[-1, 1]$" refers to the sum of three uniform $[-1, 1]$ random variables, and t_ν stands for a $t-$distribution with ν degrees of freedom. The obtained results are quite similar to these obtained with Gaussian error.

2 Simulation results

Tables 2, 3, 4 and 5 and Figures 1, 2, ..., 20 report the simulation results for the scenarios where the noise matrix $\boldsymbol{E} \in R_{n \times q}$ has i.i.d. $N(0,1)$ entries.

2.1 Case 1 with Gaussian errors

Table 2 and Figures 1 to 5 report the results for case 1 ($n = 100, p = q = 25$). They are briefly summarized as follows. In terms of estimation and prediction performances, SRRR outperforms all other methods. CCSRRR performs worse for $\rho = 0.1$ and $b = 0.2$, but for the other settings it yields comparable results to the other methods. All methods perform reasonably and similarly well for $\rho = 0.1$, while the results of RSSVD, REMMAP and SPLS are particularly poor for $b = 0.4$. In terms of variable selection, all methods, including CCSRRR, produce TPR values consistently around 1, with RSSVD, SECURE, SOFAR and SRRR experiencing difficulties for $\rho = 0.9$ and $b = 0.2$. CCSRRR yields among the best FPR values in most of the combinations. On the other hand, SARRS, SIER, MRCE and REMMAP struggle with the FPR values - they hardly produce sparse solutions.

In terms of computational time, CCSRRR and SECURE are the fastest methods, far ahead of the others, with SIER being the slowest one. For this reason, times are reported for CCSRRR and SECURE only, Figure 5.

ρ	b	Method	MSPE	SD	$\Delta(\boldsymbol{C})$	SD	Nnz	SD	TPR	SD	FPR	SD	Time	SD
0.10	0.20	SRRR	1.081	0.011	0.003	0.000	0.647	0.076	1.000	0.000	0.118	0.189	0.305	0.088
		SARRS	1.109	0.014	0.004	0.001	0.996	0.028	1.000	0.000	0.990	0.071	0.754	0.063
		SIER	1.114	0.012	0.004	0.000	0.998	0.012	1.000	0.000	0.994	0.031	129.017	44.681
		SECURE	1.175	0.042	0.007	0.002	0.604	0.015	1.000	0.000	0.032	0.051	0.115	0.036
		MRCE	1.280	0.021	0.011	0.001	0.875	0.053	1.000	0.000	1.000	0.000	5.513	0.532
		RSSVD	1.169	0.030	0.007	0.001	0.610	0.017	1.000	0.000	0.038	0.057	1.305	0.020
		REMMAP	1.349	0.003	0.014	0.000	0.657	0.001	1.000	0.000	1.000	0.000	1.005	0.315
		SPLS	1.203	0.044	0.008	0.002	0.639	0.052	0.993	0.020	0.108	0.117	1.837	0.079
		SOFAR	1.183	0.040	0.007	0.002	0.620	0.032	1.000	0.000	0.078	0.098	3.457	0.151
		CCSRRR	1.402	0.285	0.016	0.011	0.519	0.050	1.000	0.000	0.000	0.000	0.003	0.010
	0.40	SRRR	1.084	0.009	0.003	0.000	0.679	0.083	1.000	0.000	0.198	0.206	0.621	0.372
		SARRS	1.109	0.015	0.004	0.001	1.000	0.000	1.000	0.000	1.000	0.000	0.741	0.041
		SIER	1.116	0.013	0.004	0.000	0.998	0.010	1.000	0.000	0.994	0.024	128.651	14.622
		SECURE	1.181	0.047	0.007	0.002	0.608	0.019	1.000	0.000	0.032	0.059	0.161	0.034
		MRCE	1.280	0.020	0.011	0.001	0.891	0.015	1.000	0.000	1.000	0.000	7.415	0.759
		RSSVD	1.136	0.022	0.005	0.001	0.609	0.018	1.000	0.000	0.034	0.056	1.305	0.028
		REMMAP	1.355	0.010	0.015	0.000	0.675	0.001	1.000	0.000	1.000	0.000	1.138	0.050
		SPLS	1.286	0.122	0.011	0.005	0.630	0.044	0.997	0.013	0.080	0.105	1.791	0.074
		SOFAR	1.131	0.030	0.005	0.001	0.607	0.018	1.000	0.000	0.026	0.053	3.294	0.179
		CCSRRR	1.173	0.065	0.007	0.003	0.551	0.016	1.000	0.000	0.000	0.000	0.002	0.006
0.50	0.20	SRRR	1.081	0.013	0.004	0.001	0.639	0.072	1.000	0.000	0.098	0.181	0.512	0.073
		SARRS	1.109	0.015	0.006	0.001	1.000	0.002	1.000	0.000	1.000	0.000	1.443	0.130
		SIER	1.115	0.012	0.006	0.001	0.995	0.013	1.000	0.000	0.988	0.033	138.159	18.652
		SECURE	1.184	0.034	0.011	0.002	0.574	0.021	1.000	0.000	0.038	0.064	0.322	0.149
		MRCE	1.269	0.038	0.017	0.003	0.790	0.119	1.000	0.000	1.000	0.000	5.107	0.555
		RSSVD	1.235	0.042	0.013	0.002	0.584	0.027	1.000	0.000	0.046	0.065	1.306	0.015
		REMMAP	1.254	0.000	0.016	0.000	0.582	0.003	1.000	0.000	0.998	0.014	1.187	0.057
		SPLS	1.220	0.044	0.013	0.003	0.672	0.107	0.963	0.060	0.236	0.217	1.810	0.070
		SOFAR	1.256	0.073	0.016	0.005	0.552	0.035	1.000	0.000	0.018	0.039	3.811	0.179
		CCSRRR	1.151	0.067	0.008	0.003	0.561	0.052	1.000	0.000	0.050	0.076	0.002	0.006
	0.40	SRRR	1.081	0.013	0.005	0.001	0.654	0.083	1.000	0.000	0.136	0.208	0.454	0.090
		SARRS	1.110	0.016	0.007	0.001	0.996	0.028	1.000	0.000	0.990	0.071	1.607	0.147
		SIER	1.116	0.015	0.006	0.001	0.998	0.008	1.000	0.000	0.996	0.020	139.917	78.122
		SECURE	1.223	0.066	0.013	0.004	0.609	0.023	1.000	0.000	0.044	0.073	0.187	0.025
		MRCE	1.396	0.628	0.022	0.027	0.867	0.018	1.000	0.000	1.000	0.000	15.690	1.861
		RSSVD	1.189	0.029	0.010	0.002	0.615	0.022	1.000	0.000	0.044	0.061	1.303	0.018
		REMMAP	1.307	0.001	0.020	0.000	0.638	0.002	1.000	0.000	1.000	0.000	0.807	0.369
		SPLS	1.372	0.120	0.026	0.009	0.622	0.065	0.981	0.038	0.084	0.142	1.810	0.059
		SOFAR	1.147	0.035	0.009	0.002	0.615	0.028	1.000	0.000	0.046	0.079	3.614	0.151
		CCSRRR	1.360	0.315	0.014	0.012	0.558	0.035	1.000	0.000	0.000	0.000	0.003	0.006
0.90	0.20	SRRR	1.088	0.018	0.024	0.006	0.646	0.071	0.933	0.059	0.214	0.147	1.709	0.484
		SARRS	1.113	0.016	0.031	0.005	0.996	0.028	1.000	0.000	0.990	0.071	6.876	0.777
		SIER	1.115	0.013	0.026	0.003	0.970	0.051	0.989	0.028	0.942	0.107	116.137	20.379
		SECURE	1.159	0.023	0.043	0.007	0.467	0.060	0.889	0.079	0.076	0.085	0.362	0.065
		MRCE	1.177	0.013	0.051	0.004	0.560	0.020	1.000	0.000	1.000	0.000	10.380	0.831
		RSSVD	1.341	0.058	0.054	0.008	0.443	0.071	0.911	0.080	0.140	0.123	1.366	0.047
		REMMAP	1.174	0.001	0.047	0.001	0.371	0.011	1.000	0.000	0.608	0.057	1.746	0.601
		SPLS	1.207	0.023	0.051	0.008	0.934	0.082	0.981	0.052	0.862	0.168	1.891	0.099
		SOFAR	1.232	0.045	0.055	0.008	0.442	0.075	0.919	0.074	0.216	0.143	4.487	0.222
		CCSRRR	1.197	0.053	0.050	0.008	0.815	0.118	0.989	0.028	0.708	0.261	0.002	0.005
	0.40	SRRR	1.082	0.010	0.022	0.003	0.635	0.042	1.000	0.000	0.088	0.106	1.160	0.172
		SARRS	1.113	0.016	0.036	0.007	1.000	0.000	1.000	0.000	1.000	0.000	6.301	0.715
		SIER	1.113	0.014	0.027	0.004	0.997	0.011	1.000	0.000	0.992	0.027	82.236	7.090
		SECURE	1.229	0.034	0.068	0.011	0.566	0.030	0.988	0.026	0.060	0.067	0.604	0.200
		MRCE	1.236	0.022	0.075	0.007	0.643	0.016	1.000	0.000	1.000	0.000	15.748	1.687
		RSSVD	1.404	0.054	0.059	0.011	0.647	0.052	1.000	0.000	0.224	0.161	1.327	0.019
		REMMAP	1.228	0.001	0.082	0.000	0.595	0.002	1.000	0.000	0.898	0.014	2.888	0.114
		SPLS	1.329	0.035	0.112	0.018	0.893	0.116	0.964	0.070	0.786	0.202	1.858	0.084
		SOFAR	1.202	0.040	0.054	0.011	0.656	0.059	0.999	0.009	0.262	0.163	4.859	0.285
		CCSRRR	1.489	0.361	0.039	0.012	0.592	0.068	1.000	0.000	0.144	0.130	0.003	0.006

Table 2: Simulation scenario 1: $n = 100, p = 25, q = 25, s = 15, r = 5$. Model evaluation based on 50 replications using various performance measures.

