IMPLICIT SURFACE TRIAGULATIONS USING THE GAUSS MAP

CARLOS VALERO
MATHEMATICS DEPARTMENT
UNIVERSITY OF GUANAJUATO

Introduction

Efficient surface triangulation is a fundamental task in computational geometry, computer graphics, and scientific computing. Traditional approaches often rely on explicit or parametric surface representations, which can be limiting when dealing with complex topologies or evolving geometries. In contrast, implicit surfaces offer a compact and flexible framework for defining surfaces as level sets of scalar functions, enabling natural handling of intricate structures and topological changes.

In this work, we explore a novel approach to surface triangulation by inverting the Gauss map on implicit surfaces. The Gauss map, which relates points on a surface to their unit normals on the sphere, encodes critical geometric information. By inverting this map, we generate a sampling of the surface guided by the distribution of normals, resulting in a triangulation that is both geometrically faithful and computationally efficient.

This method leverages the differential properties of the implicit function defining the surface, avoiding the need for global parametrization. As such, it is particularly suited for complex geometries where parametric methods are impractical or fail to capture essential features. We demonstrate the effectiveness of this approach through examples and discuss its advantages compared to traditional meshing techniques.

Inverse Normal Map using Point Clouds

An implicit surface X in 3-dimensional is a surface that can be defined as set of points $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation

$$f(x, y, z) = 0,$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar valued function in the x, y, z variables. For every point (x, y, z) in X the normal vector to the surface is given by

$$N(x, y, z) = \nabla f(x, y, z)/|\nabla f(x, y, z)|.$$

Hence n defines a map, known as the Normal Map or the Gauss Map, of the form

$$N : X \to S^2$$

where the unit sphere S^2 is given by

$$S^2 = \{(n_x, n_y, n_z) \in \mathbb{R}^3 | n_x^2 + n_y^2 + n_z^2 = 1\}.$$

We wish to map a regular triangulation of the sphere S^2 to a triangulation of X by "inverting" the normal map n. The problem is that this map is not in general

FIGURE 0.1. Dense point cloud (half a million points) obtained as the inverse image of the normal map on the torus.

invertible. In any case, we consider the inverse images on a triangles T in the triangulation of S^2, i.e the sets of the form

$$N^{-1}(T) = \{x \in X | n(p) \in T\},$$

We will see that the collections of these sets, obtained by varying T over all the triangles of the sphere, will allow us to construct a triangulation of X that has more triangles in zone of "high curvature" and less triangles in zones of "low curvature".

Problem 1. Given a triangle $T \subset S^2$, how can we compute $N^{-1}(T)$?

We give brute force approximate solution to the above problem using point clouds as follows. A triangle T of the sphere is determined by three vertices n_1, n_2 and n_3 (which must be unit vectors). The the triangle T as a set is the set of point n in $\mathbb{R}^3$ that satisfy the equations

$$\begin{aligned} n \cdot n &= 1, \\ n \cdot (n_1 \times n_2) &\geq 0, \\ n \cdot (n_2 \times n_3) &\geq 0, \\ n \cdot (n_3 \times n_1) &\geq 0. \end{aligned}$$

Hence $N^{-1}(T)$ is consists of the set of points $p \in \mathbb{R}^3$ that satisfy the formulas

$$f(p) = 0,$$

and

$$\begin{aligned} \nabla f(p) \cdot (n_1 \times n_2) &\geq 0, \\ \nabla f(p) \cdot (n_2 \times n_3) &\geq 0, \\ \nabla f(p) \cdot (n_3 \times n_1) &\geq 0. \end{aligned}$$

(1) We represent the surface $X = f^{-1}(0)$ as a point cloud by first triangulating X (e.g. using the marching cubes algorithm) and then obtain regularly sampled points $\{p_i\}_{i=1}^k$from it.
(2) From the previous step we can obtain the vectors

$$g_i = \nabla f(p_i)$$

so that equations BLA becomes

$$\begin{aligned} g_i \cdot (n_1 \times n_2) &\geq 0, \\ g_i \cdot (n_2 \times n_3) &\geq 0, \\ g_i \cdot (n_3 \times n_1) &\geq 0. \end{aligned}$$

If $\{i_j\}_{j=1}^m$ are the indices for which the above equations hold the the points

$$\{p_{i_j}\}_{j=1}^m$$

is a point cloud representing the solution to the desired equations.

