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Integral Equations

A compendium

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Contents

1 Preface

This compendium focuses on fundamental notions and statements of the theory of linear operators and integral operators. The Fredholm theory is set forth both in the Banach and Hilbert spaces. The elements of the theory of harmonic functions are presented, including the reduction of boundary value problems to integral equations and potential theory, as well as some methods of solution to integral equations. Some methods of numerical solution are also considered.

The compendium contains many examples that illustrate most important notions and the solution techniques.

The material is largely based on the books *Elements of the Theory of Functions and Functional Analysis* by A. Kolmogorov and S. Fomin (Dover, New York, 1999), *Linear Integral Equations* by R. Kress (2nd edition, Springer, Berlin, 1999), and *Logarithmic Integral Equations in Electromagnetics* by Y. Shestopalov, Y. Smirnov, and E. Chernokozhin (VSP, Utrecht, 2000).

2 Notion and examples of integral equations (IEs). Fredholm IEs of the first and second kind

Consider *an integral equation (IE)*

$$f(x) = \phi(x) + \lambda \int_a^b K(x,y)\phi(y)dy. \tag{1}$$

This is *an IE of the 2nd kind*. Here $K(x,y)$ is a given function of two variables (*the kernel*) defined in the square

$$\Pi = \{(x,y) : \, a \le x \le b, \, a \le y \le b\},$$

$f(x)$ is a given function, $\phi(x)$ is a sought-for (unknown) function, and λ is a parameter.

We will often assume that $f(x)$ is a continuous function.

The IE

$$f(x) = \int_a^b K(x,y)\phi(y)dy \tag{2}$$

constitutes an example of *an IE of the 1st kind*.

We will say that equation (1) (resp. (2)) is *the Fredholm IE of the 2nd kind* (resp. 1st kind) if the the kernel $K(x,y)$ satisfies one of the following conditions:

(i) $K(x,y)$ is continuous as a function of variables (x,y) in the square Π;

(ii) $K(x,y)$ is maybe discontinuous for some (x,y) but the double integral

$$\int_a^b \int_a^b |K^2(x,y)|dxdy < \infty, \tag{3}$$

i.e., takes a finite value (converges in the square Π in the sense of the definition of convergent Riemann integrals).

An IE

$$f(x) = \phi(x) + \lambda \int_0^x K(x,y)\phi(y)dy, \tag{4}$$

is called *the Volterra IE of the 2nd kind*. Here the kernel $K(x,y)$ may be continuous in the square $\Pi = \{(x,y) : \, a \le x \le b, \, a \le y \le b\}$ for a certain $b > a$ or discontinuous and satisfying condition (3); $f(x)$ is a given function; and λ is a parameter,

Example 1 *Consider a Fredholm IE of the 2nd kind*

$$x^2 = \phi(x) - \int_0^1 (x^2 + y^2)\phi(y)dy, \tag{5}$$

where the kernel $K(x,y) = (x^2 + y^2)$ is defined and continuous in the square $\Pi_1 = \{(x,y) : \, 0 \le x \le 1, \, 0 \le y \le 1\}$, $f(x) = x^2$ is a given function, and the parameter $\lambda = -1$.

Example 2 *Consider an IE of the 2nd kind*

$$f(x) = \phi(x) - \int_0^1 \ln|x-y|\phi(y)dy, \tag{6}$$

were the kernel $K(x,y) = \ln|x-y|$ is discontinuous (unbounded) in the square Π_1 along the line $x = y$, $f(x)$ is a given function, and the parameter $\lambda = -1$. Equation (6) a Fredholm IE of the 2nd kind because the double integral

$$\int_0^1 \int_0^1 \ln^2|x-y|dxdy < \infty, \tag{7}$$

i.e., takes a finite value—converges in the square Π_1.

2.1 Primitive function

Below we will use some properties of *the primitive function* (or (general) antiderivative) $F(x)$ defined, for a function $f(x)$ defined on an interval I, by

$$F'(x) = f(x). \tag{8}$$

The primitive function $F(x)$ for a function $f(x)$ (its antiderivative) is denoted using the integral sign to give the notion of *the indefinite integral* of $f(x)$ on I,

$$F(x) = \int f(x)dx. \tag{9}$$

Theorem 1 *If $f(x)$ is continuous on $[a,b]$ then the function*

$$F(x) = \int_a^x f(x)dx. \tag{10}$$

(the antiderivative) is differentiable on $[a,b]$ and $F'(x) = f(x)$ for each $x \in [a,b]$.

Example 3 *$F_0(x) = arctan\sqrt{x}$ is a primitive function for the function*

$$f(x) = \frac{1}{2\sqrt{x}(1+x)}$$

because

$$F_0'(x) = \frac{dF_0}{dx} = \frac{1}{2\sqrt{x}(1+x)} = f(x). \tag{11}$$

3 Examples of solution to integral equations and ordinary differential equations

Example 4 *Consider a Volterra IE of the 2nd kind*

$$f(x) = \phi(x) - \lambda \int_0^x e^{x-y}\phi(y)dy, \tag{12}$$

where the kernel $K(x,y) = e^{x-y}$, $f(x)$ is a given continuously differentiable function, and λ is a parameter. Differentiating (12) we obtain

$$\begin{aligned} f(x) &= \phi(x) - \lambda \int_0^x e^{x-y}\phi(y)dy \\ f'(x) &= \phi'(x) - \lambda[\phi(x) + \int_0^x e^{x-y}\phi(y)dy]. \end{aligned}$$

Subtracting termwise we obtain an ordinary differential equatin (ODE) of the first order

$$\phi'(x) - (\lambda + 1)\phi(x) = f'(x) - f(x) \tag{13}$$

with respect to the (same) unknown function $\phi(x)$*. Setting* $x = 0$ *in (12) we obtain the initial condition for the unknown function* $\phi(x)$

$$\phi(0) = f(0). \tag{14}$$

Thus we have obtained the initial value problem for $\phi(x)$

$$\phi'(x) - (\lambda + 1)\phi(x) = F(x), \quad F(x) = f'(x) - f(x), \tag{15}$$

$$\phi(0) = f(0). \tag{16}$$

Integrating (15) subject to the initial condition (16) we obtain the Volterra IE (12), which is therefore equivalent to the initial value problem (15) and (16).

Solve (12) by the method of successive approximations. Set

$$\begin{aligned} K(x, y) &= e^{x-y} \quad 0 \le y \le x \le 1, \\ K(x, y) &= 0 \quad 0 \le x \le y \le 1. \end{aligned}$$

Write two first successive approximations:

$$\phi_0(x) = f(x), \quad \phi_1(x) = f(x) + \lambda \int_0^1 K(x, y)\phi_0(x)dy, \tag{17}$$

$$\phi_2(x) = f(x) + \lambda \int_0^1 K(x, y)\phi_1(y)dy = \tag{18}$$

$$f(x) + \lambda \int_0^1 K(x, y)f(y)dy + \lambda^2 \int_0^1 K(x, t)dt \int_0^1 K(t, y)f(y)dy. \tag{19}$$

Set

$$K_2(x, y) = \int_0^1 K(x, t)K(t, y)dt = \int_y^x e^{x-t}e^{t-y}dt = (x - y)e^{x-y} \tag{20}$$

to obtain (by changing the order of integration)

$$\phi_2(x) = f(x) + \lambda \int_0^1 K(x, y)f(y)dy + \lambda^2 \int_0^1 K_2(x, y)f(y)dy. \tag{21}$$

The third approximation

$$\begin{aligned} \phi_3(x) &= f(x) + \lambda \int_0^1 K(x, y)f(y)dy + \\ &+ \lambda^2 \int_0^1 K_2(x, y)f(y)dy + \lambda^3 \int_0^1 K_3(x, y)f(y)dy, \end{aligned} \tag{22}$$

where

$$K_3(x, y) = \int_0^1 K(x, t)K_2(t, y)dt = \frac{(x - y)^2}{2!}e^{x-y}. \tag{23}$$

The general relationship has the form

$$\phi_n(x) \quad = \quad f(x) + \sum_{m=1}^{n} \lambda^m \int_0^1 K_m(x,y)f(y)dy, \tag{24}$$

where

$$K_1(x,y) = K(x,y), \quad K_m(x,y) = \int_0^1 K(x,t)K_{m-1}(t,y)dt = \frac{(x-y)^{m-1}}{(m-1)!}e^{x-y}. \tag{25}$$

Assuming that successive approximations (24) converge we obtain the (unique) solution to IE (12) by passing to the limit in (24)

$$\phi(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_0^1 K_m(x,y)f(y)dy. \tag{26}$$

Introducing the notation for the resolvent

$$\Gamma(x,y,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,y). \tag{27}$$

and changing the order of summation and integration in (26) we obtain the solution in the compact form

$$\phi(x) = f(x) + \lambda \int_0^1 \Gamma(x,y,\lambda)f(y)dy. \tag{28}$$

Calculate the resolvent explicitly using (25):

$$\Gamma(x,y,\lambda) = e^{x-y} \sum_{m=1}^{\infty} \lambda^{m-1} \frac{(x-y)^{m-1}}{(m-1)!} = e^{x-y}e^{\lambda(x-y)} = e^{(\lambda+1)(x-y)}, \tag{29}$$

so that (28) gives the solution

$$\phi(x) = f(x) + \lambda \int_0^1 e^{(\lambda+1)(x-y)} f(y)dy. \tag{30}$$

Note that according to (17)

$$\begin{aligned} \Gamma(x,y,\lambda) &= e^{(\lambda+1)(x-y)}, \quad 0 \le y \le x \le 1, \\ \Gamma(x,y,\lambda) &= 0, \quad 0 \le x \le y \le 1. \end{aligned}$$

and the series (27) converges for every λ.

4 Reduction of ODEs to the Volterra IE and proof of the unique solvability using the contraction mapping

4.1 Reduction of ODEs to the Volterra IE

Consider an ordinary differential equatin (ODE) of the first order

$$\frac{dy}{dx} = f(x,y) \tag{31}$$

with the initial condition

$$y(x_0) = y_0, \tag{32}$$

where $f(x, y)$ is defined and continuous in a two-dimensional domain G which contains the point (x_0, y_0).

Integrating (31) subject to (32) we obtain

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt, \tag{33}$$

which is called *the Volterra integral equation of the second kind* with respect to the unknown function $\phi(t)$. This equation is equivalent to the initial value problem (31) and (32). Note that this is generally a *nonlinear* integral equation with respect to $\phi(t)$.

Consider now a linear ODE of the second order with variable coefficients

$$y'' + A(x)y' + B(x)y = g(x) \tag{34}$$

with the initial condition

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \tag{35}$$

where $A(x)$ and $B(x)$ are given functions continuous in an interval G which contains the point x_0. Integrating y'' in (34) we obtain

$$y'(x) = -\int_{x_0}^{x} A(t)y'(x)dx - \int_{x_0}^{x} B(x)y(x)dx + \int_{x_0}^{x} g(x)dx + y_1. \tag{36}$$

Integrating the first integral on the right-hand side in (36) by parts yields

$$y'(x) = -A(x)y(x) - \int_{x_0}^{x}(B(x) - A'(x))y(x)dx + \int_{x_0}^{x} g(x)dx + A(x_0)y_0 + y_1. \tag{37}$$

Integrating a second time we obtain

$$y(x) = -\int_{x_0}^{x} A(x)y(x)dx - \int_{x_0}^{x}\int_{x_0}^{x}(B(t) - A'(t))y(t)dtdx + \int_{x_0}^{x}\int_{x_0}^{x} g(t)dtdx + [A(x_0)y_0 + y_1](x - x_0) + y_0. \tag{38}$$

Using the relationship

$$\int_{x_0}^{x}\int_{x_0}^{x} f(t)dtdx = \int_{x_0}^{x}(x - t)f(t)dt, \tag{39}$$

we transform (38) to obtain

$$y(x) = -\int_{x_0}^{x}\{A(t) + (x - t)[(B(t) - A'(t))]\}y(t)dt + \int_{x_0}^{x}(x - t)g(t)dt + [A(x_0)y_0 + y_1](x - x_0) + y_0. \tag{40}$$

Separate the known functions in (40) and introduce the notation for the *kernel function*

$$K(x, t) = -A(t) + (t - x)[(B(t) - A'(t))], \tag{41}$$

$$f(x) = \int_{x_0}^{x}(x - t)g(t)dt + [A(x_0)y_0 + y_1](x - x_0) + y_0. \tag{42}$$

Then (40) becomes

$$y(x) = f(x) + \int_{x_0}^{x} K(x, t)y(t))dt, \tag{43}$$

which is the Volterra IE of the second kind with respect to the unknown function $\phi(t)$. This equation is equivalent to the initial value problem (34) and (35). Note that here we obtain a *linear* integral equation with respect to $y(x)$.

Example 5 *Consider a homogeneous linear ODE of the second order with constant coefficients*

$$y'' + \omega^2 y = 0 \tag{44}$$

and the initial conditions (at $x_0 = 0$)

$$y(0) = 0, \quad y'(0) = 1. \tag{45}$$

We see that here

$$A(x) \equiv 0, \quad B(x) \equiv \omega^2, \quad g(x) \equiv 0, \quad y_0 = 0, \quad y_1 = 1,$$

if we use the same notation as in (34) and (35).

Substituting into (40) and calculating (41) and (42),

$$\begin{aligned} K(x,t) &= \omega^2(t-x), \\ f(x) &= x, \end{aligned}$$

we find that the IE (43), equivalent to the initial value problem (44) and (45), takes the form

$$y(x) = x + \int_0^x (t-x)y(t)dt. \tag{46}$$

4.2 Principle of contraction mappings

A metric space is a pair of a set X consisting of elements (points) and a distance which is a nonnegative function satisfying

$$\begin{aligned} \rho(x,y) &= 0 \quad \text{if and only if} \quad x = y, \\ \rho(x,y) &= \rho(y,x) \quad \text{(Axiom of symmetry)}, \\ \rho(x,y) + \rho(y,z) &\geq \rho(x,z) \quad \text{(Triangle axiom)} \end{aligned} \tag{47}$$

A metric space will be denoted by $R = (X, \rho)$.

A sequence of points x_n in a metric space R is called *a fundamental sequence* if it satisfies the Cauchy criterion: for arbitrary $\epsilon > 0$ there exists a number N such that

$$\rho(x_n, x_m) < \epsilon \quad \text{for all} \quad n, m \geq N. \tag{48}$$

Definition 1 *If every fundamental sequence in the metric space R converges to an element in R, this metric space is said to be complete.*

Example 6 *The (Euclidian) n-dimensional space R^n of vectors $\bar{a} = (a_1, \ldots, a_n)$ with the Euclidian metric*

$$\rho(\bar{a}, \bar{b}) = \|\bar{a} - \bar{b}\|_2 = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

is complete.

Example 7 *The space $C[a,b]$ of continuous functions defined in a closed interval $a \leq x \leq b$ with the metric*

$$\rho(\phi_1, \phi_2) = \max_{a \leq x \leq b} |\phi_1(x) - \phi_2(x)|$$

is complete. Indeed, every fundamental (convergent) functional sequence $\{\phi_n(x)\}$ of continuous functions converges to a continuous function in the $C[a,b]$-metric.

Example 8 *The space $C^{(n)}[a,b]$ of componentwise continuous n-dimensional vector-functions $\overline{f} = (f_1, \dots, f_n)$ defined in a closed interval $a \leq x \leq b$ with the metric*

$$\rho(\overline{f}^1, \overline{f}^2) = \| \max_{a \leq x \leq b} |f_i^1(x) - f_i^2(x)| \|_2$$

is complete. Indeed, every componentwise-fundamental (componentwise-convergent) functional sequence $\{\bar{\phi}_n(x)\}$ of continuous vector-functions converges to a continuous function in the above-defined metric.

Definition 2 *Let R be an arbitrary metric space. A mapping A of the space B into itself is said to be a contraction if there exists a number $\alpha < 1$ such that*

$$\rho(Ax, Ay) \leq \alpha \rho(x, y) \quad \textit{for any two points} \quad x, y \in R. \tag{49}$$

Every contraction mapping is continuous:

$$x_n \to x \quad \text{yields} \quad Ax_n \to Ax \tag{50}$$

Theorem 2 *Every contraction mapping defined in a complete metric space R has one and only one fixed point; that is the equation $Ax = x$ has only one solution.*

We will use the principle of contraction mappings to prove Picard's theorem.

Theorem 3 *Assume that a function $f(x,y)$ satisfies the Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \quad \textit{in} \quad G. \tag{51}$$

Then on a some interval $x_0 - d < x < x_0 + d$ there exists a unique solution $y = \phi(x)$ to the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{52}$$

Proof. Since function $f(x,y)$ is continuous we have $|f(x,y)| \leq k$ in a region $G' \subset G$ which contains (x_0, y_0). Now choose a number d such that

$$\begin{aligned} (x,y) &\in G' \quad \text{if} \quad |x_0 - x| \leq d,\ |y_0 - y| \leq kd \\ Md &< 1. \end{aligned}$$

Consider the mapping $\psi = A\phi$ defined by

$$\psi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t)) dt, \quad |x_0 - x| \leq d. \tag{53}$$

Let us prove that this is a contraction mapping of the complete set $C^* = C[x_0 - d, x_0 + d]$ of continuous functions defined in a closed interval $x_0 - d \le x \le x_0 + d$ (see Example 7). We have

$$|\psi(x) - y_0| = \left|\int_{x_0}^{x} f(t, \phi(t))dt\right| \le \int_{x_0}^{x} |f(t, \phi(t))|\, dt \le k \int_{x_0}^{x} dt = k(x - x_0) \le kd. \tag{54}$$

$$|\psi_1(x) - \psi_2(x)| \le \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_2(t))|\, dt \le Md \max_{x_0 - d \le x \le x_0 + d} |\phi_1(x)| - \phi_2(x). \tag{55}$$

Since $Md < 1$, the mapping $\psi = A\phi$ is a contraction.

From this it follows that the operator equation $\phi = A\phi$ and consequently the integral equation (33) has one and only one solution.

This result can be generalized to systems of ODEs of the first order.

To this end, denote by $\bar{f} = (f_1, \ldots, f_n)$ an n-dimensional vector-function and write the initial value problem for a system of ODEs of the first order using this vector notation

$$\frac{d\bar{y}}{dx} = \bar{f}(x, \bar{y}), \quad \bar{y}(x_0) = \bar{y}_0. \tag{56}$$

where the vector-function $\bar{f}$ is defined and continuous in a region G of the $n+1$-dimensional space R^{n+1} such that G contains the '$n+1$-dimensional' point $x_0, \bar{y}_0$. We assume that f satisfies the the Lipschitz condition (51) termwise in variables $\bar{y}_0$. the initial value problem (56) is equivalent to a system of IEs

$$\bar{\phi}(x) = \bar{y}_0 + \int_{x_0}^{x} \bar{f}(t, \bar{\phi}(t))dt. \tag{57}$$

Since function $\bar{f}(t, \bar{y})$ is continuous we have $\|f(x, y)\| \le K$ in a region $G' \subset G$ which contains $(x_0, \bar{y}_0)$. Now choose a number d such that

$$\begin{aligned} (x, \bar{y}) &\in G' \quad \text{if} \quad |x_0 - x| \le d,\ \|\bar{y}_0 - \bar{y}\| \le Kd \\ Md &< 1. \end{aligned}$$

Then we consider the mapping $\bar{\psi} = A\bar{\phi}$ defined by

$$\bar{\psi}(x) = \bar{y}_0 + \int_{x_0}^{x} \bar{f}(t, \bar{\phi}(t))dt, \quad |x_0 - x| \le d. \tag{58}$$

and prove that this is a contraction mapping of the complete space $\bar{C}^* = \bar{C}[x_0 - d, x_0 + d]$ of continuous (componentwise) vector-functions into itself. The proof is a termwise repetitiion of the reasoning in the scalar case above.

5 Unique solvability of the Fredholm IE of the 2nd kind using the contraction mapping. Neumann series

5.1 Linear Fredholm IE of the 2nd kind

Consider a Fredholm IE of the 2nd kind

$$f(x) = \phi(x) + \lambda \int_a^b K(x, y)\phi(y)dy, \tag{59}$$

where *the kernel* $K(x,y)$ is a continuous function in the square

$$\Pi = \{(x,y):\ a \le x \le b,\ a \le y \le b\},$$

so that $|K(x,y)| \le M$ for $(x,y) \in \Pi$. Consider the mapping $g = Af$ defined by

$$g(x) = \phi(x) + \lambda \int_a^b K(x,y)\phi(y)dy, \tag{60}$$

of the complete space $C[a,b]$ into itself. We have

$$\begin{aligned} g_1(x) &= \lambda \int_a^b K(x,y) f_1(y)dy + \phi(x), \\ g_2(x) &= \lambda \int_a^b K(x,y) f_2(y)dy + \phi(x), \end{aligned}$$

$$\begin{aligned} \rho(g_1, g_2) &= \max_{a \le x \le b} |g_1(x) - g_2(x)| \le \\ &\le |\lambda| \int_a^b |K(x,y)||f_1(y) - f_2(y)|dy \le |\lambda| M(b-a) \max_{a \le y \le b} |f_1(y)| - f_2(y)| \qquad (61) \\ &< \rho(f_1, f_2) \quad \text{if} \quad |\lambda| < \frac{1}{M(b-a)} \qquad (62) \end{aligned}$$

Consequently, the mapping A is a contraction if

$$|\lambda| < \frac{1}{M(b-a)}, \tag{63}$$

and (59) has a unique solution for sufficiently small $|\lambda|$ satisfying (63).

The successive approximations to this solution have the form

$$f_n(x) = \phi(x) + \lambda \int_a^b K(x,y) f_{n-1}(y)dy. \tag{64}$$

5.2 Nonlinear Fredholm IE of the 2nd kind

The method is applicable to nonlinear IEs

$$f(x) = \phi(x) + \lambda \int_a^b K(x,y,f(y))dy, \tag{65}$$

where the kernel K and ϕ are continuous functions, and

$$|K(x,y,z_1) - K(x,y,z_2)| \le M|z_1 - z_2|. \tag{66}$$

If λ satisfies (63) we can prove in the same manner that the mapping $g = Af$ defined by

$$g(x) = \phi(x) + \lambda \int_a^b K(x,y,f(y))dy, \tag{67}$$

of the complete space $C[a,b]$ into itself is a contraction because for this mapping, the inequality (61) holds.

5.3 Linear Volterra IE of the 2nd kind

Consider a Volterra IE of the 2nd kind

$$f(x) = \phi(x) + \lambda \int_a^x K(x,y)\phi(y)dy, \tag{68}$$

where *the kernel* $K(x,y)$ is a continuous function in the square Π for some $b > a$, so that $|K(x,y)| \le M$ for $(x,y) \in \Pi$.

Formulate a generalization of the principle of contraction mappings.

Theorem 4 *If A is a continuous mapping of a complete metric space R into itself such that the mapping A^n is a contraction for some n, then the equation $Ax = x$ has one and only one solution.*

In fact if we take an arbitrary point $x \in R$ and consider the sequence $A^{kn}x$, $k = 0, 1, 2, \ldots$, the repetition of the proof of the classical principle of contraction mappings yields the convergence of this sequence. Let $x_0 = \lim_{k\to\infty} A^{kn}x$, then $Ax_0 = x_0$. In fact $Ax_0 = \lim_{k\to\infty} A^{kn}Ax$. Since the mapping A^n is a contraction, we have

$$\rho(A^{kn}Ax, A^{kn}x) \le \alpha\rho(A^{(k-1)n}Ax, A^{(k-1)n}x) \le \ldots \le \alpha^k\rho(Ax, x).$$

Consequently,

$$\lim_{k\to\infty} \rho(A^{kn}Ax, A^{kn}x) = 0,$$

i.e., $Ax_0 = x_0$.

Consider the mapping $g = Af$ defined by

$$g(x) = \phi(x) + \lambda \int_a^x K(x,y)\phi(y)dy, \tag{69}$$

of the complete space $C[a,b]$ into itself.

