T1.1

ⓐ Prove that

if $T, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ bounded on bounded interval

s.t. $T(x) = 2T\left(\frac{x}{2}\right) + g(x) \quad \forall x \geq 1$

$g(x) = \Theta(x)$

$\Rightarrow T(x) = \Theta(x \log x)$

INTUITION:

$$T(x) = 2T\left(\frac{x}{2}\right) + g(x) \sim 2\left(2T\left(\frac{x}{4}\right) + g\left(\frac{x}{2}\right)\right) + g(x)$$

$$= 2^2\, T\left(\frac{x}{2^2}\right) + 2^1 \cdot g\left(\frac{x}{2^1}\right) + 2^0 \cdot g\left(\frac{x}{2^0}\right)$$

$$\sim 2^2\left(2T\left(\frac{x}{2^3}\right) + g\left(\frac{x}{2^2}\right)\right) + 2^1 \cdot g\left(\frac{x}{2^1}\right) + 2^0 \cdot g\left(\frac{x}{2^0}\right)$$

$$= 2^3\, T\left(\frac{x}{2^3}\right) + 2^2\, g\left(\frac{x}{2^2}\right) + 2^1 \cdot g\left(\frac{x}{2^1}\right) + 2^0 \cdot g\left(\frac{x}{2^0}\right)$$

$$\dots = 2^n\, T\left(\frac{x}{2^n}\right) + \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right)$$

as suggested by the hypotheses

We can prove by induction on $n \geq 1$ that

$$\forall x > 1, \quad T(x) = 2^n\, T\left(\frac{x}{2^n}\right) + \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right) \qquad (*)$$

proof: base case: $n = 1$, $T(x) = 2T\left(\frac{x}{2}\right) + 2^0 \cdot g\left(\frac{x}{2^0}\right)$ ✓ (by hypothesis)

inductive step: Assume it's true for n and prove it for $n+1$

$$\text{Hp: } T(x) = 2^n\, T\left(\frac{x}{2^n}\right) + \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right) \quad \text{(INDUCTIVE HYPOTHESIS)}$$

Now we simply use the hypothesis given by the exercise

$$T(x) \sim 2^n \cdot \left(2T\left(\frac{x}{2^{n+1}}\right) + g\left(\frac{x}{2^n}\right)\right) + \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right)$$

$$= 2^{n+1}\, T\left(\frac{x}{2^{n+1}}\right) + 2^n g\left(\frac{x}{2^n}\right) + \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right)$$

$$= 2^{n+1}\, T\left(\frac{x}{2^{n+1}}\right) + \sum_{k=0}^{n} 2^k g\left(\frac{x}{2^k}\right) \qquad \square$$

So now we have proven (*) to be true.

We will, very likely, compute this formula for $n = \lceil \log_2(x) \rceil$ because in this way we can obtain $T\left(\frac{x}{2^k}\right) = T(\xi) \leq 1$ for $\xi \leq 1$.

Therefore, we let $n = \lceil \log_2(x) \rceil$.

NB: $\log_2(x) \leq n < \log_2(x) + 1 \implies x \leq 2^n < 2 \cdot x \implies \frac{1}{2} \leq \frac{x}{2^n} < 1$ (**)

Now we use the hypotheses that:

(i) $g(x) = \mathcal{O}(x)$

(ii) T is a bounded map on bounded interval

Therefore, let $C, N > 0$ be s.t.

- $|g(x)| \leq C \cdot x \quad \forall x \geq N$ (for (i))
- $|T(x)| \leq C \quad \forall x \in [\frac{1}{2}, 1]$ (for (ii) and (**))

Let's prove that $T(x) = \mathcal{O}(x \log_2 x)$

$$|T(x)| \overset{(*)}{=} \left| 2^n T\left(\frac{x}{2^n}\right) + \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right) \right| \overset{\substack{\text{TRIANGULAR} \\ \text{INEQUALITY}}}{\leq} \underbrace{\left| 2^n T\left(\frac{x}{2^n}\right) \right|}_{A} + \underbrace{\left| \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right) \right|}_{B}$$

$$A = \left| 2^n T\left(\frac{x}{2^n}\right) \right| = 2^n \left| T\left(\frac{x}{2^n}\right) \right| \leq 2^n \cdot C \overset{n = \lceil \log_2(x) \rceil}{\underset{(**)}{<}} 2x \cdot C$$