The process just described above can encounter problems at the singular points of the normal map , which are the points at which the Gaussian Curvature $\kappa : X \to \mathbb{R}$ vanishes. For and implicit surface we have that

$$\kappa(p) = \frac{1}{|\nabla f(p)|^2} \det\left(W(p)\right),$$

where

$$W(p) = \begin{pmatrix} w_{11}(p) & w_{12}(p) \\ w_{21}(p) & w_{22}(p) \end{pmatrix} \text{ for } w_{ij}(p) = \left(\nabla^2 f(p) T_i\right) \cdot T_j$$

and T_1, T_2 any pair orthonormal vectors tangential to X at p.

Examples

An advantage of using implicit surfaces (instead of parametric ones) is that we can blend them to create complex geometries from simple ones. For smooth functions $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$, we can represent the smooth union of the surfaces $X = f^{-1}(0)$and $Y = g^{-1}(0)$ by the formula

$$f_k(p) + g_k(p) - f_k(p) g_k(p) - 1/2 = 0$$

where

$$f_k = H_k \circ f, g_k = H_k \circ g$$

and

$$H_k(s) = \frac{1}{1 + \exp(-ks)}$$

is a scaled sigmoid function; the parameter k controls the how smooth is the surface obtained joining the X and Y. Observe that H_∞ is the Heaviside function

$$H_\infty(s) = \begin{cases} 0 & s < 0 \\ 1/2 & s = 0 \\ 1 & s > 0 \end{cases}$$

Similarly, the smooth intersection an difference of X and Y can be represented by the equations

$$f_k(p) g_k(p) - 1/2 = 0$$

and

$$f_k(p)(1 - g_k(p)) - 1/2 = 0.$$

Inverse Gauss Map

To obtain an optimal "subdivision" of a surface we compute the inverse of the normal map (defined above). We construct a triangulation of the sphere as follows (see Figure 0.2):

(1) Start with a regular tetrahedron inscribed in the unit sphere. Its four vertices A, B, C, D lie on the sphere.
(2) For each edge with endpoints u, v, compute the midpoint

$$m = \frac{u + v}{2}.$$

(3) Project this midpoint radially onto the sphere by normalization:

$$\hat{m} = \frac{u + v}{\|u + v\|}.$$

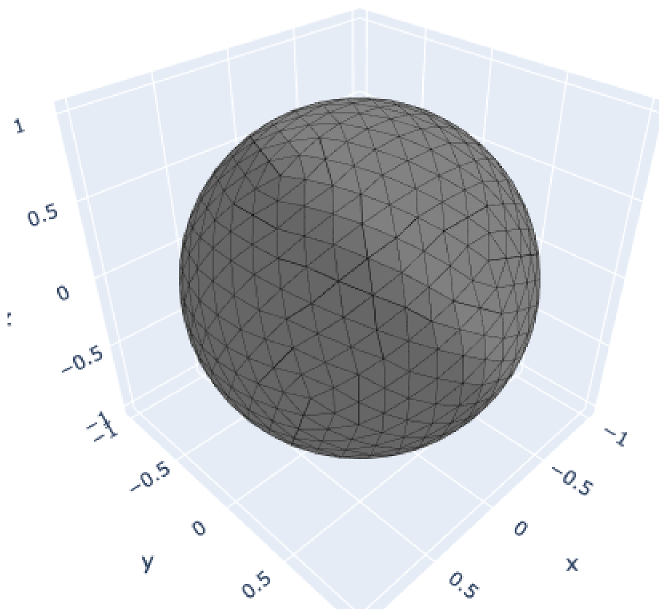

Figure 0.2. Regular sphere triangulation

(4) For each triangular face, say $\triangle ABC$, use the three edge midpoints $\hat{m}_{AB}, \hat{m}_{BC}, \hat{m}_{CA}$ to subdivide the face into four smaller triangles:

$$\triangle A\,\hat{m}_{AB}\,\hat{m}_{CA}, \quad \triangle B\,\hat{m}_{BC}\,\hat{m}_{AB}, \quad \triangle C\,\hat{m}_{CA}\,\hat{m}_{BC}, \quad \triangle \hat{m}_{AB}\,\hat{m}_{BC}\,\hat{m}_{CA}.$$

(5) Replace the original mesh with these smaller spherical triangles. Repeating the procedure k times yields a regular subdivision of the sphere with $4^{\,k+1}$ triangular faces.