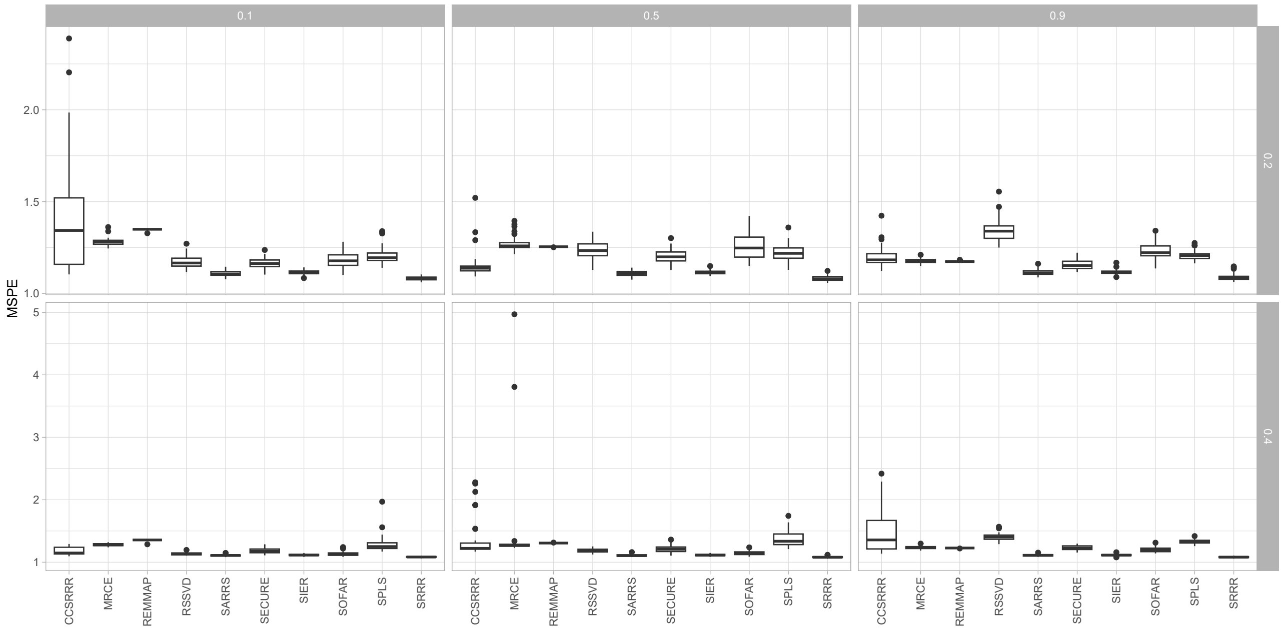

Figure 1: Simulation results for Gaussian errors under case 1: Mean squared prediction error (MSPE). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.2, 0.4$

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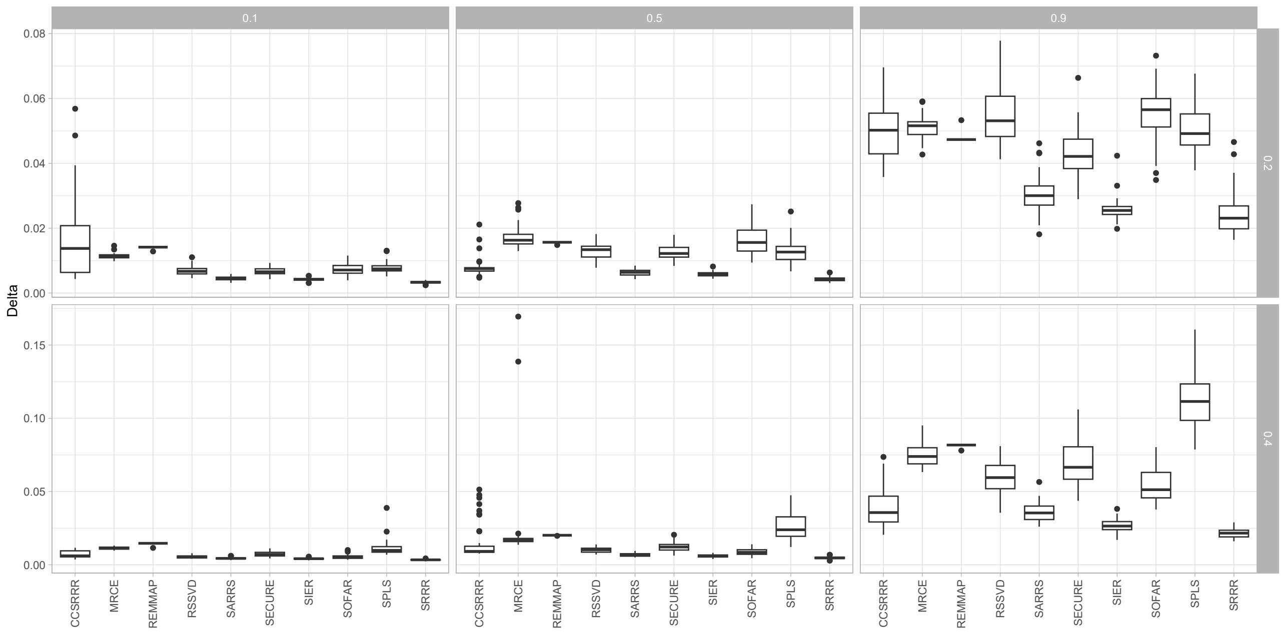

Figure 2: Simulation results for Gaussian errors under case 1: Estimation accuracy. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.2, 0.4$

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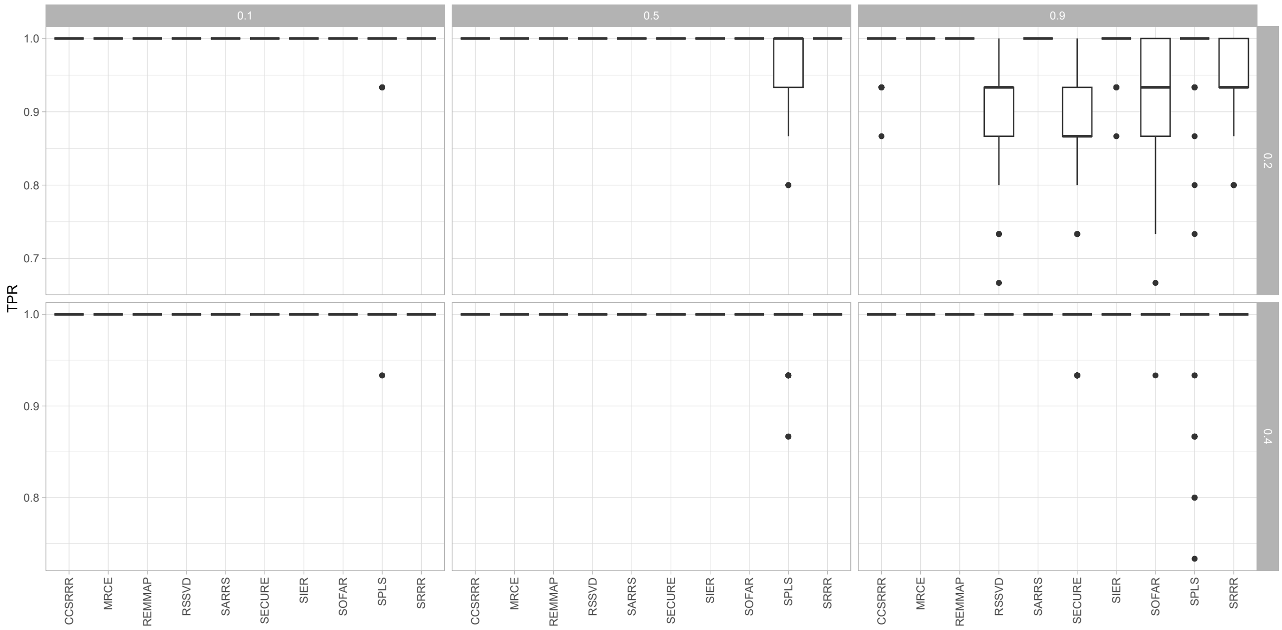

Figure 3: Simulation results for Gaussian errors under case 1: True positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.2, 0.4$. We want the TPR to be close to 1.

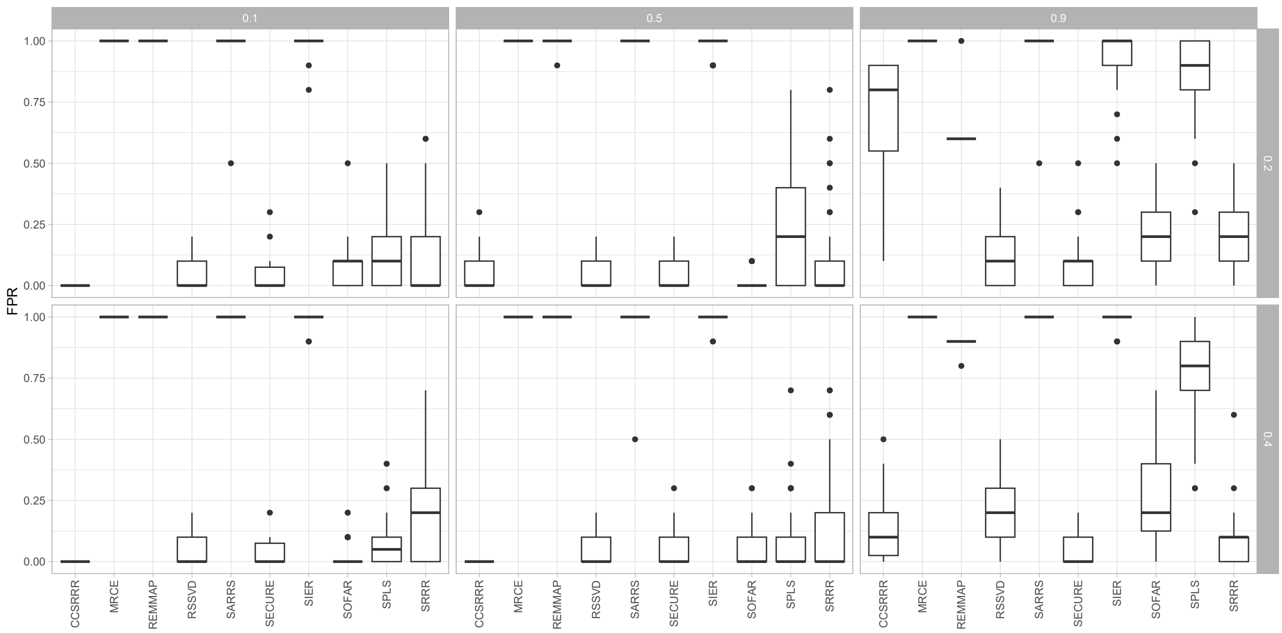

Figure 4: Simulation results for Gaussian errors under case 1: False positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.2, 0.4$. We want the FPR to be close to 0.

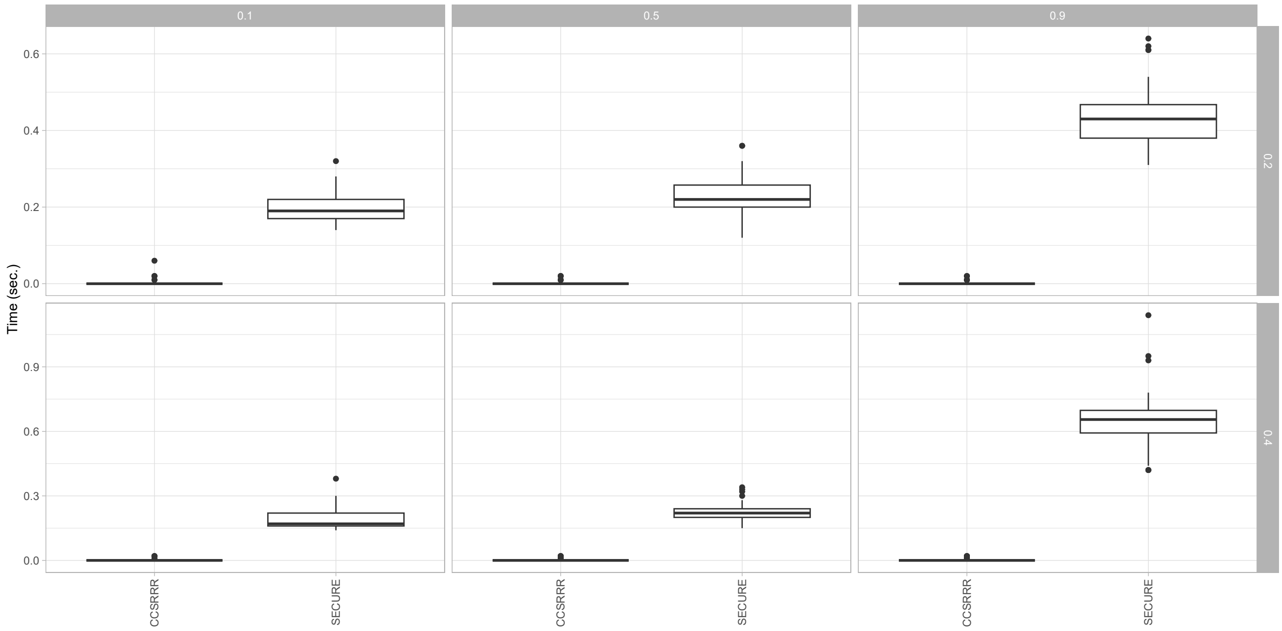

Figure 5: Simulation results for Gaussian errors under case 1: Computational time for SeCURE and CSRRR2 (all other methods are much slower). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.2, 0.4$

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2.2 Case 2 with Gaussian errors

Table 3 and Figures 6 to 10 report the results for case 2 ($n = 30, p = 100, q = 10$). They are briefly summarized as follows. In terms of estimation and prediction performances, all methods perform significantly worse than in case 1. SARRS and SIER are the best performers for $\rho = 0.1$ and $\rho = 0.5$, while SECURE and SOFAR are best for $\rho = 09$. CCSRRR performs better than REMMAP in all settings except $\rho = 0.1$ and $b = 0.5$, while it is better than most of the methods for $\rho = 0.9$.

