A STUDY OF POSITIVE THOMPSON KNOTS VIA MACHINE LEARNING: THE NUMBER OF CONNECTED COMPONENTS

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Abstract.

1. Preliminaries and notations.

There are several equivalent definitions of the Thompson groups F and of the Brown-Thompson group F_3. In this section we review the definitions that are most appropriate for our work in this paper, namely the one that use tree diagrams. For further information we refer to [13, 8] and [11].

An element of F is given by a pair of rooted, planar, binary trees (T_+, T_-) with the same number of leaves. As usual, we draw a pair of trees in the plane with one tree upside down on top of the other. Similarly, the elements of F_3 admit a description in terms of pairs of ternary trees. Two pairs of ternary trees are equivalent if they differ by a pair of opposing carets, see Figure 1. Thanks to this equivalence relation, the following rule defines the multiplication in both F and F_3:

Figure 1. Pairs of opposing carets in F and F_3.

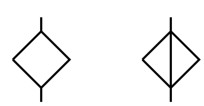


Figure 2. The monomorphism $\iota : F \to F_3$ is obtained by turning every trivalent vertex of a binary tree diagram into a 4-valent one and connecting the new edges in the only possible planar way.

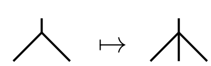
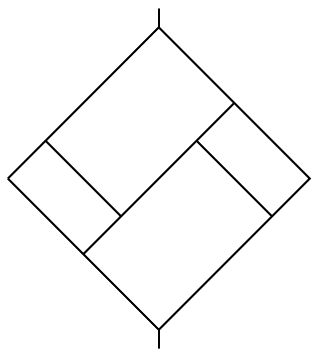
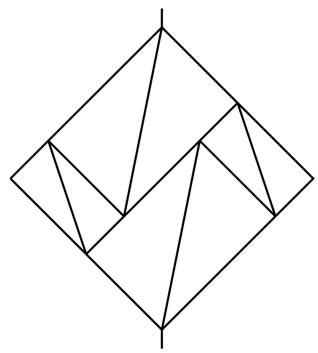
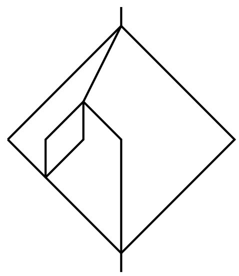


FIGURE 3. A element of F, its image under the injection $\iota : F \to F_3$, and an element in $F_3 \setminus \iota(F)$.

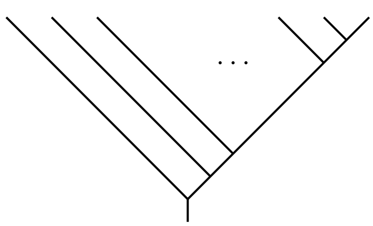
$Y =$  $\iota(Y) =$ 

$Z =$ 

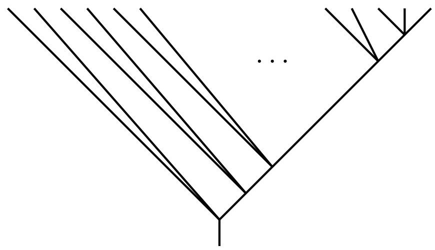
$(T_+, T) \cdot (T, T_-) := (T_+, T_-)$. The trivial element is represented by any pair (T, T) and the inverse of (T_+, T_-) is just (T_-, T_+).

There is a natural injection $\iota : F \hookrightarrow F_3$. Given $(T_+, T_-) \in F$, firstly, add a new leaf to the middle of each vertex (thus turning every trivalent vertex into a 4-valent one, see Figure 2). Then, join the new edges in the only planar way. This yields an element of F_3. We provide an example in Figure 3.

The positive monoid F_+ of F consists of the element whose bottom tree may be chosen of the following form

$T_- =$ 

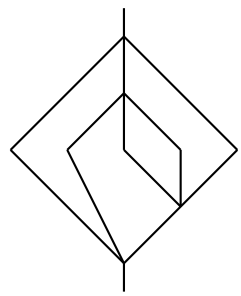
Similarly, the positive monoid $F_{3,+}$ of F_3 is given by the ternary tree diagrams whose bottom tree is The positive elements of F_3 are those whose bottom tree is of the form



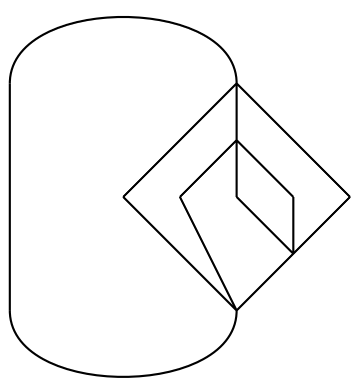
Note that $\iota(F_+)$ is contained in $F_{3,+}$.

