EXPLORING THE ITERATION OF LINEAR FUNCTIONS

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1. Introduction

In this paper we will examine the behavior of sequences that are generated by the iteration of linear functions. By taking successive iteration of a general function we are able to obtain more accurate approximate solutions to functions of the form $f(x) = x$. We will also consider the geometry involved in the iteration of linear functions. For example, we will see that when we take the iteration of the linear function $f(x) = -x_0+1$, it does not grow without bound, but oscillates between 1 and 0. When we examine the geometry of this case we will see that the function $f(x) = x$ intersects the function $f(x) = -x_0+1$ at a ninety degree angle. Viewing this behavior geometrically we will see that under iteration the linear function $f(x) = -x_0 + 1$ will produce the graph of a square and will travel around the square indefinitely. Another important reason to study the behavior of sequences that are generated by the iteration of linear functions is that it is good practice to get acquainted with theorems and definitions that are critical to real analysis and to bring richness to our understanding of calculus. We will begin by giving all of our basic definitions and then supplementing that understanding with specific examples. We have chosen to include both an informal and a formal definition of convergence. We will reconsider the formal definition after we state our theorem and its corollaries.

Definition 1. A linear equation in one variable is an equation that can be written in the form $f(x) = ax + b$.

Definition 2. A sequence is a real-valued function f with domain the set of positive integers. If $f(n) = a_n$, the sequence is denoted $\{a_n\}$ or $\{\ a_1, a_2, \ldots a_n, \ldots\}$.

Definition 3. A sequence $\{a_n\}$ is said to be convergent if there is a number L such that $\lim_{n\to\infty} a_n = L$. In other words, the values of a_n

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grow closer to L as n increases. If a limit L does not exist we say that a sequence is divergent.

DEFINITION 4. A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ (or $a_n \to L$) as $n \to \infty$ if for every$\epsilon > 0$ there is a corresponding integer N_ϵ such that if $n > N_\epsilon$ then $\mid a_n - L \mid< \epsilon$. If a sequence has limit L we say that it is convergent. Otherwise, when such an L does not exist, we say the sequence is divergent.

The iteration of a linear function produces a sequence. Using our definitions, let us now consider some specific examples of sequences that are generated by the iteration of linear functions. Recall that a linear function is defined as any linear equation in one variable that can be written in the form $f(x) = ax + b$. Given a linear function $f(x) = ax + b$ and an initial choice of x, say x_0, we can construct a sequence of numbers defined as the following:

$$
\begin{aligned}
x_1 &= ax_0 + b,\\
x_2 &= ax_1 + b,\\
x_3 &= ax_2 + b,\\
&\vdots\\
x_{n+1} &= ax_n + b,\\
&\vdots
\end{aligned}
$$

As we will see, behavior of the sequence $\{x_n\}$, depends on the choice of linear function under iteration. Moreover, the behavior will be dependent on the choices of a, b and x_0. We will also see a pattern emerge when we take iterations of the sequence $\{x_n\}$ given by $x_n = ax_{n-1} + b$ and we will consider this in more detail after providing some examples, specific and general.

Our first example is one of increasing convergence for the given function $f(x) = \frac{1}{4}x+1, x_0 = 1$. This particular linear function will generate a linear sequence that will converge to $\frac{4}{3}$. The term that has the most influence is the slope, which is given by a. When iterated, this particular linear function will exhibit increasing convergence and , in general, for any type of convergence to occur we need the condition where $0 <\mid a \mid< 1$. Now let us consider the example $f(x) = -\frac{1}{4}x + 1, x_0 = 1$. When we take the negative of our value for a we see decreasing convergence. For decreasing convergence to occur we need to satisfy the following conditions, $0 < a < 1$ and $b < 0$. For oscillating convergence to occur will need the following conditions, $-1 < a < 0$ and $b > 0$. We have one more example of convergence to discuss before we move onto our examples of divergence and that is that case of constant convergence. If we let $a = 0$ and $b = 1$ our function will become $f(x) = 1$ and under iteration this function will only have the value of b and will thus converge to b.