5.4 Neumann series

Consider an IE

$$f(x) = \phi(x) - \lambda \int_a^b K(x,y)\phi(y)dy, \tag{70}$$

In order to determine the solution by the method of successive approximations and obtain the Neumann series, rewrite (70) as

$$\phi(x) = f(x) + \lambda \int_a^b K(x,y)\phi(y)dy, \tag{71}$$

and take the right-hand side $f(x)$ as the zero approximation, setting

$$\phi_0(x) = f(x). \tag{72}$$

Substitute the zero approximation into the right-hand side of (73) to obtain the first approximation

$$\phi_1(x) = f(x) + \lambda \int_a^b K(x,y)\phi_0(y)dy, \tag{73}$$

and so on, obtaining for the $(n+1)$st approximation

$$\phi_{n+1}(x) = f(x) + \lambda \int_a^b K(x,y)\phi_n(y)dy. \tag{74}$$

Assume that the kernel $K(x,y)$ is a bounded function in the square $\Pi = \{(x,y) : a \le x \le b,\ a \le y \le b\}$, so that

$$|K(x,y)| \le M, \quad (x,y) \in \Pi.$$

or even that there exists a constant C_1 such that

$$\int_a^b |K(x,y)|^2 dy \le C_1 \quad \forall x \in [a,b], \tag{75}$$

The the following statement is valid.

Theorem 5 *Assume that condition (75) holds Then the sequence ϕ_n of successive approximations (74) converges uniformly for all λ satisfying*

$$\lambda \le \frac{1}{B}, \quad B = \int_a^b \int_a^b |K(x,y)|^2 dx dy. \tag{76}$$

The limit of the sequence ϕ_n is the unique solution to IE (70).

Proof. Write two first successive approximations (74):

$$\phi_1(x) = f(x) + \lambda \int_a^b K(x,y)f(y)dy, \tag{77}$$

$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,y)\phi_1(y)dy = \tag{78}$$

$$f(x) + \lambda \int_a^b K(x,y)f(y)dy + \lambda^2 \int_a^b K(x,t)dt \int_a^b K(t,y)f(y)dy. \tag{79}$$

Set

$$K_2(x,y) = \int_a^b K(x,t)K(t,y)dt \tag{80}$$

to obtain (by changing the order of integration)

$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,y)f(y)dy + \lambda^2 \int_a^b K_2(x,y)f(y)dy. \tag{81}$$

In the same manner, we obtain the third approximation

$$\begin{aligned} \phi_3(x) &= f(x) + \lambda \int_a^b K(x,y)f(y)dy + \\ &+ \lambda^2 \int_a^b K_2(x,y)f(y)dy + \lambda^3 \int_a^b K_3(x,y)f(y)dy, \end{aligned} \tag{82}$$

where

$$K_3(x,y) = \int_a^b K(x,t)K_2(t,y)dt. \tag{83}$$

The general relationship has the form

$$\phi_n(x) \quad = \quad f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,y) f(y) dy, \tag{84}$$

where

$$K_1(x,y) = K(x,y), \quad K_m(x,y) = \int_a^b K(x,t) K_{m-1}(t,y) dt. \tag{85}$$

and $K_m(x,y)$ is called *the mth iterated kernel*. One can easliy prove that the iterated kernels satisfy a more general relationship

$$K_m(x,y) = \int_a^b K_r(x,t) K_{m-r}(t,y) dt, \quad r = 1, 2, \ldots, m-1 \quad (m = 2, 3, \ldots). \tag{86}$$

Assuming that successive approximations (84) converge we obtain the (unique) solution to IE (70) by passing to the limit in (84)

$$\phi(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,y) f(y) dy. \tag{87}$$

In order to prove the convergence of this series, write

$$\int_a^b |K_m(x,y)|^2 dy \le C_m \quad \forall x \in [a,b], \tag{88}$$

and estimate C_m. Setting $r = m-1$ in (86) we have

$$K_m(x,y) = \int_a^b K_{m-1}(x,t) K(t,y) dt, \quad (m = 2, 3, \ldots). \tag{89}$$

Applying to (89) the Schwartz inequality we obtain

$$|K_m(x,y)|^2 = \int_a^b |K_{m-1}(x,t)|^2 \int_a^b |K(t,y)|^2 dt. \tag{90}$$

Integrating (90) with respect to y yields

$$\int_a^b |K_m(x,y)|^2 dy \le B^2 \int_a^b |K_{m-1}(x,t)|^2 \le B^2 C_{m-1} \quad (B = \int_a^b \int_a^b |K(x,y)|^2 dx dy) \tag{91}$$

which gives

$$C_m \le B^2 C_{m-1}, \tag{92}$$

and finally the required estimate

$$C_m \le B^{2m-2} C_1. \tag{93}$$

Denoting

$$D = \sqrt{\int_a^b |f(y)|^2 dy} \tag{94}$$

and applying the Schwartz inequality we obtain

$$\left| \int_a^b K_m(x,y) f(y) dy \right|^2 \le \int_a^b |K_m(x,y)|^2 dy \int_a^b |f(y)|^2 dy \le D^2 C_1 B^{2m-2}. \tag{95}$$

Thus the common term of the series (87) is estimated by the quantity

$$D\sqrt{C_1}|\lambda|^m B^{m-1}, \tag{96}$$

so that the series converges faster than the progression with the denominator $|\lambda|B$, which proves the theorem.

If we take the first n terms in the series (87) then the resulting error will be not greater than

$$D\sqrt{C_1}\frac{|\lambda|^{n+1}B^n}{1-|\lambda|B}. \tag{97}$$

5.5 Resolvent

Introduce the notation for *the resolvent*

$$\Gamma(x,y,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K_m(x,y) \tag{98}$$

Changing the order of summation and integration in (87) we obtain the solution in the compact form

$$\phi(x) = f(x) + \lambda \int_a^b \Gamma(x,y,\lambda) f(y) dy. \tag{99}$$

One can show that the resolvent satisfies the IE

$$\Gamma(x,y,\lambda) = K(x,y) + \lambda \int_a^b K(x,t)\Gamma(t,y,\lambda) dt. \tag{100}$$

6 IEs as linear operator equations. Fundamental properties of completely continuous operators

6.1 Banach spaces

Definition 3 *A complete normed space is said to be a Banach space, B.*

Example 9 *The space $C[a,b]$ of continuous functions defined in a closed interval $a \le x \le b$ with the (uniform) norm*

$$\|f\|_C = \max_{a\le x\le b} |f(x)|$$

[and the metric

$$\rho(\phi_1,\phi_2) = \|\phi_1 - \phi_2\|_C = \max_{a\le x\le b} |\phi_1(x)| - \phi_2(x)|]$$

is a normed and complete space. Indeed, every fundamental (convergent) functional sequence $\{\phi_n(x)\}$ of continuous functions converges to a continuous function in the $C[a,b]$-metric.

Definition 4 *A set M in the metric space R is said to be compact if every sequence of elements in M contains a subsequence that converges to some $x \in R$.*

Definition 5 *A family of functions $\{\phi\}$ defined on the closed interval $[a, b]$ is said to be uniformly bounded if there exists a number $M > 0$ such that*

$$|\phi(x)| < M$$

for all x and for all ϕ in the family.

Definition 6 *A family of functions $\{\phi\}$ defined on the closed interval $[a, b]$ is said to be equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$|\phi(x_1) - \phi(x_2)| < \epsilon$$

for all x_1, x_2 such that $|x_1) - x_2| < \delta$ and for all ϕ in the given family.

Formulate the most common criterion of compactness for families of functions.

Theorem 6 *(Arzela's theorem). A necessary and sufficient condition that a family of continuous functions defined on the closed interval $[a, b]$ be compact is that this family be uniformly bounded and equicontinuous.*

6.2 IEs and completely continuous linear operators

Definition 7 *An operator A which maps a Banach space E into itself is said to be completely continuous if it maps and arbitrary bounded set into a compact set.*

Consider an IE of the 2nd kind

$$f(x) = \phi(x) + \lambda \int_a^b K(x, y)\phi(y)dy, \tag{101}$$

which can be written in the operator form as

$$f = \phi + \lambda A\phi. \tag{102}$$

Here the linear (integral) operator A defined by

$$f(x) = \int_a^b K(x, y)\phi(y)dy \tag{103}$$

and considered in the (Banach) space $C[a, b]$ of continuous functions in a closed interval $a \leq x \leq b$ represents an extensive class of completely continuous linear operators.

Theorem 7 *Formula (103) defines a completely continuous linear operator in the (Banach) space $C[a, b]$ if the function $K(x, y)$ is to be bounded in the square*

$$\Pi = \{(x, y) : \, a \leq x \leq b, \, a \leq y \leq b\},$$

and all points of discontinuity of the function $K(x, y)$ lie on the finite number of curves

$$y = \psi_k(x), \quad k = 1, 2, \ldots n, \tag{104}$$

where $\psi_k(x)$ are continuous functions.

Proof. To prove the theorem, it is sufficient to show (according to Arzela's theorem) that the functions (103) defined on the closed interval $[a, b]$ (i) are continuous and the family of these functions is (ii) uniformly bounded and (iii) equicontinuous. Below we will prove statements (i), (ii), and (iii).

Under the conditions of the theorem the integral in (103) exists for arbitrary $x \in [a, b]$. Set

$$M = \sup_{\Pi} |K(x, y)|,$$

and denote by G the set of points (x, y) for which the condition

$$|y - \psi_k(x)| < \frac{\epsilon}{12Mn}, \quad k = 1, 2, \ldots n, \tag{105}$$

holds. Denote by F the complement of G with respect to Π. Since F is a closed set and $K(x, y)$ is continuous on F, there exists a $\delta > 0$ such that for points (x', y) and (x'', y) in F for which $|x' - x''| < \delta$ the inequality

$$|K(x', y) - K(x'', y)| < (b - a)\frac{1}{3}\epsilon \tag{106}$$

holds. Now let x' and x'' be such that $|x' - x''| < \delta$ holds. Then

$$|f(x') - f(x'')| \le \int_a^b |K(x', y) - K(x'', y)||\phi(y)|dy \tag{107}$$

can be evaluated by integrating over the sum A of the intervals

$$\begin{aligned} |y - \psi_k(x')| &< \frac{\epsilon}{12Mn}, \quad k = 1, 2, \ldots n, \\ |y - \psi_k(x'')| &< \frac{\epsilon}{12Mn}, \quad k = 1, 2, \ldots n, \end{aligned}$$

and over the complement B of A with respect to the closed interval $[a, b]$. The length of A does not exceed $\frac{\epsilon}{3M}$. Therefore

$$\int_A |K(x', y) - K(x'', y)||\phi(y)|dy \le \frac{2\epsilon}{3}\|\phi\|. \tag{108}$$

The integral over the complement B can be obviously estimated by

$$\int_B |K(x', y) - K(x'', y)||\phi(y)|dy \le \frac{\epsilon}{3}\|\phi\|. \tag{109}$$

Therefore

$$|f(x') - f(x'')| \le \epsilon\|\phi\|. \tag{110}$$

Thus we have proved the continuity of $f(x)$ and the equicontinuity of the functions $f(x)$ (according to Definition (6)) corresponding to the functions $\phi(x)$ with bounded norm $\|\phi\|$. The uniform boundedness of $f(x)$ (according to Definition (5)) corresponding to the functions $\phi(x)$ with bounded norms follows from the inequality

$$|\phi(x)| \le \int_B |K(x, y)||\phi(y)|dy \le M(b - a)\|\phi\|. \tag{111}$$

6.3 Properties of completely continuous operators

Theorem 8 *If $\{A_n\}$ is a sequence of completely continuous operators on a Banach space E which converges in (operator) norm to an operator A then the operator A is completely continuous.*

Proof. Consider an arbitrary bounded sequence $\{x_n\}$ of elements in a Banach space E such that $\|x_n\| < c$. It is necessary to prove that sequence $\{Ax_n\}$ contains a convergent subsequence.

The operator A_1 is completely continuous and therefore we can select a convergent subsequence from the elements $\{A_1 x_n\}$. Let

$$x_1^{(1)},\, x_2^{(1)},\, \ldots,\, x_n^{(1)},\, \ldots, \tag{112}$$

be the inverse images of the members of this convergent subsequence. We apply operator A_2 to each members of subsequence (112)). The operator A_2 is completely continuous; therefore we can again select a convergent subsequence from the elements $\{A_2 x_n^{(1)}\}$, denoting by

$$x_1^{(2)},\, x_2^{(2)},\, \ldots,\, x_n^{(2)},\, \ldots, \tag{113}$$

the inverse images of the members of this convergent subsequence. The operator A_3 is also completely continuous; therefore we can again select a convergent subsequence from the elements $\{A_3 x_n^{(2)}\}$, denoting by

$$x_1^{(3)},\, x_2^{(3)},\, \ldots,\, x_n^{(3)},\, \ldots, \tag{114}$$

the inverse images of the members of this convergent subsequence; and so on. Now we can form a diagonal subsequence

$$x_1^{(1)},\, x_2^{(2)},\, \ldots,\, x_n^{(n)},\, \ldots. \tag{115}$$

Each of the operators A_n transforms this diagonal subsequence into a convergent sequence. If we show that operator A also transforms (115)) into a convergent sequence, then this will establish its complete continuity. To this end, apply the Cauchy criterion (48) and evaluate the norm

$$\|Ax_n^{(n)} - Ax_m^{(m)}\|. \tag{116}$$

We have the chain of inequalities

$$\begin{aligned} \|Ax_n^{(n)} - Ax_m^{(m)}\| &\leq \|Ax_n^{(n)} - A_k x_n^{(n)}\| + \\ &+ \|A_k x_n^{(n)} - A_k x_m^{(m)}\| + \|A_k x_m^{(m)} - Ax_m^{(m)}\| \leq \\ &\leq \|A - A_k\|(\|x_m^{(m)}\| + \|x_m^{(m)}\|) + \|A_k x_n^{(n)} - A_k x_m^{(m)}\|. \end{aligned}$$

Choose k so that $\|A - A_k\| < \frac{\epsilon}{2c}$ and N such that

$$\|A_k x_n^{(n)} - A_k x_m^{(m)}\| < \frac{\epsilon}{2}$$

holds for arbitrary $n > N$ and $m > N$ (which is possible because the sequence $\|A_k x_n^{(n)}\|$ converges). As a result,

$$\|Ax_n^{(n)} - Ax_m^{(m)}\| < \epsilon, \tag{117}$$

i.e., the sequence $\|Ax_n^{(n)}\|$ is fundamental.

Theorem 9 *(Superposition principle). If A is a completely continuous operator and B is a bounded operator then the operators AB and BA are completely continuous.*

Proof. If M is a bounded set in a Banach space E, then the set BM is also bounded. Consequently, the set ABM is compact and this means that the operator AB is completely continuous. Further, if M is a bounded set in E, then AM is compact. Then, because B is a continuous operator, the set BAM is also compact, and this completes the proof.

Corollary *A completely continuous operator A cannot have a bounded inverse in a finite-dimensional space E.*

Proof. In the contrary case, the identity operator $I = AA^{-1}$ would be completely continuous in E which is impossible.

Theorem 10 *The adjoint of a completely operator is completely continuous.*

7 Elements of the spectral theory of linear operators. Spectrum of a completely continuous operator

In this section we will consider bounded linear operators which map a Banach space E into itself.

Definition 8 *The number λ is said to be a characteristic value of the operator A if there exists a nonzero vector x (characteristic vector) such that $Ax = \lambda x$.*

In an n-dimensional space, this definition is equivalent to the following:
the number λ is said to be a characteristic value of the operator A if it is a root of the characteristic equation

$$Det\,|A - \lambda I| = 0.$$

The totality of those values of λ for which the inverse of the operator $A - \lambda I$ does not exist is called *the spectrum* of the operator A.

The values of λ for which the operator $A - \lambda I$ has the inverse are called *regular values* of the operator A.

Thus the spectrum of the operator A consists of all nonregular points.

The operator

$$R_\lambda = (A - \lambda I)^{-1}$$

is called *the resolvent* of the operator A.

In an infinite-dimensional space, the spectrum of the operator A necessarily contains all the characteristic values and maybe some other numbers.

In this connection, the set of characteristic values is called *the point spectrum* of the operator A. The remaining part of the spectrum is called *the continuous spectrum* of the operator A. The latter also consists, in its turn, of two parts: (1) those λ for which $A - \lambda I$ has an unbounded inverse with domain dense in E (and this part is called the continuous spectrum); and (2) the remainder of the spectrum: those λ for which $A - \lambda I$ has a (bounded) inverse whose domain is not dense in E, this part is called *the residual spectrum.*

If the point λ is regular, i.e., if the inverse of the operator $A - \lambda I$ exists, then for sufficiently small δ the operator $A - (\lambda + \delta)I$ also has the inverse, i.e., the point $\delta + \lambda$ is regular also. Thus the regular points form an open set. Consequently, the spectrum, which is a complement, is a closed set.

Theorem 11 *The inverse of the operator $A - \lambda I$ exists for arbitrary λ for which $|\lambda| > \|A\|$.*

Proof. Since

$$A - \lambda I = -\lambda \left(I - \frac{1}{\lambda} A \right),$$

the resolvent

$$R_\lambda = (A - \lambda I)^{-1} = -\frac{1}{\lambda} \left(I - \frac{A}{\lambda} \right)^{-1} = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{A^k}{\lambda^k}.$$

This operator series converges for $\|A/\lambda\| < 1$, i.e., the operator $A - \lambda I$ has the inverse.

Corollary. *The spectrum of the operator A is contained in a circle of radius $\|A\|$ with center at zero.*

Theorem 12 *Every completely continuous operator A in the Banach space E has for arbitrary $\rho > 0$ only a finite number of linearly independent characteristic vectors which correspond to the characteristic values whose absolute values are greater than ρ.*

Proof. Assume, ad abs, that there exist an infinite number of linearly independent characteristic vectors x_n, $n = 1, 2, \ldots$, satisfying the conditions

$$Ax_n = \lambda_n x_n, \quad |\lambda_n| > \rho > 0 \quad (n = 1, 2, \ldots). \tag{118}$$

Consider in E the subspace E_1 generated by the vectors x_n $(n = 1, 2, \ldots)$. In this subspace the operator A has a bounded inverse. In fact, according to the definition, E_1 is the totality of elements in E which are of the form vectors

$$x = \alpha_1 x_1 + \ldots + \alpha_k x_k, \tag{119}$$

and these vectors are everywhere dense in E_1. For these vectors

$$Ax = \alpha_1 \lambda_1 x_1 + \ldots + \alpha_k \lambda_k x_k, \tag{120}$$

and, consequently,

$$A^{-1}x = (\alpha_1/\lambda_1)x_1 + \ldots + (\alpha_k/\lambda_k)x_k \tag{121}$$

and

$$\|A^{-1}x\| < \frac{1}{\rho}\|x\|.$$

Therefore, the operator A^{-1} defined for vectors of the form (119) by means of (120) can be extended by continuity (and, consequently, with preservation of norm) to all of E_1. The operator

A, being completely continuous in the Banach space E, is completely continuous in E_1. According to the corollary to Theorem (9), a completely continuous operator cannot have a bounded inverse in infinite-dimensional space. The contradiction thus obtained proves the theorem.

Corollary. *Every nonzero characteristic value of a completely continuous operator A in the Banach space E has only finite multiplicity and these characteristic values form a bounded set which cannot have a single limit point distinct from the origin of coordinates.*

We have thus obtained a characterization of the point spectrum of a completely continuous operator A in the Banach space E.

One can also show that a completely continuous operator A in the Banach space E cannot have a continuous spectrum.

8 Linear operator equations: the Fredholm theory

8.1 The Fredholm theory in Banach space

Theorem 13 *Let A be a completely continuous operator which maps a Banach space E into itself. If the equation $y = x - Ax$ is solvable for arbitrary y, then the equation $x - Ax = 0$ has no solutions distinct from zero solution.*

Proof. Assume the contrary: there exists an element $x_1 \neq 0$ such that $x_1 - Ax_1 = Tx_1 = 0$ Those elements x for which $Tx = 0$ form a linear subspace E_1 of the banach space E. Denote by E_n the subspace of E consisting of elements x for which the powers $T^n x = 0$.

It is clear that subspaces E_n form a nondecreasing subsequence

$$E_1 \subseteq E_2 \subseteq \ldots E_n \subseteq \ldots.$$

We shall show that the equality sign cannot hold at any point of this chain of inclusions. In fact, since $x_1 \neq 0$ and equation $y = Tx$ is solvable for arbitrary y, we can find a sequence of elements $x_1, x_2, \ldots, x_n, \ldots$ distinct from zero such that

$$\begin{aligned} Tx_2 &= x_1, \\ Tx_3 &= x_2, \\ \ldots, & \\ Tx_n &= x_{n-1}, \\ \ldots. & \end{aligned}$$

The element x_n belongs to subspace E_n for each n but does not belong to subspace E_{n-1}. In fact,

$$T^n x_n = T^{n-1} x_{n-1} = \ldots = Tx_1 = 0,$$

but

$$T^{n-1} x_n = T^{n-2} x_{n-1} = \ldots = Tx_2 = x_1 \neq 0.$$

All the subspaces E_n are linear and closed; therefore for arbitrary n there exists an element $y_{n+1} \in E_{n+1}$ such that

$$\|y_{n+1}\| = 1 \quad \text{and} \quad \rho(y_{n+1}, E_n) \geq \frac{1}{2},$$

where $\rho(y_{n+1}, E_n)$ denotes the distance from y_{n+1} to the space E_n:

$$\rho(y_{n+1}, E_n) = \inf\{\|y_{n+1} - x\|,\ x \in E_n\}.$$

Consider the sequence $\{Ay_k\}$. We have (assuming $p > q$)

$$\|Ay_p - Ay_q\| = \|y_p - (y_q + Ty_p - Ty_q)\| \geq \frac{1}{2},$$

since $y_q + Ty_p - Ty_q \in E_{p-1}$. It is clear from this that the sequence $\{Ay_k\}$ cannot contain any convergent subsequence which contradicts the complete continuity of the operator A. This contradiction proves the theorem

Corollary 1. *If the equation $y = x - Ax$ is solvable for arbitrary y, then this equation has unique solution for each y, i.e., the operator $I - A$ has an inverse in this case.*

We will consider, together with the equation $y = x - Ax$ the equation $h = f - A^*f$ which is *adjoint* to it, where A^* is the adjoint to A and h, f are elements of the Banach space $\bar{E}$, the conjugate of E.

Corollary 2. *If the equation $h = f - A^*f$ is solvable for all h, then the equation $f - A^*f = 0$ has only the zero solution.*

It is easy to verify this statement by recalling that the operator adjoint to a completely continuous operator is also completely continuous, and the space $\bar{E}$ conjugate to a Banach space is itself a Banach space.

Theorem 14 *A necessary and sufficient condition that the equation $y = x - Ax$ be solvable is that $f(y) = 0$ for all f for which $f - A^*f = 0$.*

Proof. 1. If we assume thast the equation $y = x - Ax$ is solvable, then

$$f(y) = f(x) - f(Ax) = f(y) = f(x) - A^*f(x)$$

i.e., $f(y) = 0$ for all f for which $f - A^*f = 0$.

2. Now assume that $f(y) = 0$ for all f for which $f - A^*f = 0$. For each of these functionals we consider the set L_f of elements for which f takes on the value zero. Then our assertion is equivalent to the fact that the set $\cap L_f$ consists only of the elements of the form $x - Ax$. Thus, it is necessary to prove that an element y_1 which cannot be represented in the form $x - Ax$ cannot be contained in $\cap L_f$. To do this we shall show that for such an element y_1 we can construct a functional f_1 satisfying

$$f_1(y_1) \neq 0, \quad f_1 - A^*f_1 = 0.$$

These conditions are equivalent to the following:

$$f_1(y_1) \neq 0, \quad f_1(x - Ax) = 0 \quad \forall x.$$

In fact,

$$(f_1 - A^* f_1)(x) = f_1(x) - A^* f_1(x) = f_1(x) - f_1(Ax) = f_1(x - Ax).$$

Let G_0 be a subspace consisting of all elements of the form $x - Ax$. Consider the subspace $\{G_0, y_1\}$ of the elements of the form $z + \alpha y_1$, $z \in G_0$ and define the linear functional by setting

$$f_1(z + \alpha y_1) = \alpha.$$

Extending the linear functional to the whole space E (which is possible by virtue of the Hahn–Banach theorem) we do obtain a linear functional satisfying the required conditions. This completes the proof of the theorem.

Corollary *If the equation $f - A^* f = 0$ does not have nonzero solution, then the equation $x - Ax = y$ is solvable for all y.*

Theorem 15 *A necessary and sufficient condition that the equation $f - A^* f = h$ be solvable is that $h(x) = 0$ for all x for which $x - Ax = 0$.*

Proof. 1. If we assume that the equation $f - A^* f = h$ is solvable, then

$$h(x) = f(x) - A^* f(x) = f(x - Ax)$$

i.e., $h(x) = 0$ if $x - Ax = 0$.

