Damped Chaotic Pendulum

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Contents

1 Abstract

This report investigates the behaviour of a torsional pendulum system, focusing on its damped and driven oscillatory motion, resonance behaviour, and chaotic dynamics. The experimental setup includes a copper wheel attached to a spring which provides restoring torque, an eddy current brake for damping, and a driving motor. The experiments examine undriven oscillations, measure under, critical and overdamping, explore resonance in driven oscillations, and observe chaotic motion in a double-well potential system. Key measurements include phase portraits, frequency spectra, and amplitude response curves. Numerous limitations were found at the time of modelling chaotic motion, but significant observations of its nature could still be drawn and studied. The results provide insights into damping effects, resonance phenomena, and transitions to chaotic behaviour, demonstrating principles of damping and chaotic motion.

2 Introduction

The importance of the oscillatory motion of pendulums is studied. A torsion pendulum is used to explore a range of phenomena including simple harmonic motion, damping effects, resonance, and chaotic dynamics. This is done by using Pohl's or torsional pendulum, which consists of a copper wheel connected to a spring, and subjecting it to damping forces from an eddy current brake and periodic driving forces from a motor. We adjust these parameters to obtain small, large, and critical damping, and the respective plots to model the motion. We also analyze how the system exhibits driven oscillations and chaotic motion in a double-well potential. These observations allow for a deeper understanding of key principles of oscillatory motion, energy transfer, and nonlinear dynamics.

3 Theory

3.1 Torsion Pendulum

The torsional pendulum is formed by a large copper wheel with rotational inertia I that is displaced from equilibrium. It experiences a restorative force caused by a coiled spring. With torsion constant k and angle of displacement θ, the equation of motion is:

$$I\ddot{\theta} = -k\theta \tag{1}$$

The torsional pendulum used for this experiment also included an eddy current break, which produces damping torque on the pendulum. It may also be driven employing a DC motor attached to an eccentric drive wheel. The wheel also had a small hanging mass attached.

If we also take into consideration the damping the damping torque produced by the eddy current break, we have the additional term $-\lambda\dot{\theta}$. Because the wheel moves through a perpendicular magnetic field it generates eddy currents. These tend to oppose the motion of the conductor or changes in the magnetic field that induces them. This opposition can manifest as a damping force that slows down moving objects. If we also take into consideration the driving torque produced by the driving motor, we add the term $Acos(\omega_D t)$ to the motion equation.

$$I\ddot{\theta} = -k\theta - \lambda\dot{\theta} \ + Acos(\omega_D t) \tag{2}$$

3.2 Damping

Damping in mechanical oscillations is the dissipation that occurs after the loss of energy in a system. In the case of small damping, the amplitude of the oscillation slowly decreases over time. The damping force is relatively weak, so in other words, the system experiences a gradual loss of energy over time. The equation of physical motion follows:

$$\theta = ae^{-\frac{\gamma}{2}t}cos(\omega t + \phi). \tag{3}$$

Large damping, or overdamping, is when the system returns slowly to the equilibrium position, with a significant energy decay. The damping force in said case is very strong, which causes it to not oscillate at all. Critical damping is similar but with an extreme damping force. In this case, the oscillation reaches equilibrium as fast as possible.

The ideal sketches for no damping, critical damping, and overdamping can be seen in the following Figure:

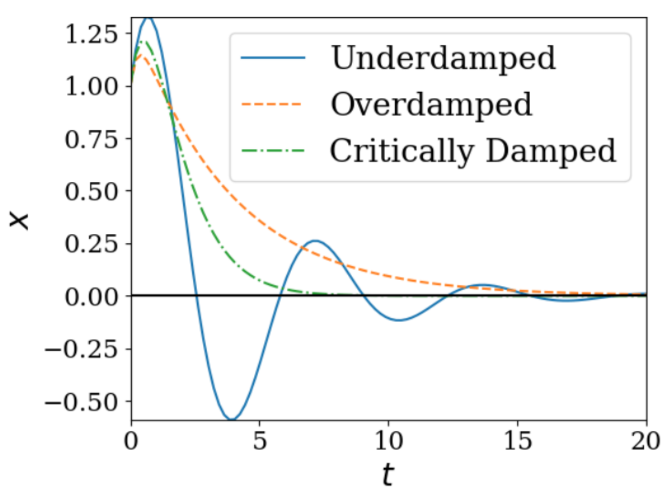

Figure 1: Damping types, Ling (2024)

3.3 Damped and Driven Oscillations

The damped-driven oscillator is a system where the pendulum experiences the decaying force of damping while being driven by an external that adds energy to the system. The equation of motion to describe this solution is given by,

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2\theta = \frac{A}{I}e^{i\omega_D t}. \tag{4}$$

In this type of motion, resonance occurs. This is when the oscillating system is driven by an external force at a frequency that matches its natural frequency. If no damping is present in the system, then the resonance would tend towards infinity. Otherwise, this is prevented by damping which limits the maximum amplitude. This causes the resonance to broaden and have a reduced peak sharpness. In critically damped or overdamped systems, resonance effects are minimized or suppressed altogether because the system cannot oscillate freely (Yartsev, 2024).

3.4 Chaos

Chaotic motion is the trivial and complex trajectories of a deterministic system, which is easily affected by initial conditions. This behaviour exhibited by the pendulum could be predicted in principle by knowing the exact conditions at one time, so the notion of randomness and determinism coexist at the same time (Hilborn, 2000). To create this state, the pendulum is modified to create a double potential well and through this, get two equilibrium points. By causing the system to experience under-damping while setting a current input, 2-period oscillations can be obtained as seen in ideal phase portrait in Figure 2. By reducing the current input we can cause the pendulum to lose its periodic motion and obtain chaotic behaviour.

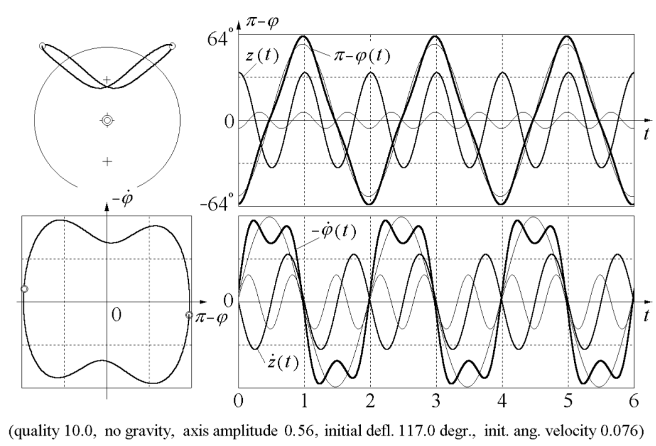

Figure 2: Stationary period-2 oscillations occurring over the upper boundary of dynamic stability, phase portrait in down left corner (Butikov, 2022).

4 Experimental Method

The Pohl's torsion pendulum is set up as seen in Figure **??**. This allows for rotational oscillations, which experience damping due to the connected eddy break current and motor. The latter gets a constant steady current supply by switching both coarse and fine voltage knobs of the power supply and increasing the current up to 1.5 A, and then to the desired value.