Now let us consider some example of divergence. For increasing divergence to occur, we need the general condition of $\mid a \mid > 1$ and $a > b$. Under iteration, the function, $f(x) = 2x + 1, x_0 = 1$ produces a sequence that diverges to infinity. Likewise the linear function $f(x) = 2x - 50, x_0 = -50$ will give decreasing divergence, and in general, it is true for all cases when $\mid a \mid > 1$ and $a < b$. Lastly we have two special cases of divergence when $a = 1$ and $a = -1$. An example of such a function would be $f(x) = x_0 + 1$. Here $b > 0$, and under iteration, $f(x) = x_0 + 1$ produces increasing divergence. If we let $b = -1$, producing the linear function $f(x) = x_0 - 1$, we will create divergence to negative infinity when $b < 0$. Lastly we will consider the behavior of $f(x) = -x_0 + 1$ under iteration. This linear function does not exhibit either increasing or decreasing divergence under iteration, but as we shall examine in our conclusion, oscillates between $- \mid x_0 + b \mid$ and therefore fails to converge.

These examples provide insight into the behavior of sequences that are generated by the iteration of linear functions but do not represent the closed form solution and all of its corollaries that we would like to obtain. In order to prove our theorem we need to begin with our definition that states that a function $f(x) = ax + b$ produces a sequence $x_{n+1} = ax_n + b$ under iteration for some initial choice for x, say x_0. We can see the pattern emerge when we take iteration of the sequence $\{x_n\}$ given by $x_n = ax_{n-1} + b, n = 1, 2, \ldots, n-1$ with a starting value of x_0.

$$x_0$$

$$x_1 = ax_0 + b$$

$$x_2 = a(ax_0 + b) + b = a^2x_0 + ab + b = a^2x_0 + b(1 + a)$$

$$x_3 = a(a^2x_0 + ab + b) + b = a^3x_0 + b(a^2 + a + 1)$$

$$\vdots$$

$$x_n = a^nx_0 + b(a^{n-1} + a^{n-2} + \ldots + 1)$$

From this generalization we are able to construct our theorem. Given $x_n = a^nx_0 + b(a^{n-1} + a^{n-2} + \ldots + 1)$, show that a closed form exists for sequences that are generated by the iteration of linear functions.

2. Proofs and Theorems

Proposition 2.1. *If $f(x) = ax + b$ and x_0 is a given real number then $x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1})$.*

First let us consider a proof by induction to show that $x_n = a^n x_0 + b(a^{n-1} + a^{n-2} + \ldots + 1)$ is equivalent to $x_{n+1} = a^{n+1} + b(a^n + a^{n+1} + \ldots + 1)$.

Proof. Given:$x_n = a^n x_0 + b(a^{n-1} + a^{n-2} + \ldots + 1)$, show that it holds for n. Let $n = 1$ such that,

$$\begin{aligned} x_1 &= a^1 x_0 + b(a^{1-1} + a^{1-2} + \ldots + 1) \\ &= ax_0 + b\left(\frac{1 - a^{-1}}{1 - a}\right) \\ &= ax_0 + b. \end{aligned}$$

Now let us assume that the property is true for n, then it follows that,

$$\begin{aligned} x_{n+1} &= f(x_n) = ax_n + b \\ &= a(a^n x_0 + \ldots + b) \\ &= a^n(a)x_0 + b(a^n + a^{n-1} + \ldots + 1) \\ &= a^{n+1}x_0 + b(1 + \ldots + a^n). \end{aligned}$$

This expression is exactly the proposition for $n + 1$ and thus the expressions for n and $n + 1$ are equivalent. □

All though our proof does state that our generalization holds for all n we still are lacking a closed form expression for x_n iterations of n, a, b and x_0 that we wish to establish. For this expression we need to construct our theorem.