Regarding the computational times, again CCSRRR is the fastest, followed by SECURE and MRCE being the slowest.

ρ	b	Method	MSPE	SD	$\Delta(\boldsymbol{C})$	SD	Nnz	SD	TPR	SD	FPR	SD	Time	SD
0.10	0.50	SRRR	9.305	1.132	0.085	0.011	0.986	0.098	0.996	0.028	0.985	0.110	5.047	0.524
		SARRS	6.677	2.422	0.058	0.025	0.244	0.105	0.751	0.221	0.154	0.095	0.342	0.047
		SIER	6.985	1.683	0.059	0.017	0.497	0.204	0.815	0.128	0.441	0.224	6.690	1.219
		SECURE	7.271	2.589	0.064	0.026	0.072	0.045	0.503	0.240	0.048	0.044	1.821	0.445
		MRCE	8.693	1.837	0.078	0.019	0.151	0.100	0.757	0.217	0.448	0.285	17.910	1.774
		RSSVD	7.317	1.658	0.064	0.017	1.000	0.000	1.000	0.000	1.000	0.000	1.427	0.040
		REMMAP	7.846	0.186	0.070	0.002	0.236	0.009	0.932	0.009	0.725	0.035	1.199	0.254
		SPLS	8.840	2.086	0.076	0.020	0.360	0.277	0.636	0.219	0.312	0.293	1.866	0.059
		SOFAR	6.730	1.538	0.058	0.016	0.440	0.075	0.843	0.102	0.445	0.087	6.696	1.457
		CCSRRR	9.317	0.778	0.084	0.008	0.496	0.192	0.813	0.148	0.553	0.225	0.001	0.005
	1.00	SRRR	19.119	6.492	0.182	0.066	0.805	0.284	0.917	0.133	0.785	0.313	4.693	0.616
		SARRS	15.148	4.895	0.135	0.045	0.184	0.124	0.619	0.328	0.107	0.100	0.350	0.050
		SIER	14.793	4.599	0.133	0.043	0.455	0.233	0.756	0.175	0.401	0.251	5.974	0.985
		SECURE	20.496	6.256	0.199	0.063	0.092	0.044	0.548	0.162	0.075	0.045	1.806	0.244
		MRCE	18.552	7.711	0.178	0.077	0.232	0.112	0.868	0.145	0.607	0.271	28.971	3.394
		RSSVD	20.688	5.729	0.198	0.056	1.000	0.000	1.000	0.000	1.000	0.000	1.428	0.036
		REMMAP	57.072	3.811	0.564	0.039	0.077	0.030	0.541	0.057	0.210	0.073	4.264	0.968
		SPLS	22.323	6.404	0.206	0.061	0.437	0.335	0.667	0.266	0.397	0.353	1.825	0.091
		SOFAR	20.821	4.963	0.201	0.050	0.545	0.074	0.877	0.077	0.532	0.086	6.217	0.301
		CCSRRR	29.265	2.012	0.284	0.019	0.413	0.035	0.827	0.075	0.461	0.051	0.002	0.005
0.50	0.50	SRRR	3.389	0.236	0.035	0.003	1.000	0.000	1.000	0.000	1.000	0.000	5.356	0.627
		SARRS	3.555	1.225	0.034	0.011	0.196	0.104	0.675	0.258	0.112	0.088	0.333	0.066
		SIER	4.040	0.667	0.039	0.006	0.434	0.256	0.712	0.193	0.385	0.273	7.803	1.400
		SECURE	4.310	1.069	0.040	0.010	0.033	0.018	0.416	0.107	0.019	0.019	2.164	0.326
		MRCE	3.942	1.062	0.036	0.010	0.114	0.056	0.783	0.153	0.439	0.199	16.644	1.547
		RSSVD	7.289	1.636	0.072	0.016	1.000	0.000	1.000	0.000	1.000	0.000	1.417	0.044
		REMMAP	8.236	0.516	0.111	0.002	0.081	0.001	0.665	0.009	0.142	0.005	3.149	0.893
		SPLS	3.461	0.361	0.037	0.005	0.300	0.319	0.509	0.308	0.263	0.327	1.945	0.065
		SOFAR	3.917	0.693	0.037	0.008	0.316	0.081	0.813	0.084	0.362	0.096	6.743	0.440
		CCSRRR	6.020	0.519	0.058	0.005	0.218	0.100	0.779	0.180	0.405	0.211	0.001	0.005
	1.00	SRRR	20.869	2.931	0.257	0.030	0.975	0.124	0.992	0.042	0.972	0.139	4.534	0.645
		SARRS	6.841	2.844	0.082	0.030	0.264	0.095	0.773	0.204	0.175	0.086	0.359	0.058
		SIER	8.502	2.582	0.101	0.033	0.415	0.125	0.849	0.135	0.338	0.132	7.577	7.888
		SECURE	13.747	4.597	0.175	0.057	0.076	0.042	0.495	0.113	0.063	0.045	2.049	0.309
		MRCE	11.998	5.551	0.152	0.063	0.182	0.083	0.812	0.164	0.537	0.233	20.896	2.619
		RSSVD	20.317	4.886	0.232	0.055	1.000	0.000	1.000	0.000	1.000	0.000	1.419	0.037
		REMMAP	25.523	2.700	0.387	0.018	0.131	0.011	0.861	0.038	0.326	0.022	6.304	0.462
		SPLS	16.645	5.100	0.223	0.043	0.215	0.263	0.480	0.233	0.168	0.273	1.758	0.047
		SOFAR	15.257	3.556	0.189	0.042	0.497	0.089	0.847	0.104	0.549	0.091	6.186	0.359
		CCSRRR	22.275	1.801	0.271	0.016	0.297	0.050	0.795	0.116	0.410	0.077	0.004	0.007
0.90	0.50	SRRR	2.189	0.373	0.054	0.010	0.777	0.333	0.901	0.168	0.755	0.364	5.990	0.670
		SARRS	2.341	0.440	0.054	0.008	0.182	0.045	0.759	0.121	0.081	0.042	0.253	0.016
		SIER	2.030	0.260	0.037	0.006	0.276	0.146	0.649	0.155	0.210	0.155	6.481	0.834
		SECURE	1.880	0.281	0.035	0.006	0.013	0.010	0.227	0.138	0.012	0.016	1.806	0.603
		MRCE	2.118	0.196	0.038	0.004	0.049	0.042	0.465	0.303	0.258	0.216	30.457	2.539
		RSSVD	3.898	0.416	0.093	0.011	1.000	0.000	1.000	0.000	1.000	0.000	1.652	0.082
		REMMAP	3.876	0.092	0.153	0.000	0.043	0.001	0.729	0.028	0.083	0.003	2.354	0.086
		SPLS	2.027	0.382	0.051	0.006	0.164	0.164	0.573	0.226	0.092	0.171	1.745	0.393
		SOFAR	1.783	0.217	0.030	0.005	0.095	0.080	0.543	0.217	0.170	0.148	6.075	0.420
		CCSRRR	2.099	0.176	0.034	0.003	0.332	0.108	0.872	0.101	0.476	0.170	0.002	0.005
	1.00	SRRR	5.497	1.923	0.295	0.080	0.365	0.190	0.755	0.156	0.296	0.202	5.945	0.869
		SARRS	4.490	0.584	0.196	0.014	0.119	0.033	0.603	0.084	0.034	0.033	0.242	0.015
		SIER	4.761	0.904	0.213	0.034	0.372	0.131	0.829	0.114	0.291	0.140	5.518	0.691
		SECURE	2.808	0.382	0.125	0.023	0.063	0.017	0.521	0.064	0.012	0.016	2.778	0.284
		MRCE	3.573	0.543	0.150	0.020	0.114	0.043	0.804	0.089	0.403	0.210	28.461	2.796
		RSSVD	10.477	2.260	0.330	0.058	1.000	0.000	1.000	0.000	1.000	0.000	1.590	0.042
		REMMAP	10.719	0.025	0.574	0.004	0.063	0.005	0.663	0.028	0.116	0.010	2.820	1.113
		SPLS	7.353	2.109	0.387	0.037	0.224	0.186	0.561	0.231	0.165	0.191	1.542	0.066
		SOFAR	3.173	0.563	0.123	0.019	0.325	0.089	0.823	0.074	0.329	0.120	6.901	0.418
		CCSRRR	6.164	1.301	0.153	0.010	0.170	0.039	0.728	0.080	0.136	0.049	0.002	0.006

Table 3: Simulation scenario 2: $n = 30, p = 100, q = 10, s = 15, r = 2$. Model evaluation based on 50 replications using various performance measures.

