T 7.1

Let $\Psi(x,y) = \#$ of y-smooth integers in $\{1,\dots,x\}$

Let $f: \mathbb{N}_{\geq 2} \to \mathbb{R}$ be a function s.t. $f(y) \geq 1 \ \forall y$ & $f(y) = y^{1+o(1)}$ for $y \to +\infty$.

Let $y = L_x(\frac{1}{2}, v)$ for some $v > 0$

<u>Th</u> $\dfrac{x f(y)}{\Psi(x,y)} \sim L_x\left(\frac{1}{2}, g(v) + o(1)\right)$ for $x \to \infty$

for some function $g: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ s.t. $g(v) \geq \sqrt{2} \ \forall v > 0$.

Recall the L-notation:

$\forall t \in [0,1], \forall \gamma \in \mathbb{R}_{>0}$

$$L_x(t,\gamma) := \exp\left((\gamma + o(1))(\log x)^t (\log\log x)^{1-t}\right) \quad \text{as } x \to \infty$$

Some preliminary facts:

1) $y = L_x(\frac{1}{2}, v) = \exp\left((v + o(1))(\log x \cdot \log\log x)^{\frac{1}{2}}\right)$

2) $\dfrac{\Psi(x,y)}{x} \sim \rho(u)$ as shown heuristically by Dickman for $y \sim x^{1/u}$

3) $f(y) = y^{1+o(1)} = L_x\left(\frac{1}{2}, v(1+o(1))\right) = L_x(\frac{1}{2}, v)$ as $y \to \infty$

4) For a theorem seen in class, we have that $\log \rho(u) = -(1+o(1)) \cdot u \log u$ as $u \to +\infty$
So, we have $\rho(u) = \exp\left(-(1+o(1)) \cdot u \log u\right)$

5) $y \sim x^{1/u} \iff 1/u = \log_x y = \dfrac{\log(y)}{\log(x)} \iff u = \dfrac{\log(x)}{\log(y)}$

So, we have $y \sim x^{1/u} \iff u = \dfrac{\log(x)}{(v+o(1))(\log x \cdot \log\log x)^{1/2}} = \dfrac{1}{v + o(1)}\left(\dfrac{\log(x)}{\log\log(x)}\right)^{1/2}$

So, now we have that

$$\frac{x f(y)}{\Psi(x,y)} = f(y) \cdot \left(\frac{\Psi(x,y)}{x}\right)^{-1} \overset{(2)}{\sim} f(y) \cdot (\rho(u))^{-1} \overset{(3),(4)}{=} L_x(\tfrac{1}{2}, v) \cdot \exp\left((1+o(1)) \cdot u \log u\right) \qquad \text{(EQ. 1)}$$

Now we do some calculations on $\exp((1+o(1))\cdot u \log u)$

$$\exp((1+o(1))\cdot u\log u) \overset{(5)}{=} \exp\left((1+o(1))\cdot\frac{1}{v+o(1)}\left(\frac{\log(x)}{\log\log(x)}\right)^{1/2}\cdot\underbrace{\log\left(\frac{1}{v+o(1)}\left(\frac{\log(x)}{\log\log(x)}\right)^{1/2}\right)}_{(*)}\right)$$

also removing infinitesimal terms

$$\overset{(*)}{=} \exp\left(\frac{1}{2v}\left(\frac{\log(x)}{\log\log(x)}\right)^{1/2}\log\log(x)\right)$$

$$= \exp\left\{\frac{1}{2v}(\log(x)\cdot\log\log(x))^{1/2}\right\}$$

$$=: L_x\left(\tfrac{1}{2},\tfrac{1}{2v}\right)$$

$$(*)\ \log\left(\frac{1}{v+o(1)}\left(\frac{\log(x)}{\log\log(x)}\right)^{1/2}\right) = \underbrace{\log\left(\frac{1}{v+o(1)}\right)}_{\sim 1} + \frac{1}{2}\left(\underbrace{\log\log(x) - \log\log\log(x)}_{\sim \log\log(x)}\right)$$

$$\sim \frac{1}{2}\log\log(x)$$

So, from (EQ. 1) we have:

$$\frac{x f(y)}{\Psi(x,y)} = L_x\left(\tfrac{1}{2},v\right)\cdot\exp((1+o(1))\cdot u\log u) = L_x\left(\tfrac{1}{2},v\right)\cdot L_x\left(\tfrac{1}{2},\tfrac{1}{2v}\right) = L_x\left(\tfrac{1}{2},v+\tfrac{1}{2v}\right)$$

$$= L_x\left(\tfrac{1}{2}, g(v)+o(1)\right) \quad \text{where } g(v)\geq\sqrt{2}\ \forall v>0 \quad \square$$

Because $v+\frac{1}{2v}\geq\sqrt{2}\ \forall v>0$.

Indeed $\frac{d}{dv}\left(v+\frac{1}{2v}\right) = 1-\frac{1}{2v^2} = \frac{2v^2-1}{2v^2}\geq 0 \iff v\leq -\frac{1}{\sqrt{2}}$ or $v\geq\frac{1}{\sqrt{2}}$.
Since we consider only $v>0$, this means that $v+\frac{1}{2v}$ for us has a MINIMUM in $v=\frac{1}{\sqrt{2}}$.
Furthermore $v+\frac{1}{2v}\Big|_{v=\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} = \sqrt{2}$.