2. Now assume that $h(x) = 0$ for all x such that $x - Ax = 0$. We shall show that the equation $f - A^* f = h$ is solvable. Consider the set F of all elements of the form $y = x - Ax$. Construct a functional f on F by setting

$$f(Tx) = h(x).$$

This equation defines in fact a linear functional. Its value is defined uniquely for each y because if $Tx_1 = Tx_2$ then $h(x_1) = h(x_2)$. It is easy to verify the linearity of the functional and extend it to the whole space E. We obtain

$$f(Tx) = T^* f(x) = h(x).$$

i.e., the functional is a solution of equation $f - A^* f = h$. This completes the proof of the theorem.

Corollary *If the equation $x - Ax = 0$ does not have nonzero solution, then the equation $f - A^* f = h$ is solvable for all h.*

The following theorem is the converse of Thereom 13.

Theorem 16 *If the equation $x - Ax = 0$ has $x = 0$ for its only solution, then the equation $x - Ax = y$ has a solution for all y.*

Proof. If the equation $x - Ax = 0$ has only one solution, then by virtue of the corollary to the preceding theorem, the equation $f - A^* f = h$ is solvable for all h. Then, by Corollary 2 to Thereom 13, the equation $f - A^* f = 0$ has $f = 0$ for its only solution. Hence Corollary to Thereom 14 implies that the equation $x - Ax = y$ has a solution for all y.

Thereoms 13 and 16 show that for the equation

$$x - Ax = y \tag{122}$$

only the following two cases are possible:
(i) Equation (122) has a unique solution for each y, i.e., the operator $I - A$ has an inverse.
(ii) The corresponding homogeneous equation $x - Ax = 0$ has a nonzero solution, i.e., the number $\lambda = 1$ is a characteristic value for the operator A.

All the results obtained above for the equation $x - Ax = y$ remain valid for the equation $x - \lambda Ax = y$ (the adjoint equation is $f - \bar{\lambda} A^* f = h$). It follows that either the operator $I - \lambda A$ has an inverse or the number $1/\lambda$ is a characteristic value for the operator A. In other words, in the case of a completely continuous operator, an arbitrary number is either a regular point or a characteristic value. Thus *a completely continuous operator has only a point spectrum.*

Theorem 17 *The dimension* n *of the space* N *whose elements are the solutions to the equation* $x - Ax = 0$ *is equal to the dimension of the subspace* N^* *whose elements are the solutions to the equation* $f - A^* f = 0$.

This theorem is proved below for operators in the Hilbert space.

8.2 Fredholm theorems for linear integral operators and the Fredholm resolvent

One can formulate the Fredholm theorems for linear integral operators

$$A\phi(x) = \int_a^b K(x, y)\phi(y)dy \tag{123}$$

in terms of the Fredholm resolvent.

The numbers λ for which there exists the Fredholm resolvent of a Fredholm IE

$$\phi(x) - \lambda \int_a^b K(x, y)\phi(y)dy = f(x) \tag{124}$$

will be called regular values. The numbers λ for which the Fredholm resolvent $\Gamma(x, y, \lambda)$ is not defined (they belong to the point spectrum) will be characteristic values of the IE (124), and they coincide with the poles of the resolvent. The inverse of characteristic values are eigenvalues of the IE.

Formulate, for example, the first Fredholm theorem in terms of regular values of the resolvent.

If λ is a regular value, then the Fredholm IE (124) has one and only one solution for arbitrary $f(x)$. This solution is given by formula (99),

$$\phi(x) = f(x) + \lambda \int_a^b \Gamma(x, y, \lambda) f(y)dy. \tag{125}$$

In particular, if λ is a regular value, then the homogeneous Fredholm IE (124)

$$\phi(x) - \lambda \int_a^b K(x,y)\phi(y)dy = 0 \tag{126}$$

has only the zero solution $\phi(x) = 0$.

9 IEs with degenerate and separable kernels

9.1 Solution of IEs with degenerate and separable kernels

An IE

$$\phi(x) - \lambda \int_a^b K(x,y)\phi(y)dy = f(x) \tag{127}$$

with *a degenerate kernel*

$$K(x,y) = \sum_{i=1}^{n} a_i(x)b_i(y) \tag{128}$$

can be represented, by changing the order of summation and integration, in the form

$$\phi(x) - \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(y)\phi(y)dy = f(x). \tag{129}$$

Here one may assume that functions $a_i(x)$ (and $b_i(y)$) are linearly independent (otherwise, the numer of terms in (128)) can be reduced).

It is easy to solve such IEs with degenerate and separable kernels. Denote

$$c_i = \int_a^b b_i(y)\phi(y)dy, \tag{130}$$

which are unknown constants. Equation (129)) becomes

$$\phi(x) = f(x) + \lambda \sum_{i=1}^{n} c_i a_i(x), \tag{131}$$

and the problem reduces to the determination of unknowns c_i. To this end, substitute (131) into equation (129) to obtain, after simple algebra,

$$\sum_{i=1}^{n} a_i(x) \left\{ c_i - \int_a^b b_i(y) \left[f(y) + \lambda \sum_{k=1}^{n} c_k a_k(y) \right] dy \right\} = 0. \tag{132}$$

Since functions $a_i(x)$ are linearly independent, equality (132) yields

$$c_i - \int_a^b b_i(y) \left[f(y) + \lambda \sum_{k=1}^{n} c_k a_k(y) \right] dy = 0, \quad i = 1, 2, \dots n. \tag{133}$$

Denoting

$$f_i = \int_a^b b_i(y) f(y) dy, \quad a_{ik} = \int_a^b b_i(y) a_k(y) dy \tag{134}$$

and changing the order of summation and integration, we rewrite (133) as

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i, \quad i = 1, 2, \dots n, \tag{135}$$

which is a linear equation system of order n with respect to unknowns c_i.

Note that the same system can be obtained by multiplying (131) by $a_k(x)$ ($k = 1, 2, \dots n$) and integrating from a to b

System (135) is equivalent to IE (129) (and (127), (128)) in the following sense: if (135) is uniquely solvable, then IE (129) has one and only one solution (and vice versa); and if (135) has no solution, then IE (129) is not solvable (and vice versa).

The determinant of system (135) is

$$D(\lambda) = \begin{vmatrix} 1 - \lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1 - \lambda a_{22} & \dots & -\lambda a_{2n} \\ . & . & \dots & . \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1 - \lambda a_{nn} \end{vmatrix}. \tag{136}$$

$D(\lambda)$ is a polynomial in powers of λ of an order not higher than n. Note that $D(\lambda)$ is not identically zero because $D(0) = 1$. Thus there exist not more than n different numbers $\lambda = \lambda_k^*$ such that $D(\lambda_k^*) = 0$. When $\lambda = \lambda_k^*$, then system (135) together with IE (129) are either not solvable or have infinitely many solutions. If $\lambda \neq \lambda_k^*$, then system (135) and IE (129) are uniquely solvable.

Example 10 *Consider a Fredholm IE of the 2nd kind*

$$\phi(x) - \lambda \int_0^1 (x + y) \phi(y) dy = f(x). \tag{137}$$

Here

$$K(x, y) = x + y = \sum_{i=1}^{2} a_i(x) b_i(y) = a_1(x) b_1(y) + a_2(x) b_2(y),$$

where

$$a_1(x) = x, \; b_1(y) = 1, \quad a_2(x) = 1, \, b_2(y) = y,$$

is a degenerate kernel, $f(x)$ is a given function, and λ is a parameter.

Look for the solution to (137) in the form (131) (note that here $n = 2$)

$$\phi(x) = f(x) + \lambda \sum_{i=1}^{2} c_i a_i(x) = f(x) + \lambda (c_1 a_1(x) + c_2 a_2(x)) = f(x) + \lambda (c_1 x + c_2). \tag{138}$$

Denoting

$$f_i = \int_0^1 b_i(y) f(y) dy, \quad a_{ik} = \int_0^1 b_i(y) a_k(y) dy, \tag{139}$$

so that

$$
\begin{aligned}
a_{11} &= \int_0^1 b_1(y)a_1(y)dy = \int_0^1 ydy = \frac{1}{2}, \\
a_{12} &= \int_0^1 b_1(y)a_2(y)dy = \int_0^1 dy = 1, \\
a_{21} &= \int_0^1 b_2(y)a_1(y)dy = \int_0^1 y^2dy = \frac{1}{3}, \\
a_{22} &= \int_0^1 b_2(y)a_2(y)dy = \int_0^1 ydy = \frac{1}{2},
\end{aligned} \tag{140}
$$

and

$$
\begin{aligned}
f_1 &= \int_0^1 b_1(y)f(y)dy = \int_0^1 f(y)dy, \\
f_2 &= \int_0^1 b_2(y)f(y)dy = \int_0^1 yf(y)dy,
\end{aligned} \tag{141}
$$

and changing the order of summation and integration, we rewrite the corresponding equation (133) as

$$c_i - \lambda \sum_{k=1}^{2} a_{ik}c_k = f_i, \quad i = 1, 2, \tag{142}$$

or, explicitly,

$$
\begin{aligned}
(1-\lambda/2)c_1 - \lambda c_2 &= f_1, \\
-(\lambda/3)c_1 + (1-\lambda/2)c_2 &= f_2,
\end{aligned}
$$

which is a linear equation system of order 2 with respect to unknowns c_i.

The determinant of system (143) is

$$D(\lambda) = \begin{vmatrix} 1-\lambda/2 & -\lambda \\ -\lambda/3 & 1-\lambda/2 \end{vmatrix} = 1 + \lambda^2/4 - \lambda - \lambda^2/3 = 1 - \lambda - \lambda^2/12 = -\frac{1}{12}(\lambda^2 + 12\lambda - 12) \tag{143}$$

is a polynomial of order 2. ($D(\lambda)$ is not identically, and $D(0) = 1$). There exist not more than 2 different numbers $\lambda = \lambda_k^*$ such that $D(\lambda_k^*) = 0$. Here there are exactly two such numbers; they are

$$
\begin{aligned}
\lambda_1^* &= -6 + 4\sqrt{3}, \\
\lambda_2^* &= -6 - 4\sqrt{3}.
\end{aligned} \tag{144}
$$

If $\lambda \neq \lambda_k^*$, then system (143) and IE (137) are uniquely solvable.

To find the solution of system (143) apply Cramer's rule and calculate the determinants of system (143)

$$
\begin{aligned}
D_1 &= \begin{vmatrix} f_1 & -\lambda \\ f_2 & 1-\lambda/2 \end{vmatrix} = f_1(1-\lambda/2) + f_2\lambda = \\
&= (1-\lambda/2)\int_0^1 f(y)dy + \lambda\int_0^1 yf(y)dy = \int_0^1 [(1-\lambda/2) + y\lambda]f(y)dy,
\end{aligned} \tag{145}
$$

$$\begin{aligned} D_1 &= \begin{vmatrix} f_1 & -\lambda \\ f_2 & 1-\lambda/2 \end{vmatrix} = f_1(1-\lambda/2) + f_2\lambda = \\ &= (1-\lambda/2)\int_0^1 f(y)dy + \lambda\int_0^1 yf(y)dy = \int_0^1 [(1-\lambda/2)+y\lambda]f(y)dy, \end{aligned} \tag{146}$$

$$\begin{aligned} D_2 &= \begin{vmatrix} (1-\lambda/2) & f_1 \\ -\lambda/3 & f_2 \end{vmatrix} = f_2(1-\lambda/2) + f_1\lambda/3 = \\ &= (1-\lambda/2)\int_0^1 yf(y)dy + (\lambda/3)\int_0^1 f(y)dy = \int_0^1 [y(1-\lambda/2)+\lambda/3]f(y)dy, \end{aligned} \tag{147}$$

Then, the solution of system (143) is

$$\begin{aligned} c_1 &= \frac{D_1}{D} = -\frac{12}{\lambda^2+12\lambda-12}\int_0^1 [(1-\lambda/2)+y\lambda]f(y)dy = \\ &= \frac{1}{\lambda^2+12\lambda-12}\int_0^1 [(6\lambda-12)-12y\lambda]f(y)dy, \end{aligned} \tag{148}$$

$$\begin{aligned} c_2 &= \frac{D_2}{D} = -\frac{12}{\lambda^2+12\lambda-12}\int_0^1 [y(1-\lambda/2)+\lambda/3]f(y)dy = \\ &= \frac{1}{\lambda^2+12\lambda-12}\int_0^1 [y(6\lambda-12)-4\lambda]f(y)dy \end{aligned} \tag{149}$$

According to (138), the solution of the IE (137) is

$$\begin{aligned} \phi(x) &= f(x) + \lambda(c_1x+c_2) = \\ &= f(x) + \frac{\lambda}{\lambda^2+12\lambda-12}\int_0^1 [6(\lambda-2)(x+y)-12xy-4\lambda]f(y)dy. \end{aligned} \tag{150}$$

When $\lambda = \lambda_k^*$ $(k = 1, 2)$, then system (143) together with IE (137) are not solvable.

Example 11 *Consider a Fredholm IE of the 2nd kind*

$$\phi(x) - \lambda\int_0^{2\pi} \sin x \cos y \phi(y)dy = f(x). \tag{151}$$

Here

$$K(x,y) = \sin x \cos y = a(x)b(y)$$

is a degenerate kernel, $f(x)$ is a given function, and λ is a parameter.

Rewrite IE (151) in the form (131) (here $n = 1$)

$$\phi(x) = f(x) + \lambda c \sin x, \quad c = \int_0^{2\pi} \cos y \phi(y)dy. \tag{152}$$

Multiplying the first equality in (152) by $b(x) = \cos x$ and integrating from 0 to 2π we obtain

$$c = \int_0^{2\pi} f(x)\cos x dx \tag{153}$$

Therefore the solution is

$$\phi(x) = f(x) + \lambda c \sin x = f(x) + \lambda \int_0^{2\pi} \sin x \cos y f(y) dy. \tag{154}$$

Example 12 *Consider a Fredholm IE of the 2nd kind*

$$x^2 = \phi(x) - \int_0^1 (x^2 + y^2)\phi(y) dy, \tag{155}$$

where $K(x,y) = x^2 + y^2$ *is a degenerate kernel* $f(x) = x^2$ *is a given function, and the parameter* $\lambda = 1$.

9.2 IEs with degenerate kernels and approximate solution of IEs

Assume that in an IE

$$\phi(x) - \lambda \int_a^b K(x,y)\phi(y) dy = f(x) \tag{156}$$

the kernel $K(x, y$ and $f(x)$ are continuous functions in the square

$$\Pi = \{(x,y) : \ a \le x \le b, \ a \le y \le b\}.$$

Then $\phi(x)$ will be also continuous. Approximate IE (156) with an IE having a degenerate kernel. To this end, replace the integral in (156) by a finite sum using, e.g., the rectangle rule

$$\int_a^b K(x,y)\phi(y) dy \approx h \sum_{k=1}^{n} K(x, y_k)\phi(y_k), \tag{157}$$

where

$$h = \frac{b-a}{n}, \qquad y_k = a + kh. \tag{158}$$

The resulting approximation takes the form

$$\phi(x) - \lambda h \sum_{k=1}^{n} K(x, y_k)\phi(y_k) = f(x) \tag{159}$$

of an IE having a degenerate kernel. Now replace variable x in (159) by a finite number of values $x = x_i$, $i = 1, 2, \ldots, n$. We obtain a linear equation system with unknowns $\phi_k = \phi(x_k)$ and right-hand side containing the known numbers $f_i = f(x_i)$

$$\phi_i - \lambda h \sum_{k=1}^{n} K(x_i, y_k)\phi_k = f_i, \quad i = 1, 2, \ldots, n. \tag{160}$$

We can find the solution $\{\phi_i\}$ of system (160) by Cramer's rule,

$$\phi_i = \frac{D_i^{(n)}(\lambda)}{D^{(n)}(\lambda)}, \quad i = 1, 2, \ldots, n, \tag{161}$$

calculating the determinants

$$D^{(n)}(\lambda) = \begin{vmatrix} 1 - h\lambda K(x_1, y_1) & -h\lambda K(x_1, y_2) & \dots & -h\lambda K(x_1, y_n) \\ -h\lambda K(x_2, y_1) & 1 - h\lambda K(x_2, y_2) & \dots & -h\lambda K(x_2, y_n) \\ \cdot & \cdot & \dots & \cdot \\ -h\lambda K(x_n, y_1) & -h\lambda K(x_n, y_2) & \dots & 1 - h\lambda K(x_n, y_n) \end{vmatrix}. \tag{162}$$

and so on.

To reconstruct the approximate solution $\tilde{\phi}^{(n)}$ using the determined $\{\phi_i\}$ one can use the IE (159) itself:

$$\phi(x) \approx \tilde{\phi}^{(n)} = \lambda h \sum_{k=1}^{n} K(x, y_k)\phi_k + f(x). \tag{163}$$

One can prove that $\tilde{\phi}^{(n)} \to \phi(x)$ as $n \to \infty$ where $\phi(x)$ is the exact solution to IE (159).

9.3 Fredholm's resolvent

As we have already mentioned the resolvent $\Gamma(x, y, \lambda)$ of IE (159) satisfies the IE

$$\Gamma(x, y, \lambda) = K(x, y) + \lambda \int_a^b K(x, t)\Gamma(t, y, \lambda)dt. \tag{164}$$

Solving this IE (164) by the described approximate method and passing to the limit $n \to \infty$ we obtain the resolvent as a ratio of two power series

$$\Gamma(x, y, \lambda) = \frac{D(x, y, \lambda)}{D(\lambda)}, \tag{165}$$

where

$$D(x, y, \lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n(x, y)\lambda^n, \tag{166}$$

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_n \lambda^n, \tag{167}$$

$$B_0(x, y) = K(x, y),$$

$$B_n(x, y) = \int_a^b \dots \int_a^b \begin{vmatrix} K(x, y) & K(x, y_1) & \dots & K(x, y_n) \\ K(y_1, y) & K(y_1, y_1) & \dots & K(y_1, y_n) \\ \cdot & \cdot & \dots & \cdot \\ K(y_n, y) & K(y_n, y_1) & \dots & K(y_n, y_n) \end{vmatrix} dy_1 \dots dy_n; \tag{168}$$

$$c_0 = 1,$$

$$c_n = \int_a^b \dots \int_a^b \begin{vmatrix} K(y_1,y_1) & K(y_1,y_2) & \dots & K(y_1,y_n) \\ K(y_2,y_1) & K(y_2,y_2) & \dots & K(y_2,y_n) \\ \cdot & \cdot & \dots & \cdot \\ K(y_n,y_1) & K(y_n,y_2) & \dots & K(y_n,y_n) \end{vmatrix} dy_1 \dots dy_n. \tag{169}$$

$D(\lambda)$ is called the Fredholm determinant, and $D(x,y,\lambda)$, the first Fredholm minor.

If the integral

$$\int_a^b \int_a^b |K(x,y)|^2 dx dy < \infty, \tag{170}$$

then seires (167) for the Fredholm determinant converges for all complex λ, so that $D(\lambda)$ is an entire function of λ. Thus (164)–(167) define the resolvent on the whole complex plane λ, with an exception of zeros of $D(\lambda)$ which are poles of the resolvent.

We may conclude that for all complex λ that do not coincide with the poles of the resolvent, the IE

$$\phi(x) - \lambda \int_a^b K(x,y)\phi(y)dy = f(x) \tag{171}$$

is uniquely solvable, and its solution is given by formula (99)

$$\phi(x) = f(x) + \lambda \int_a^b \Gamma(x,y,\lambda) f(y) dy. \tag{172}$$

For practical computation of the resolvent, one can use the following recurrent relationships:

$$\begin{aligned} B_0(x,y) &= K(x,y), \qquad c_0 = 1, \\ c_n &= \int_a^b B_{n-1}(y,y)dy, \quad n = 1,2,\dots, \qquad (173) \\ B_n(x,y) &= c_n K(x,y) - n\int_a^b K(x,t)B_{n-1}(t,y)dt, \quad n = 1,2,\dots. \qquad (174) \end{aligned}$$

Example 13 *Find the resolvent of the Fredholm IE (137) of the 2nd kind*

$$\phi(x) - \lambda \int_0^1 (x+y)\phi(y)dy = f(x). \tag{175}$$

We will use recurrent relationships (173) and (174). Here

$$K(x,y) = x + y$$

is a degenerate kernel. We have $c_0 = 1$, $B_i = B_i(x,y)$ $(i = 0,1,2)$, and

$$\begin{aligned} B_0 &= K(x,y) = x + y, \\ c_1 &= \int_0^1 B_0(y,y)dy = \int_0^1 2y dy = 1, \\ B_1 &= c_1 K(x,y) - \int_0^1 K(x,t)B_0(t,y)dt = \\ &= x + y - \int_0^1 (x+t)(t+y)dt = \frac{1}{2}(x+y) - xy - \frac{1}{3}, \\ c_2 &= \int_0^1 B_1(y,y)dy = \int_0^1 \left(\frac{1}{2}(y+y) - y^2 - \frac{1}{3}\right) dy = -\frac{1}{6}, \\ B_2 &= c_2 K(x,y) - \int_0^1 K(x,t)B_1(t,y)dt = \\ &= -\frac{1}{6}(x+y) - 2\int_0^1 (x+t)\left(\frac{1}{2}(t+y) - ty - \frac{1}{3}\right) dt = 0. \end{aligned}$$

Since $B_2(x,y) \equiv 0$, all subsequent c_3, c_4, ..., and B_3, B_4, ..., vanish (see (173) and (174)), and we obtain

$$\begin{aligned}
D(x,y,\lambda) &= \\
&= \sum_{n=0}^{1} \frac{(-1)^n}{n!} B_n(x,y)\lambda^n = B_0(x,y) - \lambda B_1(x,y) = \\
&= x + y - \lambda\left(\frac{1}{2}(x+y) - xy - \frac{1}{3}\right), \qquad (176)\\
D(\lambda) &= \\
&= \sum_{n=0}^{2} \frac{(-1)^n}{n!} c_n\lambda^n = c_0 - c_1\lambda + \frac{1}{2}c_2\lambda^2 = \\
&= 1 - \lambda - \frac{1}{12}\lambda^2 = -\frac{1}{12}(\lambda^2 + 12\lambda - 12)
\end{aligned}$$

and

$$\Gamma(x,y,\lambda) = \frac{D(x,y,\lambda)}{D(\lambda)} = -12\frac{x + y - \lambda\left(\frac{1}{2}(x+y) - xy - \frac{1}{3}\right)}{\lambda^2 + 12\lambda - 12}. \qquad (177)$$

The solution to (175) is given by formula (99)

$$\phi(x) = f(x) + \lambda\int_0^1 \Gamma(x,y,\lambda)f(y)dy,$$

so that we obtain, using (176),

$$\phi(x) = f(x) + \frac{\lambda}{\lambda^2 + 12\lambda - 12}\int_0^1 [6(\lambda - 2)(x+y) - 12xy - 4\lambda]f(y)dy, \qquad (178)$$

which coincides with formula (150).

The numbers $\lambda = \lambda_k^*$ $(k = 1,2)$ determined in (144), $\lambda_1^* = -6 + \sqrt{4}$ and $\lambda_2^* = -6 - \sqrt{4}$, are (simple, i.e., of multiplicity one) poles of the resolvent because thay are (simple) zeros of the Fredholm determinant $D(\lambda)$. When $\lambda = \lambda_k^*$ $(k = 1,2)$, then IE (175) is not solvable.

10 Hilbert spaces. Self-adjoint operators. Linear operator equations with completely continuous operators in Hilbert spaces

10.1 Hilbert space. Selfadjoint operators

A real linear space R with *the inner product* satisfying

$$\begin{aligned}
(x,y) &= (y,x) \quad \forall x,\, y \in R,\\
(x_1 + x_2, y) &= (x_1,y) + (x_2,y), \quad \forall x_1, x_2,\, y \in R,\\
(\lambda x, y) &= \lambda(x,y), \quad \forall x,\, y \in R,\\
(x,x) &\geq 0, \quad \forall x \in R, \quad (x,x) = 0 \quad \text{if and only if} \quad x = 0.
\end{aligned}$$

is called *the Euclidian space.*

In a complex Euclidian space the inner product satisfies

$$\begin{aligned}(x,y) &= \overline{(y,x)},\\ (x_1+x_2,y) &= (x_1,y)+(x_2,y),\\ (\lambda x,y) &= \lambda(x,y),\\ (x,x) &\geq 0, \quad (x,x)=0 \quad \text{if and only if} \quad x=0.\end{aligned}$$

Note that in a complex linear space

$$(x,\lambda y) = \bar{\lambda}(x,y).$$

The norm in the Euclidian space is introduced by

$$\|x\| = \sqrt{x,x}.$$

The Hilbert space H is a complete (in the metric $\rho(x,y)=\|x-y\|$) infinite-dimensional Euclidian space.

