Assignment 3: Numerical solution of an ordinary differential equation

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1 Introduction

The objective of this report was to evaluate the solutions of an ordinary differential equation (ODE) by producing using the Euler method, the improved Euler method and Runge-Kutta method. These are then compared by using a smaller step size. The function is,

$$\frac{dx}{dt} = (1+t)x + 1 - 3t + t^2. \tag{1}$$

2 Direction Field

The first part of the exercise consisted of programming a code to determine the direction vector field of the function. To do so, the np.meshgrid(x,t) and ax.quiver(T, X, dt_normalized, dx_dt_normalized) Python commands are used. The arrow directions are normalized to get uniform values in the graph. Figure 1 is the given direction field, which graphically visualizes the expected solution.

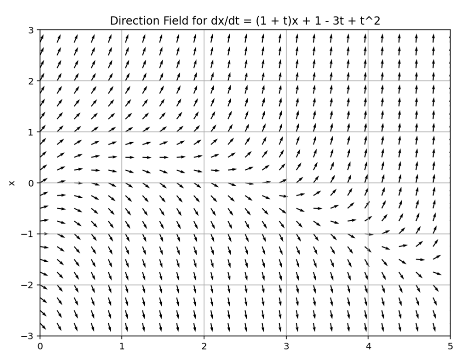

Figure 1: Direction Field

The slope of each line segment is equal to the slope of the solution. When x is much greater than zero, the directional vectors show a positive, increasing direction. When x is around 0 and below 1, these vectors are in the horizontal direction, turning and increasing vertically around t = 3 until t = 5. Below zero, the direction field shows a negative direction, decreasing vertically as t increases. Between t = 3 and t = 5 however, when the x is around -1 and 0, the directional vectors get horizontal and increase vertically. But as x decreases, so do the vectors vertically.

3 Simple Euler Method

The Euler method for solving ODEs is a simple first-order technique that approximates the solution with initial condition $x(t_0) = x_0$ and step sizes h along the x-axis. It's described by the following equation,

$$x_{i+1} = x_i + h * f(x_i, t_1). \tag{2}$$

The starting point is selected to be $x(0) = 0.0655$ with step size h = 0.04. The critical value of this function is actually $x_c = 0.065923...$, which determines the direction and rate in which x changes over time. If the initial value is chosen to be $x(0) > x_c$, then the solution increases without bounds towards $+\infty$. In the opposite case, where $x(0) < x_c$, the solution tends towards $-\infty$. In this case, the initial value is selected to be extremely close and smaller than the critical value, so in theory, it should increase without bound towards negative infinity.

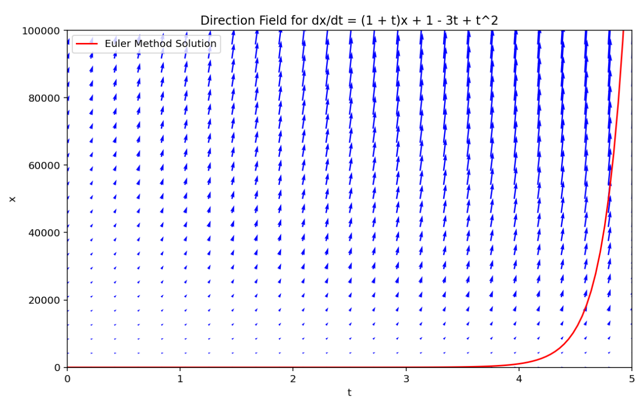

Figure 2: Euler's Method

The code describing the mentioned method yields the plot shown in Figure 2. As can be observed, it takes a great number of iterations to get the function to approximate positive infinity, which is opposite to what we should get. This can be explained by the fact that Simple Euler's Method is not the most effective technique for solving ODEs. This is because it needs very small steps to work accurately, and errors are quickly accumulated. The expected behaviour based on the critical value can be very sensitive towards numerical techniques, which is why it diverges in the wrong direction. Considering the step size may be too large, the method overestimates changes in x, which causes overshooting and divergence towards positive infinity as the error keeps building up.

4 Improved Euler's Method

The Improved Euler's Method is a second-order method based on the trapezoid rule used for integration. It's a numerical technique meant to be more accurate than the Simple Euler, as it takes the average of two slopes to solve for the solution. Through this, some of the error is reduced and more stable for larger step sizes. It follows the equation,

$$x_{i+1} = x_i + \frac{h}{2}(f(x_i, t_i) + h * f(x_i, t_i), t_{t+1}). \tag{3}$$

It can be observed in Figure 3 that while the iterations to observe the divergence

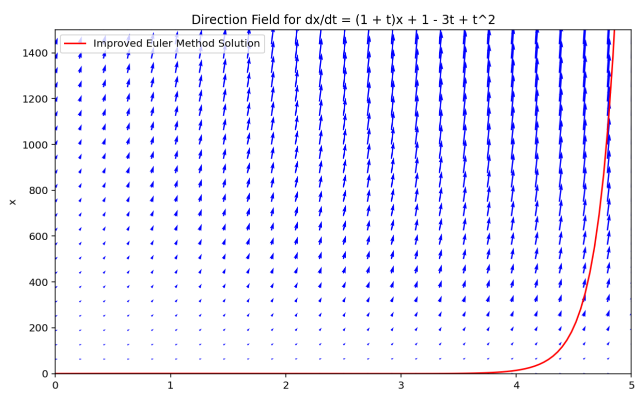

Figure 3: Improved Euler's Method

of the function are much less than in the Simple Euler Method, this one also tends to positive infinity. The initial value and step size are the same as in the previous scenario, $x(0) = 0.0655$ and h = 0.04. Because of the higher accuracy in this method, can also see that the increment without bounds happens faster than in the previously mentioned technique.