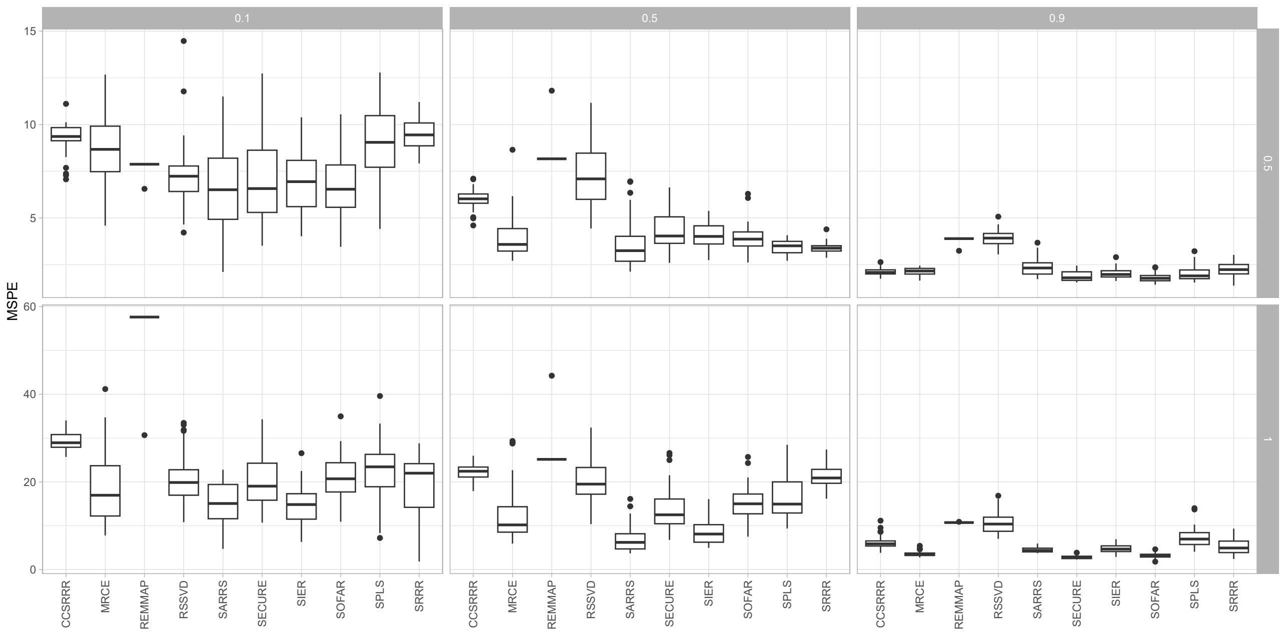

Figure 6: Simulation results for Gaussian errors under case 2: Mean squared prediction error (MSPE). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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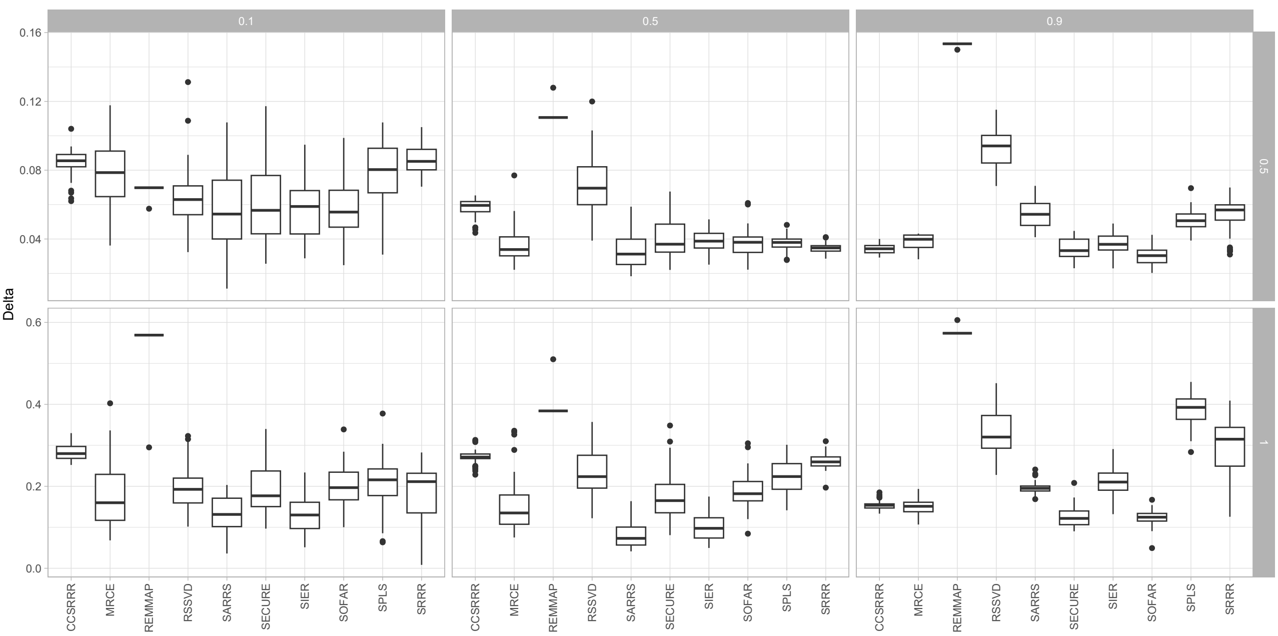

Figure 7: Simulation results for Gaussian errors under case 2: Estimation accuracy. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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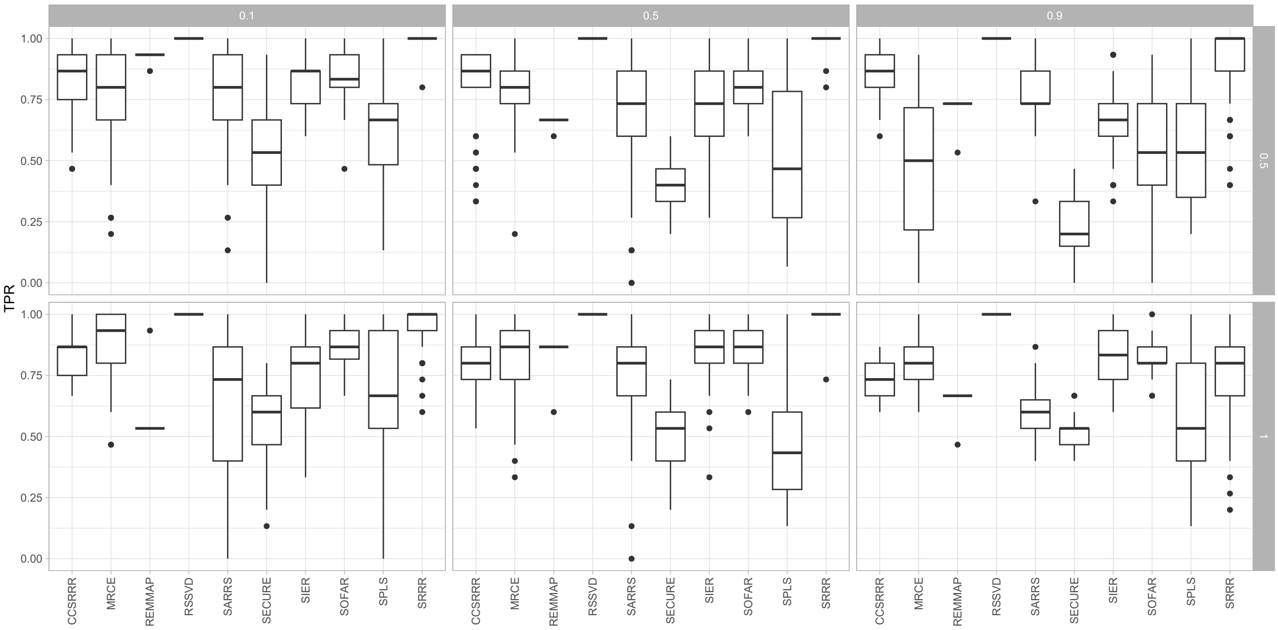

Figure 8: Simulation results for Gaussian errors under case 2: True positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$. We want the TPR to be close to 1.

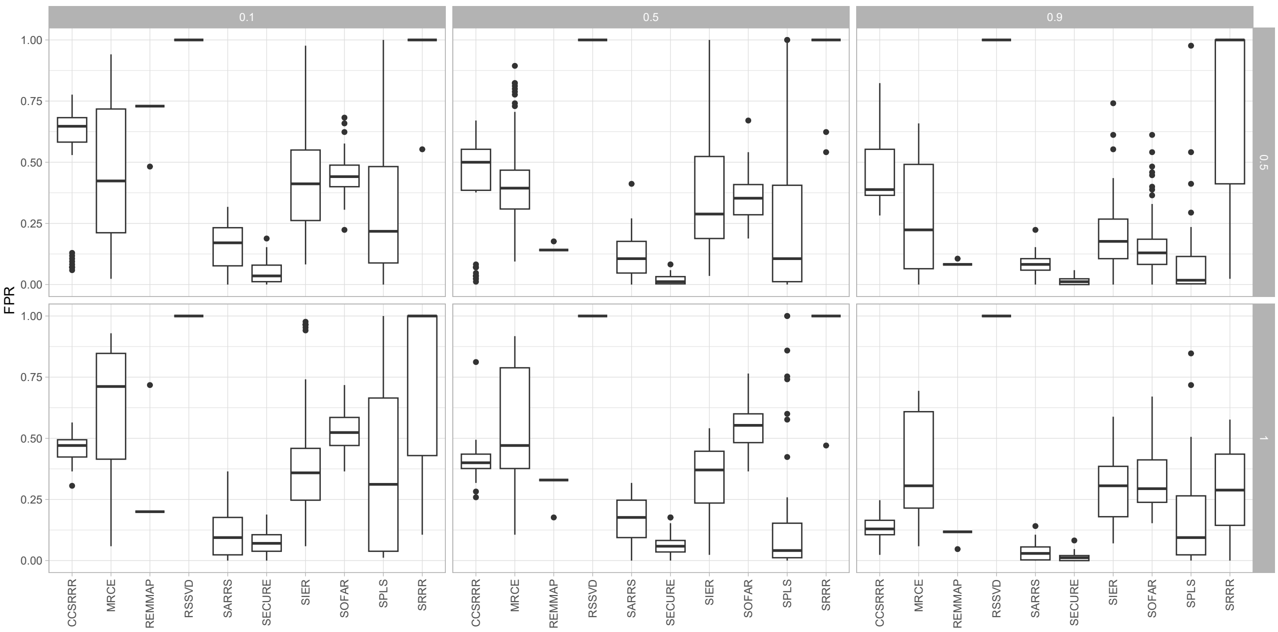

Figure 9: Simulation results for Gaussian errors under case 2: False positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$. We want the FPR to be close to 0.