Lemma 2.2. *For any $a \in \mathbb{R}$,*

$$1 + a + \ldots + a^{n-1} = \begin{cases} \dfrac{1 - a^n}{1 - a}, & a \neq 1 \\ n, & a = 1 \end{cases}$$

Proof. If $a = 1$, then $1 + \ldots + a^n = n$. If $a \neq 1$, then

$$\begin{aligned} s &= 1 + a + a^2 + \ldots + a^{n-2} + a^{n-1} \\ S - aS &= a(a^n x_0 + \ldots + b) \\ S - aS &= 1 - a^n \Leftrightarrow S(a - 1) = 1 - a^n \\ S &= \frac{1 - a^n}{1 - a}. \end{aligned}$$

□

Theorem 2.3. *If $f(x) = ax + b$ and x_0 is given then,*

$$x_n = \begin{cases} a^n \left(x_0 - \dfrac{b}{1-a} \right) + \dfrac{b}{1-a}, & a \neq 1 \\ x_0 + bn, & a = 1 \end{cases}.$$

Proof. Given our proposition $x_n = a^n x_0 + b(a^{n-1} + a^{n+2} + \ldots + 1)$, use lemma 1.1. to show that a close form exists for sequences that are generated by the iteration of linear functions when $a \neq 1$.

$$\begin{aligned} x_n &= a^n x_0 + b\left(\frac{1-a^n}{1-a}\right) \\ &= a^n x_0 + \frac{b}{1-a} - \frac{a^n b}{1-a} \\ &= a^n \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}. \end{aligned}$$

Then if $a = 1$, then it follows that $x_n = x_0 + bn$ and thus we have demonstrated both cases of our theorem. ◻

The resulting theorem is both informative and functional. It tells us that given any choice of a, n or x_0 we can produce sequences that are generated by the iteration of linear functions that exhibit various convergent and divergent behaviors. We will state these various behaviors as corollaries to our theorem. We will start by considering conditions for divergence and then consider conditions of convergence.

Corollary 2.4. *Given $x_n = a^n \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$, and $a \neq 1$.*

- *$b = 0$ when $x_n = x_0$, constant convergence.*
- *$b > 0$ given x_n, increasing divergence.*
- *$b < 0$ given x_n, decreasing divergence.*

Corollary 2.5. *Given $x_n = a^n \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$, and $a > 1$.*

- *$a_n \to \infty$ if $x_0 > \frac{b}{1-a}$ and $a > b$, increasing divergence.*
- *$a_n \to -\infty$ if $x_0 > \frac{b}{1-a}$ and $a < b$, decreasing divergence.*
- *$x_0 = \frac{b}{1-a}$, constant convergence.*

Corollary 2.6. *Given $x_n = a^n \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$, and $a = -1$.*

- *if* n *is odd* $x_n = (-1)^n \left(x_0 - \frac{b}{2}\right) + \frac{b}{2} = -x_0 + b$*, oscillating divergence.*
- *if* n *is even* $x_n = x_0 - \frac{b}{2} = x_0$*, increasing divergence.*
- *if* $x_0 = \frac{b}{2}$*, constant convergence.*

Corollary 2.7. *Given* $x_n = a^n \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$*, and* $-1 < a < 0$*.*

- $a_n \to 0$ *when* $x_0 \neq \frac{b}{1-a}$*, oscillating convergence.*
- $x_n \to \frac{b}{1-a}$ *then* $x_0 = \frac{b}{1-a}$*, constant convergence.*

Corollary 2.8. *Given* $x_n = a^n \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$*, and* $1 < a < 0$*,* $1 < a < 0$*.*

- $x_0 = \frac{b}{1-a}$*, constant convergence.*
- $x_0 > \frac{b}{1-a}$*, decreasing convergence.*
- $x_0 < \frac{b}{1-a}$*, increasing convergence.*

Corollary 2.9. $a = 0$*, constant convergence to* b*.*

Before considering our conclusion let us revisit the formal definition of convergence. Recall that the formal definition of convergence states that a sequence $\{x_n\}$ converges to $L \in \mathbb{R}$ if for every $\epsilon > 0$ there exists N_ϵ such that $\mid x_n - L \mid < \epsilon$ then $n > N_\epsilon$. To aid our understanding of this definition let us use one of our previous examples. Consider the linear function $f(x) = (1/4)x + 1$, where $x_0 = 1$. The limit of the iteration sequence associated with this function is $4/3$. Using our theorem for x_n we are now able to solve for an expression that will get n in terms of N_ϵ.

