Test Course

Valerie Hower

October 29, 2018

Contents

Dig-In:

1 Exponential functions

Exponential functions illuminated.

Exponential functions may seem somewhat esoteric at first, but they model many phenomena in the real-world.

What are exponential functions?

Definition 1. *An* ***exponential function*** *is a function of the form*

$$f(x) = a^x$$

where $a \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$.

Question 1 *Is 4^{-x} an exponential function?*

Multiple Choice:

(a) *yes* ✓

(b) *no*

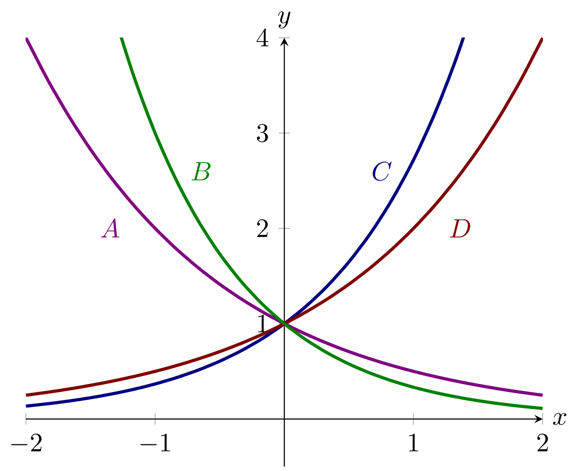
Feedback(*`attempt`*): *Note that*

$$4^{-x} = \left(4^{-1}\right)^x = \left(\frac{1}{4}\right)^x.$$

What can the graphs look like?

Graphs of exponential functions

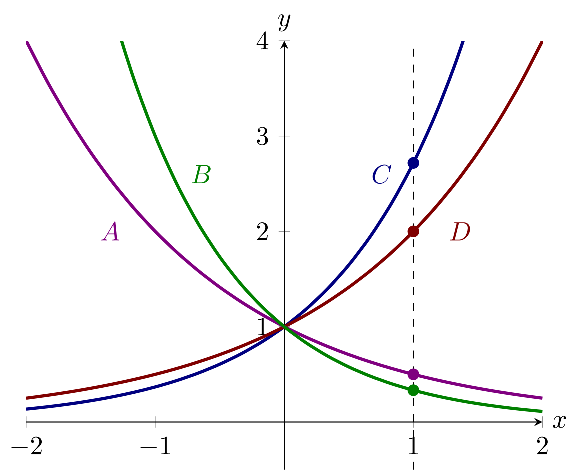
Example 1. *Here we see the the graphs of four exponential functions.*



Match the curves A, B, C, and D with the functions

$$e^x, \qquad \left(\frac{1}{2}\right)^x, \qquad \left(\frac{1}{3}\right)^x, \qquad 2^x.$$

Explanation. *One way to solve these problems is to compare these functions along the vertical line $x = 1$,*



Note

$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\left(\frac{1}{3}\right)^x$ *corresponds to* $\underset{\text{given}}{\boxed{B}}$.

- $\left(\frac{1}{2}\right)^x$ *corresponds to* $\underset{\text{given}}{\boxed{A}}$.
- 2^x *corresponds to* $\underset{\text{given}}{\boxed{D}}$.
- e^x *corresponds to* $\underset{\text{given}}{\boxed{C}}$.

Properties of exponential functions

Working with exponential functions is often simplified by applying properties of these functions. We will use these properties throughout the semester.

Properties of exponents

Let a be a positive real number with $a \neq 1$.

- $a^m \cdot a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $a^{-1} = \frac{1}{a}$

Question 2 *What exponent makes the following true?*

$$2^4 \cdot 2^3 = 2^{\boxed{7}}$$

Hint:

$$(2^4) \cdot (2^3) = (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2)$$

Example 2. *Solve the exponential equation.*

$$27^x = 9$$

Explanation. *In this example, we observe that* 27 *and* 9 *are both powers of* 3*. Once we have the same base on both sides of the equation, we may equate exponents. We use that* $27 = 3^{\overset{\boxed{3}}{\text{given}}}$ *and* $9 = 3^{\overset{\boxed{2}}{\text{given}}}$*, therefore we can write the exponential equation as*

$$3^{\overset{\boxed{3x}}{\text{given}}} = 3^{\overset{\boxed{2}}{\text{given}}}.$$

By equating exponents, we arrive at $x = \frac{2}{3}$.

Dig-In:

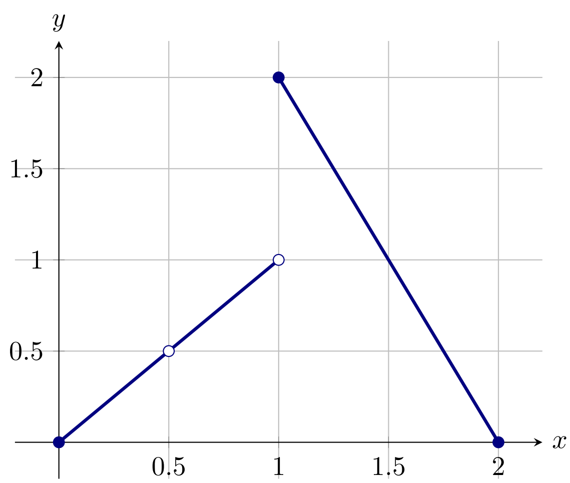
2 Continuity

The idea of continuity.