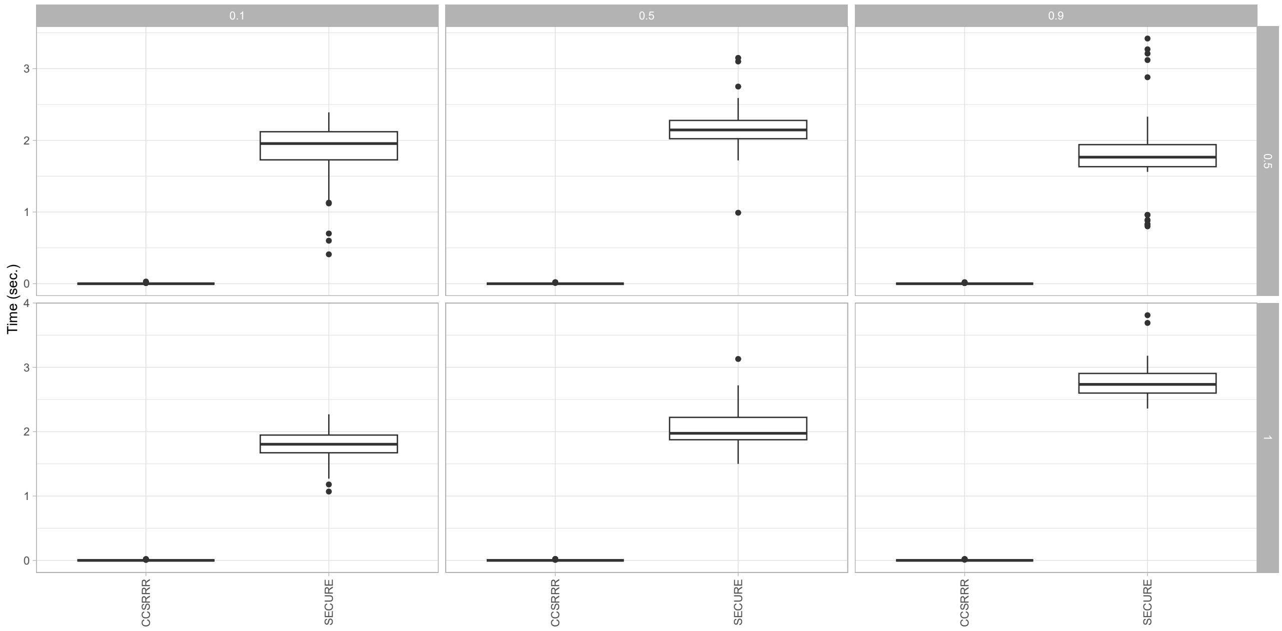

Figure 10: Simulation results for Gaussian errors under case 2: Computational time for SeCURE and CSRRR2 (all other methods are much slower). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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2.3 Case 3 with Gaussian errors

Table 4 and Figures 11 to 15 report the results for case 3 ($n = 30, p = 100, q = 100$). They are briefly summarized as follows. MRCE and REMMAP are removed from this simulation scenario, because they require too much time. The performance of the other methods in terms of estimation and prediction accuracy is similar to Case 2. CCSRRR produces pretty competitive TPR values, and rather average FPR results among the compared methods. Here, CCSRRR is again the fastest method, followed by SECURE, while the slowest one is RSSVD.

ρ	b	Method	MSPE	SD	$\Delta(\boldsymbol{C})$	SD	Nnz	SD	TPR	SD	FPR	SD	Time	SD
0.10	0.50	SRRR	6.550	0.414	0.056	0.004	1.000	0.000	1.000	0.000	1.000	0.000	5.617	0.563
		SARRS	5.141	1.724	0.042	0.017	0.256	0.139	0.721	0.303	0.174	0.120	0.412	0.054
		SIER	5.127	1.324	0.040	0.013	0.461	0.232	0.768	0.170	0.406	0.248	8.614	6.349
		SECURE	4.213	1.057	0.033	0.011	0.184	0.043	0.725	0.118	0.188	0.058	3.723	0.410
		RSSVD	5.890	1.193	0.049	0.012	1.000	0.000	1.000	0.000	1.000	0.000	40.834	1.458
		SPLS	6.220	1.285	0.051	0.013	0.265	0.243	0.535	0.216	0.217	0.254	6.938	0.232
		SOFAR	4.846	0.973	0.039	0.010	0.462	0.049	0.863	0.093	0.496	0.063	6.649	0.546
		CCSRRR	6.239	0.547	0.054	0.005	0.380	0.154	0.811	0.143	0.458	0.223	0.009	0.014
	1.00	SRRR	32.757	6.068	0.308	0.058	0.972	0.140	0.995	0.030	0.968	0.160	5.147	0.723
		SARRS	22.727	7.778	0.218	0.077	0.214	0.101	0.691	0.231	0.130	0.088	0.448	0.130
		SIER	13.127	3.909	0.118	0.037	0.445	0.201	0.785	0.165	0.385	0.213	10.322	26.220
		SECURE	12.396	3.677	0.114	0.036	0.261	0.058	0.773	0.111	0.265	0.063	3.768	0.499
		RSSVD	15.973	3.358	0.152	0.034	1.000	0.000	1.000	0.000	1.000	0.000	40.867	1.784
		SPLS	28.286	8.423	0.261	0.079	0.207	0.225	0.504	0.261	0.154	0.230	6.718	0.134
		SOFAR	15.693	2.159	0.146	0.021	0.626	0.062	0.915	0.071	0.660	0.065	5.572	0.450
		CCSRRR	18.404	1.764	0.172	0.017	0.376	0.130	0.831	0.143	0.486	0.206	0.008	0.009
0.50	0.50	SRRR	3.761	0.275	0.040	0.003	1.000	0.000	1.000	0.000	1.000	0.000	5.340	0.725
		SARRS	3.438	0.586	0.041	0.006	0.138	0.128	0.381	0.299	0.095	0.107	0.437	0.085
		SIER	3.345	0.534	0.034	0.006	0.412	0.233	0.744	0.205	0.353	0.245	6.995	1.333
		SECURE	2.778	0.768	0.023	0.008	0.139	0.044	0.700	0.096	0.118	0.061	3.969	0.373
		RSSVD	7.591	1.567	0.070	0.015	1.000	0.000	1.000	0.000	1.000	0.000	41.226	1.834
		SPLS	3.558	0.624	0.035	0.008	0.182	0.193	0.429	0.225	0.138	0.197	6.826	0.205
		SOFAR	3.726	0.935	0.031	0.008	0.393	0.057	0.865	0.088	0.457	0.076	6.605	0.402
		CCSRRR	5.547	0.743	0.046	0.006	0.280	0.132	0.789	0.130	0.326	0.203	0.011	0.009
	1.00	SRRR	22.229	7.400	0.238	0.079	0.926	0.224	0.991	0.030	0.915	0.258	5.640	0.767
		SARRS	19.927	8.473	0.231	0.070	0.238	0.091	0.833	0.173	0.133	0.087	0.383	0.082
		SIER	13.032	3.978	0.156	0.045	0.433	0.128	0.784	0.125	0.371	0.136	7.122	1.977
		SECURE	14.052	6.396	0.164	0.067	0.266	0.073	0.813	0.086	0.239	0.085	3.892	0.665
		RSSVD	35.750	8.126	0.364	0.076	1.000	0.000	1.000	0.000	1.000	0.000	40.902	1.099
		SPLS	19.019	6.037	0.219	0.059	0.236	0.169	0.599	0.186	0.172	0.174	6.706	0.159
		SOFAR	20.633	6.003	0.205	0.047	0.619	0.054	0.936	0.074	0.623	0.064	5.738	0.236
		CCSRRR	28.577	4.503	0.261	0.028	0.284	0.059	0.871	0.095	0.336	0.099	0.006	0.008
0.90	0.50	SRRR	3.416	0.719	0.112	0.022	0.854	0.288	0.933	0.149	0.841	0.314	6.806	1.039
		SARRS	2.029	0.248	0.057	0.006	0.185	0.044	0.876	0.082	0.063	0.050	0.309	0.027
		SIER	1.589	0.164	0.031	0.003	0.351	0.105	0.991	0.023	0.238	0.124	5.630	1.104
		SECURE	1.687	0.183	0.038	0.008	0.148	0.023	0.929	0.055	0.057	0.039	5.082	0.523
		RSSVD	9.793	2.849	0.225	0.043	1.000	0.000	1.000	0.000	1.000	0.000	45.820	2.760
		SPLS	3.398	0.691	0.122	0.014	0.167	0.166	0.464	0.233	0.115	0.169	7.272	0.189
		SOFAR	1.740	0.174	0.035	0.006	0.306	0.058	1.000	0.000	0.295	0.079	7.331	0.371
		CCSRRR	3.836	0.395	0.044	0.004	0.184	0.032	0.989	0.028	0.111	0.046	0.009	0.008
	1.00	SRRR	2.875	2.095	0.083	0.066	0.491	0.356	0.912	0.084	0.417	0.408	5.360	0.670
		SARRS	4.102	0.665	0.164	0.012	0.169	0.049	0.569	0.107	0.099	0.051	0.299	0.029
		SIER	3.983	0.884	0.158	0.030	0.433	0.084	0.823	0.060	0.364	0.093	6.056	2.369
		SECURE	3.367	0.944	0.120	0.025	0.181	0.050	0.681	0.091	0.131	0.059	4.721	0.376
		RSSVD	15.557	6.450	0.398	0.091	1.000	0.000	1.000	0.000	1.000	0.000	45.278	0.984
		SPLS	5.176	1.676	0.181	0.037	0.251	0.177	0.676	0.206	0.176	0.189	7.087	0.177
		SOFAR	5.266	1.471	0.140	0.023	0.508	0.056	0.945	0.052	0.509	0.070	7.238	0.528
		CCSRRR	8.497	1.762	0.175	0.020	0.217	0.082	0.785	0.127	0.227	0.123	0.008	0.009

Table 4: Simulation scenario 3: $n = 30, p = 100, q = 100, s = 15, r = 2$. Model evaluation based on 50 replications using various performance measures. MRCE and REMMAP were removed from this simulation because the computation would have taken too much time.

