Notes on the Differential Geometry of Curves

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1 Regular Curves and Arc Length

Let $I \subset \mathbb{R}$ be an interval. A curve is a differentiable map $\gamma : I \to \mathbb{R}^3$. The curve is *regular* if $\dot{\gamma}(t) \neq 0$ for all $t \in I$. Regularity ensures that each point admits a unique tangent direction described by

$$\mathbf{T}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}. \tag{1}$$

The arc-length function from $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| \, du. \tag{2}$$

If we reparametrize by s, the new curve (still denoted γ) satisfies $\|\gamma'(s)\| = 1$; primes will denote differentiation with respect to arc length in the sequel.

2 Curvature

2.1 Definition

For an arc-length parametrized curve the curvature is $\kappa(s) = \|\mathbf{T}'(s)\|$. The vector $\mathbf{T}'$ points toward the principal normal direction, and κ quantifies the instantaneous turning of the tangent.

2.2 Coordinate-Free Formula

When the curve is given in an arbitrary parameter t, we may compute

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}. \tag{3}$$

Proof. Differentiate $\mathbf{T}(t) = \dot{\gamma}/\|\dot{\gamma}\|$ using the quotient rule and separate the component parallel to $\dot{\gamma}$. Because $\mathbf{T}'$ is orthogonal to $\mathbf{T}$, only the cross-product part remains, yielding the stated expression after simplification.

2.3 Example: Circle

Consider $\gamma(\theta) = (R\cos\theta, R\sin\theta)$, a planar circle of radius R. Then $\dot{\gamma} = (-R\sin\theta, R\cos\theta)$, $\ddot{\gamma} = (-R\cos\theta, -R\sin\theta)$, and $\|\dot{\gamma} \times \ddot{\gamma}\| = R^2$. Substituting into the formula gives $\kappa = 1/R$ independent of θ. Figure 1 visualizes the tangent and normal directions.

3 Frenet Frame and Torsion

Assume $\kappa(s) \neq 0$. The principal normal is $\mathbf{N}(s) = \mathbf{T}'(s)/\kappa(s)$, and the binormal is $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$. These three vectors form an orthonormal moving frame.

3.1 Torsion

Torsion measures how the osculating plane twists:

$$\tau(s) = -\mathbf{B}'(s) \cdot \mathbf{N}(s). \tag{4}$$

In a non-arc-length parameter t we have

$$\tau(t) = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}. \tag{5}$$
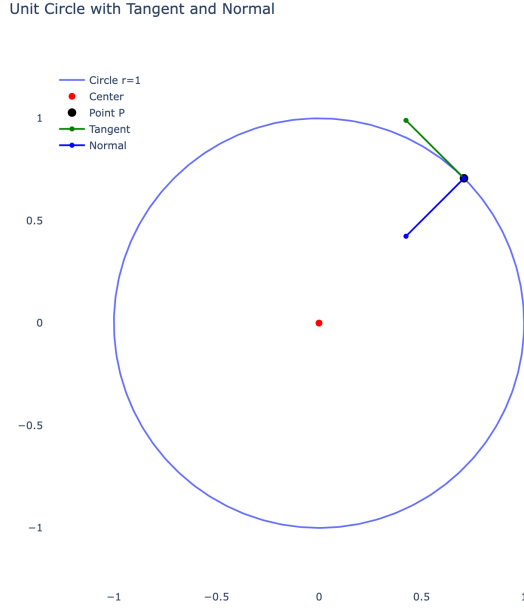
Figure 1: Unit circle with tangent $\mathbf{T}$ and inward normal $\mathbf{N}$ at a sample point. Image generated with Plotly.

Proof. Differentiate $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and expand. Since $\mathbf{T}$ and $\mathbf{N}$ are orthonormal, $\mathbf{B}' = -\tau\mathbf{N}$ for some scalar τ; solving for τ produces the dot-product formula. Writing derivatives in terms of t and using determinant identities yields the scalar triple product expression.

3.2 Frenet–Serret Equations

With s the arc length parameter the frame satisfies

$$\mathbf{T}' = \kappa\mathbf{N}, \tag{6}$$
$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}, \tag{7}$$
$$\mathbf{B}' = -\tau\mathbf{N}. \tag{8}$$

Proof. Equation (6) follows directly from the definition of $\mathbf{N}$. Differentiating $\mathbf{N}$ and decomposing into the orthonormal basis $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ yields (7). Finally, differentiating $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and substituting the first two equations gives (8).