We now review Jones's construction of knots from elements of F_3 by giving an explicit example. Consider the element of F_3

$$X = \frac{T_+}{T_-} =$$



Now join the two roots by an edge. Wolog we may suppose that the new edge passes through the point $(0, 0)$.

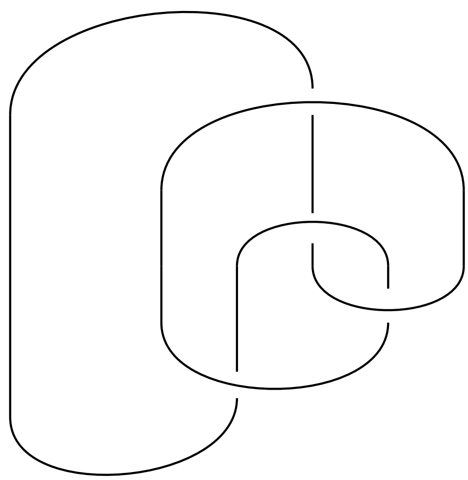


At this stage all the vertices are 4-valent, change them according to the rule displayed in Figure 4 to obtain a knot diagram.

FIGURE 4. The rules needed to turn 4-valent vertices into crossings.

Therefore, in our example we get the knot $\mathcal{L}(T_+, T_-)$

$\mathcal{L}(T_+, T_-) =$

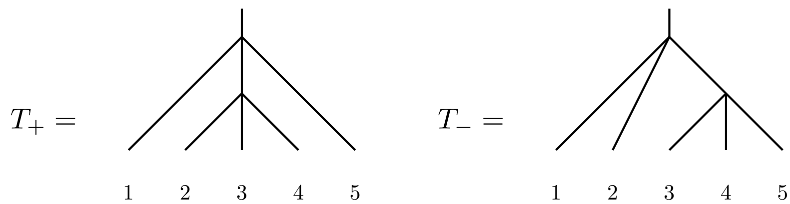


2. THOMPSON PERMUTATIONS

In this section we want to assign to each element (T_+, T_-) of the Brown-Thompson group. First we assign a map from ternary rooted trees to to permutations. More precisely, give a tree with n leaves we construct a permutation of S_{n+1}, acting on $\{0, 1, \dots n\}$, which is the product of $n + 1$ transpositions (n is always odd).

We briefly fix the notation for the permutations, [16, Chapter 3]. Given k distinct integers $i_1, \dots, i_k$ in $\{0, \dots, n\}$, the symbol $(i_1, \dots, i_k)$ represents the permutation $p : \{0, \dots, n\} \to \{0, \dots, n\}$, where $p(i_j) = i_{j+1}$ for $j < k$, $p(i_k) = i_1$, and $p(s) = s$ for all $s \in \{0, \dots, n\} \setminus \{i_1, \dots, i_k\}$. When $k = 2$, the permutation is called a transposition. A permutation of the form $(i_1, \dots, i_k)$ is called a k-cycle. Two cycles are said to be disjoint if they have no integers in common. Every permutation is the product of disjoint cycles.

Given a rooted ternary tree, say with $2n + 1$ leaves, we show how to construct a permutation on the set $\{0, \dots, 2n+1\}$ associated with the tree. We illustrate it with a couple of examples. Consider the pair of trees



where we numbered the leaves of each tree from left to right (starting from 1). We start with the tree T_+. We consider each leaf and take a path according to the rules displayed in Figure 5. Each path ends when we meet another leaf or the root (the paths for T_+ are highlighted in red in the figure below).

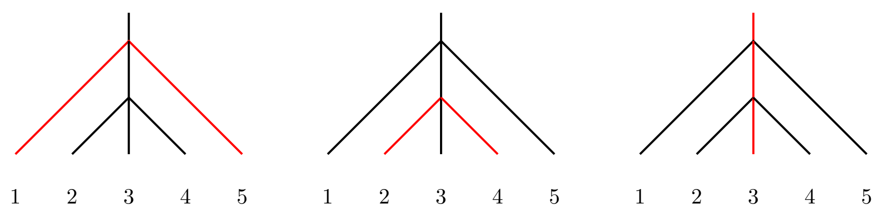
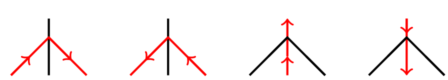
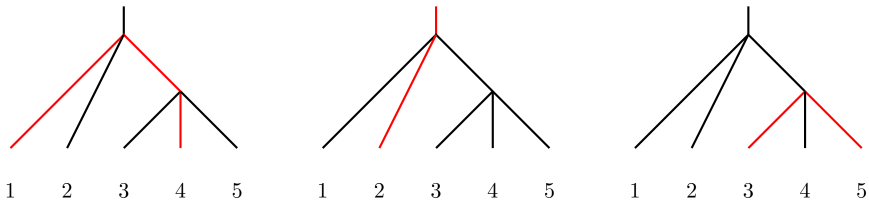