5 Runge-Kutta method

The Runge-Kutta method is a fourth-order method that uses four slope estimates to achieve a higher estimate. As before, we have step size h = 0.04, an initial point $x(0) = 0.0655$ and four slope calculations, k_1, k_2, k_3, and k_4. For this technique the following formulas are used:

$$x_{i+1} = x_i + \frac{h}{6}(f(k_1, t_1) + 2f(k_2, t_2) + 2f(k_3, t_3) + f(k_4, t_4)). \tag{4}$$

$$k_1 = x_i \tag{5}$$

$$t_1 = t_i \tag{6}$$

$$k_2 = x_i + \frac{1}{2}f(k_1, t_1) * h \tag{7}$$

$$t_2 = t_i + \frac{h}{2} \tag{8}$$

$$k_3 = x_i + \frac{1}{2}f(k_2, t_2) * h \tag{9}$$

$$t_3 = t_i + \frac{h}{2} \tag{10}$$

$$k_3 = x_i + f(k_3, t_3) * h \tag{11}$$

$$t_3 = t_i + h \tag{12}$$

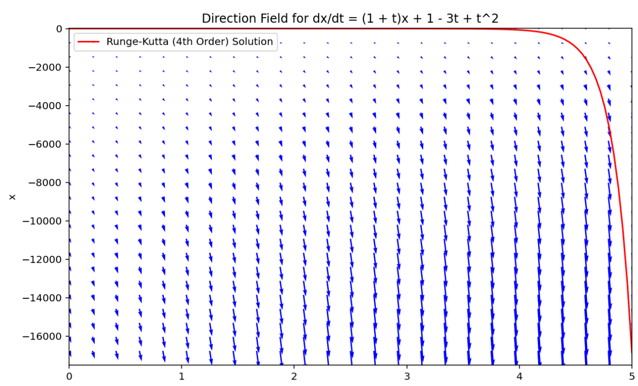

Figure 4: Runge-Kutta Method

Figure 4 shows us the resulting plot from the method. We can see that the function decreases without bound towards a negative infinitive. While we can also see that it diverges with greater x-values than the Improved Euler Method, it does so faster than the Simple Euler method. It can also be noted the greater accuracy of this method and the decreased error, as the function tends towards negative infinity as theory dictates. The errors don't accumulate in such a constructive way as in the previous techniques. This is because the use of numerous slope averages increases its precision.

6 Comparison using smaller step size

The final section of this exercise consists of repeating the process and comparing the three techniques with a smaller step size but with the same starting point. The plot that yields this result can be seen in Figure 5.

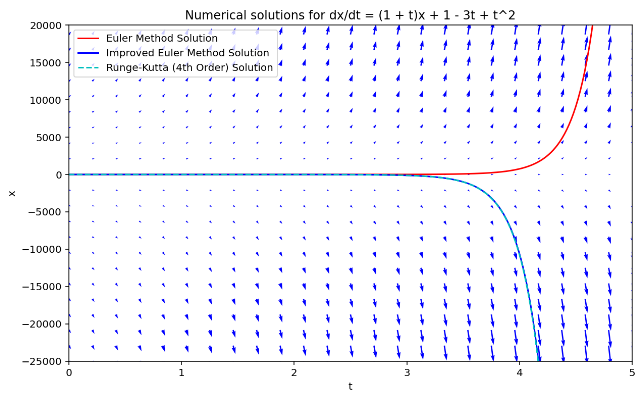

Figure 5: Comparison between the three methods with step size 0.02

It can be observed that opposite to the case with a larger step size, the method of Improved Euler diverges towards negative infinity, as it should. The Euler Method, because of its lack of accuracy, continues to increase without bound towards positive infinity, regardless of the decreased step size. This suggests that an extremely smaller value is needed for h. The Improved Euler, as stated, is shown to be just as accurate as the Runge-Kutta method, as both lines trace the same path and increase in the way that is expected.

7 Conclusion

The exercises performed explore the accuracy between the three different techniques for evaluating ODEs. Starting by using a large step size of 0.04, it could be observed that the Simple and Improved Euler's Methods did not yield the predicted result because of the accumulated error that they produced, resulting in the function overshooting and increasing without bound towards positive infinity. This is the expected result when the initial value is greater than the critical value, which is our opposite situation. This changes slightly when using a smaller step size of 0.02. The Simple Euler keeps the same behaviour, but the Improved increases in accuracy and follows the same path as Runge-Kutta towards negative infinity. This demonstrated the importance of the step size when determining a numerical solution, as a smaller value will result in a better prediction. The Runge-Kutta method is proved to be as accurate on both occasions, yielding a function that decreases towards infinity without bound.