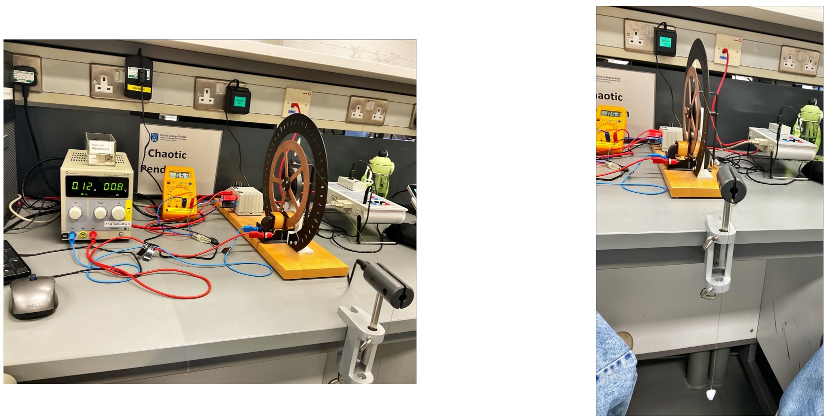

Figure 3: Experimental Setup

4.1 Experiment 1

The first part of this experimental laboratory consisted of studying the motion of the undriven pendulum with no additional damping. The damping coefficient was then calculated. To do this, the current eddy break is turned off. The analysis of the no-damping pendulum consisted of obtaining a phase portrait of the oscillations, a frequency spectrum, and θ vs. t plot. The CASSY-sensor was used to obtain these.

The same plots were then found for critical damping and overdamping. The eddy break is then turned on. Mass is removed from the hanging string, and critical damping is found using a constant input current of around 1.5 A. This current value is maintained at 1.45 A, and more mass is removed to find over-damping.

4.2 Experiment 2

In this section, we obtain a calibration curve for frequency and resonance for the driven, undamped pendulum. The goal is to then find the intrinsic damping coefficient γ_0, resonant frequency ω_0, and quality factor Q_0 through a Lorentzian function fit to the resonance curve.

First, to perform the frequency calibration the driving motor was turned on, and the fine and coarse voltage knobs of the motors were adjusted. The frequency is set to 2.0 V. A transient motion is formed as the pendulum settles down to a steady state, and a phase portrait is obtained. To obtain a frequency vs. applied voltage graph the frequency of oscillations is recorded from 2.0 V to 12 V in increments of 2.0 V. Each frequency value could also be found through the frequency spectrum.

To obtain a resonance curve, a rough estimate is recorded of when the motor is in resonance with the pendulum. The voltage is then varied from 2.0 V to 12 V again in the same increment ranges while recording the corresponding amplitudes. This value is calculated by dividing the minima and maxima by two. We plot a graph of amplitude vs. frequency to illustrate the resonance behaviour. A Lorentzian Fit function to the resonance peak is constructed to find the previously mentioned constants.

This is then repeated to find the same data for the underdamped, driven pendulum. The method for finding the frequency was changed. This was done by counting the number of oscillations over a certain time, as it resulted in more straightforward clear results.

4.3 Experiment 3

Experiment three aims to model the chaotic behaviour of a pendulum. This is first done by finding a double well potential by adding two additional weights to the wheel, as seen in Figure 4.

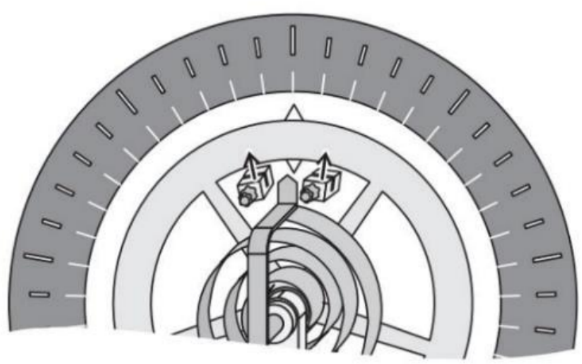

Figure 4: Pendulum with Added Weights, Trinity College Dublin (2023)

A double well potential would be found when two equilibrium points were established. This means that by moving the wheel to one side, an equilibrium point is settled. By moving it to the opposite side, a new equilibrium point could be found. Once these positions are located, a phase portrait is recorded. The eddy current break is switched on to find the 2-period oscillations. From a current value of 0.6 A, this is decreased in increments of 0.2 A allowing it time to settle into a steady state motion. The period 2 oscillations behave in such a way that the system completes its motion in two cycles of the driving period before repeating itself. Meaning, we would see it oscillating around one equilibrium point and then around the second one. By then slowing down the voltage in small increments the pendulum would lose its periodic behaviour and start exhibiting chaotic motion.

5 Results and Discussion

5.1 Experiment 1

For the first section of experiment one, we obtained the phase portrait, angle vs. time portrait, and frequency spectrum.

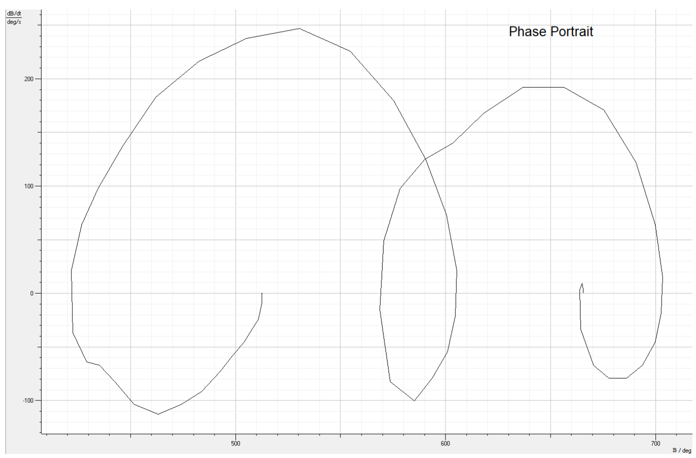

Figure 5: Phase Portrait, No Damping

Figure 5 shows the trajectory of the pendulum spiralling inwards. The amplitude decays gradually, as can be seen from the shrinking loops, which also suggests the decrease of the angular displacement and angular velocity. This can be attributed to the slight damping that arises from friction. While the ideal phase portrait of no damping would appear circular with constant oscillations that indicated constant energy, this case has to account for inevitable external factors. The angular displacement vs. time for no additional damping is shown in Figure 6. The back-and-forth movement of the pendulum is reflected in the increasing and decreasing angular displacement of the plot. The slight damping is observed again in the decaying amplitude after each cycle. The amplitude finally decreases gradually until the system reaches equilibrium and settles into a constant angular displacement. The final plot found for this part is the frequency spectrum in Figure 7. This reflects how the amplitude of the oscillations is distributed across the different frequencies. The spectrum has a prominent peak before the amplitude drops off fast. This single peak suggests that the motion of the pendulum has a low-frequency mode that dominates it. This is

