MATH 340 - Lab Instructor: Valeria Barra
LAB 10 Assignment
DUE Tuesday 04-05-2016

Numerical Integration:

Gaussian Quadrature

To approximate integrals of the form

$$I(f) = \int_{-1}^{1} f(x)dx \, ,$$

we use the nodes $\{x_1, x_2, \ldots, x_n\}$ and the weights or coefficients $\{c_1, c_2, \ldots, c_n\}$ in the Gaussian quadrature formula:

$$I_n(f) = \sum_{j=1}^{n} c_j f(x_j) \; . \tag{1}$$

To transform a generic given integral $I(f) = \int_a^b f(x)dx$ to one with endpoints of integration $[-1, 1]$ to use the formula (1) above, we need a change of variable that transforms the generic domain $x \in [a, b]$ to $t \in [-1, 1]$ (and viceversa). That is, we need to express $x = \frac{b+a+t(b-a)}{2}$, so that $-1 \leq t \leq 1$ is equivalent to $a \leq x \leq b$. We can then rewrite (1) with the function f expressed in the form:

$$I(f) = \int_{-1}^{1} f\left(\frac{(b-a)t + b + a}{2}\right) \frac{b-a}{2} dt \tag{2}$$

This is formula (5.46) in your textbook.

Problem 1:

Use the form (2) for your integrand in the Gaussian quadrature formula (1) with $n = 2, 4, 6, 8$, using nodes and weights from the table attached to this document to approximate respectively:

$$(a) \int_1^2 \ln x dx = 2\ln 2 - 1 \approx 0.38629436111989$$
$$(b) \int_0^4 \frac{x}{\sqrt{x^2+9}} dx = 2$$
$$(c) \int_0^1 xe^x dx = 1$$
$$(d) \int_{-1}^2 x^5 dx = 10.5$$

For each of the approximated integrals $I_n(f)$ calculate the error from the exact value of the integral $I(f)$, given by

$$Err_n = |I(f) - I_n(f)|$$

Show your results in a tabular form as you did for last Lab assignment. Show for each execution $(a)-(d)$ in columns n, $I_n(f)$, Err_n.
Comment on your results.

Table 5.7. Nodes and Weights of Gaussian Quadrature Formulas

n	x_i	w_i
2	±0.5773502692	1.0
3	±0.7745966692	0.5555555556
	0.0	0.8888888889
4	±0.8611363116	0.3478548451
	±0.3399810436	0.6521451549
5	±0.9061798459	0.2369268851
	±0.5384693101	0.4786286705
	0.0	0.5688888889
6	±0.9324695142	0.1713244924
	±0.6612093865	0.3607615730
	±0.2386191861	0.4679139346
7	±0.9491079123	0.1294849662
	±0.7415311856	0.2797053915
	±0.4058451514	0.3818300505
	0.0	0.4179591837
8	±0.9602898565	0.1012285363
	±0.7966664774	0.2223810345
	±0.5255324099	0.3137066459
	±0.1834346425	0.3626837834

Solving this system is a formidable problem. Thankfully, the nodes $\{x_i\}$ and weights $\{w_i\}$ have been calculated and collected in tables for the most commonly used values of n. Table 5.7 contains the solutions for $n = 2, 3, \ldots, 8$. For more complete tables, see A. Stroud and D. Secrest (1966). Most computer centers will have programs to produce these nodes and weights or to directly perform the numerical integration.

There is also another approach to the development of the numerical integration formula (5.50), using the *theory of orthogonal polynomials*. From that theory, it can be shown that the nodes $\{x_1, \ldots, x_n\}$ are the zeros of the *Legendre polynomial* of degree n on the interval $[-1, 1]$. Recall that these polynomials were introduced in Section 4.7 of Chapter 4. For example,

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right),$$

and its roots are the nodes given in (5.56). Since the Legendre polynomials are well known, the nodes $\{x_j\}$ can be found without any recourse to the nonlinear system (5.59). For an introduction to this theory, see Atkinson (1989, p. 270).

The sequence of formulas (5.50) is called the *Gaussian numerical integration* method. From its definition, $I_n(f)$ uses n nodes, and it is exact for all polynomials