In the Hilbert space H, the operator A^* *adjoint* to an operator A is defined by

$$(Ax,y) = (x,A^*y), \quad \forall x,\, y \in H.$$

The selfadjoint operator $A=A^*$ is defined from

$$(Ax,y) = (x,Ay), \quad \forall x,\, y \in H.$$

10.2 Completely continuous integral operators in Hilbert space

Consider an IE of the second kind

$$\phi(x) = f(x) + \int_a^b K(x,y)\phi(y)dy. \tag{179}$$

Assume that $K(x,y)$ is *a Hilbert–Schmidt kernel*, i.e., a square-integrable function in the square $\Pi = \{(x,y):\ a\leq x\leq b,\ a\leq y\leq b\}$, so that

$$\int_a^b\int_a^b |K(x,y)|^2 dxdy \leq \infty, \tag{180}$$

and $f(x)\in L_2[a,b]$, i.e.,

$$\int_a^b |f(x)|^2 dx \leq \infty.$$

Define a linear Fredholm (integral) operator corresponding to IE (179)

$$A\phi(x) = \int_a^b K(x,y)\phi(y)dy. \tag{181}$$

If $K(x,y)$ is a Hilbert–Schmidt kernel, then operator (181) will be called *a Hilbert–Schmidt operator.*

Rewrite IE (179) as a linear operator equation

$$\phi = A\phi(x) + f, \quad f,\, \phi \in L_2[a,b]. \tag{182}$$

Theorem 18 *Equality (182) and condition (180) define a completely continuous linear operator in the space* $L_2[a,b]$*. The norm of this operator is estimated as*

$$\|A\| \leq \sqrt{\int_a^b \int_a^b |K(x,y)|^2 dxdy}. \tag{183}$$

Proof. Note first of all that we have already mentioned (see (75)) that if condition (180) holds, then there exists a constant C_1 such that

$$\int_a^b |K(x,y)|^2 dy \leq C_1 \quad \text{almost everywhere in } [a,b], \tag{184}$$

i.e., $K(x,y) \in L_2[a,b]$ as a function of y almost for all $x \in [a,b]$. Therefore the function

$$\psi(x) = \int_a^b K(x,y)\phi(y)dy \quad \text{is defined almost everywhere in } [a,b] \tag{185}$$

We will now show that $\psi(x) \in L_2[a,b]$. Applying the Schwartz inequality we obtain

$$\begin{aligned} |\psi(x)|^2 &= \left|\int_a^b K(x,y)\phi(y)dy\right|^2 \leq \int_a^b |K(x,y)|^2 dy \int_a^b |\phi(y)|^2 dy = \\ &= \|\phi\|^2 \int_a^b |K(x,y)|^2 dy. \end{aligned} \tag{186}$$

Integrating with respect to x and replacing an iterated integral of $|K(x,y)|^2$ by a double integral, we obtain

$$\|A\phi\|^2 = \int_a^b |\psi(y)|^2 dy \leq \|\phi\|^2 \int_a^b \int_a^b |K(x,y)|^2 dxdy, \tag{187}$$

which proves (1) that $\psi(x)$ is a square-integrable function, i.e.,

$$\int_a^b |\psi(x)|^2 dx \leq \infty,$$

and estimates the norm of the operator $A: \ L_2[a,b] \to L_2[a,b]$.

Let us show that the operator $A: \ L_2[a,b] \to L_2[a,b]$ is completely continuous. To this end, we will use Theorem 8 and construct a sequence $\{A_n\}$ of completely continuous operators on the space $L_2(\Pi)$ which converges in (operator) norm to the operator A. Let $\{\psi_n\}$ be an orthonormal system in $L_2[a,b]$. Then $\{\psi_m(x)\psi_n(y)\}$ form an orthonormal system in $L_2(\Pi)$. Consequently,

$$K(x,y) = \sum_{m,n=1}^{\infty} a_{mn}\psi_m(x)\psi_n(y). \tag{188}$$

By setting

$$K_N(x,y) = \sum_{m,n=1}^{N} a_{mn}\psi_m(x)\psi_n(y). \tag{189}$$

define an operator A_N which is finite-dimensional (because it has a degenerate kernel and maps therefore $L_2[a,b]$ into a finite-dimensional space generated by a finite system $\{\psi_m\}_{m=1}^N$) and therefore completely continuous.

Now note that $K_N(x,y)$ in (189) is a partial sum of the Fourier series for $K(x,y)$, so that

$$\int_a^b \int_a^b |K(x,y) - K_N(x,y)|^2 dxdy \to 0, \quad N \to \infty. \tag{190}$$

Applying estimate (183) to the operator $A - A_N$ and using (190) we obtain

$$\|A - A_N\| \to 0, \quad N \to \infty. \tag{191}$$

This completes the proof of the theorem.

10.3 Selfadjoint operators in Hilbert space

Consider first some properties of a selfadjoint operator A ($A = A^*$) acting in the Hilbert space H.

Theorem 19 *All eigenvalues of a selfadjoint operator A in H are real.*

Proof. Indeed, let λ be an eigenvalue of a selfadjoint operator A, i.e., $Ax = \lambda x$, $x \in H$, $x \neq 0$. Then we have

$$\lambda(x,x) = (\lambda x, x) = (Ax, x) = (x, Ax) = (x, \lambda x) = \bar{\lambda}(x,x), \tag{192}$$

which yields $\lambda = \bar{\lambda}$, i.e., λ is a real number.

Theorem 20 *Eigenvectors of a selfadjoint operator A corresponding to different eigenvalues are orthogonal.*

Proof. Indeed, let λ and μ be two different eigenvalues of a selfadjoint operator A, i.e.,

$$Ax = \lambda x, \quad x \in H, \ x \neq 0; \qquad Ay = \mu y, \quad y \in H, \ y \neq 0.$$

Then we have

$$\lambda(x,y) = (\lambda x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu(x,y), \tag{193}$$

which gives

$$(\lambda - \mu)(x,y) = 0, \tag{194}$$

i.e., $(x,y) = 0$.

Below we will prove all the Fredholm theorems for integral operators in the Hilbert space.

10.4 The Fredholm theory in Hilbert space

We take the integral operator (181), assume that the kernel satisfies the Hilbert–Schmidt condition

$$\int_a^b \int_a^b |K(x,y)|^2 dxdy \leq \infty, \tag{195}$$

and write IE (179)

$$\phi(x) = \int_a^b K(x,y)\phi(y)dy + f(x) \tag{196}$$

in the operator form (182),

$$\phi = A\phi + f, \tag{197}$$

or

$$T\phi = f, \quad T = I - A. \tag{198}$$

The corresponding homogeneous IE

$$T\phi_0 = 0, \tag{199}$$

and the adjoint equations

$$T^*\psi = g, \quad T^* = I - A^*, \tag{200}$$

and

$$T^*\psi_0 = 0. \tag{201}$$

We will consider equations (198)–(201) in the Hilbert space $H = L_2[a, b]$. Note that (198)–(201) may be considered as operator equations in H with an arbitrary completely continuous operator A.

The four Fredholm theorems that are formulated below establish relations between the solutions to these four equations.

Theorem 21 *The inhomogeneous equation $T\phi = f$ is solvable if and only if f is orthogonal to every solution of the adjoint homogeneous equation $T^*\psi_0 = 0$.*

Theorem 22 (the Fredholm alternative). *One of the following situations is possible:*
(i) the inhomogeneous equation $T\phi = f$ has a unique solution for every $f \in H$
(ii) the homogeneous equation $T\phi_0 = 0$ has a nonzero solution .

Theorem 23 *The homogeneous equations $T\phi_0 = 0$ and $T^*\psi_0 = 0$ have the same (finite) number of linearly independent solutions.*

To prove the Fredholm theorems, we will use the definiton of a subspace in the Hilbert space H:
M is a linear subspace of H if M closed and $f, g \in M$ yields $\alpha f + \beta g \in M$ where α and β are arbitrary numbers,
and the following two properties of the Hilbert space H:
(i) if h is an arbitrary element in H, then the set h^{orth} of all elements $f \in H$ orthogonal to h, $h^{orth} = \{f : f \in H, (f, h) = 0\}$ form a linear subspace of H which is called the orthogonal complement,

and
(ii) if M is a (closed) linear subspace of H, then arbitrary element $f \in H$ is uniquely represented as $f = h + h'$, where $h \in M$, and $h' \in M^{orth}$ and M^{orth} is the orthogonal complement to M.

Lemma 1 *The manifold $ImT = \{y \in H : y = Tx, x \in H\}$ is closed.*

Proof. Assume that $y_n \in \operatorname{Im} T$ and $y_n \to y$. To prove the lemma, one must show that $y \in \operatorname{Im} T$, i.e., $y = Tx$. We will now prove the latter statement.

According to the definition of $\operatorname{Im} T$, there exist vectors $x_n \in H$ such that

$$y_n = Tx_n = x_n - Ax_n. \tag{202}$$

We will consider the manifold $\operatorname{Ker} T = \{y \in H : Ty = 0\}$ which is a closed linear subspace of H. One may assume that vectors x_n are orthogonal to $\operatorname{Ker} T$, i.e., $x_n \in \operatorname{Ker} T^{orth}$; otherwise, one can write

$$x_n = h_n + h'_n, \quad h_n \in \operatorname{Ker} T,$$

and replace x_n by $h_n = x_n - h'_n$ with $h'_n \in \operatorname{Ker} T^{orth}$. We may assume also that the set of norms $\|x_n\|$ is bounded. Indeed, if we assume the contrary, then there will be an unbounded subsequence $\{x_{nk}\}$ with $\|x_{nk}\| \to \infty$. Dividing by $\|x_{nk}\|$ we obtain from (202)

$$\frac{x_{nk}}{\|x_{nk}\|} - A\frac{x_{nk}}{\|x_{nk}\|} \to 0.$$

Note that A is a completely continuous operator; therefore we may assume that

$$\left\{ A\frac{x_{nk}}{\|x_{nk}\|} \right\}$$

is a convergent subsequence, which converges, e.g., to a vector $z \in H$. It is clear that $\|z\| = 1$ and $Tz = 0$, i.e., $z \in \operatorname{Ker} T$. We assume however that vectors x_n are orthogonal to $\operatorname{Ker} T$; therefore vector z must be also orthogonal to $\operatorname{Ker} T$. This contradiction means that sequence $\|x_n\|$ is bounded. Therefore, sequence $\{Ax_n\}$ contains a convergent subsequence; consequently, sequence $\{x_n\}$ will also contain a convergent subsequence due to (202). Denoting by x the limit of $\{x_n\}$, we see that, again $y = Tx$ due to (202). The lemma is proved.

Lemma 2 *The space H is the direct orthogonal sum of closed subspaces $KerT^*$ and ImT^*,*

$$KerT + ImT^* = H, \tag{203}$$

and also

$$KerT^* + ImT = H. \tag{204}$$

Proof. The closed subspaces $\operatorname{Ker} T$ and $\operatorname{Im} T^*$ are orthogonal because if $h \in \operatorname{Ker} T$ then $(h, T^*x) = (Th, x) = (0, x) = 0$ for all $x \in H$. It is left to prove there is no nonzero vector orthogonal to $\operatorname{Ker} T$ and $\operatorname{Im} T^*$. Indeed, if a vector z is orthogonal to $\operatorname{Im} T^*$, then $(Tz, x) = (z, T^*x) = 0$ for all $x \in H$, i.e., $z \in \operatorname{Ker} T$. The lemma is proved.

Lemma 2 yields the first Fredholm theorem 21. Indeed, f is orthogonal to $\operatorname{Ker} T^*$ if and only if $f \in \operatorname{Im} T^*$, i.e., there exists a vector ϕ such that $T\phi = f$.

Next, set $H^k = \operatorname{Im} T^k$ for every k, so that, in particular, $H^1 = \operatorname{Im} T$. It is clear that

$$\ldots \subset H^k \subset \ldots \subset H^2 \subset H^1 \subset H, \tag{205}$$

all the subsets in the chain are closed, and $T(H^k) = H^{k+1}$.

Lemma 3 *There exists a number j such that $H^{k+1} = H^k$ for all $k \geq j$.*

Proof. If such a j does not exist, then all H^k are different, and one can construct an orthonormal sequence $\{x_k\}$ such that $x_k \in H^k$ and x_k is orthogonal to H^{k+1}. Let $l > k$. Then

$$Ax_l - Ax_k = -x_k + (x_l + Tx_k - Tx_l).$$

Consequently, $\|Ax_l - Ax_k\| \geq 1$ because $x_l + Tx_k - Tx_l \in H^{k+1}$. Therefore, sequence $\{Ax_k\}$ does not contain a convergent subsequence, which contradicts to the fact that A is a completely continuous operator. The lemma is proved.

Lemma 4 *If $Ker T = \{0\}$ then $Im T = H$.*

Proof. If $\mathrm{Ker}\, T = \{0\}$ then T is a one-to-one operator. If in this case $\mathrm{Im}\, T \neq H$ then the chain (205) consists of different subspaces which contradicts Lemma 3. Therefore $\mathrm{Im}\, T = H$.

In the same manner, we prove that $\mathrm{Im}\, T^* = H$ if $\mathrm{Ker}\, T^* = \{0\}$.

Lemma 5 *If $Im T = H$ then $Ker T = \{0\}$.*

Proof. Since $\mathrm{Im}\, T = H$, we have $\mathrm{Ker}\, T^* = \{0\}$ according to Lemma 3. Then $\mathrm{Im}\, T^* = H$ according to Lemma 4 and $\mathrm{Ker}\, T = \{0\}$ by Lemma 2.

The totality of statements of Lemmas 4 and 5 constitutes the proof of Fredholm's alternative 22.

Now let us prove the third Fredholm's Theorem 23.

To this end, assume that $\mathrm{Ker}\, T$ is an infinite-dimensional space: $\mathrm{dimKer}\, T = \infty$. Then this space conyains an infinite orthonormal sequence $\{x_n\}$ such that $Ax_n = x_n$ and $\|Ax_l - Ax_k\| = \sqrt{2}$ if $k \neq l$. Therefore, sequence $\{Ax_n\}$ does not contain a convergent subsequence, which contradicts to the fact that A is a completely continuous operator.

Now set $\mu = \mathrm{dimKer}\, T$ and $\nu = \mathrm{dimKer}\, T^*$ and assume that $\mu < \nu$. Let $\phi_1, \ldots, \phi_\mu$ be an orthonormal basis in $\mathrm{Ker}\, T$ and $\psi_1, \ldots, \psi_\nu$ be an orthonormal basis in $\mathrm{Ker}\, T^*$. Set

$$Sx = Tx + \sum_{j=1}^{\mu} (x, \phi_j)\psi_j.$$

Operator S is a sum of T and a finite-dimensional operator; therefore all the results proved above for T remain valid for S.

Let us show that the homogeneous equation $Sx = 0$ has only a trivial solution. Indeed, assume that

$$Tx + \sum_{j=1}^{\mu} (x, \phi_j)\psi_j = 0. \tag{206}$$

According to Lemmas 2 vectors psi_j are orthogonal to all vectors of the form Tx (206) yields

$$Tx = 0$$

and

$$(x, \phi_j) = 0, quad 1 \leq j \leq \mu.$$

We see that, on the one hand, vector x must be a linear combination of vectors psi_j, and on the other hand, vector x must be orthogonal to these vectors. Therefore, $x = 0$ and equation $Sx = 0$ has only a trivial solution. Thus according to the second Fredholm's Theorem 22 there exists a vector y such that

$$Ty + \sum_{j=1}^{\mu}(y, \phi_j)\psi_j = \psi_{\mu+1}.$$

Calculating the inner product of both sides of this equality and $\psi_{\mu+1}$ we obtain

$$(Ty, \psi_{\mu+1}) + \sum_{j=1}^{\mu}(y, \phi_j)(\psi_j, \psi_{\mu+1}) = (\psi_{\mu+1}, \psi_{\mu+1}).$$

Here $(Ty, \psi_{\mu+1}) = 0$ because $Ty \in \operatorname{Im} T$ and $\operatorname{Im} T$ is orthogonal to $\operatorname{Ker} T^*$. Thus we have

$$(0 + \sum_{j=1}^{\mu}(y, \phi_j) \times 0 = 1.$$

This contradiction is obtained because we assumed $\mu < \nu$. Therefore $\mu \geq \nu$. Replacing now T by T^* we obtain in the same manner that $\mu \leq \nu$. Therefore $\mu = \nu$. The third Fredholm's theorem is proved.

To sum up, the Fredholm's theorems show that the number $\lambda = 1$ is either a regular point or an eigenvalue of finite multiplicity of the operator A in (197).

All the results obtained above for the equation (197) $\phi = A\phi + f$ (i.e., for the operator $A - I$) remain valid for the equation $\lambda\phi = A\phi + f$ (for the operator $A - \lambda I$ with $\lambda \neq 0$). It follows that in the case of a completely continuous operator, an arbitrary nonzero number is either a regular point or an eigenvalue of finite multiplicity. Thus *a completely continuous operator has only a point spectrum*. The point 0 is an exception and always belongs to the spectrum of a completely continuous operator in an infinite-dimensional (Hilbert) space but may not be an eigenvalue.

10.5 The Hilbert–Schmidt theorem

Let us formulate the important Hilbert–Schmidt theorem.

Theorem 24 *A completely continuous selfadjoint operator A in the Hilbert space H has an orthonormal system of eigenvectors $\{\psi_k\}$ corresponding to eigenvalues λ_k such that every element $\xi \in H$ is represented as*

$$\xi = \sum_k c_k\psi_k + \xi', \tag{207}$$

where the vector $\xi' \in Ker A$, i.e., $A\xi' = 0$. The representation (207) is unique, and

$$A\xi = \sum_k \lambda_k c_k \psi_k, \tag{208}$$

and

$$\lambda_k \to 0, \quad n \to \infty, \tag{209}$$

if $\{\psi_k\}$ is an infinite system.

11 IEs with symmetric kernels. Hilbert–Schmidt theory

Consider first a general a Hilbert–Schmidt operator (181).

Theorem 25 *Let A be a Hilbert–Schmidt operator with a (Hilbert–Schmidt) kernel $K(x,y)$. Then the operator A^* adjoint to A is an operator with the adjoint kernel $K^*(x,y) = \overline{K(y,x)}$.*

Proof. Using the Fubini theorem and changing the order of integration, we obtain

$$\begin{aligned}(Af,g) &= \int_a^b \left\{\int_a^b K(x,y)f(y)dy\right\}\overline{g(x)}dx = \\ &= \int_a^b\int_a^b K(x,y)f(y)\overline{g(x)}dydx = \\ &= \int_a^b \left\{\int_a^b K(x,y)\overline{g(x)}dx\right\}f(y)dy = \\ &= \int_a^b f(y)\overline{\left\{\int_a^b \overline{K(x,y)}g(x)dx\right\}}dy,\end{aligned}$$

which proves the theorem.

Consider *a symmetric* IE

$$\phi(x) - \lambda\int_a^b K(x,y)\phi(y)dy = f(x) \tag{210}$$

with *a symmetric kernel*

$$K(x,y) = \overline{K(y,x)} \tag{211}$$

for complex-valued or

$$K(x,y) = K(y,x) \tag{212}$$

for real-valued $K(x,y)$.

According to Theorem 25, an integral operator (181) is selfadjoint in the space $L_2[a,b]$, i.e., $A = A^*$, if and only if it has a symmetric kernel.

11.1 Selfadjoint integral operators

Let us formulate the properties (19) and (20) of eigenvalues and eigenfunctions (eigenvectors) for a selfadjoint integral operator acting in the Hilbert space H.

Theorem 26 *All eigenvalues of a integral operator with a symmetric kernel are real.*

Proof. Let λ_0 and ϕ_0 be an eigenvalue and eigenfunction of a integral operator with a symmetric kernel, i.e.,

$$\phi_0(x) - \lambda_0\int_a^b K(x,y)\phi_0(y)dy = 0. \tag{213}$$

Multiply equality (213) by $\bar{\phi}_0$ and integrate with respect to x between a and b to obtain

$$\|\phi_0\|^2 - \lambda_0 (K\phi_0, \phi_0) = 0, \tag{214}$$

which yields

$$\lambda_0 = \frac{\|\phi_0\|^2}{(K\phi_0, \phi_0)}. \tag{215}$$

Rewrite this equality as

$$\lambda_0 = \frac{\|\phi_0\|^2}{\mu}, \quad \mu = (K\phi_0, \phi_0), \tag{216}$$

and prove that μ is a real number. For a symmetric kernel, we have

$$\mu = (K\phi_0, \phi_0) = (\phi_0, K\phi_0). \tag{217}$$

According to the definition of the inner product

$$(\phi_0, K\phi_0) = \overline{(K\phi_0, \phi_0)}.$$

Thus

$$\mu = (K\phi_0, \phi_0) = \overline{\mu} = \overline{(K\phi_0, \phi_0)},$$

so that $\mu = \overline{\mu}$, i.e., μ is a real number and, consequently, λ_0 in (215) is a real number.

Theorem 27 *Eigenfunctions of an integral operator with a symmetric kernel corresponding to different eigenvalues are orthogonal.*

Proof. Let λ and μ and ϕ_λ and ϕ_μ be two different eigenvalues and the corresponding eigenfunctions of an integral operator with a symmetric kernel, i.e.,

$$\begin{aligned} \phi_\lambda(x) &= \lambda \int_a^b K(x,y)\phi_\lambda(y)dy = 0, && (218)\\ \phi_\mu(x) &= \lambda \int_a^b K(x,y)\phi_\mu(y)dy = 0. && (219) \end{aligned}$$

Then we have

$$\lambda(\phi_\lambda, \phi_\mu) = (\lambda\phi_\lambda, \phi_\mu) = (A\phi_\lambda, \phi_\mu) = (\phi_\lambda, A\phi_\mu) = (\phi_\lambda, \mu\phi_\mu) = \mu(\phi_\lambda, \phi_\mu), \tag{220}$$

which yields

$$(\lambda - \mu)(\phi_\lambda, \phi_\mu) = 0, \tag{221}$$

i.e., $(\phi_\lambda, \phi_\mu) = 0$.

One can also reformulate the Hilbert–Schmidt theorem 24 for integral operators.

Theorem 28 *Let* $\lambda_1, \lambda_2, \dots,$ *and* $\phi_1, \phi_2, \dots,$ *be eigenvalues and eigenfunctions (eigenvectors) of an integral operator with a symmetric kernel and let* $h(x) \in L_2[a,b]$, *i.e.,*

$$\int_a^b |h(x)|^2 dx \le \infty.$$

If $K(x,y)$ *is a Hilbert–Schmidt kernel, i.e., a square-integrable function in the square* $\Pi = \{(x,y) : \, a \le x \le b, \, a \le y \le b\}$, *so that*

$$\int_a^b \int_a^b |K(x,y)|^2 dx dy < \infty, \tag{222}$$

then the function

$$f(x) = Ah = \int_a^b K(x,y) h(y) dy \tag{223}$$

is decomposed into an absolutely and uniformly convergent Fourier series in the orthonornal system $\phi_1, \phi_2, \dots,$

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x), \quad f_n = (f, \phi_n).$$

The Fourier coefficients f_n *of the function* $f(x)$ *are coupled with the Fourier coefficients* h_n *of the function* $h(x)$ *by the relationships*

$$f_n = \frac{h_n}{\lambda_n}, \quad h_n = (h, \phi_n),$$

so that

$$f(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n} \phi_n(x) \qquad (f_n = (f, \phi_n), \quad h_n = (h, \phi_n)). \tag{224}$$

Setting $h(x) = K(x,y)$ in (224), we obtain

$$K_2(x,y) = \sum_{n=1}^{\infty} \frac{\omega_n(y)}{\lambda_n} \phi_n(x), \quad \omega_n(y) = (K(x,y), \phi_n(x)), \tag{225}$$

and $\omega_n(y)$ are the Fourier coefficients of the kernel $K(x,y)$. Let us calculate $\omega_n(y)$. We have, using the formula for the Fourier coefficients,

$$\omega_n(y) = \int_a^b K(x,y) \overline{\phi_n(x)} dx, \tag{226}$$

or, because the kernel is symmetric,

$$\omega_n(y) = \int_a^b \overline{K(x,y) \phi_n(x)} dx, \tag{227}$$

Note that $\phi_n(y)$ satisfies the equation

$$\phi_n(x) = \lambda_n \int_a^b K(x,y) \phi_n(y) dy. \tag{228}$$

Replacing x by y abd vice versa, we obtain

$$\frac{1}{\lambda_n}\phi_n(y) = \int_a^b K(y,x)\phi_n(x)dx, \tag{229}$$

so that

$$\omega_n(y) = \frac{1}{\lambda_n}\overline{\phi_n(y)}.$$

Now we have

$$K_2(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\overline{\phi_n(y)}}{\lambda_n^2}. \tag{230}$$

In the same manner, we obtain

$$K_3(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\overline{\phi_n(y)}}{\lambda_n^3},$$

and, generally,

$$K_m(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\overline{\phi_n(y)}}{\lambda_n^m}. \tag{231}$$

which is bilinear series for kernel $K_m(x,y)$.