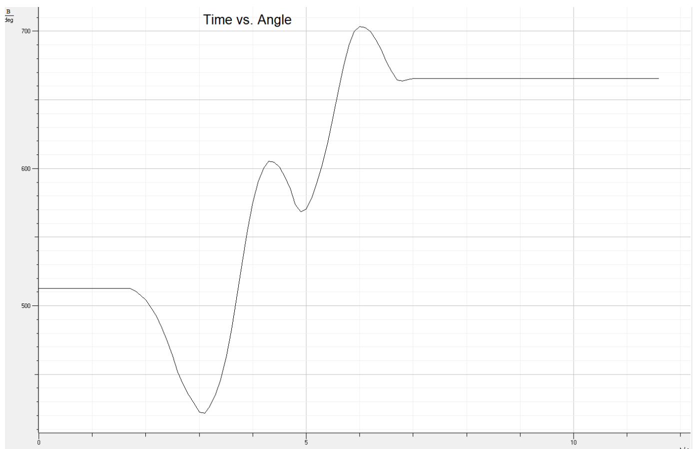

Figure 6: θ vs. t, No Damping

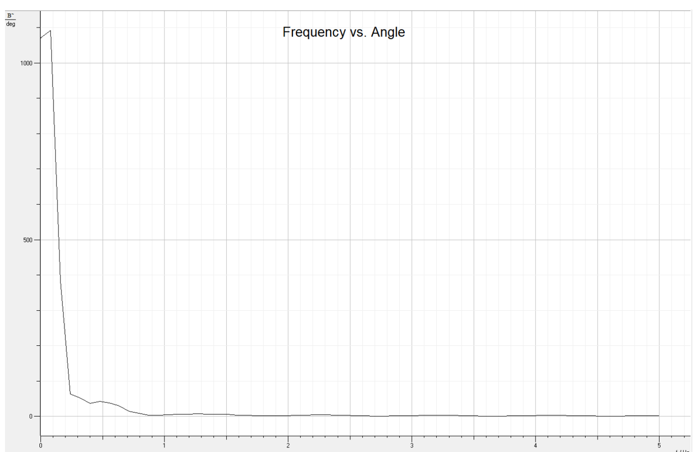

Figure 7: Frequency Spectrum, No Damping

consistent with the presence of damping, which damps out higher frequencies more quickly.

To find the intrinsic damping coefficient, we can see how the last section of the angle vs. time plot resembles an exponential function, so we build its respective fit to find the damping constant. The term from equation (3) for small damping gives us the term $ae^{-\frac{\gamma}{2}t}$, with an exponent term that can be extracted from the exponential model as R0 to then find the desired constant. A fit is built and shown in Figure 8. The fit gives us an equation with the value $-\frac{\gamma_0}{2}$ = -2.643 $\pm$ 0.214. Solving for the intrinsic value coefficient, we find that γ_0 = 1.322 $\pm$ 0.107.

The oscillatory motion is repeated for critical damping and overdamping.

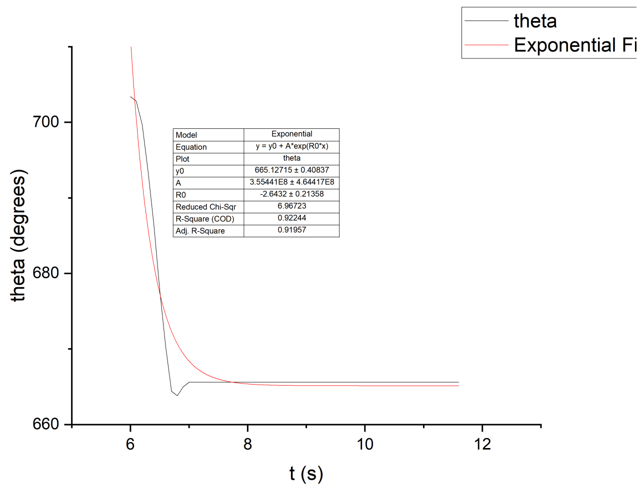

Figure 8: θ vs. t Fitting

Critical damping yields figures 9, 10, and 11.

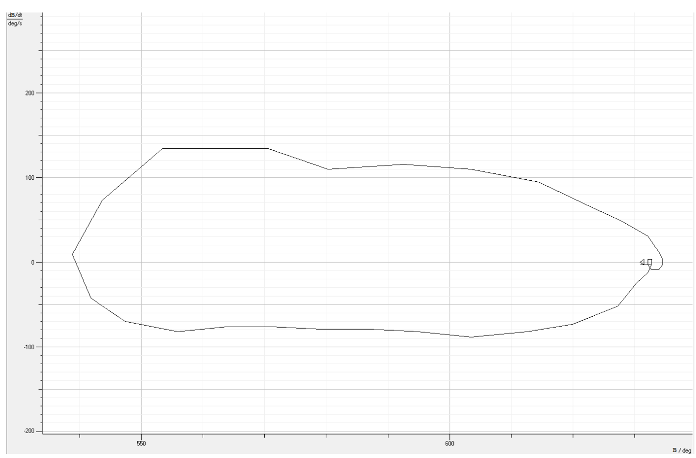

Figure 9: Phase Portrait, Critical Damping

The phase portrait for critical damping shows the motion of the pendulum from its displacement to the occurrence of damping. The top rounded curve of the portrait represents the initial displacement of the pendulum which is then released to experience critical damping. This is reflected in the path traced seen as the lower curve of the elongated circle. The motion returns to equilibrium as quickly as possible without oscillating. We can see the indication of the decrease in displacement and velocity as the line moves from the initial position towards

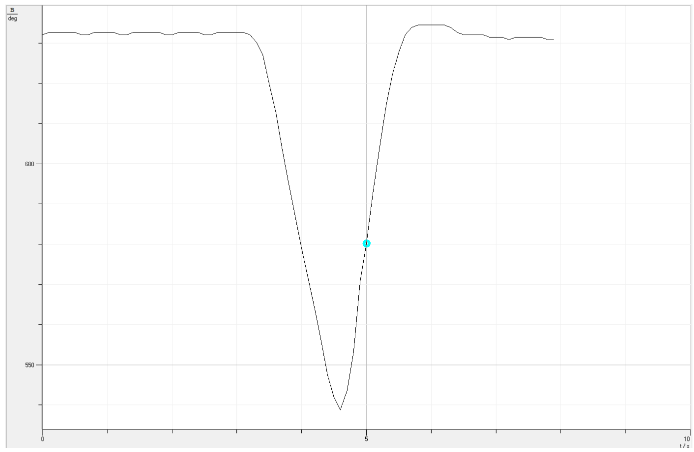

Figure 10: θ vs. t, Critical Damping

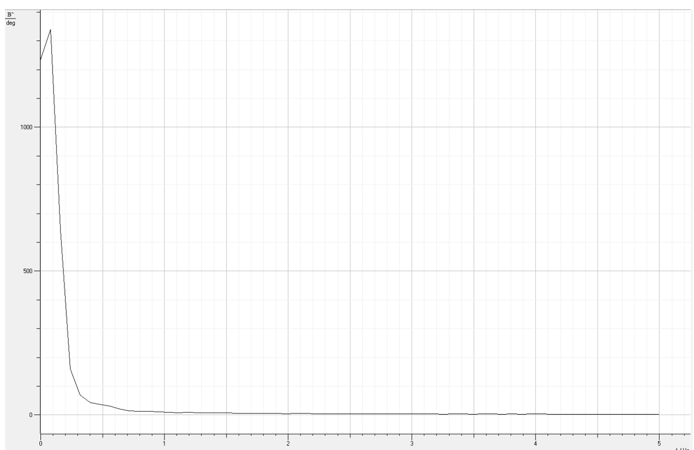

Figure 11: Frequency Spectrum, Critical Damping

the origin. The phenomena may not be as pronounced as expected in theory by the limiting factors of the experiment. The apparatus lacked force for recording a meaningful critical damping occurrence, so a very small mass had to be used with the maximum input current that was possible. In an ideal scenario, the phase portrait would show a more pronounced curve as the pendulum displaced to the origin.

