Throwing Darts and the Poisson Distribution

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1 Introduction

This report discusses the Poisson Distribution and its relevance when modelling situations with random processes. First, the code is built to model the mentioned distribution with three different mean values. A comparison is drawn from the plot and a table that displays the relevant statistical values. The second part regards the process of throwing random darts at a board divided into L sections. The simulation is repeated for different numbers of trials and regions.

2 Poisson Distribution

The Poisson Distribution is a discrete probability distribution that can predict the occurrence of a given number of events in a defined time interval. It can be modelled by the following equation, where $< n >$ is the mean number and n is the number of events.

$$P(n) = \frac{< n >^n}{n!} e^{-<n>} \tag{1}$$

We start the computational experiment by modelling said distribution with mean values $< n > = 1$, 5, 10. Figures 1, 2, and 3 show the resulting plots.

We can see from the figures how they shift according to the mean, which is seen clearly in the centre. It can also be observed that the probability of the mean also decreases as its number increases.

To continue the study of this Poisson Distribution, a code is written to find the sum of the probabilities, the variance, and the standard deviation. The variance is found by subtracting the sum of n times probability P(n) squared and n squared times P(n). The standard deviation is the square root of the variance. The respective output is given in Table 1. It can be verified that the distribution for the three mean values is normalized, as the sum of the probabilities for each mean equals 1, as theory dictates it should.

The table compares the values for the three different means. With its increasing value, we also see an increase in the variance and standard deviation. When the mean is low, it's less likely to see multiple events occurring. Most of

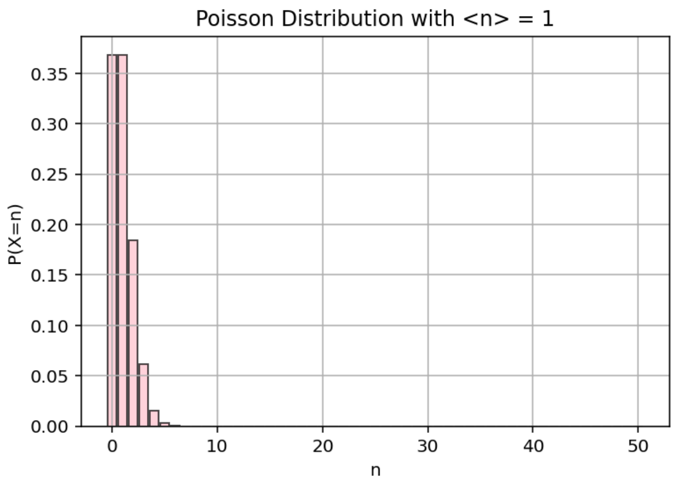


Figure 1: Distribution with $< n > = 1$

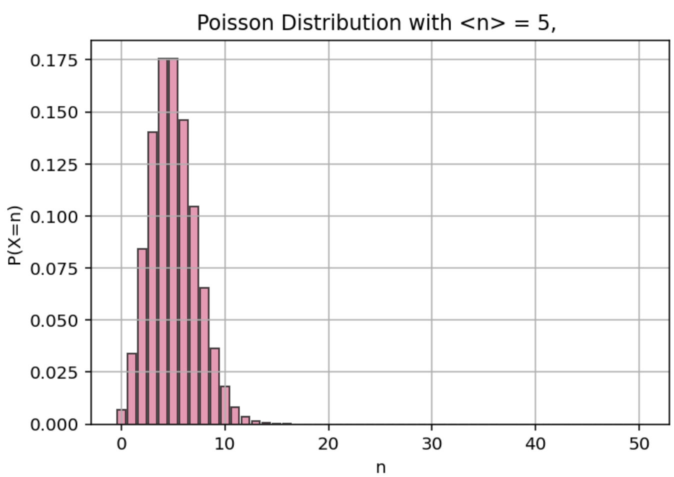


Figure 2: Distribution with $< n > = 5$

Mean	Sum P(n)	Sum n*P(n)	Sum n^2*P(n)	Variance	Std Dev
1	1	1	2	1	1
5	1	5	30	5	2.23607
10	1	10	110	10	3.16228

Table 1: Distribution Values

the probability is concentrated around 0 or 1. On the opposite side, a higher mean has a more spread-out distribution centred around the new mean. This

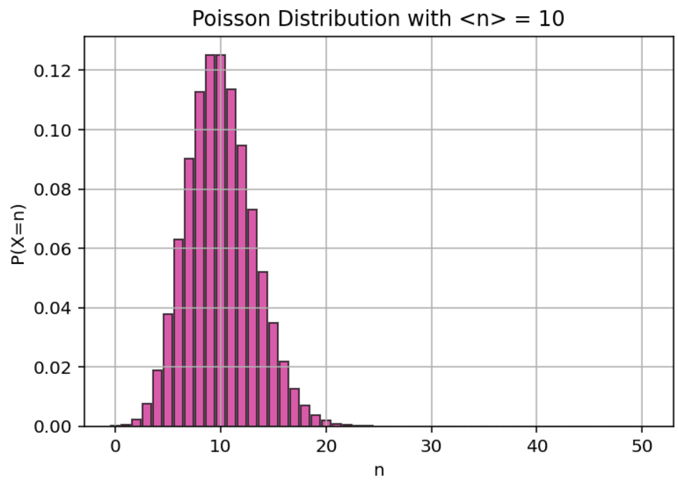


Figure 3: Distribution with $< n > = 10$

is also supported by the presented plots.