Therefore $n < \log_2(x) + 1$
Hence $2^n < 2^{\log_2 x} \cdot 2 = 2x$

$$B = \left| \sum_{k=0}^{n-1} 2^k g\left(\frac{x}{2^k}\right) \right| \leq \sum_{k=0}^{n-1} 2^k \left| g\left(\frac{x}{2^k}\right) \right| \leq \sum_{k=0}^{n-1} \cancel{2^k} \cdot C \cdot \frac{x}{\cancel{2^k}} = C \cdot \sum_{k=0}^{n-1} x = C \cdot n \cdot x$$

$$< C \cdot (\log_2(x) + 1) \cdot x = C x \log_2(x) + C \cdot x$$

Therefore

$$|T(x)| < 2C \cdot x + C \cdot x + C \cdot x \log_2(x)$$

$$\implies T(x) = \mathcal{O}(x \log_2(x)) \quad \square$$

(b)

Let $a, b \geq 2$ be $\in \mathbb{N}$

If $T: \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ is s.t. $T(n) \leq a\, T(\lceil \frac{n}{b} \rceil)$ $\forall n \geq 1$

$\Rightarrow T(n) = O(n^{\log_b(a)})$

INTUITION:

$$T(n) \leq a\, T(\lceil n/b \rceil) \leq a^2\, T(\lceil \lceil n/b \rceil / b \rceil)$$

So the "ceiling" function $r(x) := \lceil \frac{x}{b} \rceil$ gets composed as an argument of $T: \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$.

Therefore, we have

$$T(n) \leq a \cdot T(r(n)) \leq a^2\, T(r^2(n)) \leq \ldots \leq a^k\, T(r^k(n)) \leq \ldots \qquad \forall k \geq 1$$

where $r^k(n) = \underbrace{r \circ \ldots \circ r}_{k \text{ times}} (n)$.

At a certain point, say for $k = l_m$, we will have $r^{l_m}(n) = 1$.

Assume l_m is the smallest integer for which that happens.

We have $r^{l_m}(n) > 0$, due to it being obtained via several divisions and applying the ceiling function.

The idea is that for $k = l_m - 1$ we have $r^{l_m - 1}(n) = \alpha > 1$, but then $0 < \frac{\alpha}{b} < 1$. At that point, however, $r^{l_m} = \lceil \frac{\alpha}{b} \rceil = \alpha' = 1$. (due to ceiling properties)

Next, $r^{l_m + 1} = \lceil \frac{\alpha'}{b} \rceil = 1$ as well for the same reason.

So, $\forall k \geq l_m$, $r^k(n) = 1$.

From this argument, we have

$$T(n) \leq a^{l_m} \cdot T(1). \qquad (***)$$

Moreover, we have the following property from the "nested divisions" section on the Wikipedia page of "floor and ceiling functions":

Let $n \in \mathbb{N}_{>0}$, $m, x \in \mathbb{R}$. Then $\left\lceil \frac{\lceil \frac{x}{m} \rceil}{n} \right\rceil = \left\lceil \frac{x}{mn} \right\rceil$ (P)

INTUITION: $r^2(n) = \left\lceil \frac{r(n)}{b} \right\rceil = \left\lceil \frac{\lceil n/b \rceil}{b} \right\rceil \overset{(P)}{=} \left\lceil \frac{n}{b^2} \right\rceil < \frac{n}{b^2} + 1 \overset{b \geq 2}{<} \frac{n}{b}$

Therefore we have $r^k(n) < \frac{n}{b^{k-1}}$ $\forall k \geq 2$ and prove it by induction over $k \geq 2$

proof: base: $k=2$, proven in the INTUITION above

inductive step: Hp: $r^{k-1}(n) < \frac{n}{b^{k-2}}$ Th: $r^k(n) < \frac{n}{b^{k-1}}$

$$r^k(n) = \left\lceil \frac{r^{k-1}(n)}{b} \right\rceil < \left\lceil \frac{n}{b^{k-1}} \cdot \frac{1}{b} \right\rceil = \left\lceil \frac{n}{b^k} \right\rceil < \frac{n}{b^k} + 1 \overset{b \geq 2}{<} \frac{n}{b^{k-1}} \quad \square$$

Thus,

$$1 = r^{l_m}(n) < \frac{n}{b^{l_m - 1}} \Rightarrow b^{l_m - 1} < n \Rightarrow l_m - 1 < \log_b(n) \quad (**)$$

So now we have

$$T(n) \overset{(***)}{\leq} a^{l_m} T(1) = a \cdot a^{l_m - 1} \cdot T(1) \overset{(**)}{<} a \cdot a^{\log_b(n)} \cdot T(1)$$

$$= a \cdot T(1) \cdot b^{\log_b(a^{\log_b n})} = a \cdot T(1) \cdot b^{\log_b a \cdot \log_b n} =$$

$$= a \cdot T(1) \cdot (b^{\log_b n})^{\log_b a} = a \cdot T(1) \cdot n^{\log_b(a)}$$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b(a)})$ $\square$