For kernel $K(x,y)$, the bilinear series is

$$K(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\overline{\phi_n(y)}}{\lambda_n}. \tag{232}$$

This series may diverge in the sense of C-norm (i.e., uniformly); however it alwyas converges in L_2-norm.

11.2 Hilbert–Schmidt theorem for integral operators

Consider a Hilbert–Schmidt integral operator A with a symmetric kernel $K(x,y)$. In this case, the following conditions are assumed to be satisfied:

$$\begin{array}{ll} \text{i} & A\phi(x) = \int_a^b K(x,y)\phi(y)dy, \\ \text{ii} & \int_a^b \int_a^b |K(x,y)|^2 dxdy \leq \infty, \\ \text{iii} & K(x,y) = \overline{K(y,x)}. \end{array} \tag{233}$$

According to Theorem 18, A is a completely continuous selfadjoint operator in the space $L_2[a,b]$ and we can apply the Hilbert–Schmidt theorem 24 to prove the following statement.

Theorem 29 *If 1 is not an eigenvalue of the operator A then the IE (210), written in the operator form as*

$$\phi(x) = A\phi + f(x), \tag{234}$$

has one and only one solution for every f. If 1 is an eigenvalue of the operator A, then IE (210) (or (234)) is solvable if and only if f is orthogonal to all eigenfunctions of the operator A corresponding to the eigenvalue $\lambda = 1$, and in this case IE (210) has infinitely many solutions.

Proof. According to the Hilbert–Schmidt theorem, a completely continuous selfadjoint operator A in the Hilbert space $H = L_2[a, b]$ has an orthonormal system of eigenvectors $\{\psi_k\}$ corresponding to eigenvalues λ_k such that every element $\xi \in H$ is represented as

$$\xi = \sum_k c_k \psi_k + \xi', \tag{235}$$

where $\xi' \in \operatorname{Ker} A$, i.e., $A\xi' = 0$. Set

$$f = \sum_k b_k \psi_k + f', \quad Af' = 0, \tag{236}$$

and look for a solution to (234) in the form

$$\phi = \sum_k x_k \psi_k + \phi', \quad A\phi' = 0. \tag{237}$$

Substituting (236) and (237) into (234) we obtain

$$\sum_k x_k \psi_k + \phi' = \sum_k x_k \lambda_k \psi_k + \sum_k b_k \psi_k + f', \tag{238}$$

which holds if and only if

$$\begin{aligned} f' &= \phi', \\ x_k(1 - \lambda_k) &= b_k \quad (k = 1, 2, \ldots) \end{aligned}$$

i.e., when

$$\begin{aligned} f' &= \phi', \\ x_k &= \frac{b_k}{1 - \lambda_k}, \quad \lambda_k \neq 1 \\ b_k &= 0, \quad \lambda_k = 1. \end{aligned} \tag{239}$$

The latter gives the necessary and sufficient condition for the solvability of equation (234). Note that the coefficients x_n that correspond to those n for which $\lambda_n = 1$ are arbitrary numbers. The theorem is proved.

12 Harmonic functions and Green's formulas

Denote by $D \in \mathrm{R}^2$ a two-dimensional domain bounded by the closed smooth contour Γ.

A twice continuously differentiable real-valued function u defined on a domain D is called *harmonic* if it satisfies Laplace's equation

$$\Delta u = 0 \quad \text{in} \quad D, \tag{240}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{241}$$

is called *Laplace's operator* (Laplacian), the function $u = u(\mathbf{x})$, and $\mathbf{x} = (x, y) \in \mathrm{R}^2$. We will also use the notation $\mathbf{y} = (x_0, y_0)$.

The function

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|} \tag{242}$$

is called *the fundamental solution of Laplace's equation*. For a fixed $\mathbf{y} \in \mathrm{R}^2$, $\mathbf{y} \neq \mathbf{x}$, the function $\Phi(\mathbf{x}, \mathbf{y})$ is harmonic, i.e., satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{in} \quad D. \tag{243}$$

The proof follows by straightforward differentiation.

12.1 Green's formulas

Let us formulate Green's theorems which leads to the second and third Green's formulas.

Let $D \in \mathrm{R}^2$ be a (two-dimensional) domain bounded by the closed smooth contour Γ and let n_y denote the unit normal vector to the boundary Γ directed into the exterior of Γ and corresponding to a point $\mathbf{y} \in \Gamma$. Then for every function u which is once continuously differentiable in the closed domain $\bar{D} = D + \Gamma$, $u \in C^1(\bar{D})$, and every function v which is twice continuously differentiable in $\bar{D}$, $v \in C^2(\bar{D})$, Green's first theorem (Green's first formula) is valid

$$\int\int_D (u \Delta v + \operatorname{grad} u \cdot \operatorname{grad} v) d\mathbf{x} = \int_\Gamma u \frac{\partial v}{\partial n_y} dl_y, \tag{244}$$

where $\cdot$ denotes the inner product of two vector-functions. For $u \in C^2(\bar{D})$ and $v \in C^2(\bar{D})$, Green's second theorem (Green's second formula) is valid

$$\int\int_D (u \Delta v - v \Delta u) d\mathbf{x} = \int_\Gamma \left(u \frac{\partial v}{\partial n_y} - v \frac{\partial u}{\partial n_y} \right) dl_y, \tag{245}$$

Proof. Apply Gauss theorem

$$\int\int_D \operatorname{div} A d\mathbf{x} = \int_\Gamma n_y \cdot A dl_y \tag{246}$$

to the vector function (vector field) $A = (A_1.A_2) \in C^1(\bar{D})$ (with the components A_1. A_2 being once continuously differentiable in the closed domain $\bar{D} = D + \Gamma$, $A_i \in C^1(\bar{D})$, $i = 1, 2$) defined by

$$A = u \cdot \operatorname{grad} v = \left[u\frac{\partial v}{\partial x}, \, u\frac{\partial v}{\partial y} \right], \tag{247}$$

the components are

$$A_1 = u\frac{\partial v}{\partial x}, \quad A_2 = u\frac{\partial v}{\partial y}. \tag{248}$$

We have

$$\operatorname{div} A = \operatorname{div}(u \operatorname{grad} v) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y} \right) = \tag{249}$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} = \operatorname{grad} u \cdot \operatorname{grad} v + u\Delta v. \tag{250}$$

On the other hand, according to the definition of the normal derivative,

$$n_y \cdot A = n_y \cdot (u \operatorname{grad} v) = u(n_y \cdot \operatorname{grad} v) = u\frac{\partial v}{\partial n_y}, \tag{251}$$

so that, finally,

$$\int\!\!\int_D \operatorname{div} A d\mathbf{x} = \int\!\!\int_D (u\Delta v + \operatorname{grad} u \cdot \operatorname{grad} v) d\mathbf{x}, \quad \int_\Gamma n_y \cdot A \, dl_y = \int_\Gamma u\frac{\partial v}{\partial n_y} dl_y, \tag{252}$$

which proves Green's first formula. By interchanging u and v and subtracting we obtain the desired Green's second formula (245).

Let a twice continuously differentiable function $u \in C^2(\bar{D})$ be harmonic in the domain D. Then *Green's third theorem (Green's third formula)* is valid

$$u(\mathbf{x}) = \int_\Gamma \left(\Phi(\mathbf{x}, \mathbf{y})\frac{\partial u}{\partial n_y} - u(\mathbf{y})\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \right) dl_y, \quad \mathbf{x} \in \quad D. \tag{253}$$

Proof. For $\mathbf{x} \in D$ we choose a circle

$$\Omega(\mathbf{x}, r) = \{\mathbf{y} : \, |\mathbf{x} - \mathbf{y}| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r\}$$

of radius r (the boundary of a vicinity of a point $\mathbf{x} \in D$) such that $\Omega(\mathbf{x}, r) \in D$ and direct the unit normal n to $\Omega(\mathbf{x}, r)$ into the exterior of $\Omega(\mathbf{x}, r)$. Then we apply Green's second formula (245) to the harmonic function u and $\Phi(\mathbf{x}, \mathbf{y})$ (which is also a harmonic function) in the domain $\mathbf{y} \in D : \, |\mathbf{x} - \mathbf{y}| > r$ to obtain

$$0 = \int\!\!\int_{D \setminus \Omega(\mathbf{x}, r)} (u\Delta\Phi(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y})\Delta u) d\mathbf{x} = \tag{254}$$

$$\int_{\Gamma \cup \Omega(\mathbf{x}, r)} \left(u(\mathbf{y})\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} - \Phi(\mathbf{x}, \mathbf{y})\frac{\partial u(\mathbf{y})}{\partial n_y} \right) dl_y. \tag{255}$$

Since on $\Omega(\mathbf{x}, r)$ we have

$$\operatorname{grad}_y \Phi(\mathbf{x}, \mathbf{y}) = \frac{n_y}{2\pi r}, \tag{256}$$

a straightforward calculation using the mean-value theorem and the fact that

$$\int_\Gamma \frac{\partial v}{\partial n_y} dl_y = 0 \tag{257}$$

for a harmonic in D function v shows that

$$\lim_{r\to 0} \int_{\Omega(\mathbf{x},r)} \left(u(\mathbf{y}) \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} - \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u(\mathbf{y})}{\partial n_y} \right) dl_y = u(\mathbf{x}), \tag{258}$$

which yields (253).

12.2 Properties of harmonic functions

Theorem 30 *Let a twice continuously differentiable function v be harmonic in a domain D bounded by the closed smooth contour Γ. Then*

$$\int_\Gamma \frac{\partial v(\mathbf{y})}{\partial n_y} dl_y = 0. \tag{259}$$

Proof follows from the first Green's formula applied to two harmonic functions v and $u = 1$.

Theorem 31 (Mean Value Formula) *Let a twice continuously differentiable function u be harmonic in a ball*

$$B(\mathbf{x}, r) = \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \le r\}$$

of radius r (a vicinity of a point $\mathbf{x}$) with the boundary $\Omega(\mathbf{x}, r) = \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| = r\}$ (a circle) and continuous in the closure $\bar{B}(\mathbf{x}, r)$. Then

$$u(\mathbf{x}) = \frac{1}{\pi r^2} \int_{B(\mathbf{x},r)} u(\mathbf{y}) dy = \frac{1}{2\pi r} \int_{\Omega(\mathbf{x},r)} u(\mathbf{y}) dl_y, \tag{260}$$

i.e., the value of u in the center of the ball is equal to the integral mean values over both the ball and its boundary.

Proof. For each $0 < \rho < r$ we apply (259) and Green's third formula to obtain

$$u(\mathbf{x}) = \frac{1}{2\pi\rho} \int_{|\mathbf{x}-\mathbf{y}|=\rho} u(\mathbf{y}) dl_y, \tag{261}$$

whence the second mean value formula in (260) follows by passing to the limit $\rho \to r$. Multiplying (261) by ρ and integrating with respect to ρ from 0 to r we obtain the first mean value formula in (260).

Theorem 32 (Maximum–Minimum Principle) *A harmonic function on a domain cannot attain its maximum or its minimum unless it is constant.*

Corollary. *Let $D \in \mathrm{R}^2$ be a two-dimensional domain bounded by the closed smooth contour Γ and let u be harmonic in D and continuous in $\bar{D}$. Then u attains both its maximum and its minimum on the boundary.*

13 Boundary value problems

Formulate *the interior Dirichlet problem*: find a function u that is harmonic in a domain D bounded by the closed smooth contour Γ, continuous in $\bar{D} = D \cup \Gamma$ and satisfies the Dirichlet boundary condition:

$$\Delta u = 0 \quad \text{in} \quad D, \tag{262}$$

$$u|_{\Gamma} = -f, \tag{263}$$

where f is a given continuous function.

Formulate *the interior Neumann problem*: find a function u that is harmonic in a domain D bounded by the closed smooth contour Γ, continuous in $\bar{D} = D \cup \Gamma$ and satisfies the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = -g, \tag{264}$$

where g is a given continuous function.

Theorem 33 *The interior Dirichlet problem has at most one solution.*

Proof. Assume that the interior Dirichlet problem has two solutions, u_1 and u_2. Their difference $u = u_1 - u_2$ is a harmonic function that is continuous up to the boundary and satisifes the homogeneous boundary condition $u = 0$ on the boundary Γ of D. Then from the maximum–minimum principle or its corollary we obtain $u = 0$ in D, which proves the theorem.

Theorem 34 *Two solutions of the interior Neumann problem can differ only by a constant. The exterior Neumann problem has at most one solution.*

Proof. We will prove the first statement of the theorem. Assume that the interior Neumann problem has two solutions, u_1 and u_2. Their difference $u = u_1 - u_2$ is a harmonic function that is continuous up to the boundary and satisifes the homogeneous boundary condition $\frac{\partial u}{\partial n_y} = 0$ on the boundary Γ of D. Suppose that u is not constant in D. Then there exists a closed ball B contained in D such that

$$\int\!\!\int_B |\mathrm{grad}\, u|^2 d\mathbf{x} > 0.$$

From Green's first formula applied to a pair of (harmonic) functions u, $v = u$ and the interior D_h of a surface $\Gamma_h = \{\mathbf{x} - hn_x : \mathbf{x} \in \Gamma\}$ parallel to the Ds boundary Γ with sufficiently small $h > 0$,

$$\int\int_{D_h} (u\Delta u + |\mathrm{grad}\, u|^2) d\mathbf{x} = \int_{\Gamma} u \frac{\partial u}{\partial n_y} dl_y, \tag{265}$$

we have first

$$\int\int_{B} |\mathrm{grad}\, u|^2 d\mathbf{x} \leq \int\int_{D_h} |\mathrm{grad}\, u|^2 d\mathbf{x}, \tag{266}$$

(because $B \in D_h$); next we obtain

$$\int\int_{D_h} |\mathrm{grad}\, u|^2 d\mathbf{x} \leq \int_{\Gamma} u \frac{\partial u}{\partial n_y} dl_y = 0. \tag{267}$$

Passing to the limit $h \to 0$ we obtain a contradiction

$$\int\int_{B} |\mathrm{grad}\, u|^2 d\mathbf{x} \leq 0.$$

Hence, the difference $u = u_1 - u_2$ must be constant in D.

14 Potentials with logarithmic kernels

In the theory of boundary value problems, the integrals

$$u(\mathbf{x}) = \int_{C} E(\mathbf{x}, \mathbf{y}) \xi(\mathbf{y}) dl_y, \quad v(\mathbf{x}) = \int_{C} \frac{\partial}{\partial \mathbf{n}_y} E(\mathbf{x}, \mathbf{y}) \eta(\mathbf{y}) dl_y \tag{268}$$

are called *the potentials*. Here, $\mathbf{x} = (x, y)$, $\mathbf{y} = (x_0, y_0) \in \mathrm{R}^2$; $E(\mathbf{x}, \mathbf{y})$ is the fundamental solution of a second-order elliptic differential operator;

$$\frac{\partial}{\partial \mathbf{n}_y} = \frac{\partial}{\partial n_y}$$

is the normal derivative at the point $\mathbf{y}$ of the closed piecewise smooth boundary C of a domain in R^2; and $\xi(\mathbf{y})$ and $\eta(\mathbf{y})$ are sufficiently smooth functions defined on C. In the case of Laplace's operator Δu,

$$g(\mathbf{x}, \mathbf{y}) \; = \; g(\mathbf{x} - \mathbf{y}) \; = \; \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}. \tag{269}$$

In the case of the Helmholtz operator $\mathcal{H}(k) = \Delta + k^2$, one can take $E(\mathbf{x}, \mathbf{y})$ in the form

$$E(\mathbf{x}, \mathbf{y}) = \mathcal{E}(\mathbf{x} - \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|),$$

where $H_0^{(1)}(z) = -2ig(z) + h(z)$ is the Hankel function of the first kind and zero order (one of the so-called cylindrical functions) and $\frac{1}{2} g(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}$ is the kernel of the two-dimensional single layer potential; the second derivative of $h(z)$ has a logarithmic singularity.

14.1 Properties of potentials

Statement 1. Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Then the kernel of the double-layer potential

$$V(\mathbf{x},\mathbf{y}) = \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y}, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}, \tag{270}$$

is a continuous function on Γ for $\mathbf{x}$, $\mathbf{y} \in \Gamma$.

Proof. Performing differentiation we obtain

$$V(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \frac{\cos\theta_{\mathbf{x},\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|}, \tag{271}$$

where $\theta_{\mathbf{x},\mathbf{y}}$ is the angle between the normal vector $\mathbf{n}_y$ at the integration point $\mathbf{y} = (x_0, y_0)$ and vector $\mathbf{x}-\mathbf{y}$. Choose the origin $O = (0,0)$ of (Cartesian) coordinates at the point $\mathbf{y}$ on curve Γ so that the Ox axis goes along the tangent and the Oy axis, along the normal to Γ at this point. Then one can write the equation of curve Γ in a sufficiently small vicinity of the point $\mathbf{y}$ in the form

$$y = y(x).$$

The asumption concerning the smoothness of Γ means that $y_0(x_0)$ is a differentiable function in a vicinity of the origin $O = (0,0)$, and one can write a segment of the Taylor series

$$y = y(0) + xy'(0) + \frac{x^2}{2} y''(\eta x) = \frac{1}{2} x^2 y''(\eta x) \quad (0 \le \eta < 1)$$

because $y(0) = y'(0) = 0$. Denoting $r = |\mathbf{x}-\mathbf{y}|$ and taking into account that $\mathbf{x} = (x,y)$ and the origin of coordinates is placed at the point $\mathbf{y} = (0,0)$, we obtain

$$r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} = \sqrt{x^2 + \frac{1}{4}x^4 (y''(\eta x))^2} = x\sqrt{1 + \frac{1}{4}x^2 (y''(\eta x))^2};$$

$$\cos\theta_{\mathbf{x},\mathbf{y}} = \frac{y}{r} = \frac{1}{2} \frac{xy''(\eta x)}{\sqrt{1 + x^2(1/4)(y''(\eta x))^2}},$$

and

$$\frac{\cos\theta_{\mathbf{x},\mathbf{y}}}{r} = \frac{y}{r^2} = \frac{1}{2} \frac{y''(\eta x)}{1 + x^2(1/4)(y''(\eta x))^2}.$$

The curvature K of a plane curve is given by

$$K = \frac{y''}{(1+(y')^2)^{3/2}},$$

which yields $y''(0) = K(\mathbf{y})$, and, finally,

$$\lim_{r\to 0} \frac{\cos\theta_{\mathbf{x},\mathbf{y}}}{r} = \frac{1}{2} K(\mathbf{y})$$

which proves the continuity of the kernel (270).

Statement 2 (Gauss formula) Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. For the double-layer potential with a constant density

$$v^0(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}, \tag{272}$$

where the (exterior) unit normal vector n of Γ is directed into the exterior domain $\mathrm{R}^2 \setminus \bar{D}$, we have

$$\begin{aligned} v^0(\mathbf{x}) &= -1, \quad \mathbf{x} \in D, \\ v^0(\mathbf{x}) &= -\frac{1}{2}, \quad \mathbf{x} \in \Gamma, \\ v^0(\mathbf{x}) &= 0, \quad \mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}. \end{aligned} \tag{273}$$

Proof follows for $\mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}$ from the equality

$$\int\limits_{\Gamma} \frac{\partial v(\mathbf{y})}{\partial n_y} dl_y = 0 \tag{274}$$

applied to $v(\mathbf{y}) = \Phi(\mathbf{x},\mathbf{y})$. For $\mathbf{x} \in D$ it follows from Green's third formula

$$u(\mathbf{x}) = \int\limits_{\Gamma} \left(\Phi(\mathbf{x},\mathbf{y}) \frac{\partial u}{\partial n_y} - u(\mathbf{y}) \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y} \right) dl_y, \quad \mathbf{x} \in \quad D, \tag{275}$$

applied to $u(\mathbf{y}) = 1$ in D.

Note also that if we set

$$v^0(\mathbf{x}') = -\frac{1}{2}, \quad \mathbf{x}' \in \Gamma,$$

we can also write (273) as

$$v^0_{\pm}(\mathbf{x}') = \lim_{h \to \pm 0} v(\mathbf{x} + h n_{x'}) = v^0(\mathbf{x}') \pm \frac{1}{2}, \quad \mathbf{x}' \in \Gamma. \tag{276}$$

Corollary. Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Introduce the single-layer potential with a constant density

$$u^0(\mathbf{x}) = \int\limits_{\Gamma} \Phi(\mathbf{x},\mathbf{y}) dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{277}$$

For the normal derivative of this single-layer potential

$$\frac{\partial u^0(\mathbf{x})}{\partial n_x} = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_x} dl_y, \tag{278}$$

where the (exterior) unit normal vector n_x of Γ is directed into the exterior domain $\mathrm{R}^2 \setminus \bar{D}$, we have

$$\begin{aligned} \frac{\partial u^0(\mathbf{x})}{\partial n_x} &= 1, \quad \mathbf{x} \in D, \\ \frac{\partial u^0(\mathbf{x})}{\partial n_x} &= \frac{1}{2}, \quad \mathbf{x} \in \Gamma, \\ \frac{\partial u^0(\mathbf{x})}{\partial n_x} &= 0, \quad \mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}. \end{aligned} \tag{279}$$

Theorem 35 *Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential*

$$v(\mathbf{x}) = \int\limits_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}, \tag{280}$$

with a continuous density φ can be continuously extended from D to $\bar{D}$ and from $R^2 \setminus \bar{D}$ to $R^2 \setminus D$ with the limiting values on Γ

$$v_\pm(\mathbf{x}') = \int\limits_\Gamma \frac{\partial \Phi(\mathbf{x}', \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y \pm \frac{1}{2} \varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{281}$$

or

$$v_\pm(\mathbf{x}') = v(\mathbf{x}') \pm \frac{1}{2} \varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{282}$$

where

$$v_\pm(\mathbf{x}') = \lim_{h \to \pm 0} v(\mathbf{x} + h n_{x'}). \tag{283}$$

Proof.
1. Introduce the function

$$I(\mathbf{x}) = v(\mathbf{x}) - v^0(\mathbf{x}) = \int\limits_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} (\varphi(\mathbf{y}) - \varphi_0) dl_y \tag{284}$$

where $\varphi_0 = \varphi(\mathbf{x}')$ and prove its continuity at the point $\mathbf{x}' \in \Gamma$. For every $\epsilon > 0$ and every $\eta > 0$ there exists a vicinity $C_1 \subset \Gamma$ of the point $\mathbf{x}' \in \Gamma$ such that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| < \eta, \quad \mathbf{x}' \in C_1. \tag{285}$$

Set

$$I = I_1 + I_2 = \int\limits_{C_1} \ldots + \int\limits_{\Gamma \setminus C_1} \ldots .$$

Then

$$|I_1| \le \eta B_1,$$

where B_1 is a constant taken according to

$$\int_{\Gamma} \left| \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \right| dl_y \leq B_1 \quad \forall \mathbf{x} \in \overline{D}, \tag{286}$$

and condition (286) holds because kernel (270) is continuous on Γ.

Taking $\eta = \epsilon / B_1$, we see that for every $\epsilon > 0$ there exists a vicinity $C_1 \subset \Gamma$ of the point $\mathbf{x}' \in \Gamma$ (i.e., $\mathbf{x}' \in C_1$) such that

$$|I_1(\mathbf{x})| < \epsilon \quad \forall \mathbf{x} \in \overline{D}. \tag{287}$$

Inequality (287) means that $I(\mathbf{x})$ is continuous at the point $\mathbf{x}' \in \Gamma$.

