T. 2. 1

Prove the correctness of RECURSIVE-DFT, supposing $n = 2^k$

Algorithm 2 RECURSIVE-DFT

Require: An integer $n = 2^k$ and a vector $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$.
Ensure: $\mathtt{DFT}_{\omega_n}(\mathbf{a}) = \widehat{\mathbf{a}} = (\widehat{a_0}, \widehat{a_1}, \ldots, \widehat{a_{n-1}})$.

$\mathbf{a}^{\text{even}} := (a_0, a_2, \ldots, a_{n-2})$
$\mathbf{a}^{\text{odd}} := (a_1, a_3, \ldots, a_{n-1})$
if $n = 2$ **then**
  $\widehat{\mathbf{a}^{\text{even}}} := \mathbf{a}^{\text{even}}$
  $\widehat{\mathbf{a}^{\text{odd}}} := \mathbf{a}^{\text{odd}}$
else
  $\widehat{\mathbf{a}^{\text{even}}} :=$ RECURSIVE-DFT$(n/2, \mathbf{a}^{\text{even}})$
  $\widehat{\mathbf{a}^{\text{odd}}} :=$ RECURSIVE-DFT$(n/2, \mathbf{a}^{\text{odd}})$
end if
$\omega_n = e^{\frac{2\pi i}{n}}$
$w = 1$
for $i = 0, \ldots, 2^{k-1} - 1$ **do**
  $\widehat{a_i} = \widehat{a_i^{\text{even}}} + w\widehat{a_i^{\text{odd}}}$
  $\widehat{a_{i+2^{k-1}}} = \widehat{a_i^{\text{even}}} - w\widehat{a_i^{\text{odd}}}$.
  $w := w \cdot \omega_n$
end for
return $\widehat{\mathbf{a}} = (\widehat{a_0}, \ldots, \widehat{a_{n-1}})$.

Some preliminary facts:

- $$\hat{a}_j = f(\omega_n^j) = \sum_{t=0}^{n-1} a_t \omega_n^{jt} \qquad \text{with } f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \quad (*)$$

- $$\omega_n = \exp\left(\frac{2\pi i}{n}\right), \qquad \omega_{n/2} = \omega_n^2 \quad (**)$$

- $$f(x) = \left(a_0 + a_2 x^2 + \ldots + a_{n-2} x^{n-2}\right) + x\left(a_1 + a_3 x^2 + \ldots + a_{n-1} x^{n-2}\right)$$
  $$= f_{\text{even}}(x^2) + x \cdot f_{\text{odd}}(x^2) \quad (***)$$

  with $f_{\text{even}}(y) = a_0 + a_2 y + \ldots + a_{n-2} y^{\frac{n-2}{2}}$ $= \frac{n}{2} - 1$
  $f_{\text{odd}}(y) = a_1 + a_3 y + \ldots + a_{n-1} y^{\frac{n-2}{2}}$

IN GENERAL we have

$$\hat{a}_i^{\text{even}} = f_{\text{even}}(\omega_{n/2}^i) = f_{\text{even}}(\omega_n^{2i})$$
$$\hat{a}_i^{\text{odd}} = f_{\text{odd}}(\omega_{n/2}^i) = f_{\text{odd}}(\omega_n^{2i})$$

We want to prove that lines 13-14 of RECURSIVE-DFT actually yield $\hat{a}_i$ and $\hat{a}_{i+n/2}$ for $i \in \{0, \dots, n/2 - 1\}$.

For line 13 we have:

$$\hat{a}_i^{even} + \omega_n^i \hat{a}_i^{odd} = f_{even}(\omega_n^{2i}) + \omega_n^i f_{odd}(\omega_n^{2i}) \overset{(***)}{=} f(\omega_n^i) \overset{(*)}{=} \hat{a}_i \quad \checkmark$$

For line 14 we have:

$$\hat{a}_i^{even} - \omega_n^i \hat{a}_i^{odd} = f_{even}(\omega_n^{2i}) - \omega_n^i f_{odd}(\omega_n^{2i}) \overset{(**)}{=} f_{even}(\omega_{n/2}^i) - \omega_n^i f_{odd}(\omega_{n/2}^i)$$

$$= f_{even}(\omega_n^{2i} \cdot \underset{\omega_n^n}{\underset{\shortparallel}{1}}) + \underset{\omega_n^i \cdot \underbrace{\omega_n^{n/2}}_{=-1}}{\underset{\shortparallel}{\omega_n^{i+n/2}}} \cdot f_{odd}(\omega_n^{2i} \cdot \underset{\omega_n^n}{\underset{\shortparallel}{1}})$$

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$$= f_{even}\left(\omega_n^{2(i+\frac{n}{2})}\right) + \omega_n^{i+n/2} \cdot f_{odd}\left(\omega_n^{2(i+\frac{n}{2})}\right)$$

$$= f\left(\omega_n^{i+n/2}\right)$$

$$\overset{(*)}{=} \hat{a}_{i+n/2} \quad \checkmark$$

Therefore, lines 13-14 yield what they should $\forall i \in \{0, \dots, \frac{n}{2}-1\}$ and thus the correctness of the algorithm is proven. $\square$