MATH 340 - Lab Instructor: Valeria Barra

LAB 8 Assignment

DUE Tuesday 03-22-2016

Numerical Differentiation:

First Order Derivative

To approximate the first order derivative of a function $f(x)$ at a point x_0, we can use the **forward finite difference**:

$$f'(x_0) \approx D_f(f(x_0)) = \frac{f(x_0+h) - f(x_0)}{h} \tag{1}$$

where h is the stepsize (difference between two consecutive grid points $h = x_i - x_{i-1}$). The notation used here with the subscript $_f$ means that the finite difference used is a forward difference. This numerical approximation is of first order of accuracy, meaning that the error between the approximated discrete derivative and the actual derivative, namely $Err_f = |f'(x_0) - D_f(f(x_0))|$ depends on h. So halving the stepsize h will roughly halve the error as well.

We can use a more accurate approximation of the first order derivative, using a centered difference:

$$f'(x_0) \approx D_c(f(x_0)) = \frac{f(x_0+h) - f(x_0-h)}{2h} \tag{2}$$

where again, the notation used here with the subscript $_c$ means that the finite difference used is now a centered difference. This approximation is of second order of accuracy, meaning that the error $Err_c = |f'(x_0) - D_c(f(x_0))|$ depends on h^2; so halving the stepsize h will roughly divide by 4 the error.

Problem 1:

Use both the forward difference $D_f(f(x_0))$ and the centered one $D_c(f(x_0))$ to approximate the first order derivative of the function $f(x) = e^x$ at the point $x_0 = 0$, with $h = 10^{-n}$, with $n = 1, 2, \ldots, 9$. To compute the errors Err_f and Err_c you need the actual value of $f'(x_0)$. Use the built-in `diff` in `MATLAB`, that works

with symbolic functions; do not calculate this by hand. Print your results in a table with the following columns: h, $D_f(f(x_0))$, Err_f, $D_c(f(x_0))$, Err_c, $Ratio_{f_n}$, $Ratio_{c_n}$, where $Ratio_{f_n}$, $Ratio_{c_n}$ are respectively the ratio of two consecutive errors for different n's using the forward scheme: $Ratio_{f_n} = \frac{Err_{f_n}}{Err_{f_{n-1}}}$ and the ratio of two consecutive errors for different n's using the centered scheme: $Ratio_{c_n} = \frac{Err_{c_n}}{Err_{c_{n-1}}}$ (for $n \geq 2$ of course). Compare your table with table of example 5.3 on page 247 of your textbook. To compare with the book's results, note that you need 14 digits of precision.
To print out your numerical results in tabular form (the border lines don't matter) use the `fprintf` command and make sure to end it with a new line `\n`.

Second Order Derivative

To approximate the second order derivative of a function $f(x)$ at a point x_0, we can use the **centered finite difference of second order**:

$$f''(x_0) \approx D_c^{(2)}(f(x_0)) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2} \tag{3}$$

This approximation is of second order of accuracy, meaning that the error $Err_c = |f''(x_0) - D_c^{(2)}(f(x_0))|$ depends on h^2; so halving the stepsize h will roughly divide by 4 the error.

Errors due to round-off

If we assume that the evaluation of the function at the grid points $f(x_i)$ has some round off errors, it can be written as a truncated version $\widehat{f}(x_i)$, and this leads to a truncated version of the second derivative:

$$f''(x_0) \approx D_c^{(2)}(\widehat{f}(x_0)) = \frac{\widehat{f}(x_0 - h) - 2\widehat{f}(x_0) + \widehat{f}(x_0 + h)}{h^2} \tag{4}$$

Problem 2:

Use both the exact version of the evaluations (3) and the truncated one (4) to approximate the second order derivative of $f(x) = \cos x$ at the point $x_0 = \pi/6$, with $h = \frac{1}{2^n}$, with $n = 1, 2, \ldots, 9$. To truncate the evaluations $f(x_i)$ with 6 digits of precision, use `round(f(x_i)*10^6)/10^6`, or if you use recent versions of Matlab you can take advantage of the command `round(f(x_i),n)` where `n` represents how many decimal digits you want to truncate at. Print your results in a table with the following columns: h, $D_c^{(2)}(f(x_0))$, $D_c^{(2)}(\widehat{f}(x_0))$, $ErrD_c^{(2)}(f)$, $ErrD_c^{(2)}(\widehat{f})$, $RatioD_c^{(2)}(f)$, $RatioD_c^{(2)}(\widehat{f})$, where the ratios are defined as above.

Finally, plot in the same figure both $ErrD_c(f)$, $ErrD_c(\widehat{f})$ versus a vector of all h's. You can constraint the axes to better visualize curves closely. What happens to $ErrD_c(\widehat{f})$? Write your conclusions on your results.