IMPLICIT SURFACE TRIAGULATIONS USING THE GAUSS MAP

CARLOS VALERO
MATHEMATICS DEPARTMENT
UNIVERSITY OF GUANAJUATO

Introduction

BLA

Inverse Normal Map using Point Clouds

An implicit surface X in 3-dimensional is a surface that can be defined as set of points $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation

$$f(x, y, z) = 0,$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar valued function in the x, y, z variables. For every point (x, y, z) in X the normal vector to the surface is given by

$$N(x, y, z) = \nabla f(x, y, z)/|\nabla f(x, y, z)|.$$

Hence n defines a map, known as the Normal Map or the Gauss Map, of the form

$$N : X \to S^2$$

where the unit sphere S^2 is given by

$$S^2 = \{(n_x, n_y, n_z) \in \mathbb{R}^3 | n_x^2 + n_y^2 + n_z^2 = 1\}.$$

We wish to map a regular triangulation of the sphere S^2 to a triangulation of X by "inverting" the normal map n. The problem is that this map is not in general invertible. In any case, we consider the inverse images on a triangles T in the triangulation of S^2, i.e the sets of the form

$$N^{-1}(T) = \{x \in X | n(p) \in T\},$$

We will see that the collections of these sets, obtained by varying T over all the triangles of the sphere, will allow us to construct a triangulation of X that has more triangles in zone of "high curvature" and less triangles in zones of "low curvature".

Problem 1. Given a triangle $T \subset S^2$, how can we compute $N^{-1}(T)$?

We give brute force approximate solution to the above problem using point clouds as follows. A triangle T of the sphere is determined by three vertices n_1, n_2 and n_3 (which must be unit vectors). The the triangle T as a set is the set of point n in

FIGURE 0.1. Dense point cloud (half a million points) obtained as the inverse image of the normal map on the torus.

$\mathbb{R}^3$ that satisfy the equations

$$\begin{aligned} n \cdot n &= 1, \\ n \cdot (n_1 \times n_2) &\geq 0, \\ n \cdot (n_2 \times n_3) &\geq 0, \\ n \cdot (n_3 \times n_1) &\geq 0. \end{aligned}$$

Hence $N^{-1}(T)$ is consists of the set of points $p \in \mathbb{R}^3$ that satisfy the formulas

$$f(p) = 0,$$

and

$$\begin{aligned} \nabla f(p) \cdot (n_1 \times n_2) &\geq 0, \\ \nabla f(p) \cdot (n_2 \times n_3) &\geq 0, \\ \nabla f(p) \cdot (n_3 \times n_1) &\geq 0. \end{aligned}$$

(1) We represent the surface $X = f^{-1}(0)$ as a point cloud by first triangulating X (e.g. using the marching cubes algorithm) and then obtain regularly sampled points $\{p_i\}_{i=1}^k$from it.
(2) From the previous step we can obtain the vectors

$$g_i = \nabla f(p_i)$$

so that equations BLA becomes

$$\begin{aligned} g_i \cdot (n_1 \times n_2) &\geq 0, \\ g_i \cdot (n_2 \times n_3) &\geq 0, \\ g_i \cdot (n_3 \times n_1) &\geq 0. \end{aligned}$$

If $\{i_j\}_{j=1}^m$ are the indices for which the above equations hold the the points

$$\{p_{i_j}\}_{j=1}^m$$

is a point cloud representing the solution to the desired equations.

The process just described above can encounter problems at the singular points of the normal map (PICTURE AND EXPLAIN FOLDINGS), which are the points at which the Gaussian Curvature $\kappa : X \to \mathbb{R}$ vanishes. For and implicit surface we have that

$$\kappa(p) = \frac{1}{|\nabla f(p)|^2} \det(W(p)),$$

where

$$W(p) = \begin{pmatrix} w_{11}(p) & w_{12}(p) \\ w_{21}(p) & w_{22}(p) \end{pmatrix} \text{ for } w_{ij}(p) = \left(\nabla^2 f(p) T_i\right) \cdot T_j$$

and T_1, T_2 any pair orthonormal vectors tangential to X at p.

Examples

An advantage of using implicit surfaces (instead of parametric ones) is that we can blend them to create complex geometries from simple ones. For smooth functions $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$, we can represent the smooth union of the surfaces $X = f^{-1}(0)$and $Y = g^{-1}(0)$ by the formula

$$f_k(p) + g_k(p) - f_k(p)g_k(p) - 1/2 = 0$$

where

$$f_k = H_k \circ f, g_k = H_k \circ g$$

and

$$H_k(s) = \frac{1}{1 + \exp(-ks)}$$

is a scaled sigmoid function; the parameter k controls the how smooth is the surface obtained joining the X and Y. Observe that H_∞ is the Heaviside function

$$H_\infty(s) = \begin{cases} 0 & s < 0 \\ 1/2 & s = 0 \\ 1 & s > 0 \end{cases}$$

Similarly, the smooth intersection an difference of X and Y can be represented by the equations

$$f_k(p)g_k(p) - 1/2 = 0$$

and

$$f_k(p)(1 - g_k(p)) - 1/2 = 0.$$

MAKE A FEW PICTURES!

Alignment energy

$$E_n(x, y, z) = |\nabla f(x, y, z) \times n|^2 + f(x, y, z)^2$$

Observe that $E(x, y, z) = 0$ exactly when (x, y, z) is in the surface $f(x, y, z) = 0$ and $\nabla f(x, y, z)$ is parallel to n. This last conditions is equivalent to $g(x, y, z) = \pm n$. Numerically, we can solve the equation $E_n(x, y, z) = 0$ by minimizing E_n. This can be done by taking an initial condition $(x_{0,}y_0, z_0)$ and applying gradient descent (or some other more sophisticated numerical techique), In any case, if (x_0, y_0, z_0) is so that $f(x_0, y_0, z_0)$ is small and $\nabla f(x_0, y_0, z_0)$ is close to parallel to n and $\nabla f(x_0, y_0, z_0) \cdot n > 0$ we can expect that the minimization of E_0 with such an initial condition will provide us with a point (x, y, z) such that

$$f(x, y, z) = 0 \text{ and } g(x, y, z) = n.$$