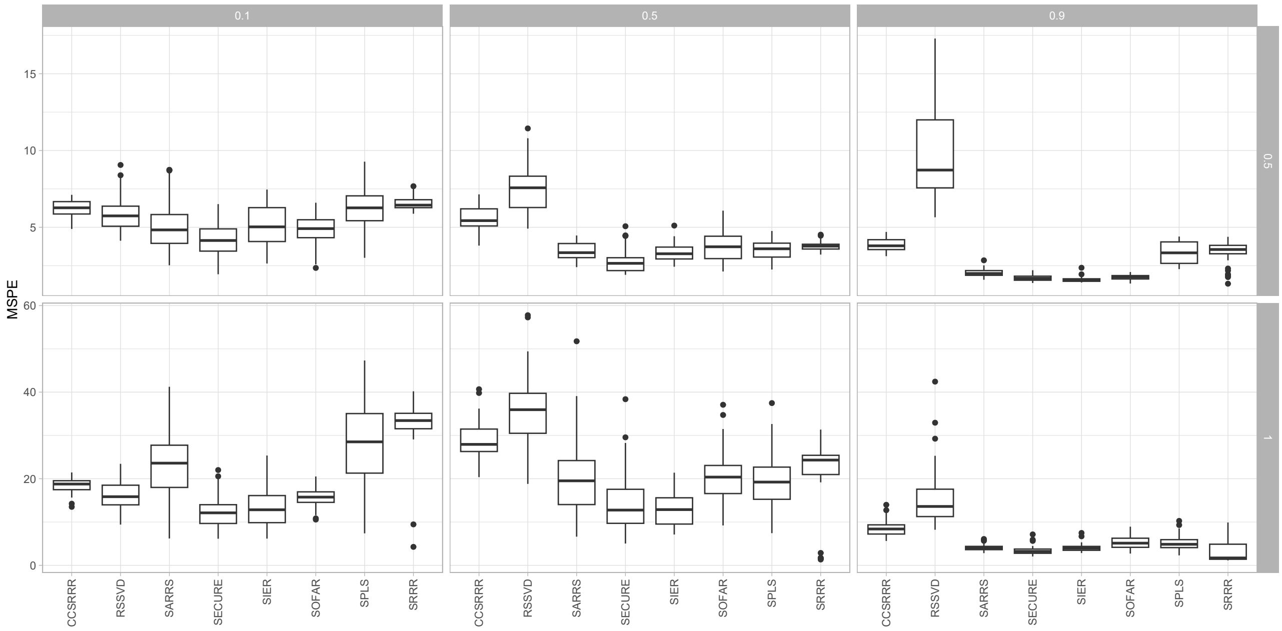

Figure 11: Simulation results for Gaussian errors under case 3: Mean squared prediction error (MSPE). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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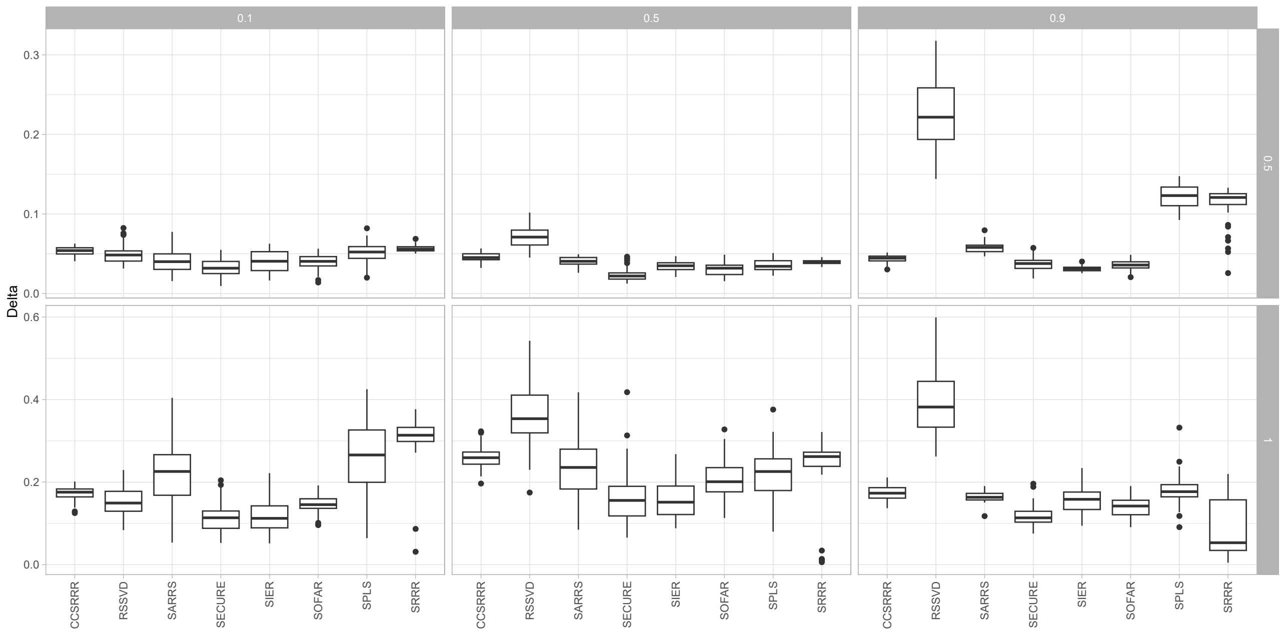

Figure 12: Simulation results for Gaussian errors under case 3: Estimation accuracy. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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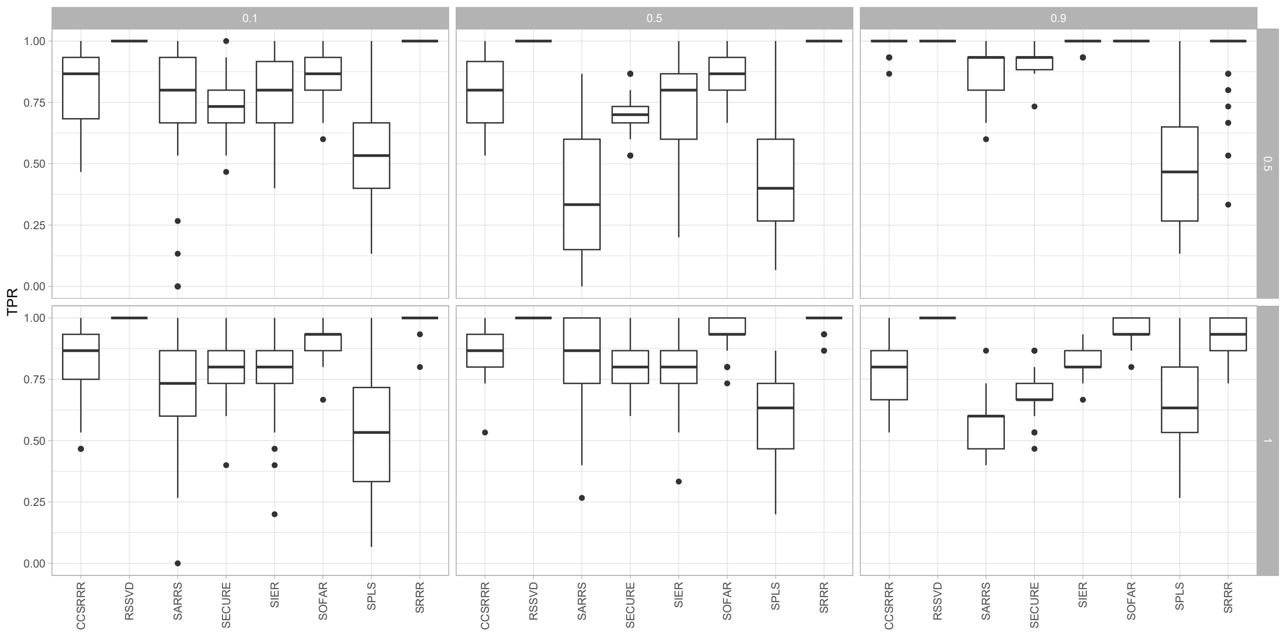

Figure 13: Simulation results for Gaussian errors under case 3: True positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$. We want the TPR to be close to 1.

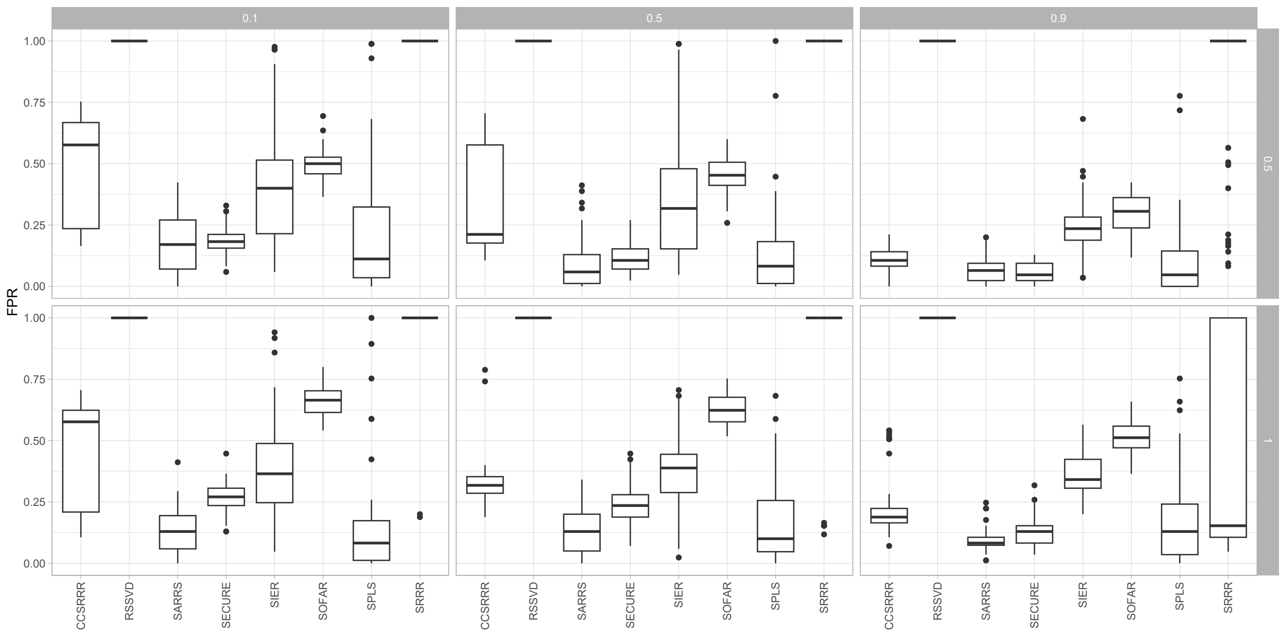

Figure 14: Simulation results for Gaussian errors under case 3: False positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$. We want the FPR to be close to 0.

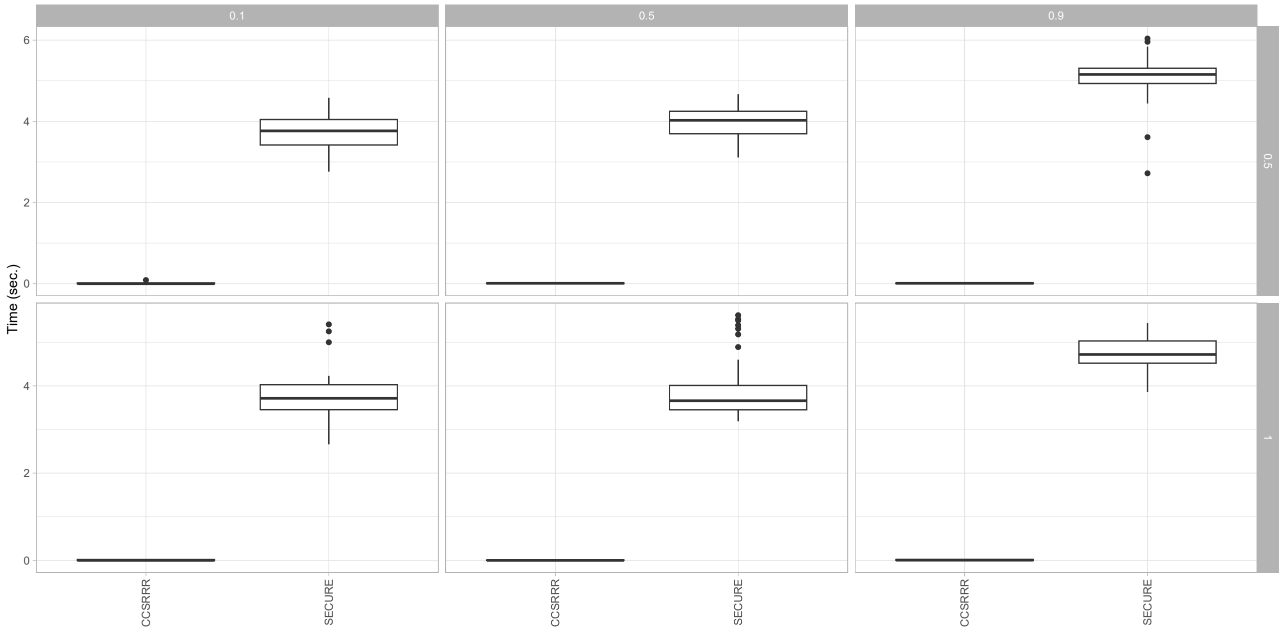

Figure 15: Simulation results for Gaussian errors under case 3: Computational time for SeCURE and CSRRR2 (all other methods are much slower). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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2.4 Case 4 with Gaussian errors

Table 5 and Figures 16 to 20 report the results for case 4 ($n = 30, p = 100, q = 1000$). They are briefly summarized as follows. In this case MRCE, REMMAP and RSSVD are removed because they require too much time. The box-plots of MSPE show that CCSRRR performs reasonably good for $b = 0.5$, and less convincing for $b = 1.0$. Here, SECURE produces the worst results for $\rho = 0.1$ and $\rho = 0.5$. The box-plots of estimation accuracy also show satisfactory performance for CCSRRR for $b = 0.5$. For $b = 1.0$, all methods are rather disappointing, except possibly SRRR. The box-plots of TPR show that CCSRRR is among the best performs for all values of ρ and b. The box-plots of FPR show that CCSRRR achieves very reasonable results. The box-plots of computational times show that CCSRRR is always considerably faster than SECURE. It is still the second fastest after CCSRRR, but for the price of very bad results. The attempts to choose better estimation parameters result in significant increase of the computational time.