The angular displacement vs. time plot in Figure 10 shows how the angular displacement decreases rapidly after an initial release and moves toward zero without oscillating. This again demonstrates the critically damping behaviour, as the system returns to equilibrium as quickly as possible without crossing over the equilibrium position. While the overall behaviour towards the end tail of the curve indicated clear critical damping, there are small fluctuations present that could indicate minor external disturbances or imperfections of the mechanism. The frequency spectrum in Figure 11 has a very small peak before decaying. Since there is no oscillatory motion like in the small damping case, there is no single frequency at which energy is concentrated, resulting in a smaller and broader peak.

Figures 12, 13, and 14 are the plots obtained for overdamping.

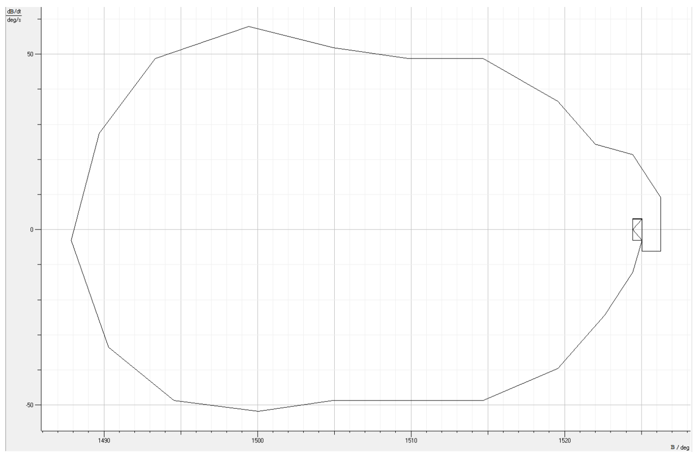

Figure 12: Phase Portrait, Overdamping

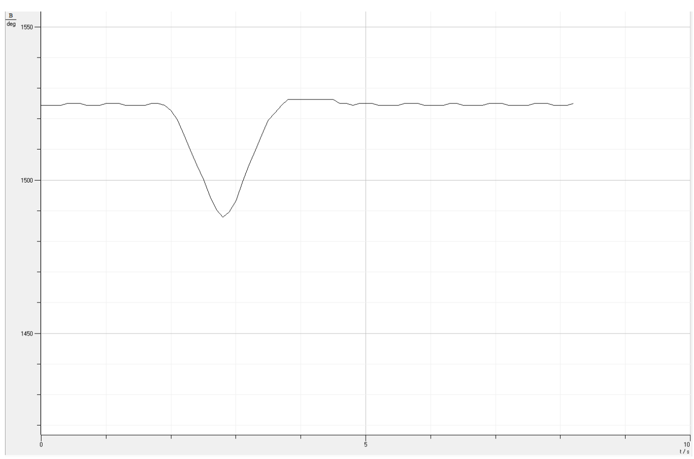

Figure 13: θ vs. t, Overdamping

The overdamping Phase Portrait in Figure 12 exhibits a similar shape as in the critical damping case, with the difference that we can see a slower approach to the equilibrium. The final path towards the origin is not as pronounced as

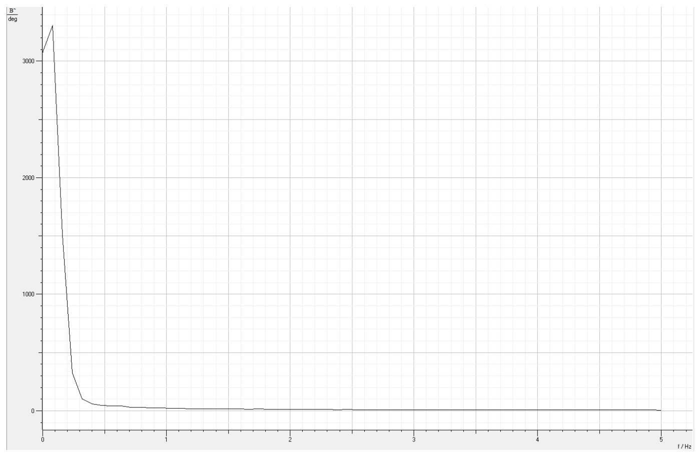

Figure 14: Frequency Spectrum, Overamping

in the critically damped case, but more elongated and gradual. The absence of oscillations and the gradual motion towards equilibrium indicate efficient but slower energy dissipation. While energy is dissipated steadily, the damping force is so strong that it resists rapid movement, leading to a prolonged return to rest. As is the case with critical damping, the lack of ideal conditions prevents the phase portrait from being as clear and distinctive as an ideal case, but the characteristic behaviour can still be observed.

Angular displacement in Figure 13 also reflects the slower return to equilibrium in a longer, steeper slope towards the end. The corresponding frequency spectrum does not show a clear, distinct peak as seen in the no additional damping case. The amplitude decreases rapidly with increasing frequency, indicating that the system has low-frequency components. This again is characteristic of strong damping, which quickly dissipates energy and prevents oscillatory motion.

5.2 Experiment 2

The system's attractor for the undamped driven oscillator yields the phase portrait seen in Figure 15. The trajectory shows the transient motion before reaching a steady state. We can observe big oscillations in the spiral's outer loops, which tend to loop inwards as time progresses. This eventually converges to a periodic motion in the centre of the spiral. The final pattern at the centre represents the steady-state oscillation, showing consistent amplitude and frequency. Another factor to note is the degree of symmetry that indicates the driving force is relatively consistent and not erratic. The corresponding frequency spectrum is seen in Figure 16. To find the frequency of oscillation of the pendulum one can find the differences between the consecutive maximums.