4 Examples

4.1 Circular Helix

Let $\gamma(t) = (a \cos t, a \sin t, bt)$ with constants $a > 0$, $b \neq 0$. Computations show

$$\kappa = \frac{a}{a^2 + b^2}, \qquad \tau = \frac{b}{a^2 + b^2}, \tag{9}$$

both constant. Thus the helix bends and twists uniformly. Figure 2 depicts the Frenet frame obtained numerically via Plotly.
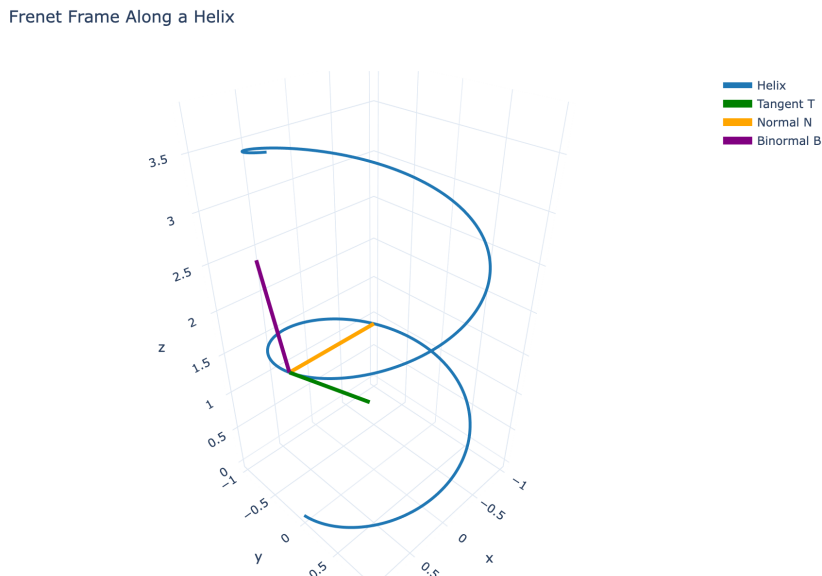

Figure 2: Space curve $\gamma(t) = (\cos t, \sin t, 0.3t)$ with tangent $\mathbf{T}$, normal $\mathbf{N}$, and binormal $\mathbf{B}$ at a sample point. Generated with Plotly.

Proof of constants. Compute derivatives $\dot{\gamma} = (-a \sin t, a \cos t, b)$ and $\ddot{\gamma} = (-a \cos t, -a \sin t, 0)$. Substitute into the general formulas to obtain the stated constants after simplification.

4.2 Planar Curves and Signed Curvature

For a plane curve expressed as $y = y(x)$ with $y' = dy/dx$, the signed curvature is

$$k(x) = \frac{y''}{(1 + (y')^2)^{3/2}}, \tag{10}$$

where the sign indicates orientation relative to the positive normal.

Proof. Parameterize the curve as $\gamma(x) = (x, y(x))$ and evaluate the cross-product formula while keeping track of orientation.

5 Osculating Circles and Radius of Curvature

The osculating circle at s_0 shares position, tangent, and curvature with the curve. Its radius is $\rho = 1/\kappa(s_0)$ and center $\mathbf{C} = \gamma(s_0) + \rho \mathbf{N}(s_0)$. Higher curvature corresponds to a smaller osculating circle.

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6 Existence and Uniqueness Theorem

Given smooth functions $\kappa(s) > 0$ and $\tau(s)$, there exists (up to rigid motions) a unique space curve whose Frenet invariants equal κ and τ.

Proof Sketch. The Frenet–Serret system (6)–(8) forms a linear ODE on $SO(3)$ with smooth coefficients. Solving for $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ with specified initial orthonormal data and integrating $\gamma'(s) = \mathbf{T}(s)$ produces a curve realizing the invariants. Any two solutions differ by a rigid motion because the system preserves inner products.

7 Computational Remarks

- Numerical differentiation amplifies noise; smooth or fit splines before computing κ and τ.

- For discrete curves, use finite differences and re-normalize tangents to mimic arc-length parameterization.

- The binormal becomes unstable when κ is nearly zero; switch to alternative frames (e.g., parallel transport frames) for trajectories with inflection points.

8 Further Reading

Standard references include Do Carmo's *Differential Geometry of Curves and Surfaces* and Struik's *Lectures on Classical Differential Geometry*. Modern applications appear in robotics motion planning and computer graphics, where curvature constraints regulate smooth paths.