ρ	b	Method	MSPE	SD	$\Delta(\boldsymbol{C})$	SD	Nnz	SD	TPR	SD	FPR	SD	Time	SD
		SRRR	6.806	8.181	0.058	0.082	0.501	0.318	0.993	0.020	0.414	0.373	91.367	17.254
		SARRS	7.582	2.928	0.067	0.029	0.643	0.116	0.993	0.047	0.582	0.133	1.281	0.246
		SIER	13.967	3.278	0.126	0.031	0.731	0.175	0.965	0.093	0.690	0.196	36.692	5.312
	0.50	SECURE	24.332	1.235	0.234	0.012	0.003	0.005	0.099	0.140	0.003	0.007	1.268	0.762
		SPLS	15.797	3.780	0.143	0.036	0.262	0.223	0.705	0.196	0.184	0.239	34.769	0.965
		SOFAR	22.233	3.312	0.213	0.033	0.080	0.107	0.385	0.417	0.128	0.171	36.252	5.934
0.10		CCSRRR	17.518	0.969	0.166	0.009	0.663	0.082	0.985	0.034	0.629	0.093	0.051	0.012
		SRRR	30.371	26.983	0.293	0.269	0.656	0.372	1.000	0.000	0.596	0.438	95.918	24.169
		SARRS	19.703	6.756	0.195	0.071	0.658	0.129	1.000	0.000	0.598	0.151	1.466	0.446
		SIER	35.623	7.611	0.342	0.075	0.693	0.167	0.955	0.072	0.647	0.189	32.275	3.128
	1.00	SECURE	80.746	4.390	0.827	0.045	0.003	0.008	0.055	0.130	0.004	0.013	0.964	0.700
		SPLS	39.976	11.320	0.379	0.104	0.207	0.151	0.653	0.171	0.128	0.158	33.839	0.618
		SOFAR	51.497	4.141	0.522	0.043	0.911	0.037	1.000	0.000	0.945	0.036	34.077	5.098
		CCSRRR	58.417	3.474	0.593	0.036	0.688	0.138	0.969	0.065	0.673	0.165	0.048	0.006
		SRRR	2.010	2.009	0.014	0.025	0.317	0.125	0.980	0.043	0.201	0.146	122.721	28.234
		SARRS	4.759	1.091	0.054	0.014	0.684	0.114	1.000	0.000	0.628	0.134	1.369	0.378
		SIER	7.437	1.260	0.086	0.015	0.619	0.217	0.909	0.126	0.568	0.237	35.076	3.513
	0.50	SECURE	16.849	1.591	0.163	0.009	0.004	0.004	0.129	0.131	0.001	0.004	1.459	1.036
		SPLS	8.915	1.725	0.113	0.020	0.175	0.105	0.620	0.187	0.096	0.100	33.686	0.594
		SOFAR	14.124	2.896	0.142	0.021	0.076	0.067	0.487	0.317	0.107	0.098	37.044	4.843
0.50		CCSRRR	10.635	0.992	0.114	0.007	0.502	0.104	0.948	0.059	0.451	0.118	0.050	0.007
		SRRR	9.600	14.285	0.117	0.177	0.407	0.221	0.953	0.068	0.311	0.258	143.549	38.007
		SARRS	21.958	8.949	0.300	0.102	0.621	0.117	1.000	0.000	0.554	0.137	1.573	0.375
		SIER	27.760	7.241	0.331	0.071	0.681	0.143	0.977	0.044	0.628	0.164	30.389	3.177
	1.00	SECURE	81.014	2.829	0.739	0.016	0.001	0.002	0.015	0.059	0.000	0.002	0.795	0.374
		SPLS	30.134	8.402	0.411	0.094	0.202	0.159	0.585	0.185	0.134	0.161	33.649	0.609
		SOFAR	34.882	4.206	0.408	0.039	0.871	0.041	0.996	0.016	0.906	0.046	34.741	5.198
		CCSRRR	43.836	3.653	0.491	0.025	0.582	0.113	0.976	0.037	0.543	0.137	0.049	0.007
		SRRR	1.231	0.050	0.011	0.004	0.291	0.031	1.000	0.000	0.165	0.037	285.743	65.200
		SARRS	5.154	0.946	0.181	0.014	0.444	0.112	1.000	0.000	0.346	0.132	0.914	0.155
		SIER	2.641	0.252	0.098	0.009	0.573	0.119	1.000	0.000	0.497	0.140	23.611	4.602
	0.50	SECURE	8.982	0.668	0.141	0.005	0.006	0.003	0.191	0.084	0.000	0.002	1.937	1.024
		SPLS	4.377	0.559	0.157	0.013	0.270	0.200	0.712	0.211	0.192	0.212	37.152	0.793
		SOFAR	3.652	0.898	0.103	0.011	0.205	0.085	0.884	0.102	0.204	0.101	39.587	6.075
0.90		CCSRRR	3.526	0.308	0.106	0.004	0.355	0.042	0.939	0.057	0.267	0.051	0.050	0.007
		SRRR	2.442	1.410	0.101	0.091	0.316	0.048	0.973	0.052	0.201	0.057	270.848	74.599
		SARRS	13.563	2.788	0.660	0.048	0.503	0.161	0.993	0.047	0.416	0.188	0.945	0.256
		SIER	7.468	1.870	0.295	0.047	0.620	0.121	0.999	0.009	0.553	0.142	21.550	2.451
	1.00	SECURE	30.793	1.825	0.603	0.011	0.010	0.003	0.256	0.074	0.000	0.002	1.464	0.639
		SPLS	18.188	4.156	0.833	0.070	0.213	0.175	0.571	0.213	0.150	0.181	36.965	0.804
		SOFAR	11.517	4.702	0.447	0.073	0.309	0.185	0.912	0.129	0.344	0.224	39.220	3.355
		CCSRRR	11.152	1.180	0.458	0.024	0.426	0.141	0.972	0.041	0.352	0.172	0.049	0.009

Table 5: Simulation scenario 4: $n = 30, p = 100, q = 1000, s = 15, r = 5$. Model evaluation based on 50 replications using various performance measures. MRCE, REMMAP and RSSVD were removed from this simulation because they would have taken too much time.

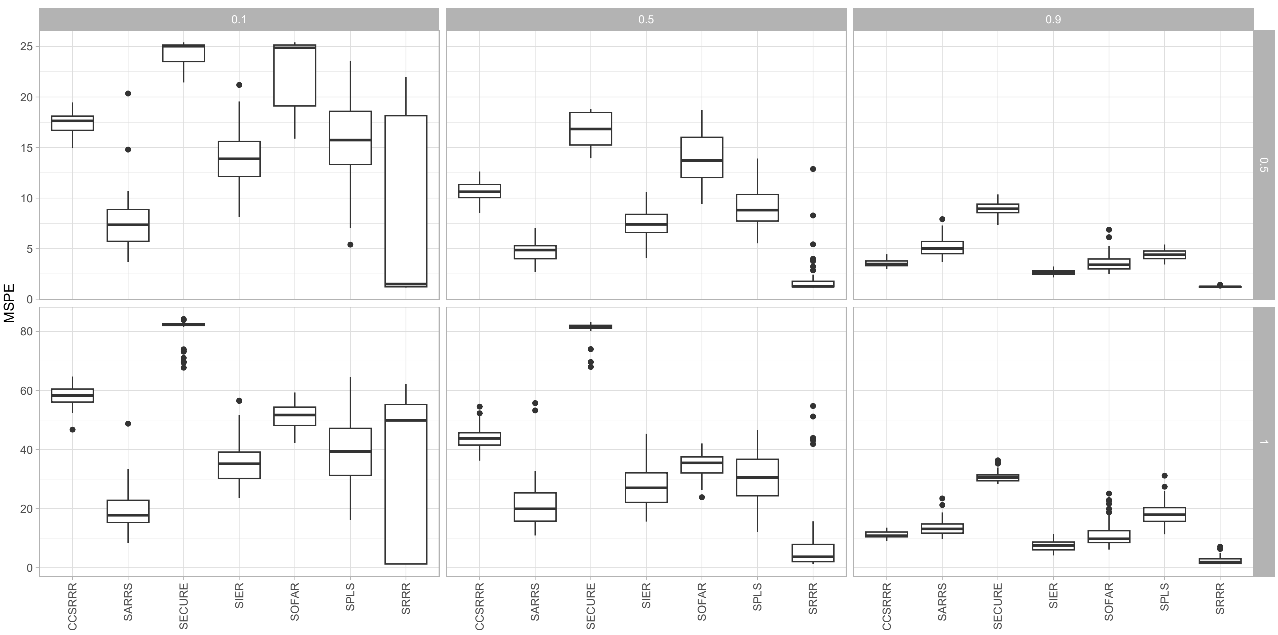

Figure 16: Simulation results for Gaussian errors under case 4: Mean squared prediction error (MSPE). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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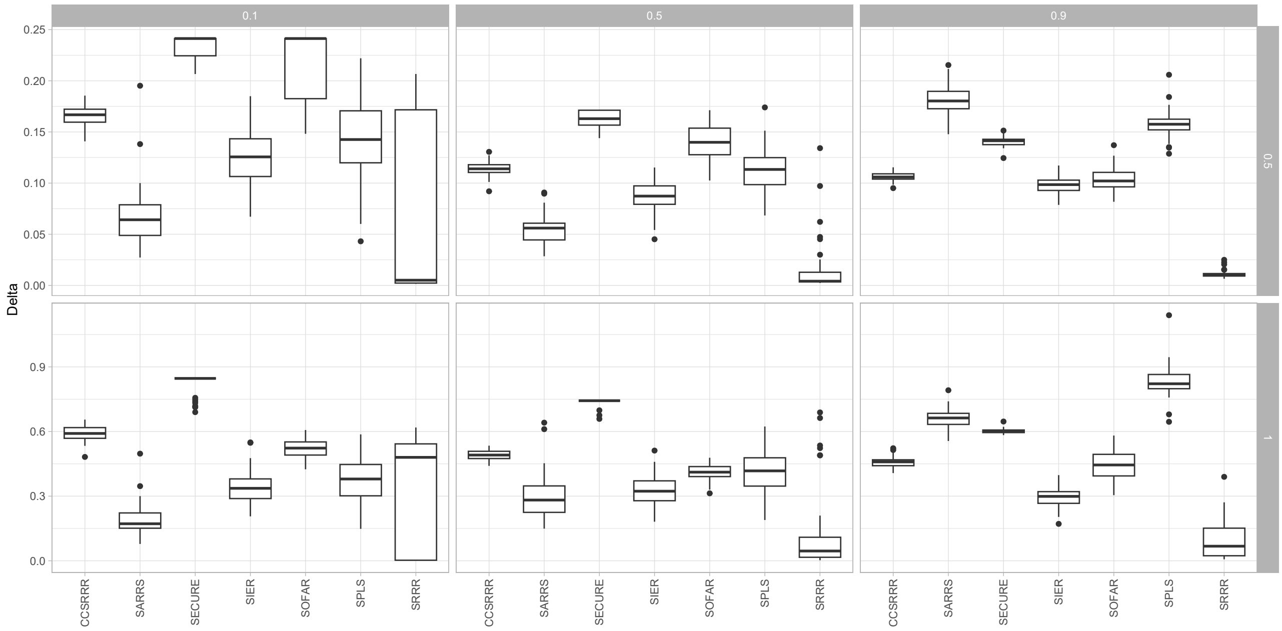