We can create points that belong to each of these triangles and compute by brute force the inverse points using the Gauss Map to obtain a surface as in Figure 0.3

We can use level set methods to compute the edges of these triangles (see Figure 0.4)

If we apply this procedure to a double torus, we get Figure 0.5 .

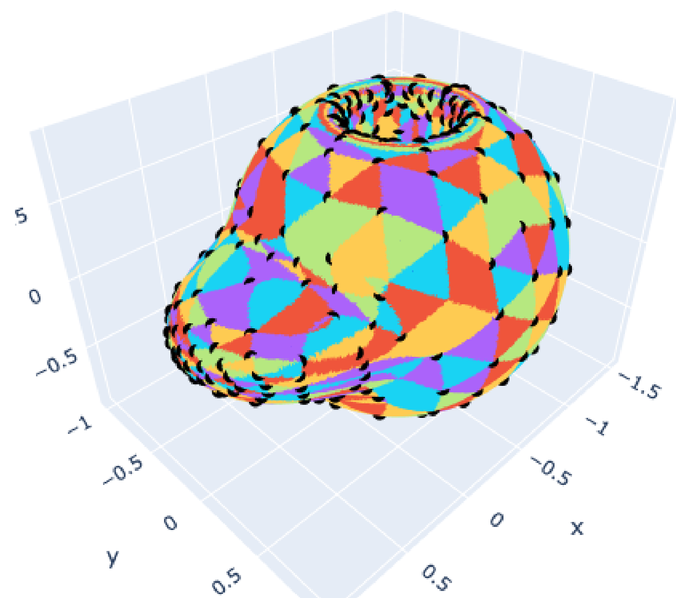

Figure 0.3. Blended surface curvature regions

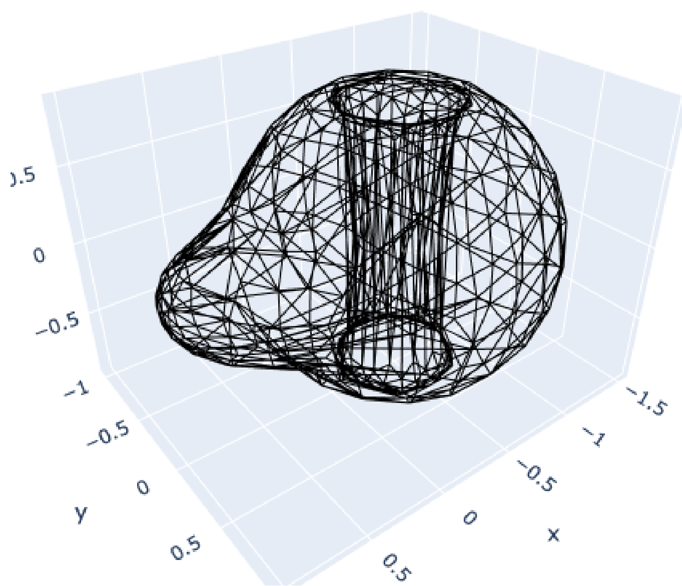

Figure 0.4. Edges as level sets

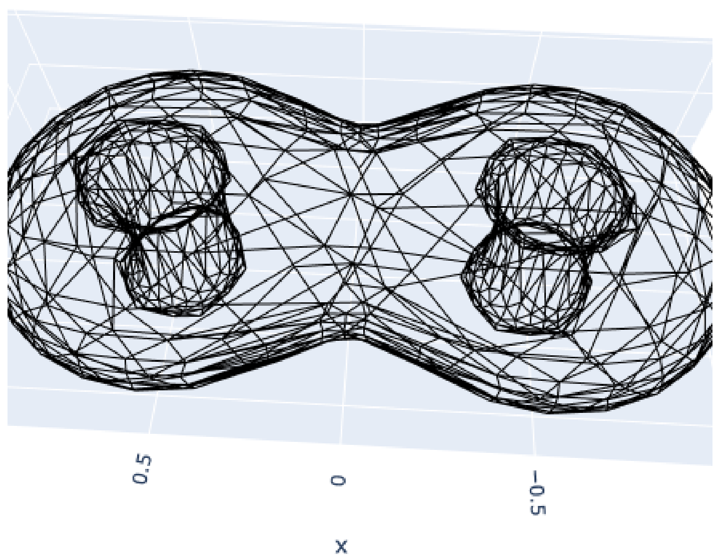

FIGURE 0.5. Optimal triangulation of double torus