$$\mid x_n - 4/3 \mid < \epsilon$$
$$\mid (1/4)^n (x_0 - 4/3) + 4/3 - 4/3 \mid < \epsilon$$
$$(1/4)^n (x_0 - 4/3) < \epsilon$$
$$(1/4)^n < \frac{\epsilon}{x_0 - 4/3}$$
$$n \ln(1/4) < \ln\left(\frac{\epsilon}{x_0 - 4/3}\right)$$

$$-n \ln 4 < \ln \left(\frac{\epsilon}{x_0 - 4/3} \right)$$

$$n > \frac{\ln \frac{\epsilon}{x_0 - 4/3}}{\ln 4} = N_\epsilon$$

What this expression is telling us is that given any value of ϵ we can determine how many iterations we need to get with our limit. For example if we let $\epsilon = .001$ we will get the following expression,

$$n > \frac{\ln \mid \frac{.001}{1-4/3} \mid}{\ln 1/4} = N_\epsilon.$$

Therefore we get the following value for $N_\epsilon = 4.1904 < n$. If we want to get within 1/1000 within our limit we need to let $n = 5$. This expression also provides understanding as to why some sequences of linear functions diverge under iteration. If a limit does not exist and $\mid a \mid > 1$ then x_n will not approach any singular value unless $x_0 = \frac{b}{1-a}$. We can now see the power and precision of the definition of convergence using N_ϵ. Instead of simply stating the conditions for convergence to exist, it tells us how far out we need to go to get close to the limit.

3. Conclusion

Now that we have established our theorem and all of its corollaries we can now revisit some of our examples, the formal definition of convergence and the geometry of linear functions under iteration. Beginning with the geometry, if we take a convergent linear iteration sequence, the value to which it will converge will be the intersection of the line $y = ax+b$ and the line $y = x$, which will always be $\frac{b}{1-a}$ and is true for all convergent linear iteration sequences. We see, algebra aside, that our geometric results provide a specialized understanding to a wide range of behaviors. We must note that the only value that a linear iteration sequence can converge to is $\frac{b}{1-a}$ because a linear iteration sequence can have at most one solution. If, for example, we consider the quadratic iteration sequence $f(x) = 2x(1-x)$ we could have at most two values for which it converges.

When we reconsider our algebraic argument we examine the behaviors that are produced by sequences created by linear functions under iteration, given various values for a, b and x_0. In particular, we notice the strong effect the a term has on behavior, for example, if given $-1 < a < 1$ we will always see a type of convergence. Given our closed form solution that we obtained in our theorem stating that if $a \neq 1$, then $x_n = a^n \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$, otherwise if $a = 1$ then $x_n = x_0 + b_n$, we can clearly see that if $x_0 = \frac{b}{1-a}$, then a_n will be multiplied by zero and only the $\frac{b}{1-a}$ term will remain, and thus constant convergence will occur. Additionally, when considering the case when $a > 1$, if $x_0 > \frac{b}{1-a}$ and $a > b$, increasing divergence will result. The only condition that needs to be changed for decreasing divergence to occur is that $a < b$. Here we can see a dependence on both the value of a and its relationship to the values of b and x_0.

We can appreciate from the proof of our theorem, that we now have a general form that will allow us to understand behavior through analysis of variables and constants. From our geometry we have a broad and intuitive understanding of behavior as well. Moreover, we see that it is not a far reach to extend our geometrical understanding to the iteration of sequences that are not linear. Lastly, we have furthered our understanding of our formal definition of convergence, which enables us to find N_ϵ in terms of n, which tells us what value of n we need to use to be within a certain value of the limit, L. Taken in context with our theorem and its corollaries, these three types of understanding provide us with a rigorous means of examining the behavior of sequences that are generated by the iteration of linear functions.

References

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