Data was obtained to find the graph of frequency vs. applied voltage through

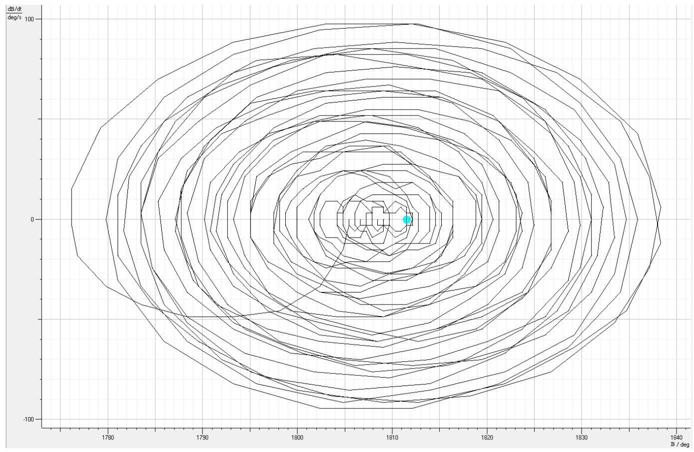

Figure 15: Attractor for Undamped Driven Oscillator

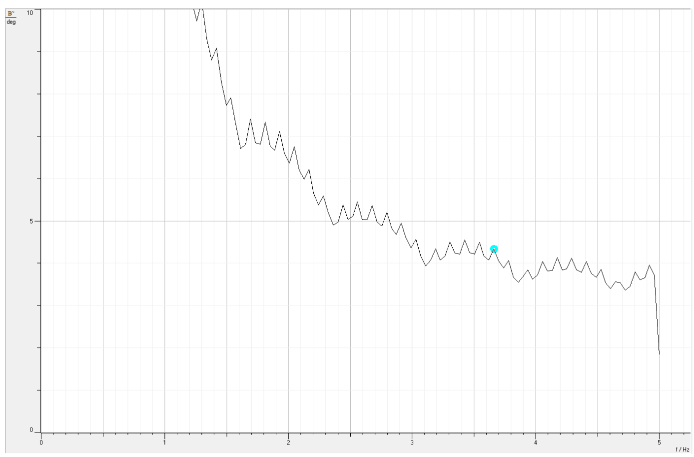

Figure 16: Frequency Spectrum for Attractor

increasing voltage from 2.0 V to 12 V. The mathematical relationship between the mentioned variables results in Figure 17.

The plot shows an increasing trend where the frequency of oscillation rises as the applied voltage is increased. In an ideal undamped system, where the driving force is directly proportional to the applied voltage, we would expect a linear relationship between the two variables. A higher voltage implies a stronger driving force, which increases the energy supplied to the system and, therefore, raises the frequency of oscillation.

Figure 18 illustrates the resonance behaviour. Frequency was measured again for each voltage value. The point at which the motor is in resonance with the pendulum was roughly estimated to be around 8 V.

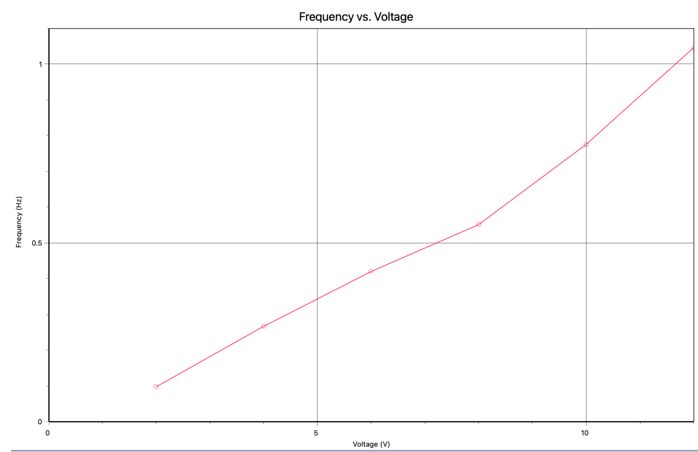

Figure 17: Frequency vs. Applied Voltage

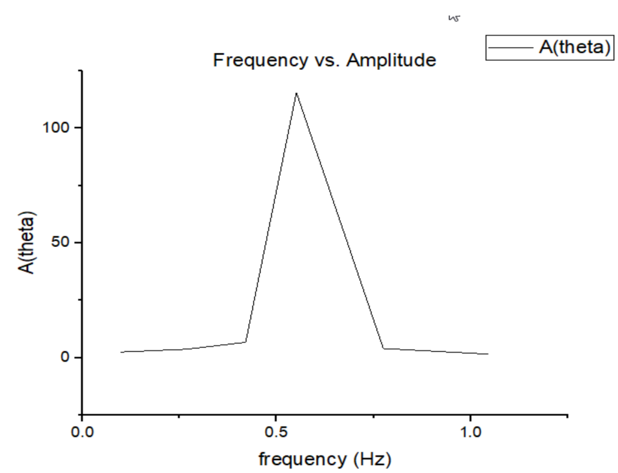

Figure 18: Resonance Behaviour, No Damping

The plot shows a sharp peak at a specific frequency where the amplitude reaches its maximum. This indicates resonance, where the driving frequency matches the natural frequency of the system, resulting in maximum energy transfer and the highest amplitude of oscillation. This value displays the classic behaviour of resonance in an undamped oscillator, as the energy input from the driving force accumulates over time, leading to a large amplitude response. The symmetrical shape we can see is typical of resonance curves, where the amplitude rises as the driving frequency approaches the natural frequency and falls off as the frequency moves away from resonance. As the driving frequency moves away from the resonant frequency (to either side of the peak), the amplitude of oscillation decreases rapidly. This indicates that the system is not as easily driven at frequencies that differ from the natural frequency. Since energy losses due to damping are minimal, the system can build up a significant amplitude at the resonant frequency which explains the big and narrow peak we observe in the graph.

A Python code (Appendix) was constructed to find the Lorentizan fit and calculate the intrinsic damping coefficient γ_0, resonant frequency ω_0, and quality factor Q_0. In this case, the maximum amplitude gives us 127.16°. The resonant frequency $\omega_0 = 0.5614$ and the damping coefficient $\gamma_0 = 0.03611$. The quality factor, using equation $Q_0 = \frac{\omega_0}{\gamma}$, is $Q_0 = 15.55$. A higher Q value indicates lower rates of energy loss, meaning less decay. This relatively high value indicates that the system is moderately resonant and that the system dissipates energy more slowly than in other cases, like critical or overdamping.

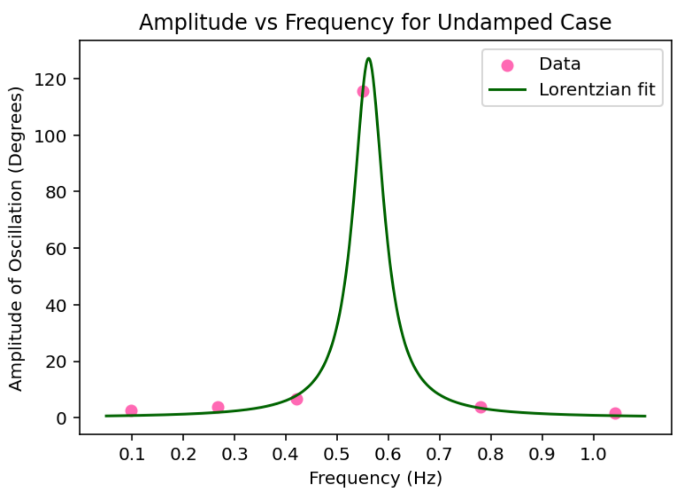

Figure 19: Lorentzian Fit, No Damping

For the following section, a more effective method was found by counting the oscillations per time and using these to calculate the frequency. The Frequency vs. Applied Voltage is seen in Figure 20, and the Resonance Behaviour in Figure 21.