3 Throwing Darts

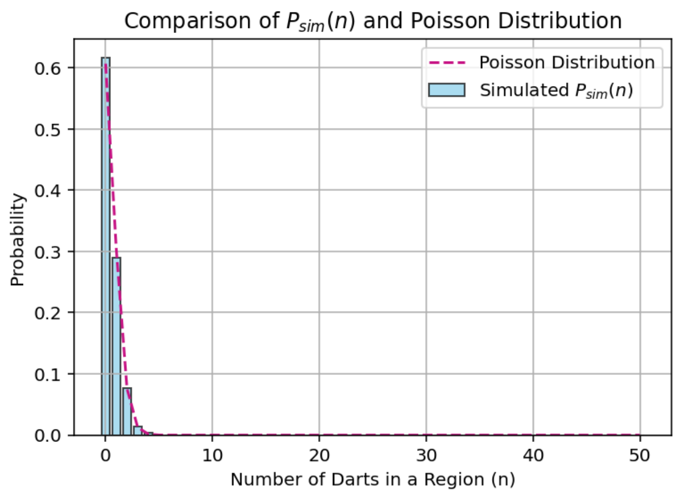
The second part of this computational exercise simulates the event of throwing N darts in a space with L elements in T trials. The probability of the simulation is given by,

$$P_{sim}(n) = \frac{H(n)}{LT} \tag{2}$$

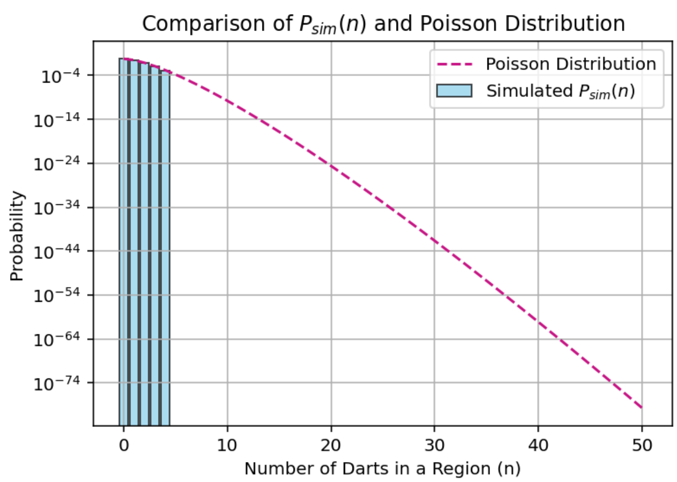
To study this situation, the respective code is constructed to model it and plot a distribution that displays the probability of the simulated number of darts that would fall in a region with L = 100 elements. In Figure 4(a) we can see the yielding plot, which is almost exactly outlined by the theoretical Poisson distribution. This is expected as darts are uniformly and independently thrown across regions.

From the data with the log y-scale, we can observe that the simulated values of probability follow the Poisson distribution until around 10^{-4}. This represents the smallest probability the simulation can reliably estimate. From this point, the simulation starts to diverge significantly from the Poisson distribution curve, mainly because of insufficient trials or low probability events in the distribution tail.

The final part of the exercise consisted of repeating the simulations with T = 100, 1000, and 10000. For T = 100, the smallest non-zero P(n) observed is 3.00000e-04. For T = 1000, the P(n) observed is 1.00000e-05. For T = 10000, it's 1.00000e-06. The calculations were replicated for L = 5, N = 50 and T = 10, 1000, 10000. For T = 100, the observed probability was observed to be



(a) Comparison between methods



(b) Comparison in Vertical Log Scale

Figure 4: Comparison of Poisson Distribution and Numerical Simulation

3.00000e-04. T = 1000 gave 1.00000e-05. Finally, T = 10000 yielded 1.00000e-06. For L = 100, it can be noted that the smallest probability decreases as the number of trials increases. More trials give a better chance to capture rare events accurately. When the L value is reduced, the probabilities also decrease with more trials, although the values are greater than those with fewer elements. When L is reduced to 5, the average number of darts per region $< n >= \frac{N}{L}$ becomes larger. This results in a higher threshold for the smallest probabilities that can be simulated since lower dart counts (like 0 or 1) will rarely occur.

4 Conclusion

In this coding experiment, the use of the Poisson distribution was studied. The dart-throwing experiment was simulated to examine how random events distribute across fixed regions. Our simulated results were compared to the theoretical Poisson distribution. Our results show that the precision of the simulated distribution is enhanced by increasing the number of trials T, allowing us to probe smaller probabilities more accurately. More trials lead to the capture of rare events resulting in a more accurate estimate of the probability distribution that closely matches the theoretical model. Decreasing the number of regions L reduces the likelihood of regions with few or no darts. Because each region is more likely to receive a larger number of darts, it becomes more likely for regions to have a moderate or high number of darts, which raises the smaller probability possible.