2. Now, if we take $v_{\pm}(\mathbf{x}')$ for $\mathbf{x}' \in \Gamma$ and consider the limits $v_{\pm}(\mathbf{x}') = \lim_{h \to \pm 0} v(\mathbf{x} + h n_y)$ from inside and outside contour Γ using (276), we obtain

$$v_{-}(\mathbf{x}') = I(\mathbf{x}') + \lim_{h \to -0} v^0(\mathbf{x}' - h n_{x'}) = v(\mathbf{x}') - \frac{1}{2}\varphi(\mathbf{x}'), \tag{288}$$

$$v_{+}(\mathbf{x}') = I(\mathbf{x}') + \lim_{h \to +0} v^0(\mathbf{x}' + h n_{x'}) = v(\mathbf{x}') + \frac{1}{2}\varphi(\mathbf{x}'), \tag{289}$$

which prove the theorem.

Corollary. Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Introduce the single-layer potential

$$u(\mathbf{x}) = \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}. \tag{290}$$

with a continuous density φ. The normal derivative of this single-layer potential

$$\frac{\partial u(\mathbf{x})}{\partial n_x} = \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_x} \varphi(\mathbf{y}) dl_y \tag{291}$$

can be continuously extended from D to $\bar{D}$ and from $\mathrm{R}^2 \setminus \bar{D}$ to $\mathrm{R}^2 \setminus D$ with the limiting values on Γ

$$\left.\frac{\partial u(\mathbf{x}')}{\partial n_x}\right|_{\pm} = \int_{\Gamma} \frac{\partial \Phi(\mathbf{x}', \mathbf{y})}{\partial n_{x'}} \varphi(\mathbf{y}) dl_y \mp \frac{1}{2}\varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{292}$$

or

$$\left.\frac{\partial u(\mathbf{x}')}{\partial n_x}\right|_{\pm} = \frac{\partial u(\mathbf{x}')}{\partial n_x} \mp \frac{1}{2}\varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{293}$$

where

$$\frac{\partial u(\mathbf{x}')}{\partial n_{x'}} = \lim_{h \to \pm 0} n_{x'} \cdot \operatorname{grad} v(\mathbf{x}' + h n_{x'}). \tag{294}$$

14.2 Generalized potentials

Let $S_\Pi(\Gamma) \in \mathrm{R}^2$ be a domain bounded by the closed piecewise smooth contour Γ. We assume that a rectilinear interval Γ_0 is a subset of Γ, so that $\Gamma_0 = \{\mathbf{x} : y = 0, x \in [a, b]\}$.

Let us say that functions $U_l(\mathbf{x})$ are *the generalized single layer (SLP)* $(l = 1)$ *or double layer (DLP)* $(l = 2)$ *potentials* if

$$U_l(\mathbf{x}) = \int_\Gamma K_l(\mathbf{x}, t)\varphi_l(t)dt, \quad \mathbf{x} = (x, y) \in S_\Pi(\Gamma),$$

where

$$K_l(\mathbf{x}, t) = g_l(\mathbf{x}, t) + F_l(\mathbf{x}, t) \quad (l = 1, 2),$$

$$g_1(\mathbf{x}, t) = g(x, y^0) = \frac{1}{\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}^0|}, \quad g_2(\mathbf{x}, t) = \frac{\partial}{\partial y_0} g(\mathbf{x}, \mathbf{y}^0) \quad [\mathbf{y}^0 = (t, 0)],$$

$F_{1,2}$ are smooth functions, and we shall assume that for every closed domain $S_{0\Pi}(\Gamma) \subset S_\Pi(\Gamma)$, the following conditions hold

i) $F_1(\mathbf{x}, t)$ is once continuously differentiable with respect to the variables of $\mathbf{x}$ and continuous in t;

ii) $F_2(\mathbf{x}, t)$ and

$$F_2^1(\mathbf{x}, t) = \frac{\partial}{\partial y} \int_q^t F_2(x, s)ds, \quad q \in \mathrm{R}^1,$$

are continuous.

We shall also assume that the densities of the generalized potentials $\varphi_1 \in L_2^{(1)}(\Gamma)$ and $\varphi_2 \in L_2^{(2)}(\Gamma)$, where functional spaces $L_2^{(1)}$ and $L_2^{(2)}$ are defined in Section 3.

15 Reduction of boundary value problems to integral equations

Green's formulas show that each harmonic function can be represented as a combination of single- and double-layer potentials. For boundary value problems we will find a solution in the form of one of these potentials.

Introduce integral operators K_0 and K_1 acting in the space $C(\Gamma)$ of continuous functions defined on contour Γ

$$K_0(\mathbf{x}) = 2 \int_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \mathbf{x} \in \Gamma \tag{295}$$

and

$$K_1(\mathbf{x}) = 2 \int_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_x} \psi(\mathbf{y}) dl_y, \quad \mathbf{x} \in \Gamma. \tag{296}$$

The kernels of operators K_0 and K_1 are continuous on Γ As seen by interchanging the order of integration, K_0 and K_1 are adjoint as integral operators in the space $C(\Gamma)$ of continuous functions defined on curve Γ.

Theorem 36 *The operators $I - K_0$ and $I - K_1$ have trivial nullspaces*

$$N(I - K_0) = \{0\}, \quad N(I - K_1) = \{0\}, \tag{297}$$

The nullspaces of operators $I + K_0$ and $I + K_1$ have dimension one and

$$N(I + K_0) = \mathit{span}\{1\}, \quad N(I + K_1) = \mathit{span}\{\psi_0\} \tag{298}$$

with

$$\int_\Gamma \psi_0 dl_y \neq 0. \tag{299}$$

Proof. Let ϕ be a solution to the homogeneous integral equation $\phi - K_0\phi = 0$ and define a double-layer potential

$$v(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \mathbf{x} \in D \tag{300}$$

with a continuous density ϕ. Then we have, for the limiting values on Γ,

$$v_\pm(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y \pm \frac{1}{2}\varphi(\mathbf{x}) \tag{301}$$

which yields

$$2v_-(\mathbf{x}) = K_0\varphi(\mathbf{x}) - \varphi = 0. \tag{302}$$

From the uniqueness of the interior Dirichlet problem (Theorem 33) it follows that $v = 0$ in the whole domain D. Now we will apply the contunuity of the normal derivative of the double-layer potential (300) over a smooth contour Γ,

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_+ - \left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_- = 0, \quad \mathbf{y} \in \Gamma, \tag{303}$$

where the internal and extrenal limiting values with respect to Γ are defined as follows

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_- = \lim_{\mathbf{y} \to \Gamma, \mathbf{y} \in D} \frac{\partial v(\mathbf{y})}{\partial n_y} = \lim_{h \to 0} n_y \cdot \operatorname{grad} v(\mathbf{y} - h n_y) \tag{304}$$

and

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_+ = \lim_{\mathbf{y} \to \Gamma, \mathbf{y} \in \mathrm{R}^2 \setminus \bar{D}} \frac{\partial v(\mathbf{y})}{\partial n_y} = \lim_{h \to 0} n_y \cdot \operatorname{grad} v(\mathbf{y} + h n_y). \tag{305}$$

Note that (303) can be written as

$$\lim_{h \to 0} n_y \cdot [\operatorname{grad} v(\mathbf{y} + h n_y) - \operatorname{grad} v(\mathbf{y} - h n_y)] = 0. \tag{306}$$

From (304)–(306) it follows that

$$\left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_+ - \left.\frac{\partial v(\mathbf{y})}{\partial n_y}\right|_- = 0, \quad \mathbf{y} \in \Gamma. \tag{307}$$

Using the uniqueness for the exterior Neumann problem and the fact that $v(\mathbf{y}) \to 0$, $|\mathbf{y}| \to \infty$, we find that $v(\mathbf{y}) = 0$, $\mathbf{y} \in \mathrm{R}^2 \setminus \bar{D}$. Hence from (301) we deduce $\varphi = v_+ - v_- = 0$ on Γ. Thus $N(I - K_0) = \{0\}$ and, by the Fredholm alternative, $N(I - K_1) = \{0\}$.

Theorem 37 *Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential*

$$v(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y, \quad \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x} \in D, \tag{308}$$

with a continuous density φ is a solution of the interior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(\mathbf{x}) - 2\int\limits_{\Gamma} \frac{\partial\Phi\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y = -2f(x), \quad \mathbf{x} \in \Gamma. \tag{309}$$

Proof follows from Theorem 14.1.

Theorem 38 *The interior Dirichlet problem has a unique solution.*

Proof The integral equation $\phi - K\phi = 2f$ of the interior Dirichlet problem has a unique solution by Theorem 36 because $N(I-K) = \{0\}$.

Theorem 39 *Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential*

$$u(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y, \quad \mathbf{x} \in R^2 \setminus \bar{D}, \tag{310}$$

with a continuous density φ is a solution of the exterior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(\mathbf{x}) + 2\int\limits_{\Gamma} \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\varphi(\mathbf{y})dl_y = 2f(x), \quad \mathbf{x} \in \Gamma. \tag{311}$$

Here we assume that the origin is contained in D.

Proof follows from Theorem 14.1.

Theorem 40 *The exterior Dirichlet problem has a unique solution.*

Proof. The integral operator $\tilde{K} : C(\Gamma) \to C(\Gamma)$ defined by the right-hand side of (310) is compact in the space $C(\Gamma)$ of functions continuous on Γ because its kernel is a continuous function on Γ. Let φ be a solution to the homogeneous integral equation $\varphi + \tilde{K}\varphi = 0$ on Γ and define u by (310). Then $2u = \varphi + \tilde{K}\varphi = 0$ on Γ, and by the uniqueness for the exterior Dirichlet problem it follows that $u \in\in \mathrm{R}^2 \setminus \bar{D}$, and $\int_\Gamma \varphi dl_y = 0$. Therefore $\varphi + K\varphi = 0$ which means according to Theorem 36 (because $N(I+K) = \text{span}\,\{1\}$), that $\varphi =$ *const* on Γ. Now $\int_\Gamma \varphi dl_y = 0$ implies that $\varphi = 0$ on Γ, and the existence of a unique solution to the integral equation (311) follows from the Fredholm property of its operator.

Theorem 41 *Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The single-layer potential*

$$u(\mathbf{x}) = \int\limits_{\Gamma} \Phi(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})dl_y, \quad \mathbf{x} \in D, \tag{312}$$

with a continuous density ψ is a solution of the interior Neumann problem provided that ψ is a solution of the integral equation

$$\psi(\mathbf{x}) + 2\int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_x}\psi(\mathbf{y})dl_y = 2g(x), \quad \mathbf{x} \in \Gamma. \tag{313}$$

Theorem 42 *The interior Neumann problem is solvable if and only if*

$$\int_{\Gamma} \psi dl_y = 0 \tag{314}$$

is satisfied.

16 Functional spaces and Chebyshev polynomials

16.1 Chebyshev polynomials of the first and second kind

The Chebyshev polynomials of the first and second kind, T_n and U_n, are defined by the following recurrent relationships:

$$T_0(x) = 1, \quad T_1(x) = x; \tag{315}$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, ...; \tag{316}$$

$$U_0(x) = 1, \quad U_1(x) = 2x; \tag{317}$$

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots . \tag{318}$$

We see that formulas (316) and (318) are the same and the polynomials are different because they are determined using different initial conditions (315) and (317).

Write down few first Chebyshev polynomials of the first and second kind; we determine them directly from formulas (315)–(318):

$$\begin{aligned}
T_0(x) =& \ 1, \\
T_1(x) =& \ x, \\
T_2(x) =& \ 2x^2 - 1, \\
T_3(x) =& \ 4x^3 - 3x, \\
T_4(x) =& \ 8x^4 - 8x^2 + 1, \\
T_5(x) =& \ 16x^5 - 20x^3 + 5x, \\
T_6(x) =& \ 32x^6 - 48x^4 + 18x^2 - 1;
\end{aligned} \tag{319}$$

$$\begin{aligned}
U_0(x) =& \ 1, \\
U_1(x) =& \ 2x, \\
U_2(x) =& \ 4x^2 - 1, \\
U_3(x) =& \ 8x^3 - 4x, \\
U_4(x) =& \ 16x^4 - 12x^2 + 1, \\
U_5(x) =& \ 32x^5 - 32x^3 + 6x, \\
U_6(x) =& \ 64x^6 - 80x^4 + 24x^2 - 1.
\end{aligned} \tag{320}$$

We see that T_n and U_n are polynomials of degree n.

Let us breifly summarize some important properties of the Chebyshev polynomials.

If $|x| \le 1$, the following convenient representations are valid

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, ...; \tag{321}$$

$$U_n(x) = \frac{1}{\sqrt{1-x^2}} \sin\left((n+1) \arccos x\right), \quad n = 0, 1, ... \tag{322}$$

At $x = \pm 1$ the right-hand side of formula (322) should be replaced by the limit.

Polynomials of the first and second kind are coupled by the relationships

$$U_n(x) = \frac{1}{n+1} T'_{n+1}(x), \quad n = 0, 1, ... \tag{323}$$

Polynomials T_n and U_n are, respectively, even functions for even n and odd functions for odd n:

$$\begin{aligned} T_n(-x) = &\ (-1)^n\, T_n(x), \\ U_n(-x) = &\ (-1)^n\, U_n(x). \end{aligned} \tag{324}$$

One of the most important properties is the *orthogonality* of the Chebyshev polynomials in the segment $[-1, 1]$ with a certain weight function (*a weight*). This property is expressed as follows:

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x = \begin{cases} 0, & n \ne m \\ \pi/2, & n = m \ne 0 \\ \pi, & n = m = 0 \end{cases}; \tag{325}$$

$$\int_{-1}^{1} U_n(x) U_m(x) \sqrt{1-x^2}\, \mathrm{d}x = \begin{cases} 0, & n \ne m \\ \pi/2, & n = m \end{cases}. \tag{326}$$

Thus in the segment $[-1, 1]$, the Chebyshev polynomials of the first kind are orthogonal with the weight $1/\sqrt{1-x^2}$ and the Chebyshev polynomials of the second kind are orthogonal with the weight $\sqrt{1-x^2}$.

16.2 Fourier–Chebyshev series

Formulas (325) and (326) enable one to decompose functions in the Fourier series in Chebyshev polynomials (*the Fourier–Chebyshev series*),

$$f(x) = \frac{a_0}{2} T_0(x) + \sum_{n=1}^{\infty} a_n T_n(x), \quad x \in [-1, 1], \tag{327}$$

or

$$f(x) = \sum_{n=0}^{\infty} b_n U_n(x), \quad x \in [-1, 1]. \tag{328}$$

Coefficients a_n and b_n are determined as follows

$$a_n = \frac{2}{\pi} \int_{-1}^{1} f(x) T_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}, \quad n = 0, 1, ...; \tag{329}$$

$$b_n = \frac{2}{\pi}\int\limits_{-1}^{1} f(x)U_n(x)\sqrt{1-x^2}\mathrm{d}x, \quad n = 0, 1, \dots . \tag{330}$$

Note that $f(x)$ may be a complex-valued function; then the Fourier coefficients a_n and b_n are complex numbers.

Consider some examples of decomposing functions in the Fourier–Chebyshev series

$$\sqrt{1-x^2} = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}T_{2n}(x), \quad -1 \le x \le 1;$$

$$\arcsin x = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}T_{2n+1}(x), \quad -1 \le x \le 1;$$

$$\ln(1+x) = -\ln 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}T_n(x), \quad -1 < x \le 1.$$

Similar decompositions can be obtained using the Chebyshev polynomials U_n of the second kind. In the first example above, an even function $f(x) = \sqrt{1-x^2}$ is decomposed; therefore its expansion contains only coefficients multiplying $T_{2n}(x)$ with even indices, while the terms with odd indices vanish.

Since $|T_n(x)| \le 1$ according to (321), the first and second series converge for all $x \in [-1, 1]$ uniformly in this segment. The third series represents a different type of convergence: it converges for all $x \in (-1, 1]$ $(-1 < x \le 1)$ and diverges at the point $x = -1$; indeed, the decomposed function $f(x) = \ln(x+1)$ is not defined at $x = -1$.

In order to describe the convergence of the Fourier–Chebyshev series introduce the functional spaces of *functions square-integrable in the segment* $(-1, 1)$ *with the weights* $1/\sqrt{1-x^2}$ *and* $\sqrt{1-x^2}$. We shall write $f \in L_2^{(1)}$ if the integral

$$\int\limits_{-1}^{1} |f(x)|^2 \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{331}$$

converges, and $f \in L_2^{(2)}$ if the integral

$$\int\limits_{-1}^{1} |f(x)|^2 \sqrt{1-x^2}\mathrm{d}x. \tag{332}$$

converges. The numbers specified by (convergent) integrals (331) and (332) are the squared norms of the function $f(x)$ in the spaces $L_2^{(1)}$ and $L_2^{(2)}$; we will denote them by $\|f\|_1^2$ and $\|f\|_2^2$, respectively. Write the symbolic definitions of these spaces:

$$L_2^{(1)} := \{f(x) : \|f\|_1^2 := \int\limits_{-1}^{1} |f(x)|^2 \frac{\mathrm{d}x}{\sqrt{1-x^2}} < \infty\},$$

$$L_2^{(2)} := \{f(x) : \|f\|_2^2 := \int\limits_{-1}^{1} |f(x)|^2 \sqrt{1-x^2}\mathrm{d}x < \infty\}.$$

Spaces $L_2^{(1)}$ and $L_2^{(2)}$ are (infinite-dimensional) Hilbert spaces with the inner products

$$(f,g)_1 = \int_{-1}^{1} f(x)\overline{g(x)}\frac{\mathrm{d}x}{\sqrt{1-x^2}},$$

$$(f,g)_2 = \int_{-1}^{1} f(x)\overline{g(x)}\sqrt{1-x^2}\,\mathrm{d}x.$$

Let us define two more functional spaces of differentiable functions defined in the segment $(-1,1)$ whose first derivatives are square-integrable in $(-1,1)$ with the weights $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$:

$$\tilde{W}_2^1 := \{f(x):\ f \in L_2^{(1)},\ f' \in L_2^{(2)}\},$$

and

$$\widehat{W}_2^1 := \{f(x):\ f \in L_2^{(1)},\ f'(x)\sqrt{1-x^2} \in L_2^{(2)}\};$$

they are also Hilbert spaces with the inner products

$$(f,g)_{\tilde{W}_2^1} = \int_{-1}^{1} f(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} \int_{-1}^{1} \overline{g(x)}\frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int_{-1}^{1} f'(x)\overline{g'(x)}\sqrt{1-x^2}\,\mathrm{d}x,$$

$$(f,g)_{\widehat{W}_2^1} = \int_{-1}^{1} f(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} \int_{-1}^{1} \overline{g(x)}\frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int_{-1}^{1} f'(x)\overline{g'(x)}\,(1-x^2)^{3/2}\,\mathrm{d}x$$

and norms

$$\|f\|^2_{\tilde{W}_2^1} = \left|\int_{-1}^{1} f(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}}\right|^2 + \int_{-1}^{1} |f'(x)|^2\sqrt{1-x^2}\,\mathrm{d}x,$$

$$\|f\|^2_{\widehat{W}_2^1} = \left|\int_{-1}^{1} f(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}}\right|^2 + \int_{-1}^{1} |f'(x)|^2\,(1-x^2)^{3/2}\,\mathrm{d}x.$$

If a function $f(x)$ is decomposed in the Fourier–Chebyshev series (327) or (328) and is an elemet of the space $L_2^{(1)}$ or $L_2^{(2)}$ (that is, integrals (331) and (332) converge), then the corresponding series always converge *in the root-mean-square sense*, i.e.,

$$\|f - S_N\|_1 \to 0 \quad \text{and} \quad \|f - S_N\|_2 \to 0 \quad \text{as} \quad N \to \infty,$$

where S_N are partial sums of the Fourier–Chebyshev series. The 'smoother' function $f(x)$ in the segment $[-1,1]$ the 'faster' the convergence of its Fourier–Chebyshev series. Let us explain this statement: if $f(x)$ has p continuous derivatives in the segment $[-1,1]$, then

$$\left|f(x) - \frac{a_0}{2}T_0(x) - \sum_{n=1}^{N} a_n T_n(x)\right| \le \frac{C\ln N}{N^p}, \quad x \in [-1,1], \tag{333}$$

where C is a constant that does not depend on N.

A similar statement is valid for the Fourier–Chebyshev series in polynomials U_n: if $f(x)$ has p continuous derivatives in the segment $[-1,1]$ and $p \geq 2$, then

$$\left| f(x) - \sum_{n=0}^{N} b_n U_n(x) \right| \leq \frac{C}{N^{p-3/2}}, \quad x \in [-1,1]. \tag{334}$$

We will often use the following three relationships involving the Chebyshev polynomials:

$$\int_{-1}^{1} \frac{1}{x-y} T_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi U_{n-1}(y), \quad -1 < y < 1, \quad n \geq 1; \tag{335}$$

$$\int_{-1}^{1} \frac{1}{x-y} U_{n-1}(x) \sqrt{1-x^2} \mathrm{d}x = -\pi T_n(y), \quad -1 < y < 1, \quad n \geq 1; \tag{336}$$

$$\int_{-1}^{1} \ln \frac{1}{|x-y|} T_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} \pi \ln 2, & n = 0; \\ \pi\, T_n(y)/n, & n \geq 1. \end{cases} \tag{337}$$

The integrals in (335) and (336) are improper integrals in the sense of Cauchy (a detailed analysis of such integrals is performed in [8, 9]). The integral in (337) is a standard convergent improper integral.

17 Solution to integral equations with a logarithmic singulatity of the kernel

17.1 Integral equations with a logarithmic singulatity of the kernel

Many boundary value problems of mathematical physics are reduced to *the integral equations with a logarithmic singulatity of the kernel*

$$L\varphi := \int_{-1}^{1} \ln \frac{1}{|x-y|}\, \varphi(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} = f(y), \quad -1 < y < 1 \tag{338}$$

and

$$(L+K)\varphi := \int_{-1}^{1} (\ln \frac{1}{|x-y|} + K(x,y))\, \varphi(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} = f(y), \quad -1 < y < 1. \tag{339}$$

In equations (338) and (339) $\varphi(x)$ is an unknown function and the right-hand side $f(y)$ is a given (continuous) function. The functions $\ln(1/|x-y|)$ in (338) and $\ln(1/|x-y|) + K(x,y)$ in (339) are *the kernels* of the integral equations; $K(x,y)$ is a given function which is assumed to be continuous (has no singularity) at $x = y$. The kernels go to infinity at $x = y$ and therefore referred to as functions with a (logarithmic) singulatity, and equations (338) and (339) are called integral equations with a logarithmic singulatity of the kernel. In equations (338) and (339) the weight function $1/\sqrt{1-x^2}$ is separated explicitly. This function describes the behaviour (singularty) of the solution in the vicinity of the endpoints -1 and $+1$. Linear integral operators L and $L+K$ are defined by the left-hand sides of (338) and (339).

Equation (338) is a particular case of (339) for $K(x,y) \equiv 0$ and can be solved explicitly.

Assume that the right-hand side $f(y) \in \tilde{W}_2^1$ (for simplicity, one may assume that f is a continuously differentiable function in the segment $[-1,1]$) and look for the solution $\varphi(x)$ in the space $L_2^{(1)}$. This implies that integral operators L and $L+K$ are considered as bounded linear operators

$$L : L_2^{(1)} \to \tilde{W}_2^1 \quad \text{and} \quad L+K : L_2^{(1)} \to \tilde{W}_2^1.$$

Note that if $K(x,y)$ is a sufficiently smooth function, then K is a compact (completely continuous) operator in these spaces.