Figure 20 has a linear line that follows the pattern we previously obtained with no damping. The frequency of the system rises as the voltage does too. For the underdamped oscillations, energy dissipates over time due to the loss caused by damping. The plot appears relatively linear, but any slight deviations or curvature could indicate minor nonlinear effects, such as changes in damping at different velocities (e.g., air resistance increasing with velocity) or mechanical constraints that slightly alter the system's behaviour at higher voltages.

The resonance behaviour for the underdamped system in Figure 21 yields, like in the previous case, a sharp peak at around 0.5 Hz where the amplitude

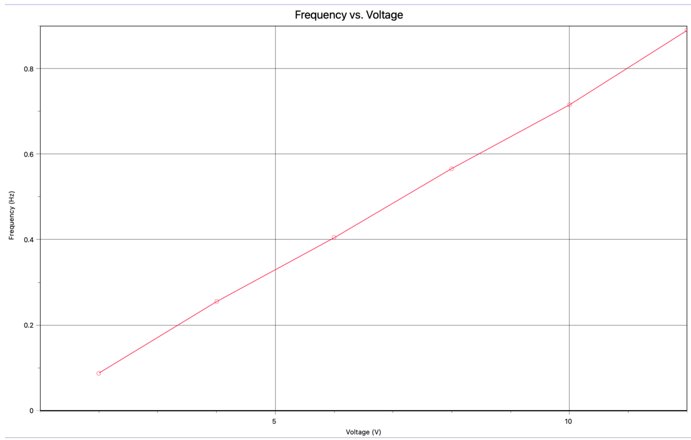

Figure 20: Frequency vs. Applied Voltage, Damping

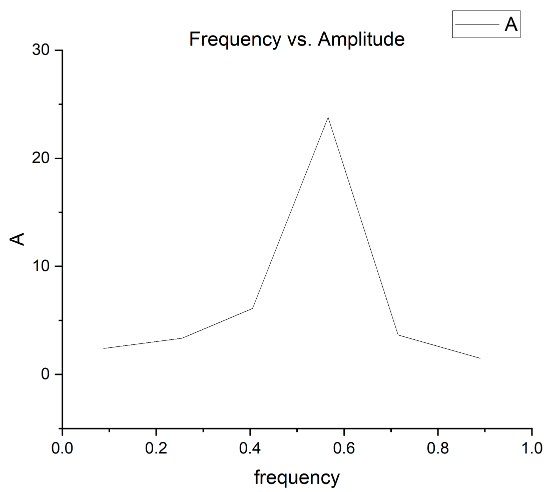

Figure 21: Resonance Behaviour, Damping

reaches its maximum value. The symmetry around the resonance value again reflects the expected pattern from the theory. When comparing the resonance behaviour of this to the no-damping case, we can observe the broader and lower which can be attributed to the energy dissipation that takes place in this situation.

Lorentzian fit for the under-damping system results in Figure 22. Different from the previous fit, we can see more data points that stand outside of the predicting curve. This graph is just a rough estimate of a Lorentzian fit which has manually selected initial parameters. The key error here is that we did not take multiple measurements around the resonance point, from 0.5 - 0.7 Hz, as this would have resulted in a better fit. The maximum amplitude value is 26.18°.

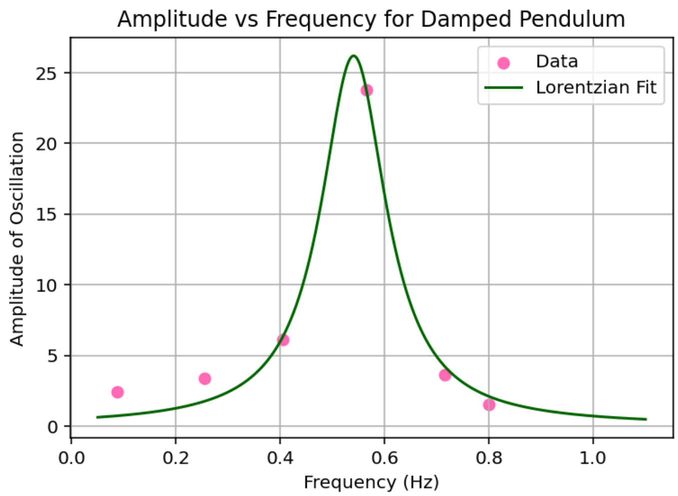

Figure 22: Lorentzian Fit, Underdamping

The resonant frequency is $\omega = 0.541$, damping coefficient $\gamma = 0.0766$, and quality factor Q = 7.063. This Q value is still high and is consistent with the idea that energy dissipates at slower rates. Nonetheless, this is a lower value than in the no-damping situation. This is because, for the underdamped pendulum, the presence of even greater amounts of damping induces energy loss, which reduces the quality factor in comparison and broadens the resonance peak.

5.3 Experiment 3

There were two different pairs of weights provided to find the double potential well. The aim was to be able to find the two equilibrium points using the smaller weights, so we'd be able to set the pendulum in 2-period motion and alter the voltage to implement chaotic motion. This was impossible to do so, as the small weights caused an extremely sensitive system regardless of the countless adjustments made to reach the two different equilibrium points. Changing the mass and the corresponding positions of the weights was attempted without result. Regardless, using the bigger weights resulted in easily finding a state where two equilibrium points could be found by pushing the wheel in their respective direction. By oscillating on one side, we got a phase portrait in Figure 23. The other side's motion is seen in the phase portrait in Figure 24.

The issue with using these weights is the limitations imposed by the driving motor, as it is not strong enough to maintain the necessary motion that is required to cause 2-period oscillations. In a rudimentary manner, a physical

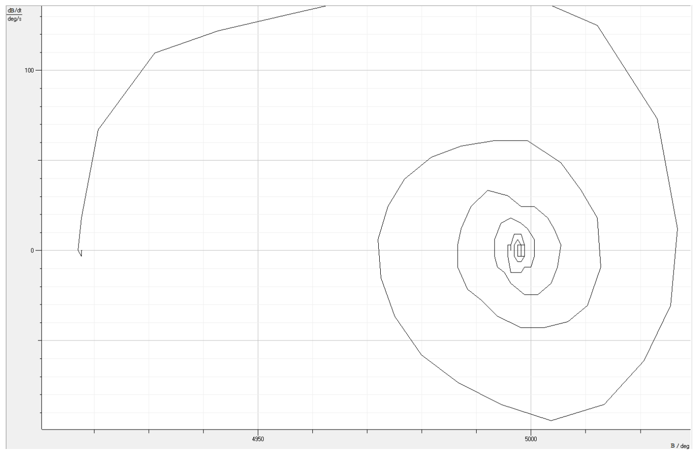

Figure 23: Big Weights, Side 1

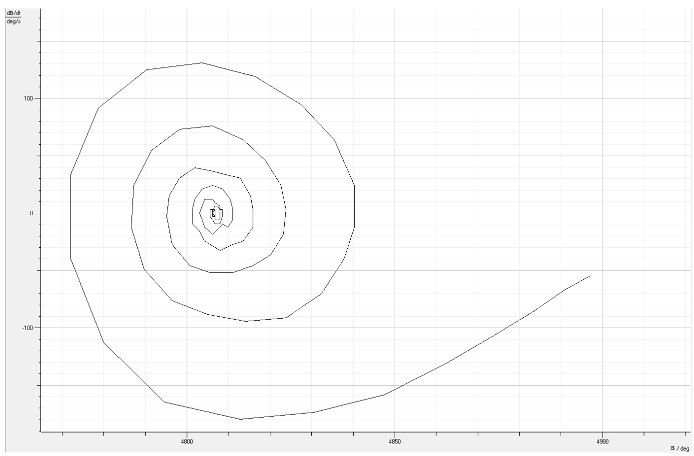

Figure 24: Big Weights, Side 2

push was caused to nonetheless model the phase portrait we would get if the set-up worked accordingly as needed. This can be followed in Figure 25.