Figure 17: Simulation results for Gaussian errors under case 4: Estimation accuracy. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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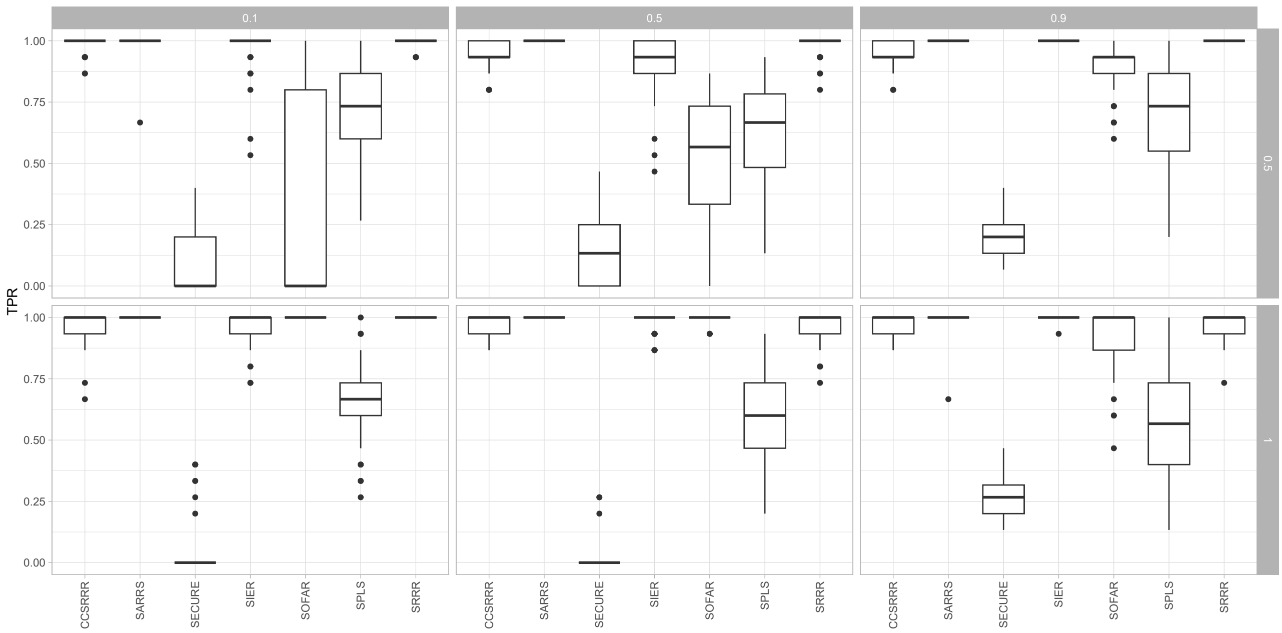

Figure 18: Simulation results for Gaussian errors under case 4: True positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$. We want the TPR to be close to 1.

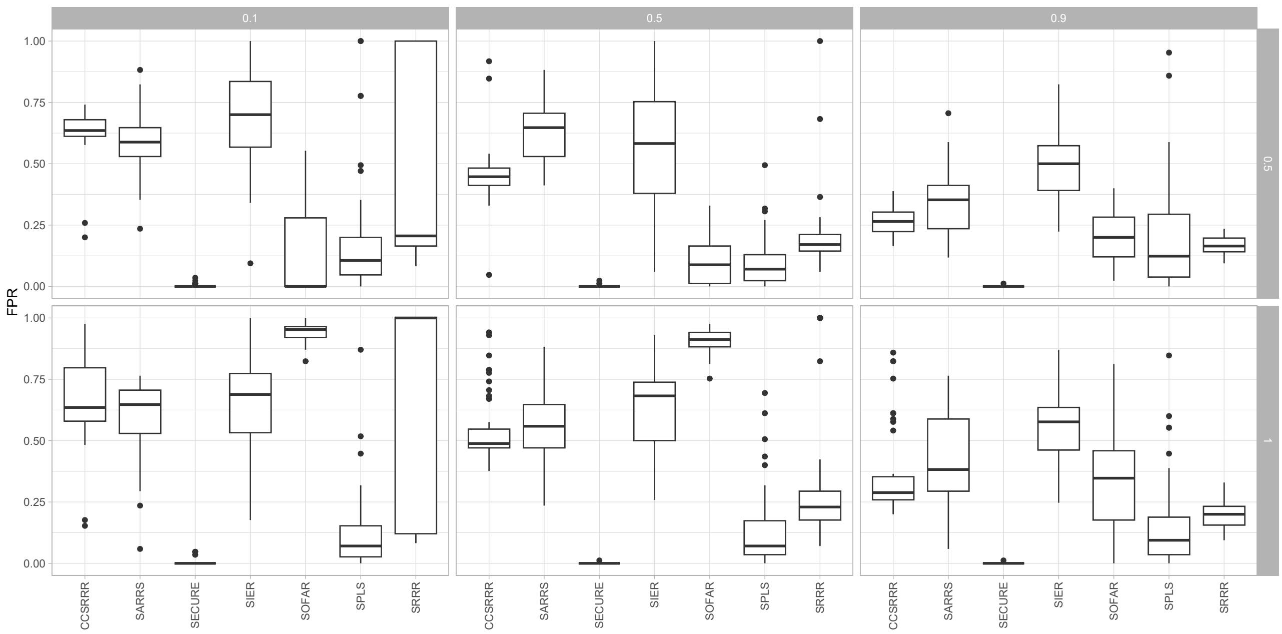

Figure 19: Simulation results for Gaussian errors under case 4: False positive rate. Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$. We want the FPR to be close to 0.

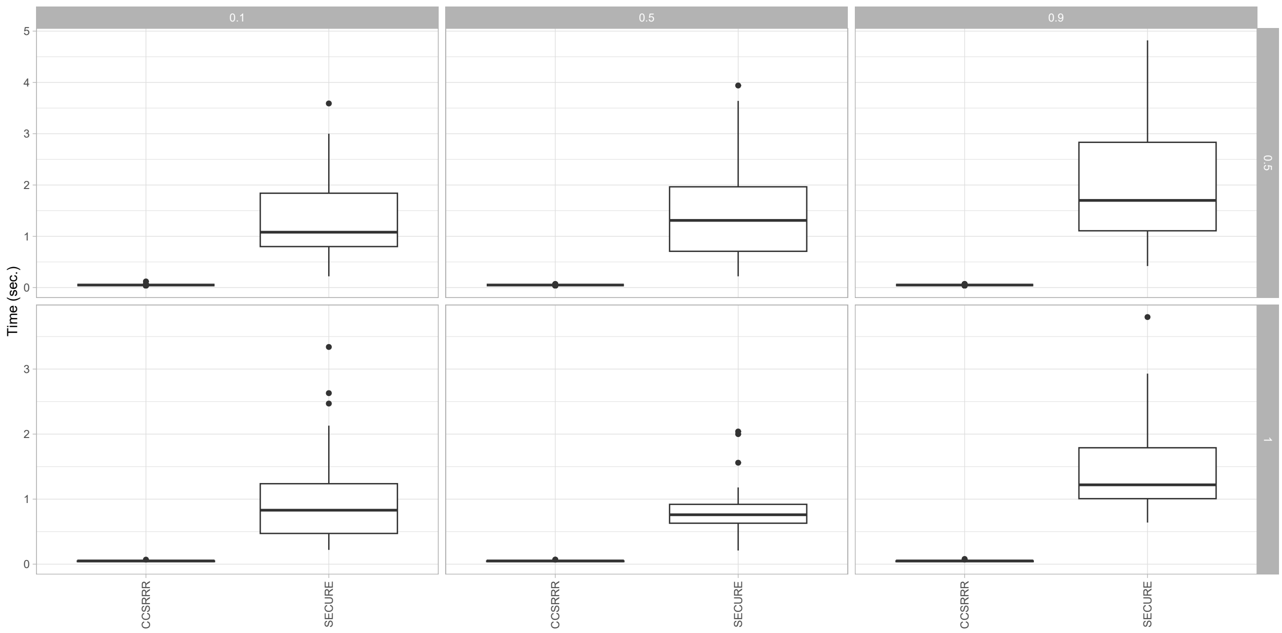

Figure 20: Simulation results for Gaussian errors under case 4: Computational time for SeCURE and CSRRR2 (all other methods are much slower). Reported results are 50 independent replications for $\rho = 0.1, 0.5, 0.9$ and $b = 0.5, 1.0$

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References

Bunea, F., She, Y., and Wegkamp, M. H. (2012). Joint variable and rank selection for parsimonious estimation of high-dimensional matrices. *The Annals of Statistics*, 40(5):2359–2388.

Chen, K., Chan, K.-S., and Stenseth, N. C. (2012). Reduced rank stochastic regression with a sparse singular value decomposition. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 74(2):203–221.

Chen, L. and Huang, J. Z. (2012). Sparse reduced-rank regression for simultaneous dimension reduction and variable selection. *Journal of the American Statistical Association*, 107(500):1533–1545.

Chung, D., Chun, H., and Keles, S. (2009). *spls: Sparse Partial Least Squares (SPLS) Regression and Classification*. R package version 2.1-0.

Fan, Y. and Tang, C. Y. (2012). Tuning parameter selection in high dimensional penalized likelihood. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 75(3):531–552.

Hilafu, H., Safo, S. E., and Haine, L. (2020). Sparse reduced-rank regression for integrating omics data. *BMC Bioinformatics*, 21(283):1–17.

Luo, R. and Qi, X. (2017). Signal extraction approach for sparse multivariate response regression. *Journal of Multivariate Analysis*, 153:83–97.

Ma, Z., Ma, Z., and Sun, T. (2020). Tuning parameter selection in high dimensional penalized likelihood. *Statistica Sinica*, 30:2179–2201.

Mishra, A., Dey, D. K., and Chen, K. (2017). Sequential co-sparse factor regression. *Journal of Computational and Graphical Statistics*, 26(4):814–825.

Peng, J., Zhu, J., Bergamaschi, A., Han, W., Noh, D.-Y., Pollack, J. R., and Wang, P. (2008). Regularized multivariate regression for identifying master predictors with application to integrative genomics study of breast cancer. *The annals of applied statistics*, 4:53–77.

Rothman, A. J., , Levina, E., and Zhu, J. (2010). Sparse multivariate regression with covariance estimation. *Journal of computational and graphical statistics*, 19(4):947–962.

Uematsu, Y., Fan, Y., Chen, K., Lv, J., and Lin, W. (2017). Sofar: Large-scale association network learning. *IEEE Transactions on Information Theory*, 65:4924–4939.