Idea 1. *A function f is **continuous at** $x = a$ if you can trace through the point $(a, f(a))$ without lifting your pen.*

YouTube link: `https://www.youtube.com/watch?v=hXFLvVFQa5k`

Question 3 *Consider the graph of the function f*



Which of the following are true?

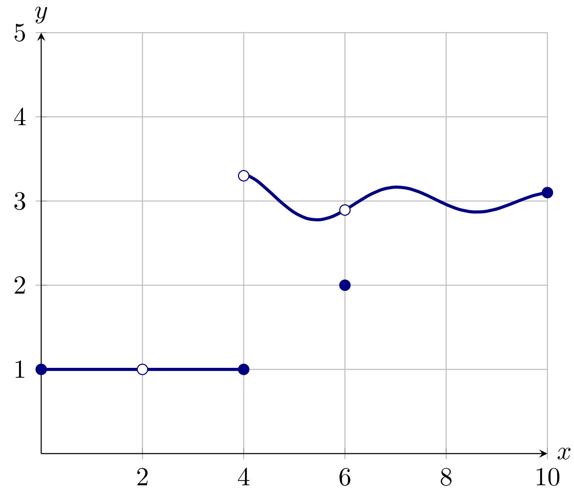
Multiple Choice:

(a) f is continuous at $x = 0.5$

(b) f is continuous at $x = 1$

(c) f is continuous at $x = 1.5$ ✓

Example 3. *Give x-values where the function below is discontinous (i.e. not continuous).*



Explanation. *To start, f is not even defined at $x = \underset{\text{given}}{\boxed{2}}$, therefore f cannot be continuous at $x = \underset{\text{given}}{\boxed{2}}$ as you must lift your pen over the hole.*

Next, from the plot above we see that $\lim_{x \to 4} f(x)$ does not exist because

$$\lim_{x \to 4^-} f(x) = \underset{\text{given}}{\boxed{1}} \qquad \textit{and} \qquad \lim_{x \to 4^+} f(x) \approx 3.3$$

This causes a "jump" in the function at $x = 4$ where you must lift your pen. This means f cannot be continuous at $x = 4$.

We also see that $\lim_{x \to 6} f(x)$ exists and $\lim_{x \to 6} f(x) \approx 2.9$. However, as $f(6) = \underset{\text{given}}{\boxed{2}}$, we must pick up our pen to color in the point $(6, f(6))$ as we trace along the graph. This means f is not continuous at $x = 6$.

Remark 1. *The following common functions are continuous at every x-value in their domains. Try visualizing some of their graphs to convince yourself of the continuity.*

Constant function $f(x) = k$ *for k a real number*

Power functions $f(x) = x, f(x) = x^2, f(x) = x^3, \cdots$ *or more generally* $f(x) = x^r$ *for $r > 0$*

Exponential function $f(x) = a^x$ *with base $a > 0$*

Logarithmic function $f(x) = \log_a(x)$ *with base $a > 0$*

Dig-In:

3 The Chain Rule

The Chain Rule. This section is adapted from Applied Calculus, by Shana Calaway, Dale Hoffman, and David Lippman, used under the Creative Commons Attribution license

The Chain Rule will let us find the derivative of a composition.

Example 4. *Find the derivative of* $y = (4x^3 + 15x)^2$

This is not a simple polynomial, so we cant use the power rule yet. This function is a product, so we could write it as $(4x^3 + 15x)(4x^3 + 15x)$ *and use the product rule. Or we could multiply/F.O.I.L it out and simply differentiate the resulting polynomial. Let's do the second way:*

$$y = (4x^3 + 15x)(4x^3 + 15x) = 16x^6 + 120x^4 + 225x^2$$
$$y' = 64x^5 + 480x^3 + 450x$$

Now suppose we want to find the derivative of $y = (4x^3+15x)^{20}$. We *could* write it as a product with 20 factors and use the product rule, or we *could* multiply it out. But I dont want to do that, do you?

We need an easier way, a rule that will handle a composition like this. The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out. It will also handle compositions where it wouldnt be possible to "multiply it out."

The Chain Rule is the most common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to. And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.

Formula 1 (Chain Rule)**.** *Here* $f(x)$ *and* $g(x)$ *are differentiable functions and* $y = f(g(x))$.

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

in words: The derivative of a composition is the derivative of the outside, with the inside staying the same, TIMES the derivative of the inside function.

Example 5. *Find the derivative of* $y = (4x^3 + 15x)^2$.

This is the same one we did before by multiplying out. This time, lets use the Chain Rule: The inside function is what appears inside the parentheses: $g(x) = \underbrace{\boxed{4x^3 + 15x}}_{\text{given}}$. *In our mind, we can replace the inside function with an "x" and read off the outside function* $f(x) = \underbrace{\boxed{x^2}}_{\text{given}}$.