17.2 Solution via Fourier–Chebyshev series

Let us describe the method of solution to integral equation (338). Look for an approximation $\varphi_N(x)$ to the solution $\varphi(x)$ of (338) in the form of a partial sum of the Fourier–Chebyshev series

$$\varphi_N(x) = \frac{a_0}{2}T_0(x) + \sum_{n=1}^{N} a_n T_n(x), \tag{340}$$

where a_n are unknown coefficients. Expand the given function $f(y)$ in the Fourier–Chebyshev series

$$f_N(y) = \frac{f_0}{2}T_0(y) + \sum_{n=1}^{N} f_n T_n(y), \tag{341}$$

where f_n are known coefficients determined according to (329). Substituting (340) and (341) into equation (338) and using formula (337), we obtain

$$\frac{a_0}{2}\ln 2 T_0(y) + \sum_{n=1}^{N} \frac{1}{n} a_n T_n(y) = \frac{f_0}{2}T_0(y) + \sum_{n=1}^{N} f_n T_n(y). \tag{342}$$

Now equate coefficients multiplying $T_n(y)$ with the same indices n:

$$\frac{a_0}{2}\ln 2 = \frac{f_0}{2}, \quad \frac{1}{n}a_n = f_n,$$

which gives unknown coefficients a_n

$$a_0 = \frac{f_0}{\ln 2}, \quad a_n = n f_n$$

and the approximation

$$\varphi_N(x) = \frac{f_0}{2\ln 2}T_0(x) + \sum_{n=1}^{N} f_n T_n(x). \tag{343}$$

The exact solution to equation (338) is obtained by a transition to the limit $N \to \infty$ in (343)

$$\varphi(x) = \frac{f_0}{2\ln 2}T_0(x) + \sum_{n=1}^{\infty} n f_n T_n(x). \tag{344}$$

In some cases one can determine the sum of series (344) explicitly and obtain the solution in a closed form. One can show that the approximation $\varphi_N(x)$ of the solution to equation (338) defined by (340) converges to the exact solution $\varphi(x)$ as $N \to \infty$ in the norm of space $L_2^{(1)}$.

Consider integral equation (339). Expand the kernel $K(x, y)$ in the double Fourier–Chebyshev series, take its partial sum

$$K(x, y) \approx K_N(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{N} k_{ij} T_i(x) T_j(y), \tag{345}$$

and repeat the above procedure. A more detailed description is given below.

Example 14 *Consider the integral equation*

$$\int_{-1}^{1} (\ln \frac{1}{|x-y|} + xy)\, \varphi(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} = y + \sqrt{1-y^2}, \quad -1 < y < 1.$$

Use the method of solution described above and set

$$\varphi_N(x) = \frac{a_0}{2} T_0(x) + \sum_{n=1}^{N} a_n T_n(x)$$

to obtain

$$y + \sqrt{1-y^2} = \frac{2}{\pi} T_0(y) + T_1(y) - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{T_{2k}(y)}{4k^2 - 1},$$

$$K(x, y) = T_1(x) T_1(y).$$

Substituting these expressions into the initial integral equation, we obtain

$$\frac{a_0}{2} \ln 2 T_0(y) + \sum_{n=1}^{N} \frac{a_n}{n} T_n(y) + \frac{\pi}{2} a_1 T_1(y) = \frac{2}{\pi} T_0(y) + T_1(y) - \frac{4}{\pi} \sum_{k=1}^{[N/2]} \frac{T_{2k}(y)}{4k^2 - 1},$$

here $[\frac{N}{2}]$ *denotes the integer part of a number. Equating the coefficients that multiply* $T_n(y)$ *with the same indices* n *we determine the unknowns*

$$a_0 = \frac{4}{\pi \ln 2}, \quad a_1 = \frac{2}{\pi}, \quad a_{2n+1} = 0, \quad a_{2n} = -\frac{8}{\pi} \frac{n}{4n^2 - 1}, \quad n \geq 1.$$

The approximate solution has the form

$$\varphi_N(x) = \frac{4}{\pi \ln 2} + \frac{2}{\pi} x - \frac{8}{\pi} \sum_{n=1}^{[N/2]} \frac{n}{4n^2 - 1} T_{2n}(x).$$

The exact solution is represented as the Fourier–Chebyshev series

$$\varphi(x) = \frac{4}{\pi \ln 2} + \frac{2}{\pi} x - \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} T_{2n}(x).$$

The latter series converges to the exact solution $\varphi(x)$ *as* $N \to \infty$ *in the norm of space* $L_2^{(1)}$*, and its sum* $\varphi \in L_2^{(1)}$.

18 Solution to singular integral equations

18.1 Singular integral equations

Consider two types of the so-called *singular integral equations*

$$S_1\varphi := \int_{-1}^{1} \frac{1}{x-y}\varphi(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = f(y), \quad -1 < y < 1; \tag{346}$$

$$(S_1 + K_1)\varphi = \int_{-1}^{1} \left(\frac{1}{x-y} + K_1(x,y)\right)\varphi(x)\,\frac{\mathrm{d}x}{\sqrt{1-x^2}} = f(y), \quad -1 < y < 1; \tag{347}$$

$$S_2\varphi := \int_{-1}^{1} \frac{1}{x-y}\varphi(x)\sqrt{1-x^2}\mathrm{d}x = f(y), \quad -1 < y < 1; \tag{348}$$

$$(S_2 + K_2)\varphi := \int_{-1}^{1} \left(\frac{1}{x-y} + K_2(x,y)\right)\varphi(x)\sqrt{1-x^2}\,\mathrm{d}x = f(y), \quad -1 < y < 1. \tag{349}$$

Equations (346) and (348) constitute particular cases of a general integral equation (347) and can be solved explicitly. Singular integral operators S_1 and $S_1 + K_1$ are considered as bounded linear operators in the weighted functional spaces introduced above, $S_1 : L_2^{(1)} \to L_2^{(2)}$, $S_1 + K_1 : L_2^{(1)} \to L_2^{(2)}$. Singular integral operators S_2 $S_2 + K_2$ are considered as bounded linear operators in the weighted functional spaces $S_2 : L_2^{(2)} \to L_2^{(1)}$, $S_2 + K_2 : L_2^{(2)} \to L_2^{(1)}$. If the kernel functions $K_1(x,y)$ and $K_2(x,y)$ are sufficiently smooth, K_1 and K_2 will be compact operators in these spaces.

18.2 Solution via Fourier–Chebyshev series

Look for an approximation $\varphi_N(x)$ to the solution $\varphi(x)$ of the singular integral equations in the form of partial sums of the Fourier–Chebyshev series. From formulas (335) and (336) it follows that one should solve (346) and (347) by expanding functions φ in the Chebyshev polynomials of the first kind and f in the Chebyshev polynomials of the second kind. In equations (348) and (349), φ should be decomposed in the Chebyshev polynomials of the second kind and f in the Chebyshev polynomials of the first kind.

Solve equation (346). Assume that $f \in L_2^{(2)}$ and look for a solution $\varphi \in L_2^{(1)}$:

$$\varphi_N(x) = \frac{a_0}{2}T_0(x) + \sum_{n=1}^{N} a_n T_n(x); \tag{350}$$

$$f_N(y) = \sum_{k=0}^{N-1} f_k U_k(y), \tag{351}$$

where coefficients f_k are calculated by formulas (330). Substituting expressions (350) and (351) into equation (346) and using (335), we obtain

$$\pi \sum_{n=1}^{N} a_n U_{n-1}(y) = \sum_{k=0}^{N-1} f_k U_k(y). \tag{352}$$

Coefficient a_0 vanishes because (see [4])

$$\int_{-1}^{1} \frac{1}{x-y} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 0, \quad -1 < y < 1. \tag{353}$$

Equating, in (352), the coefficients multiplying $U_n(y)$ with equal indices, we obtain $a_n = f_{n-1}/\pi, \quad n \geq 1$. The coefficient a_0 remains undetermined. Thus the solution to equation (346) depends on an arbitrary constant $C = a_0/2$. Write the corresponding formulas for the approximate solution,

$$\varphi_N(x) = C + \frac{1}{\pi} \sum_{n=1}^{N} f_{n-1}\, T_n(x); \tag{354}$$

and the exact solution,

$$\varphi(x) = C + \frac{1}{\pi} \sum_{n=1}^{\infty} f_{n-1} T_n(x). \tag{355}$$

The latter series converges in the $L_2^{(1)}$-norm.

In order to obtain a unique solution to equation (346) (to determine the constant C), one should specify an additional condition for function $\varphi(x)$, for example, in the form of an orthogonality condition

$$\int_{-1}^{1} \varphi(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 0. \tag{356}$$

(orthogonality of the solution to a constant in the space $L_2^{(1)}$). Then using the orthogonality (325) of the Cebyshev polynomial $T_0(x) \equiv 1$ to all $T_n(x)$, $n \geq 1$, and calculating the inner product in (355) (multiplying both sides by $1/\sqrt{1-x^2}$ and integrating from -1 to 1) we obtain

$$C = \frac{1}{\pi} \int_{-1}^{1} \varphi(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 0.$$

Equation (347) can be solved in a similar manner; however first it is convenient to decompose the kernel function $K_1(x, y)$ in the Fourier–Chebyshev series and take the partial sum

$$K_1(x, y) \approx K_N^{(1)}(x, y) = \sum_{i=0}^{N} \sum_{j=0}^{N-1} k_{ij}^{(1)} T_i(x) U_j(y), \tag{357}$$

and then proceed with the solution according to the above.

If $K_1(x, y)$ is a polynomial in powers of x and y, representation (357) will be exact and can be obtained by rearranging explicit formulas (315)–(320).

Example 15 *Solve a singular integral equation*

$$\int_{-1}^{1} \left(\frac{1}{x-y} + x^2 y^3 \right) \varphi(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = y, \quad -1 < y < 1$$

subject to condition (356). Look for an approximate solution in the form

$$\varphi_N(x) = \frac{a_0}{2}T_0(x) + \sum_{n-1}^{N} a_n T_n(x).$$

The right-hand side

$$y = \frac{1}{2}U_1(y).$$

Decompose $K_1(x, y)$ in the Fourier–Chebyshev series using (320),

$$K_1(x, y) = \left(\frac{1}{2}T_0(x) + \frac{1}{2}T_2(x)\right)\left(\frac{1}{4}U_1(y) + \frac{1}{8}U_3(y)\right).$$

Substituting the latter into equation (335) and using the orthogonality conditions (325), we obtain

$$\pi \sum_{n=1}^{N} a_n U_{n-1}(y) + \left(\frac{1}{4}U_1(y) + \frac{1}{8}U_3(y)\right)\frac{\pi(a_0 + a_2)}{4} = \frac{1}{2}U_1(y),$$

Equating the coefficients multiplying $U_n(y)$ with equal indices, we obtain

$$\pi a_1 = 0, \quad \pi a_2 + \frac{\pi(a_0 + a_2)}{16} = \frac{1}{2}, \quad \pi a_3 = 0,$$

$$\pi a_4 + \frac{\pi(a_0 + a_2)}{32} = 0; \quad \pi a_n = 0, \quad n \geq 5.$$

Finally, after some algebra, we determine the unknown coefficients

$$a_0 = C, \quad a_2 = \frac{1}{17}(\frac{8}{\pi} - C), \quad a_4 = \frac{1}{68}(2C - \frac{1}{\pi}), \quad a_n = 0, \quad n \neq 0, 2, 4,$$

where C is an arbitrary constant. In order to determine the unique solution that satisfies (356), calculate

$$C = 0, \quad a_0 = 0, \quad a_2 = \frac{8}{17\pi}, \quad a_4 = -\frac{1}{68\pi}.$$

The final form of the solution is

$$\varphi(x) = \frac{8}{17\pi}T_2(x) - \frac{1}{68\pi}T_4(x),$$

or

$$\varphi(x) = -\frac{2}{17\pi}x^4 + \frac{18}{17\pi}x^2 - \frac{33}{68\pi}.$$

Note that the approximate solution $\varphi_N(x) = \varphi(x)$ for $N \geq 4$. We see that here, one can obtain the solution explicitly as a polynomial.

Consider equations (348) and (349). Begin with (348) and look for its solution in the form

$$\varphi_N(x) = \sum_{n=0}^{N-1} b_n U_n(x). \tag{358}$$

Expand the right-hand side in the Chebyshev polynomials of the first kind

$$f_N(y) = \frac{f_0}{2} T_0(y) + \sum_{k=1}^{N} f_k T_k(y), \tag{359}$$

where coefficients f_k are calculated by (329). Substitute (358) and (359) into (348) and use (336) to obtain

$$-\pi \sum_{n=0}^{N-1} b_n T_{n+1}(y) = \frac{f_0}{2} T_0(y) + \sum_{k=1}^{N} f_k T_k(y), \tag{360}$$

where unknown coefficients b_n are determined form (360),

$$b_n = -\frac{1}{\pi} f_{n+1}, \quad n \geq 0.$$

Equality (360) holds only if $f_0 = 0$. Consequently, we cannot state that the solution to (348) exists for arbitrary function $f(y)$. The (unique) solution exists only if $f(y)$ satisfies the orthogonality condition

$$\int_{-1}^{1} f(y) \frac{\mathrm{d}y}{\sqrt{1-y^2}} = 0, \tag{361}$$

which is equivalent to the requirement $f_0 = 0$ (according to (329) taken at $n = 0$). Note that, when solving equation (348), we impose, unlike equation (356), the condition (361) for equation (346), on the known function $f(y)$. As a result, we obtain

a unique approximate solution to (348) for every $N \geq 2$

$$\varphi_N(x) = -\frac{1}{\pi} \sum_{n=0}^{N-1} f_{n+1} U_n(x), \tag{362}$$

and

a unique exact solution to (348)

$$\varphi(x) = -\frac{1}{\pi} \sum_{n=0}^{\infty} f_{n+1} U_n(x). \tag{363}$$

We will solve equations (348) and (349) under the assumption that the right-hand side $f \in L_2^{(1)}$ and $\varphi \in L_2^{(2)}$. Series (363) converges in the norm of the space $L_2^{(2)}$.

The solution of equation (349) is similar to that of (348). The difference is that $K_2(x, y)$ is approximately replaced by a partial sum of the Fourier–Chebyshev series

$$K_2(x, y) \approx K_N^{(2)}(x, y) = \sum_{i=0}^{N-1} \sum_{j=0}^{N} k_{ij}^{(2)} U_i(x) T_j(y). \tag{364}$$

Substituting (358), (359), and (364) into equation (349) we obtain a linear equation system with respect to unknowns b_n. As well as in the case of equation (348), the solution to (349) exists not for all right-hand sides $f(y)$.

The method of decomposing in Fourier–Chebyshev series allows one not only to solve equations (348) and (349) but also verify the existence of solution. Consider an example of an equation of the form (349) that has no solution (not solvable).

Example 16 *Solve the singular integral equation*

$$\int_{-1}^{1}\left(\frac{1}{x-y}+\sqrt{1-y^2}\right)\varphi(x)\sqrt{1-x^2}\,\mathrm{d}x = 1, \quad -1<y<1; \quad \varphi\in L_2^{(2)}.$$

Look for an approximate solution, as well as in the case of (348), in the form of a series

$$\varphi_N(x)=\sum_{n=0}^{N-1} b_n U_n(x).$$

Replace $K_2(x,y)=\sqrt{1-y^2}$ *by an approximate decomposition*

$$K_N^{(2)}(x,y)=\frac{2}{\pi}T_0(y)-\frac{4}{\pi}\sum_{k=1}^{[N/2]}\frac{1}{4k^2-1}T_{2k}(y).$$

The right-hand side of this equation is $1=T_0(y)$*. Substituting this expression into the equation, we obtain*

$$-\pi\sum_{n=0}^{N-1} b_n T_{n+1}(y)+\left(T_0(y)-2\sum_{k=1}^{[N/2]}\frac{1}{4k^2-1}T_{2k}(y)\right)b_0 = T_0(y).$$

Equating the coefficients that multiply $T_0(y)$ *on both sides of the equation, we obtain* $b_0=1$*; equating the coefficients that multiply* $T_1(y)$*, we have* $-\pi b_0=0$*,which yields* $b_0=0$*. This cobtradiction proves that the considered equation has no solution. Note that this circumstance is not connected with the fact that we determine an approxiate solution* $\varphi_N(x)$ *rather than the exact solution* $\varphi(x)$*. By considering infinite series (setting* $N=\infty$*) in place of finite sums we can obtain the same result.*

Let us prove the conditions for function $f(y)$ that provide the solvability of the equation

$$\int_{-1}^{1}\left(\frac{1}{x-y}+\sqrt{1-y^2}\right)\varphi(x)\sqrt{1-x^2}\,\mathrm{d}x = f(y), \quad -1<y<1$$

with the right-hand side $f(y)$ To this end, replace the decomposition of the right-hand side which was equal to one by the expansion

$$f_N(y)=\frac{f_0}{2}T_0(y)+\sum_{n=1}^{N} f_n T_n(y).$$

Equating the coefficients that multiply equal-order polynomials, we obtain

$$b_0=\frac{f_0}{2}, \quad b_0=-\frac{1}{\pi}f_1,$$

$$b_{2k-1}=-\frac{1}{\pi}\left(f_{2k}+\frac{2}{4k^2-1}b_0\right),$$

$$b_{2k}=-\frac{1}{\pi}f_{2k+1}; \quad k\geq 1.$$

We see that the equation has the unique solution if and only if

$$\frac{f_0}{2} = -\frac{1}{\pi} f_1.$$

Rewrite this condition using formula (329)

$$\frac{1}{2} \int_{-1}^{1} f(y) \frac{\mathrm{d}y}{\sqrt{1-y^2}} = -\frac{1}{\pi} \int_{-1}^{1} y\, f(y) \frac{\mathrm{d}y}{\sqrt{1-y^2}}.$$

Here, one can uniquely determine coefficient b_n, which provides the unique solvability of the equation.

19 Matrix representation of an operator in the Hilbert space and summation operators

19.1 Matrix representation of an operator in the Hilbert space

Let $\{\varphi_n\}$ be an orthonormal basis in the Hilbert space H; then each element $f \in H$ can be represented in the form of the generalized Fourier series $f = \sum\limits_{n=1}^{\infty} a_n \varphi_n$, where the sequence of complex numbers $\{a_n\}$ is such that the series $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$ converges in H. There exists a (linear) one-to-one correspondence between such sequences $\{a_n\}$ and the elements of the Hilbert space.

We will use this isomorphism to construct the matrix representation of a completely continuous operator A and a completely continuous operator-valued function $A(\gamma)$ holomorphic with respect to complex variable γ acting in H.

Assume that $g = Af$, $f \in H$, and

$$g_k = g_k(\gamma) = (A(\gamma)f, \varphi_k),$$

i.e., an infinite sequence (vector) of the Fourier coefficients $\{g_n\} = (g_1, g_2, ...)$ corresponds to an element $g \in H$ and we can write $g = (g_1, g_2, ...)$. Let $f = \sum\limits_{k=1}^{\infty} f_k \varphi_k$; then $f = (f_1, f_2, ...)$. Set $f^N = Pf = \sum\limits_{k=1}^{N} f_k \varphi_k$ and $A\varphi_n = \sum\limits_{k=1}^{\infty} a_{kn} \varphi_k$, $n = 1, 2, \ldots$. Then,

$$\lim_{N\to\infty} Af^N = \lim_{N\to\infty} \sum_{n=1}^{N} f_n \sum_{k=1}^{\infty} a_{kn}\varphi_k = Af = \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{\infty} a_{kn} f_n \Big) \varphi_k.$$

On the other side, $Af = \sum\limits_{k=1}^{\infty} g_k \varphi_k$, hence, $g_k = \sum\limits_{n=1}^{\infty} a_{kn}(Af, \varphi_k) = (g, \varphi_k)$. Now we set $A\varphi_k = g_k$ and

$$(g_k, \varphi_j) = (A\varphi_k, \varphi_j) = a_{jk},$$

where a_{jk} are the Fourier coefficients in the expansion of element g_k over the basis $\{\varphi_j\}$. Consequently,

$$A\varphi_k = g_k = \sum_{j=1}^{\infty} a_{jk}\varphi_j,$$

and the series

$$\sum_{j=1}^{\infty} |a_{jk}|^2 = \sum_{j=1}^{\infty} |(g_k, \varphi_j)|^2 < \infty; \quad k = 1, 2, \ldots.$$

Infinite matrix $\{a_{jk}\}_{j,k=1}^{\infty}$ defines operator A; that is, one can reconstruct A by this matrix and an orthonormal basis $\{\varphi_k\}$ of the Hilbert space H: find the unique element Af for each $f \in H$. In order to prove the latter assertion, we will assume that $f = \sum\limits_{k=1}^{\infty} \xi_k\varphi_k$ and $f^N = Pf$. Then, $Af^N = \sum\limits_{k=1}^{\infty} \eta_k^N\varphi_k$, where $\eta_k^N = \sum\limits_{j=1}^{N} a_{kj}\xi_j$. Applying the continuity of operator A, we have

$$C_k = (Af, \varphi_k) = \lim_{N\to\infty}(Af^N, \varphi_k) = \lim_{N\to\infty}\eta_k^N = \sum_{j=1}^{\infty} a_{kj}\xi_j.$$

Hence, for each $f \in H$,

$$Af = \sum_{k=1}^{\infty} C_k\varphi_k, \qquad C_k = \sum_{j=1}^{\infty} a_{kj}\xi_j,$$

and the latter equalities define the matrix representation of the operator A in the basis $\{\varphi_k\}$ of the Hilbert space H.

19.2 Summation operators in the spaces of sequences

In this section, we will employ the approach based on the use of Chebyshev polynomials to construct families of summation operators associated with logarithmic integral equations.

Let us introduce the linear space h_p formed by the sequences of complex numbers $\{a_n\}$ such that

$$\sum_{n=1}^{\infty} |a_n|^2 n^p < \infty, \quad p \geq 0,$$

with the inner product and the norm

$$(a, b) = \sum_{n=1}^{\infty} a_n\bar{b}_n n^p, \quad \|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^2 n^p\right)^{\frac{1}{2}}.$$

One can easily verify the validity of the following statement.

Theorem 43 *h_p, $p \geq 0$ is the Hilbert space.*

Let us introduce the summation operators

$$L = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots \\ 0 & 2^{-1} & \ldots & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & n^{-1} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

or, in other notation,

$$L = \{l_{nj}\}_{n,j=1}^{\infty}, \quad l_{nj} = 0,\ n \neq j, \quad l_{nn} = \frac{1}{n},\ n = 1, 2, \ldots,$$

and

$$L^{-1} = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots \\ 0 & 2 & \ldots & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & n & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

or

$$L^{-1} = \{l_{nj}\}_{n,j=1}^{\infty}, \quad l_{nj} = 0,\ n \neq j, \quad l_{nn} = n,\ n = 1, \ldots.$$

that play an important role in the theory of integral operators and operator-valued functions with a logarithmic singularity of the kernel.

Theorem 44 *L is a linear bounded invertible operator acting from the space h_p to the space h_{p+2}: $L: h_p \to h_{p+2}$ ($Im\, L = h_{p+2}$, $\mathrm{Ker}\, L = \{\emptyset\}$). The inverse operator $L^{-1}: h_{p+2} \to h_p$, is bounded, $p \geq 0$.*

□ Let us show that L acts from h_p to h_{p+2} and L^{-1} acts from h_{p+2} to h_p. Assume that $a \in h_p$. Then, for La we obtain

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n} \right|^2 n^{p+2} = \sum_{n=1}^{\infty} |a_n|^2 n^p < \infty.$$

If $b \in h_{p+2}$, we obtain similar relationships for $L^{-1}b$:

$$\sum_{n=1}^{\infty} |b_n n|^2 n^p = \sum_{n=1}^{\infty} |b_n|^2 n^{p+2} < \infty.$$

The norm $\|La\|_{p+2} = \|a\|_p$. The element $a = L^{-1}b$, $a \in h_p$, is the unique solution of the equation $La = b$, and $\ker L = \{\emptyset\}$. According to the Banach theorem, L^{-1} is a bounded operator. □

Note that $\|L\| = \|L^{-1}\| = 1$ and

$$L^{-1}L = LL^{-1} = E,$$

where E is the infinite unit matrix.

Assume that the summation operator A acting in h_p is given by its matrix $\{a_{nj}\}_{n,j=1}^{\infty}$ and the following condition holds:

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{n^{p+2}}{j^p} |a_{nj}|^2 \leq C^2 < \infty, \quad C = \text{const}, \ \mathrm{C} > 0.$$

Then, we can prove the following statement.

