Mandatory Assignment 1

University of Oslo
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MAT-MEK 4270 Numerical Methods for Partial Differential Equations

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1.2.3. Exact Solution

We show that

$$u(t, x, y) = \exp(\iota(k_x x + k_y y - \omega t)), \tag{0.1}$$

where ι is the imaginary unit, satisfies the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Taking second partial derivatives of the given function u with respect to t, x and y gives

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -\omega^2 u(t, x, y), \\
\frac{\partial^2 u}{\partial x^2} &= -k_x^2 u(t, x, y), \\
\frac{\partial^2 u}{\partial t^2} &= -k_y^2 u(t, x, y).
\end{aligned}$$

The wave equation is satisfied if and only if $\omega^2 = c^2(k_x + k_y)$ holds, which is exactly the relation between ω, c and $|k|^2 = k_x^2 + k_y^2$.

1.2.4. Dispersion coefficient

We assume that $m_x = m_y$ such that $k_x = k_y = k$. A discrete version of 0.1 is

$$u_{i,j}^n = \exp(\iota(kh(i + j) - \tilde{\omega} n \Delta t)),$$

where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e. the numerical approximation of the exact ω. We show that for CFL number $C = \frac{1}{\sqrt{2}}$ we get $\tilde{\omega} = \omega$.

We insert the given discrete solution into the discretized wave equation

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right).$$

We observe that the given $u_{i,j}^n$ is symmetric with respect to i and j, so we can rewrite the discretized wave equation as

$$u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} = \frac{2c^2 \Delta t^2}{h^2} \left(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n \right).$$

Using the definition of the CFL number we obtain $\frac{2c^2\Delta t^2}{h^2} = 2C^2 = 1$ and simplify

$$u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} = u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n.$$

We now use the given representation of $u_{i,j}^n$ and obtain

$$\begin{aligned}
&\exp(\iota kh(i + j)) \exp(-\iota \tilde{\omega} n \Delta t)(\exp(-\iota \tilde{\omega} \Delta t) - 2 + \exp(\iota \tilde{\omega} \Delta t)) = \\
&\exp(\iota kh(i + j)) \exp(-\iota \tilde{\omega} n \Delta t)(\exp(\iota kh) - 2 + \exp(-\iota kh)).
\end{aligned}$$

This equation is obviously satisfied if $\tilde{\omega}\Delta t = kh$. We now compare $\tilde{\omega}$ and ω:

$$\frac{\omega}{\tilde{\omega}} = \frac{\sqrt{k_x^2 + k_y^2} c \Delta t}{kh} = \frac{\sqrt{2} kc \Delta t}{kh} = \frac{\sqrt{2} c \Delta t}{h} = \sqrt{2} C = 1.$$

This proves the claim.