FIGURE 5. Rules for calculating the permutation.



We note that in every tree there exists exactly one path from a leaf, say f, to the root. For this path we consider the permutation $(0, f)$. For example, in our case we have $(1, 5)$, $(2, 4)$, $(0, 3)$, $(4, 2)$, $(5, 1)$. Since all the transpositions (but the one corresponding to the root) occur exactly twice, we set aside only one of each. Now we define the permutation $\pi(T_+) : \{0, 1, \ldots, 2n+1\} \to \{0, 1, \ldots, 2n+1\}$ to be the product of all these transpositions. We call $\pi(T_+)$ the tangled permutation associated with T_+. In this example, we get $\pi(T_+) = (1, 5)(2, 4)(0, 3)$.

For the second tree we follow the same procedure. For T_- the paths are



and the transpositions are $(1,4)$, $(0,2)$, $(3,5)$, $(4,1)$, $(5,3)$. Therefore, the permutation associated to T_- is $\pi(T_-) = (0,2)(1,4)(3,5)$. Now, the permutation associated with (T_+, T_-) is defined as $\mathcal{P}(T_+, T_-) := \pi(T_+) \circ \pi(T_-)$. We call $\mathcal{P}(T_+, T_-)$ the Thompson permutation of (T_+, T_-). In our example we have

$$\mathcal{P}(T_+, T_-) = (1,5)(2,4)(0,3)(0,2)(1,4)(3,5) = (1,4,2,0,3,5)$$

For $n \geq 1$, the symmetric group S_n is the group of permutation on the set $\{0, 1, \ldots, n-1\}$. We adopt the cycle notation for permutation. Recall that $(i_1, i_2, \ldots, i_k)$ denotes the permutation mapping i_1 to i_2, i_2 to i_3, and so on.

Here we give a formula for the permutation of the bottom tree of a positive element of F_3. Here we provide a simple formula for the Thompson permutation associated with the standard ternary bottom tree T_-^n with n leaves. Note that n is necessarily an odd number. For the smallest values of n it holds

$$\pi(T_-^n) \begin{cases} (0,2)(1,3) & \text{if } n = 3 \\ (0,2)(1,4)(3,5) & \text{if } n = 5 \\ (0,2)(1,4)(3,6)(5,7) & \text{if } n = 7 \end{cases}$$

A simple inductive argument yields the following formula

$$\pi(T_-^n) := (0,2) \left(\prod_{i=0}^{(n-3)/2} (2i+1, 2i+4) \right) (n-2, n) \in S_n \tag{1}$$

where by convention $\prod_{i=0}^{k} g_k := g_1 g_2 \cdots g_k$.

For the top tree we have the following formula. A ternary tree T' with $2n+3$ leaves can be constructed from a tree T with $2n+1$ by attaching to it a ternary caret below one of its leaves, say the k-th leaf. Let $\pi(T)$ the permutation acting on $\{0, 2n\}$. Then $\pi(T')$ is given by the following formula

$$\pi(T')(j) := \begin{cases} \pi(T)(j) & \text{if } j \leq k-1 \text{ and } \pi(T)(j) \leq k-1 \\ k+1 & \text{if } j \leq k-1 \text{ and } \pi(T)(j) = k \\ \pi(T)(j) + 2 & \text{if } j \leq k-1 \text{ and } \pi(T)(j) \geq k+1 \\ k+2 & \text{if } j = k \\ k & \text{if } j = k+2 \\ \pi(T)(j-2) + 2 & \text{if } j \geq k+3 \text{ and } \pi(T)(j-2) \geq k+1 \\ k+1 & \text{if } j \geq k+3 \text{ and } \pi(T)(j-2) = k \\ \pi(T)(j-2) & \text{if } j \geq k+3 \text{ and } \pi(T)(j-2) \leq k-1 \end{cases}$$

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