Therefore, $v+\frac{1}{2v}\geq\sqrt{2}$ ✓

T7.3

Let $L_{\alpha,C}(x) := \exp\left(C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha}\right)$ for $C>0, x>1, \alpha \in (0,1)$

<u>Th</u> $L_{\alpha,C}(x)$ is subexponential in $\log(x)$, namely:

1) $\forall \varepsilon > 0$, $L_{\alpha,C}(x) < k \cdot x^{\varepsilon}$ for x large enough

2) $\forall N > 0$, $L_{\alpha,C}(x) > k'(\log x)^{N}$ for x large enough

1) First of all, we notice that, $\forall \varepsilon > 0$:

$L_{\alpha,C}(x) < k \cdot x^{\varepsilon}$ for some x large enough $\iff$ $L_{\alpha,C}(x) \leq x^{\varepsilon}$ for some x large enough

just by chosing a larger value of x WLOG.

So we show that

$$\forall \varepsilon > 0, \ L_{\alpha,C}(x) \leq x^{\varepsilon} \text{ for } x \text{ large enough} \quad (\text{EQ. 1})$$

We have:

$$L_{\alpha,C}(x) := \exp\left(C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha}\right)$$

We have that (EQ. 1) holds iff

$$\exp\left(C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha}\right) \leq x^{\varepsilon} = \exp\left(\log(x^{\varepsilon})\right) = \exp\left(\varepsilon \cdot \log(x)\right)$$

$$\iff \exp\left(C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha} - \varepsilon \cdot \log(x)\right) \leq 1$$

$$\iff C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha} - \varepsilon \cdot \log(x) \leq 0$$

$$\iff C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha} \leq \varepsilon \cdot \log(x)$$

$$\iff \frac{C \cdot \log(x)^{\alpha} \cdot \log(\log(x))^{1-\alpha}}{\log(x)} \leq \varepsilon$$

$$\iff \frac{C \cdot \log(\log(x))^{1-\alpha}}{\log(x)^{1-\alpha}} \leq \varepsilon \iff C \cdot \left(\frac{\log(\log(x))}{\log(x)}\right)^{1-\alpha} \leq \varepsilon \quad (\text{EQ. 2})$$

We want to show that (EQ.2) holds for x large enough, $\forall \varepsilon > 0$, $\forall \alpha \in (0,1)$.
If this is the case, then (EQ. 1) also holds for x large enough, $\forall \varepsilon > 0$, $\forall \alpha \in (0,1)$.

Indeed, (EQ. 2) holds for x large enough, $\forall \varepsilon > 0$, $\forall \alpha \in (0,1)$ since

$$\frac{\log(\log(x))}{\log(x)} \xrightarrow[x \to +\infty]{} 0 \quad \text{and} \quad 1-\alpha > 0 \quad \text{because } 0 < \alpha < 1. \ \checkmark$$

2) First of all, we notice that, $\forall N > 0$:

$$\underset{\text{for some } x \text{ large enough}}{L_{\alpha,c}(x) > k'(\log x)^N} \iff \underset{\text{for some } x \text{ large enough}}{L_{\alpha,c}(x) \geq (\log x)^N}$$

just by chosing a larger value of x WLOG.

So we show that

$$\forall N > 0, \ L_{\alpha,c}(x) \geq (\log x)^N \text{ for } x \text{ large enough} \quad \text{(EQ. 3)}$$

We have that (EQ. 3) holds iff:

$$\exp\left(C \cdot \log(x)^\alpha \cdot \log(\log(x))^{1-\alpha}\right) \geq (\log x)^N = \exp\left(N \log(\log(x))\right)$$

$$\iff \exp\left(C \cdot \log(x)^\alpha \cdot \log(\log(x))^{1-\alpha} - N \log(\log(x))\right) \geq 1$$

$$\iff C \cdot \log(x)^\alpha \cdot \log(\log(x))^{1-\alpha} - N \log(\log(x)) \geq 0$$

$$\iff C \cdot \log(x)^\alpha \cdot \log(\log(x))^{1-\alpha} \geq N \log(\log(x))$$

$$\iff \frac{C \cdot \log(x)^\alpha \cdot \log(\log(x))^{1-\alpha}}{\log(\log(x))} \geq N$$

$$\iff C \cdot \left(\frac{\log(x)}{\log(\log(x))}\right)^\alpha \geq N \quad \text{(EQ. 4)}$$

We want to show that (EQ. 4) holds for x large enough, $\forall N > 0$, $\forall \alpha \in (0,1)$.
If this is the case, then (EQ. 3) also holds for x large enough, $\forall N > 0$, $\forall \alpha \in (0,1)$.

Indeed, (EQ. 4) holds for x large enough, $\forall N > 0$, $\forall \alpha \in (0,1)$ since

$$\frac{\log(x)}{\log(\log(x))} \xrightarrow[x \to +\infty]{} +\infty \quad \text{and} \quad \alpha > 0 \quad \text{from the hypothesis} \quad \square$$