We can note the general egg shape of the phase portrait. We can also discern two distinct regions, which are characteristic of oscillations within a double well potential. The two regions represent the two wells of the potential, separated by a barrier (the middle region where trajectories connect or cross over). In an ideal period-2 state, the system would follow a path that repeats itself every two cycles of the driving force, leading to a predictable and consistent pattern in the phase space over two cycles. Here, you can see that the trajectory does not immediately retrace itself after one loop; instead, it transitions across the central barrier to the opposite "well" before returning. This indicates that

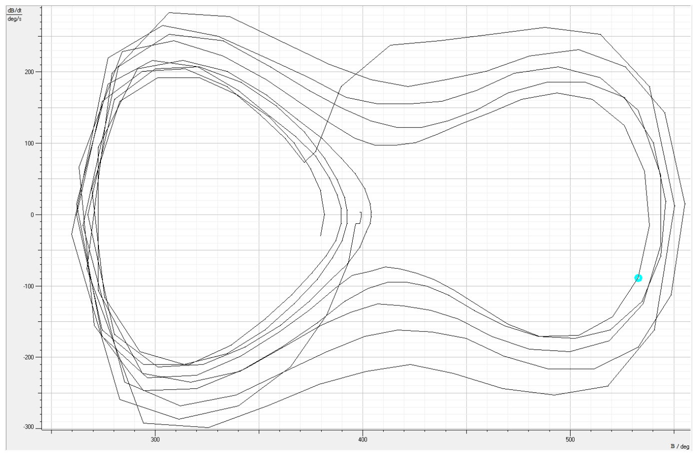

Figure 25: Phase portrait of 2-period oscillations with big weights.

the system is undergoing complex oscillatory motion and trying to establish the mentioned pattern. Because of the lack of driving force, the system does not fully stabilize in the desired state and settles into an irregular trajectory. In an ideal scenario, after getting a 2-period motion the decrease in voltage in constant intervals would eventually lead to the system achieving chaotic motions and losing periodic behaviour.

A Python code was constructed (Appendix) to model what we should've ideally obtained as a phase portrait for chaotic behaviour. The obtained simulation yields the plotted Figure 26. We can discern the extremely complex and non-repeating trajectories. It shows varied and dense paths that don't settle into a regular, periodic motion as a normal pendulum does. The overall shape of the chaotic attractor shows some degree of symmetry about the centre, exposing the underlying structure of the pendulum's dynamics. This symmetry arises from the nature of the driving force and the double-well potential. Another characteristic of chaos is the sensitivity towards initial conditions, as any switch in these would lead to different trajectories over time.

The frequency spectrum was modelled as well, in Figure 27.

We obtained a plot with a wide range of frequencies. As mentioned, this is a hallmark of chaotic motion, where the system's behaviour is irregular and non-repeating. The chaotic motion causes the pendulum to oscillate with contributions from many different frequencies. Amplitude also decreases with increasing frequency. This could indicate that because higher-frequency components have smaller amplitudes, they contribute less to the overall motion. This result varies greatly from the previous frequency spectrums obtained for periodic motion. Where we would've seen a few sharp peaks corresponding to the fundamental frequency and its harmonics, the chaotic spectrum lacks a clear, dominant single frequency. Instead, the energy is spread across a range of frequencies.

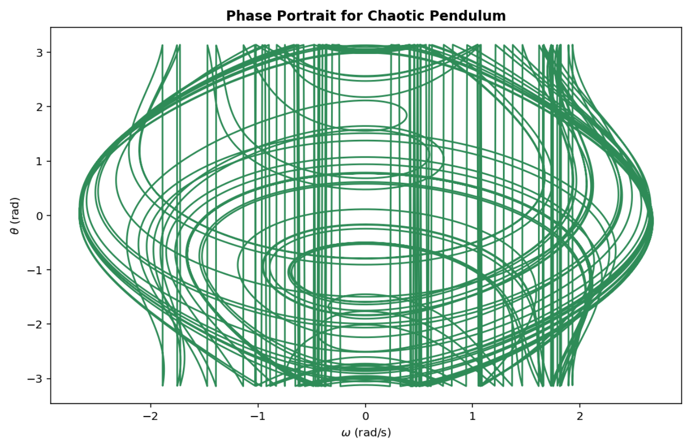

Figure 26: Phase Portrait of Chaos Motion

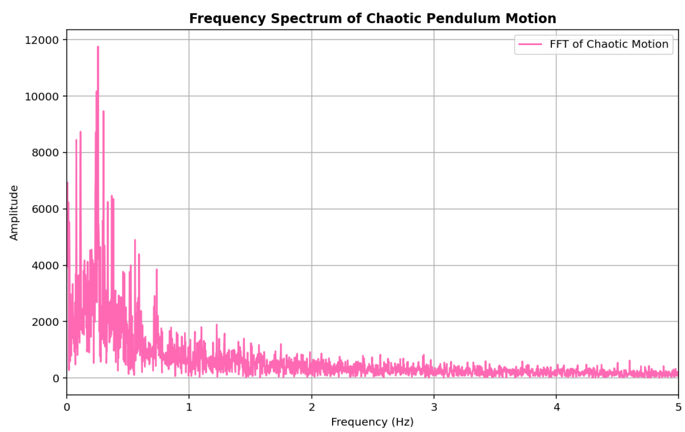

Figure 27: Frequency Spectrum of Chaos Motion

6 Conclusion

The investigation of the torsion pendulum system demonstrates the extensive variety of behaviours exhibited by a damped and driven oscillator. Through manipulation of damping forces and external driving frequencies, we observed transitions from undamped oscillations to critically damped and overdamped states. Resonance phenomena were explored, giving an insight into the conditions under which maximum amplitude response occurs and how damping limits the peak amplitude. Finally, by introducing a double-well potential, we attempted to observe complex and chaotic motion, illustrating the sensitivity of such systems to initial conditions and the underlying nonlinear dynamics. While there may have been numerous limitations at the time of studying the system, the findings still offer a comprehensive understanding of the oscillator motion of the pendulum, resonance, and chaos, while highlighting the relationship between restoring, damping, and driving forces in dynamic systems.

