1 Acceleration Controller

The control law that we designed to provide the commands at the acceleration level is:

$$\ddot{q} = J^{\#}\ddot{x} + \left(I - J^{\#}J\right)\ddot{q}_0. \tag{1}$$

where:

$$\ddot{x} = a - \dot{J}(q)\,\dot{q}.$$

$$a = \ddot{p}_d + K_d\left(\dot{p}_d - J\dot{q}\right) + K_p\left(p_d - f(q)\right), .$$

with $K_p > 0\,\mathrm{s}^{-2}$ and $K_d > 0\,\mathrm{s}^{-1}$ and diagonal matrices, and $\ddot{q}_0$ is an acceleration to be projected in the null space of J. This controller is derived from the acceleration resolution of kinematic redundancy, with the addition of PD control terms with feedforward acceleration to achieve asymptotic tracking in operational space.

1.1 Proof of stability

Consider the robot dynamic model

$$M(q)\,\ddot{q} + n(q, \dot{q}) = u.$$

where $n(q, \dot{q}) = c(q, \dot{q}) + g(q)$, but it will be used in place of $c(\cdot)$ and $g(\cdot)$ to get a more compact notation. The tracking error function and its derivatives up to the second order are defined as:

$$e = p_d(t) - f\left(q(t)\right) \qquad \dot{e} = \dot{p}_d(t) - J(q)\,\dot{q} \qquad \ddot{e} = \ddot{p}_d(t) - \ddot{r} = \ddot{p}_d(t) - \left(J(q)\,\ddot{q} + \dot{J}(q)\,\dot{q}\right).$$

If the acceleration is the one provided by (1), the dynamic model becomes

$$M[J^{\#}\left(\ddot{p}_d + K_d\dot{e} + K_p e - \dot{J}(q)\,\dot{q}\right) + \phi_N] + n(q, \dot{q}) = u.$$

where ϕ_N is any vector in the null space of J.
If no singularity is crossed, then the tracking error converges to 0.

Proof. [?] The closed-loop system is:

$$M\ddot{q} + n(q, \dot{q}) = M[J^{\#}\left(\ddot{p}_d + K_d\dot{e} + K_p e - \dot{J}(q)\,\dot{q}\right) + \phi_N] + n(q, \dot{q}).$$

By left multiplying M^{-1} (that is always well-defined because M is positive definite for any configuration), we get:

$$\ddot{q} = J^{\#}\left(\ddot{p}_d + K_d\dot{e} + K_p e - \dot{J}(q)\,\dot{q}\right) + \phi_N.$$

Taking into account that $\ddot{q} = J^{\#}\ddot{x} + \ddot{q}_N$ (where $\ddot{q}_N$ is a vector in the null space of J), leads to

$$J^{\#}\ddot{x} + \ddot{q}_N = J^{\#}\left(\ddot{p}_d + K_d\dot{e} + K_p e - \dot{J}(q)\,\dot{q}\right) + \phi_N.$$

While considering that $\ddot{x} = \ddot{r} - \dot{J}(q)\,\dot{q}$, we derive

$$\ddot{q}_N - \phi_N = J^{\#}\left(\ddot{p}_d - \ddot{r} + K_d\dot{e} + K_p e - \dot{J}(q)\,\dot{q} + \dot{J}(q)\,\dot{q}\right).$$

Left multiplying the above equation by J yields:

$$\ddot{e} + K_d\dot{e} + K_p e = 0.$$

If K_d and K_p are properly chosen, the error e converges globally to 0 and exponentially fast.[1]

1.2 Damping

The $\ddot{q}_0$ term in the controller is used, in our case, to perform velocity or momentum damping by self-motion. The results are compared with those of the controller that executes the task without damping. Therefore, we will consider three possible forms of the controller:

(i) $\ddot{q} = J^{\#}\ddot{x}$ $\quad(\ddot{q}_0 = 0)$: The pseudo-inverse of J selects, among all the joint accelerations that minimize $\|\ddot{x} - J(q)\,\ddot{q}\|^2$, the one that minimizes the norm of the joints' acceleration $\|\ddot{q}\|^2$. No damping is performed.

(ii) $\ddot{q} = J^{\#}\ddot{x} + \left(I - J^{\#}J\right)(-K_v\dot{q})$ $\quad\left(K_v > 0\,\mathrm{s}^{-1}\right)$: $\ddot{q}_0$ is a term that decreases the acceleration of the joints by a quantity that is proportional to the current joints' velocity. Since this is a null space term, damping doesn't affect the motion of the end-effector. By damping velocity, we are able to avoid undesired joint velocities or excessive joint movements, also preventing the system's state from diverging.

(iii) $\ddot{q} = J^{\#}\ddot{x} + \left(I - J^{\#}J\right)\left(-K_m M(q)\,\dot{q}\right)$ $\quad\left(K_m > 0\,\mathrm{s}^{-1}\,\mathrm{kg}^{-1}\right)$: $\ddot{q}_0$ is a term that decreases the joint's acceleration by a quantity proportional to the momentum of the joints ($M\dot{q}$). Again, this is a null space term, so the damping has no effect on the e.e and keeps the state of the system constrained within a specific range.

[1] In the Laplace domain, the linear equation $\ddot{e} + K_d\dot{e} + K_p e = 0$ becomes

$$\mathcal{L}\{e(t)\}(s) \longrightarrow \left(s^2 + K_d s + K_p\right) e(s) = 0.$$

and by the Routh criterion, we can extract the conditions on $K_{\{d,p\}}$ for exponential stability.