Theorem 45 *The operator $A : h_p \to h_{p+2}$ is completely continuous*

□ Let $\xi \in h_p$ and $A\xi = \eta$. Then,

$$\sum_{n=1}^{\infty} n^{p+2}|\eta_n|^2 = \sum_{n=1}^{\infty} n^{p+2} \left|\sum_{j=1}^{\infty} a_{nj}\xi_j\right|$$

$$\leq \|\xi\|_p^2 \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n^{p+2}\frac{|a_{nj}|^2}{j^p} \leq C^2\|\xi\|_p^2,$$

hence, $\eta \in h_{p+2}$. Now we will prove that A is completely continuous. In fact, operator A can be represented as a limit with respect to the operator norm of a sequence of finite-dimensional operators A_N, which are produced by the cut matrix A and defined by the matrices $A_N = \{a_{nj}\}_{n,j=1}^N$:

$$\|(A - A_N)\xi\|_{p+2}^2 = \sum_{n=1}^{N-1} n^{p+2} \left|\sum_{j=N}^{\infty} a_{nj}\xi_j\right|^2 + \sum_{n=N}^{\infty} n^{p+2} \left|\sum_{j=1}^{\infty} a_{nj}\xi_j\right|^2$$

$$\leq \|\xi\|_p^2 \sum_{n=N}^{\infty}\sum_{j=1}^{\infty} n^{p+2}\frac{|a_{nj}|^2}{j^p} + \|\xi\|_p^2 \sum_{n=1}^{N-1}\sum_{j=N}^{\infty} \frac{|a_{nj}|^2}{j^p} n^{p+2}$$

$$\leq \|\xi\|_p^2 \Big[\sum_{n=N}^{\infty}\Big(\sum_{j=1}^{\infty} n^{p+2}\frac{|a_{nj}|^2}{j^p}\Big) + \sum_{j=N}^{\infty}\Big(\sum_{n=1}^{\infty} n^{p+2}\frac{|a_{nj}|^2}{j^p}\Big)\Big],$$

and, consequently, the following estimates are valid with respect to the operator norm:

$$\|A - A_N\| \leq \sum_{n=N}^{\infty}\Big(\sum_{j=1}^{N} n^{p+2}\frac{|a_{nj}|^2}{j^p}\Big) + \sum_{j=N}^{\infty}\Big(\sum_{n=1}^{\infty} n^{p+2}\frac{|a_{nj}|^2}{j^p}\Big) \to 0, \quad N \to \infty. \quad \square$$

Corollary 1 *The sequence of operators A_N strongly converges to operator A: $\|A - A_N\| \to 0$, $N \to \infty$, and $\|A\| < C$, $C = \text{const}$. In order to provide the complete continuity of A, it is sufficient to require the fulfilment of the estimate*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n^{p+2}|a_{nj}|^2 < C, \quad C = \text{const}. \tag{365}$$

Definition 9 *Assume that condition (365) holds for the matrix elements of summation operator A. Then, the summation operator $F = L + A : h_p \to h_{p+2}$ is called the L-operator.*

Theorem 46 *L-operator $F = L + A : h_p \to h_{p+2}$ is a Fredholm operator;* $\operatorname{ind}(L + A) = 0$.

20 Matrix Representation of Logarithmic Integral Operators

Consider a class Φ of functions φ such that they admit in the interval $(-1, 1)$ the representation in the form of the Fourier–Chebyshev series

$$\varphi(x) = \frac{1}{2}T_0(x)\xi_0 + \sum_{n=1}^{\infty} \xi_n T_n(x); \quad -1 \leq x \leq 1, \tag{366}$$

where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials and $\xi = (\xi_0, \xi_1, ...\xi_n, ...) \in h_2$; that is,

$$\|\xi\|_2^2 = \frac{1}{2}|\xi_0|^2 + \sum_{n=1}^{\infty} |\xi_n|^2 n^2 < \infty.$$

In this section we add the coordinate ξ_0 to vector ξ and modify the corresponding formulas. It is easy to see that all results of previous sections are also valid in this case. Recall the definitions of weighted Hilbert spaces associated with logarithmic integral operators:

$$L_2^{(1)} = \{f(x) : \|f\|_1^2 = \int_{-1}^{1} |f(x)|^2 \frac{dx}{\sqrt{1-x^2}} < \infty\},$$

$$L_2^{(2)} = \{f(x) : \|f\|_2^2 = \int_{-1}^{1} |f(x)|^2 \sqrt{1-x^2} dx < \infty\},$$

$$\widetilde{W}_2^1 = \{f(x) : \ f \in L_2^{(1)}, \ f' \in L_2^{(2)}\},$$

and

$$\widehat{W}_2^1 = \{f(x) : \ f \in L_2^{(2)}, f^*(x) = f(x)\sqrt{1-x^2}, \ f^{*\prime} \in L_2^{(2)}, \ f^*(-1) = f^*(1) = 0\}.$$

Below, we will denote the weights by $p_1(x) = \sqrt{1-x^2}$ and $p_1^{-1}(x)$.

Note that the Chebyshev polynomials specify orthogonal bases in spaces $L_2^{(1)}$ and $L_2^{(2)}$. In order to obtain the matrix representation of the integral operators with a logarithmic singularity of the kernel, we will prove a similar statement for weighted space $\widetilde{W}_2^1$.

Lemma 6 *$\widetilde{W}_2^1$ and Φ are isomorphic.*

□ Assume that $\varphi \in \Phi$. We will show that $\varphi \in \widetilde{W}_2^1$. Function $\varphi(x)$ is continuous, and (366) is its Fourier–Chebyshev series with the coefficients

$$\xi_n = \frac{2}{\pi} \int_{-1}^{1} T_n(x)\varphi(x)p_1^{-1}(x)dx.$$

The series $\sum_{n=1}^{\infty} |\xi_n|^2 n^2$ converges and, due to the Riesz–Fischer theorem, there exists an element $f \in L_2^{(1)}$, such that its Fourier coefficients in the Chebyshev polynomial series are equal to $n\xi_n$; that is, setting $U_n(x) = \sin(n \arccos x)$, one obtains

$$n\xi_n = \frac{2}{\pi} \int_{-1}^{1} nT_n(x)\varphi(x)p_1^{-1}(x)dx = -\frac{2}{\pi} \int_{-1}^{1} U_n'(x)\varphi(x)dx$$

$$= -\frac{2}{\pi} \int_{-1}^{1} U_n'(x)(p_1(x)\varphi(x))\frac{dx}{p_1(x)} = \frac{2}{\pi} \int_{-1}^{1} U_n(x)f(x)\frac{dx}{p_1(x)}, \quad n \geq 1.$$

Since $f(x)$ is an integrable function, the integral $\int\limits_{-1}^{1} |f(x)|p_1^{-1}(x)dx$ converges, and

$$F(x) = \int\limits_{-1}^{x} \frac{f(t)dt}{p_1(t)}$$

is absolutely continuous (as a function in the form of an integral depending on the upper limit of integration), $F'(x) = f(x)p_1^{-1}(x)$, and

$$\frac{2}{\pi}\int\limits_{-1}^{1} U_n(x)f(x)\frac{dx}{p_1(x)} = \frac{2}{\pi}\int\limits_{-1}^{1} U_n(x)F'(x)dx = -\frac{2}{\pi}\int\limits_{-1}^{1} U_n'(x)F(x)dx.$$

The latter equalities yield

$$\int\limits_{-1}^{1} U_n'(x)[\varphi(x) - F(x)]dx = -n\int\limits_{-1}^{1} T_n(x)[\varphi(x) - F(x)]\frac{dx}{p_1(x)} = 0, \ n \geq 1.$$

Hence, $\varphi(x) - F(x) = C =$ const. The explicit form of $F'(x)$ yields $\varphi'(x) = f(x)p_1^{-1}(x)$; therefore, $\varphi'(x)p_1(x) \in L_2^{(1)}$ and $\varphi'(x) \in L_2^{(2)}$.

Now, let us prove the inverse statement. Let $\varphi \in \widetilde{W}_2^1$ and be represented by the Fourier–Chebyshev series (366). Form the Fourier series

$$p_1(x)\varphi'(x) = \sum_{n=1}^{\infty} \eta_n U_n(x), \quad |x| \leq 1,$$

where

$$\eta_n = \frac{2}{\pi}\int\limits_{-1}^{1} U_n(x)(p_1(x)\varphi'(x))\frac{dx}{p_1(x)} = -\frac{2}{\pi}\int\limits_{-1}^{1} U_n'(x)\varphi(x)dx =$$

$$= \frac{2n}{\pi}\int\limits_{-1}^{1} T_n(x)\frac{\varphi(x)dx}{p_1(x)}.$$

Now, in order to prove that $\xi \in h_2$, it is sufficient to apply the Bessel inequality to $\eta_n = n\xi_n$, where

$$\xi_n = \frac{2}{\pi}\int\limits_{-1}^{1} \varphi(x)T_n(x)\frac{dx}{p_1(x)}.$$

□

20.1 Solution to integral equations with a logarithmic singularity of the kernel

Consider the integral operator with a logarithmic singularity of the kernel

$$L\varphi = -\frac{1}{\pi}\int\limits_{-1}^{1} \ln|x - s|\varphi(s)\frac{ds}{p_1(s)}, \quad \varphi \in \widetilde{W}_2^1,$$

and the integral equation

$$L\varphi + N\varphi = f, \quad \varphi \in \widetilde{W}_2^1, \tag{367}$$

where

$$\Phi_N(x) = N\varphi = \int_{-1}^{1} N(\mathbf{q})\varphi(s)\frac{ds}{\pi p_1(x)}, \quad \mathbf{q} = (x, s),$$

and $N(\mathbf{q})$ is an arbitrary complex-valued function satisfying the continuity conditions

$$N(\mathbf{q}) \in C^1(\Pi_1); \quad \frac{\partial^2 N}{\partial x^2}(\mathbf{q}) \in L_2[\Pi_1, p_1^{-1}(x)p_1^{-1}(s)],$$
$$\Pi_1 = ([-1,1] \times [-1,1]).$$

Write the Fourier–Chebyshev expansions

$$\varphi(s) = \frac{1}{\sqrt{2}}\xi_0 T_0 + \sum_{n=1}^{\infty} \xi_n T_n(s), \quad \xi = (\xi_0, \xi_1, ..., \xi_n, ...) \in h_2, \tag{368}$$

and

$$f(x) = \frac{f_0}{2}T_0(x) + \sum_{n=1}^{\infty} f_n T_n(x),$$

where the coefficients

$$f_n = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}, \quad n = 0, 1, \ldots.$$

Function $\Phi_N(x)$ is continuous and may be thus also expanded in the Fourier–Chebyshev series

$$\Phi_N(x) = \frac{1}{2}b_0 T_0(x) + \sum_{n=1}^{\infty} b_n T_n(x), \tag{369}$$

where

$$b_n = 2\int\int_{\Pi_1} N(\mathbf{q})T_n(x)\varphi(s)\frac{dxds}{\pi^2 p_1(x)p_1(s)}. \tag{370}$$

According to Lemma 6 and formulas (368) and (369), equation (367) is equivalent in $\widetilde{W}_2^1$ to the equation

$$\begin{aligned} \ln 2\frac{\xi_0}{\sqrt{2}}T_0(x) + \sum_{n=1}^{\infty} n^{-1}\xi_n T_n(x) + \frac{b_0}{2}T_0(x) + \sum_{n=1}^{\infty} b_n T_n(x) \\ = \frac{f_0}{2}T_0(x) + \sum_{n=1}^{\infty} f_n T_n(x), \quad |x| \le 1. \end{aligned} \tag{371}$$

Note that since $\Phi_N(x)$ is a differentiable function of x in $[-1,1]$, the series in (371) converge uniformly, and the equality in (371) is an identity in $(-1,1)$.

Now, we substitute series (368) for $\varphi(x)$ into (370). Taking into account that $\{T_n(x)\}$ is a basis and one can change the integration and summation and to pass to double integrals (these statements are verified directly), we obtain an infinite system of linear algebraic equations

$$\begin{cases} \sqrt{2}\ln 2\xi_0 + b_0 = f_0, \\ n^{-1}\xi_n + b_n = f_n, \qquad n \ge 1. \end{cases} \tag{372}$$

In terms of the summation operators, this systems can be represented as

$$L\xi + A\xi = f; \quad \xi = (\xi_0, \xi_1, \dots, \xi_n, \dots) \in h_2; \quad f = (f_0, f_1, \dots, f_n, \dots) \in h_4.$$

Here, operators L and A are defined by

$$L = \{l_{nj}\}_{n,j=0}^{\infty}, \quad l_{nj} = 0,\ n \neq j; \quad l_{00} = \ln 2; \quad l_{nn} = \frac{1}{n},\ n = 1, 2, \dots;$$

$$A = \{a_{nj}\}_{n,j=0}^{\infty}, \quad a_{nj} = \varepsilon_{nj} \int\limits_{\Pi_1}\int N(\mathbf{q}) T_n(x) T_j(s) \frac{dxds}{\pi^2 p_1(x) p_1(s)}, \tag{373}$$

where

$$\varepsilon_{nj} = \begin{cases} 1, & n = j = 0, \\ 2, & n \leq 1,\ j \leq 1, \\ \sqrt{2}, & \text{for other } n, j. \end{cases}$$

Lemma 7 *Let $N(\mathbf{q})$ satisfy the continuity conditions formulated in (367). Then*

$$\sum_{n=1,j=0}^{\infty} |a_{nj}|^2 n^4 + \sum_{j=0}^{\infty} |a_{0j}|^2 \frac{1}{\ln^2 2} \leq C_0, \tag{374}$$

where

$$C_0 = \frac{1}{\ln^2 2} \int_{-1}^{1} \left| \int_{-1}^{1} \frac{N(\mathbf{q}) dx}{\pi p_1(x)} \right|^2 \frac{ds}{\pi p_1(s)} + \int\limits_{\Pi_1}\int p_1^2(x) \left| (p_1(x) N'_x)'_x \right|^2 \frac{dxds}{\pi^2 p_1(x) p_1(s)}. \tag{375}$$

□ Expand a function of two variables belonging to the weighted space $L_2[\Pi_1, p_1^{-1}(x) p_1^{-1}(s)]$ in the double Fourier–Chebyshev series using the Fourier–Chebyshev system $\{T_n(x) T_m(s)\}$:

$$-p_1(x)(p_1(x) N'_x(\mathbf{q}))'_x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm} T_n(x) T_m(s),$$

where

$$\begin{aligned} C_{nm} &= -\frac{\varepsilon_{nm}^2}{\pi} \int_{-1}^{1} T_m(s) \frac{ds}{p_1(s)} \int_{-1}^{1} T_n(x) (p_1(x) N'_x(\mathbf{q}))'_x dx \\ &= \frac{\varepsilon_{nm}^2}{\pi} \int_{-1}^{1} T_m(s) \frac{ds}{p_1(s)} \int_{-1}^{1} T'_n(x) p_1(x) N'_x(\mathbf{q}) dx \\ &= \frac{n\varepsilon_{nm}^2}{\pi^2} \int_{-1}^{1} \frac{T_m(s)}{p_1(s)} ds \int_{-1}^{1} U_n(x) N'_x(\mathbf{q}) dx \\ &= \frac{n^2 \varepsilon_{nm}^2}{\pi^2} \int\limits_{\Pi_1}\int \frac{N(\mathbf{q}) T_n(x) T_m(s)}{p_1(x) p_1(s)} dxds = n^2 \varepsilon_{nm} a_{nm}. \end{aligned}$$

The Bessel inequality yields

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n^4 \pi^2 |a_{nm}|^2 \leq \int\limits_{\Pi_1}\int p_1^2(x) |[p_1(x) N'_x(\mathbf{q})]'_x|^2 \frac{dxds}{p_1(x) p_1(s)}.$$

Let us expand the function

$$g(s) = \frac{1}{\ln 2} \int_{-1}^{1} \frac{N(\mathbf{q})dx}{\pi^2 p_1(x)}$$

in the Fourier–Chebyshev series. Again, due to the Bessel inequality, we obtain

$$\frac{1}{\ln^2 2} \sum_{m=0}^{\infty} |a_{0m}|^2 \leq \frac{1}{\ln^2 2} \int_{-1}^{1} \left| \int_{-1}^{1} \frac{N(\mathbf{q})ds}{\pi p_1(x)} \right|^2 \frac{ds}{\pi p_1(s)}. \quad \square$$

21 Galerkin methods and basis of Chebyshev polynomials

We describe the approximate solution of linear operator equations considering their projections onto finite-dimensional subspaces, which is convenient for practical calculations. Below, it is assumed that all operators are linear and bounded. Then, the general results concerning the substantiation and convergence the Galerkin methods will be applied to solving logarithmic integral equations.

Definition 10 *Let X and Y be the Banach spaces and let $A : X \to Y$ be an injective operator. Let $X_n \subset X$ and $Y_n \subset Y$ be two sequences of subspaces with* $\dim X_n = \dim Y_n = n$ *and $P_n : Y \to Y_n$ be the projection operators. The projection method generated by X_n and P_n approximates the equation*

$$A\varphi = f$$

by its projection

$$P_n A \varphi_n = P_n f.$$

This projection method is called convergent for operator A, if there exists an index N such that for each $f \in Im\, A$, the approximating equation $P_n A\varphi = P_n f$ has a unique solution $\varphi_n \in X_n$ for all $n \geq N$ and these solutions converge, $\varphi_n \to \varphi$, $n \to \infty$, to the unique solution φ of $A\varphi = f$.

In terms of operators, the convergence of the projection method means that for all $n \geq N$, the finite-dimensional operators $A_n := P_n A : X_n \to Y_n$ are invertible and the pointwise convergence $A_n^{-1} P_n A \varphi \to \varphi$, $n \to \infty$, holds for all $\varphi \in X$. In the general case, one may expect that the convergence occurs if subspaces X_n form a dense set:

$$\inf_{\psi \in X_n} \|\psi - \varphi\| \to 0, \quad n \to \infty \tag{376}$$

for all $\varphi \in X$. This property is called *the approximating property* (any element $\varphi \in X$ can be approximated by elements $\psi \in X_n$ with an arbitrary accuracy in the norm of X). Therefore, in the subsequent analysis, we will always assume that this condition is fulfilled. Since $A_n = P_n A$ is a linear operator acting in finite-dimensional spaces, the projection method reduces to solving a finite-dimensional linear system. Below, the Galerkin method will be considered as the projection method under the assumption that the above definition is valid in terms of the orthogonal projection.

Consider first the general convergence and error analysis.

Lemma 8 *The projection method converges if and only if there exists an index N and a positive constant M such that for all $n \geq N$, the finite-dimensional operators $A_n = P_n A : X_n \to Y_n$ are invertible and operators $A_n^{-1} P_n A : X \to X$ are uniformly bounded,*

$$\|A_n^{-1} P_n A\| \leq M. \tag{377}$$

The error estimate is valid

$$\|\varphi_n - \varphi\| \leq (1 + M) \inf_{\psi \in X_n} \|\psi - \varphi\|. \tag{378}$$

Relationship (378) is usually called *the quasioptimal estimate* and corresponds to the error estimate of a projection method based on the approximation of elements from X by elements of subspaces X_n. It shows that the error of the projection method is determined by the quality of approximation of the exact solution by elements of subspace X_n.

Consider the equation

$$S\varphi + K\varphi = f \tag{379}$$

and the projection methods

$$P_n S \varphi_n = P_n f \tag{380}$$

and

$$P_n (S + K) \varphi_n = P_n f \tag{381}$$

with subspaces X_n and projection operators $P_n : Y \to Y_n$.

Lemma 9 *Assume that $S : X \to Y$ is a bounded operator with a bounded inverse $S^{-1} : Y \to X$ and that the projection method (380) is convergent for S. Let $K : X \to Y$ be a compact bounded operator and $S + K$ is injective. Then, the projection method (381) converges for $S + K$.*

Lemma 10 *Assume that $Y_n = S(X_n)$ and $\|P_n K - K\| \to 0$, $n \to \infty$. Then, for sufficiently large n, approximate equation (381) is uniquely solvable and the error estimate*

$$\|\varphi_n - \varphi\| \leq M \|P_n S \varphi - S\varphi\|$$

is valid, where M is a positive constant depending on S and K.

For operator equations in Hilbert spaces, the projection method employing an orthogonal projection into finite-dimensional subspaces is called the Galerkin method.

Consider a pair of Hilbert spaces X and Y and assume that $A : X \to Y$ is an injective operator. Let $X_n \subset X$ and $Y_n \subset Y$ be the subspaces with $\dim X_n = \dim Y_n = n$. Then, $\varphi_n \in X_n$ is a solution of the equation $A\varphi = f$ obtained by the projection method generated by X_n and operators of orthogonal projections $P_n : Y \to Y_n$ if and only if

$$(A\varphi_n, g) = (f, g) \tag{382}$$

for all $g \in Y_n$. Indeed, equations (382) are equivalent to $P_n(A\varphi_n - f)$.

Equation (382) is called *the Galerkin equation.*

Assume that X_n and Y_n are the spaces of linear combinations of the basis and probe functions: $X_n = \{u_1, \dots, u_n\}$ and $Y_n = \{v_1, \dots, v_n\}$. Then, we can express φ_n as a linear combination

$$\varphi_n = \sum_{k=1}^{n} \gamma_n u_n,$$

and show that equation (382) is equivalent to the linear system of order n

$$\sum_{k=1}^{n} \gamma_k (Au_k, v_j) = (f, v_j), \quad j = 1, \dots, n, \tag{383}$$

with respect to coefficients $\gamma_1, \dots, \gamma_n$.

Thus, the Galerkin method may be specified by choosing subspaces X_n and projection operators P_n (or the basis and probe functions). We note that the convergence of the Galerkin method do not depend on a particular choice of the basis functions in subspaces X_n and probe functions in subspaces Y_n. In order to perform a theoretical analysis of the Galerkin methods, we can choose only subspaces X_n and projection operators P_n. However, in practice, it is more convenient to specify first the basis and probe functions, because it is impossible to calculate the matrix coefficients using (383).

21.1 Numerical solution of logarithmic integral equation by the Galerkin method

Now, let us apply the projection scheme of the Galerkin method to logarithmic integral equation

$$(L+K)\varphi = \int_{-1}^{1} (\ln \frac{1}{|x-y|} + K(x,y))\, \varphi(x) \, \frac{dx}{\sqrt{1-x^2}} = f(y), \quad -1 < y < 1. \tag{384}$$

Here, $K(x,y)$ is a smooth function; accurate conditions concerning the smoothness of $K(x,y)$ that guarantee the compactness of integral operator K are imposed in Section 2.

One can determine an approximate solution of equation (384) by the Galerkin method taking the Chebyshev polynomials of the first kind as the basis and probe functions. According to the scheme of the Galerkin method, we set

$$\varphi_N(x) = \frac{a_0}{2} T_0(x) + \sum_{n=1}^{N-1} a_n T_n(x),$$

$$X_N = \{T_0, \dots, T_{N-1}\}, \quad Y_N = \{T_0, \dots, T_{N-1}\}, \tag{385}$$

and find the unknown coefficients from finite linear system (383). Since $L : X_N \to Y_N$ is a bijective operator, the convergence of the Galerkin method follows from Lemma 10 applied to operator L. If operator K is compact and $L+K$ is injective, then we can also prove the convergence of the Galerkin method for operator $L+K$ using Lemma 9. Note also that quasioptimal estimate 378 in the $L_2^{(1)}$ norm is valid for $\varphi_N(x) \to \varphi(x)$, $N \to \infty$.

Theorem 47 *If operator $L+K : L_2^{(1)} \to \widetilde{W}_2^1$ is injective and operator $K : L_2^{(1)} \to \widetilde{W}_2^1$ is compact, then the Galerkin method (385) converges for operator $L+K$.*

The same technique can be applied to constructing an approximate solution to the hypersingular equation

$$(H+K)\varphi = \frac{d}{dy}\int\limits_{-1}^{1}\left(\frac{1}{x-y}+K(x,y)\right)\varphi(x)\sqrt{1-x^2}dx = f(y), \quad -1<y<1. \tag{386}$$

The conditions concerning the smoothness of $K(x,y)$ that guarantee the compactness of integral operator K are imposed in Section 2.

Choose the Chebyshev polynomials of the second kind as the basis and probe functions, i.e., set

$$X_N = \{U_0,\dots,U_{N-1}\}, \quad Y_N = \{U_0,\dots,U_{N-1}\}. \tag{387}$$

Unknown approximate solution $\varphi_N(x)$ is determined in the form

$$\varphi_N(x) = \sum_{n=0}^{N-1} b_n U_n(x). \tag{388}$$

Note that hypersingular operator H maps X_N onto Y_N and is bijective.

From Lemma 10, it follows that the Galerkin method for operator H converges. By virtue of Lemma 9, this result is also valid for operator $H+K$ under the assumption that K is compact and $H+K$ is injective. Quasioptimal formula (378) estimates the rate of convergence of approximate solution $\varphi_N(x)$ to exact one $\varphi(x)$, $N\to\infty$ in the norm of space $\widehat{W}_2^1$. Unknown coefficients a_n are found from finite linear system (383).

Theorem 48 *If operator $H+K:\widehat{W}_2^1\to L_2^{(2)}$ is injective and $K:\widehat{W}_2^1\to L_2^{(2)}$ is compact, then the Galerkin method (387) converges for operator $H+K$.*