7 References

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8 Appendix

8.1 Lorentzian Fit

```
 #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Sun Nov 10 02:52:30 2024

@author: vwitch
"""

import matplotlib.pyplot as plt
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
import numpy as np

# data for the undamped case
voltage = [1.97, 3.95, 6.04, 7.99, 10.12, 11.94]
frequency = np.array([0.098, 0.267, 0.42, 0.55, 0.78, 1.0415])  #
Frequency in Hz
amplitude = np.array([2.45, 3.7, 6.7, 115.6, 4, 1.55])           #
Amplitude in Degrees

# Plotting Frequency vs Voltage
plt.figure()
plt.scatter(voltage, frequency)
plt.xlabel('Voltage (V)')
plt.ylabel('Frequency {Hz)')
plt.title('Frequency vs Applied Voltage for Undamped Pendulum')
plt.grid(True)
plt.show()
# define the Lorentzian function
def lorentzian(f, A, f0, gamma):
    return A * gamma**2 / ((f - f0)**2 + gamma**2)

# use parameter bounds to limit the maximum amplitude
initial_guess = [max(amplitude), 0.55, 0.1]  # Start with f0 near 0.55 Hz
bounds = ([0, 0.5, 0.01], [max(amplitude) * 1.1, 0.6, 1])  # Example
bounds

# apply curve fitting with bounds on A, f0, and gamma
params_bounded, covariance_bounded = curve_fit(lorentzian, frequency,
amplitude, p0=initial_guess, bounds=bounds)

# Generate finer frequency ranges for smooth plotting
```

```
frequency_fine = np.linspace(0.05, 1.1, 1000)

# Plotting with bounded parameters
plt.scatter(frequency, amplitude, color='hotpink', label='Data')
plt.plot(frequency_fine, lorentzian(frequency_fine, *params_bounded),
color='darkgreen', label='Lorentzian fit')
plt.xlabel('Frequency (Hz)')
plt.ylabel('Amplitude of Oscillation (Degrees)')
plt.legend()
plt.title('Amplitude vs Frequency for Undamped Case')
plt.xticks([0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0])
plt.show()

# Display fitted parameters with bounds
print("Fitted parameters:")
print("Max Amplitude:", params_bounded[0])
print("Resonant Frequency (f0):", params_bounded[1])
print("Width (gamma):", params_bounded[2])



# Data from the table
voltage_damped = [2.4, 4, 6, 8, 10, 12]
frequency_damped = np.array([0.0876, 0.255, 0.405, 0.566, 0.715, 0.8])
amplitude_damped = np.array([2.4, 3.35, 6.1, 23.8, 3.65, 1.5])

# Plotting Frequency vs Voltage
plt.figure()
plt.scatter(voltage_damped, frequency_damped)
plt.xlabel("Voltage (V)")
plt.ylabel("Frequency (Hz)")
plt.title("Frequency vs Voltage for Damped Pendulum")
plt.grid(True)
plt.show()

# Use parameter bounds to limit the maximum amplitude
initial_guess = [max(amplitude_damped), 0.55, 0.1]  # Start with f0 near
0.55 Hz
bounds = ([0, 0.5, 0.01], [max(amplitude_damped) * 1.1, 0.6, 1])  #
Example bounds

# Apply curve fitting with bounds on A, f0, and gamma
params_bounded, covariance_bounded = curve_fit(lorentzian,
frequency_damped, amplitude_damped, p0=initial_guess, bounds=bounds)

# Plotting Amplitude of Oscillation vs Frequency with the fitted curve
```

```
plt.figure()
plt.scatter(frequency_damped, amplitude_damped, color='hotpink', label='Data')
plt.plot(frequency_fine, lorentzian(frequency_fine, *params_bounded),
color='darkgreen', label='Lorentzian Fit')
plt.xlabel("Frequency (Hz)")
plt.ylabel("Amplitude of Oscillation")
plt.title("Amplitude vs Frequency for Damped Pendulum")
plt.legend()
plt.grid(True)
plt.show()

# Print the fitted parameters
print("Fitted parameters with bounds for damped case:")
print("Max Amplitude:", params_bounded[0])
print("Resonant Frequency (f0):", params_bounded[1])
print("Width (gamma):", params_bounded[2])
```

8.2 Chaotic Motion

```
    #!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Sun Nov 10 04:35:29 2024

@author: vwitch
"""

import numpy as np
import matplotlib.pyplot as plt

# Parameters for potential chaos in the pendulum
k = 0.5            # Damping coefficient
phi = 0.7          # Driving frequency
A = 1.6            # Driving amplitude
theta = 0.2        # Initial angle (rad)
omega = 0.5        # Initial angular velocity (rad/s)
dt = 0.01
nsteps = 40000     # Total steps for the simulation
transient = 7000   # Ignore initial transient period

# Function defining the pendulum's dynamics
def f(theta, omega, t):
    return -np.sin(theta) - k * omega + A * np.cos(phi * t)

# Arrays to store data
theta_final = []
```

```
omega_final = []
theta_data = []  # Array to store theta at each step
time_data = []   # Array to store time values

# Runge-Kutta integration
t = 0.0  # Initialize time
for i in range(nsteps):
    # Runge-Kutta steps
    k1a = dt * omega
    k1b = dt * f(theta, omega, t)
    k2a = dt * (omega + k1b / 2)
    k2b = dt * f(theta + k1a / 2, omega + k1b / 2, t + dt / 2)
    k3a = dt * (omega + k2b / 2)
    k3b = dt * f(theta + k2a / 2, omega + k2b / 2, t + dt / 2)
    k4a = dt * (omega + k3b)
    k4b = dt * f(theta + k3a, omega + k3b, t + dt)

    # Update theta and omega
    theta += (k1a + 2 * k2a + 2 * k3a + k4a) / 6
    omega += (k1b + 2 * k2b + 2 * k3b + k4b) / 6
    t += dt

    # Ensure theta stays within -pi to pi
    if np.abs(theta) > np.pi:
        theta -= 2 * np.pi * np.sign(theta)

    # Store all theta values and corresponding time values for FFT
    theta_data.append(theta)
    time_data.append(t)

    # Store data for phase portrait after transient period
    if i > transient:
        theta_final.append(theta)
        omega_final.append(omega)

# Plot phase portrait
plt.figure(figsize=(10, 6))
plt.plot(omega_final, theta_final, markersize=0.7, color='seagreen')
plt.xlabel(r'$\omega$ (rad/s)')
plt.ylabel(r'$\theta$ (rad)')
plt.title('Phase Portrait for Chaotic Pendulum', fontweight ='bold')
plt.show()

# Perform FFT on the theta data
theta_fft = np.fft.fft(theta_data)
frequencies = np.fft.fftfreq(len(theta_data), dt)
```

```
# Get the positive frequencies and corresponding FFT magnitudes
positive_freqs = frequencies[:len(frequencies) // 2]
positive_fft = np.abs(theta_fft[:len(theta_fft) // 2])

# Plotting the FFT result
plt.figure(figsize=(10, 6))
plt.plot(positive_freqs, positive_fft, color='hotpink', label="FFT of
Chaotic Motion")
plt.xlabel("Frequency (Hz)")
plt.ylabel("Amplitude")
#plt.yscale('log')
plt.title("Frequency Spectrum of Chaotic Pendulum Motion", fontweight =
'bold')
plt.grid(True)
plt.xlim(0, 5)
plt.legend()
plt.show()
```