

Basic Algebra for Data Analysis

Exam

November 11, 2022

1 Multiple choice questions

1. If $X = X^T$ and $Y = Y^T$, which of these matrices are certainly symmetric?

- (a) $X^2 - Y^2$
- (b) $(X + Y)(X - Y)$
- (c) XYX
- (d) $XYXY$

2. Which of the following properties about any non-singular matrices A and B are true?

- (a) $(AB)^{-1} = A^{-1}B^{-1}$
- (b) $(AB)^{-1} = B^{-1}A^{-1}$
- (c) $(AB)^T = A^TB^T$
- (d) $(A^{-1})^T = (A^T)^{-1}$

3. An $m \times m$ non-singular matrix A is given and $A^T = A^{-1}$.

Suppose that we "augment" the matrix A by an $m \times m$ identity matrix I , forming the matrix $(A \ I)$. If we do Gaussian elimination on *this* matrix, we will get something like:

$$(A \ I) \rightarrow (I \ C)$$

where the first m columns are an $m \times m$ identity matrix I , and the last m columns (from the elimination steps acting on I) are some matrix C .

C is equal to

- (a) A
- (b) A^{-T}
- (c) A^{-1}
- (d) $-A$

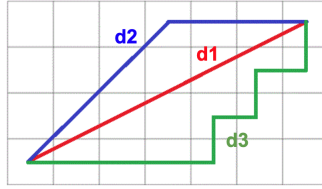
4. Consider the system of equations given below:

$$\begin{aligned} ax_1 + 3x_2 &= c \\ 10x_1 + bx_2 &= 7 \end{aligned}$$

What is true?

- (a) If $\frac{a}{c} = \frac{10}{7}$ then there are an infinite number of solutions
- (b) If $\frac{a}{c} = \frac{10}{7}$ and $\frac{a}{10} \neq \frac{3}{b}$ then there are no solutions
- (c) If $\frac{a}{10} = \frac{3}{b}$ then there exists exactly one solution

- (d) If $\frac{a}{10} \neq \frac{c}{7}$ then there exists exactly one solution
5. Consider distances $d1, d2, d3$:



What is true?

- (a) $d2$ is Manhattan distance
 - (b) $d3$ is Manhattan distance
 - (c) $d1$ is Euclidean distance
 - (d) $d2 = d3$
6. Consider the following vectors:

$$x = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

What is true about a scalar product (x, y) and euclidean norms $\|x\|_2$ and $\|y\|_2$?

- (a) $|(x, y)| = \|x\|_2 \|y\|_2$
 - (b) $|(x, y)| < \|x\|_2 \|y\|_2$
 - (c) $|(x, y)| > \|x\|_2 \|y\|_2$
7. Which of the following a singular matrix does not have?
- (a) Inverse
 - (b) Determinant
 - (c) Eigenvalues
 - (d) Transpose
8. What is true for determinants of any non-singular matrices A and B ?
- (a) $\det(4A) = 4\det(A)$
 - (b) $\det(I + A) = 1 + \det(A)$.
 - (c) The determinant of any matrix A is always the product of its pivots.
 - (d) AB and BA have the same determinant.
9. Calculate the trace of matrix A , given

$$A = \begin{pmatrix} 0 & 6 & 8 & 22 \\ -13 & 0 & 32 & 25 \\ 11 & 0 & 3 & 43 \\ 4 & 31 & 98 & -4 \end{pmatrix}$$

- (a) -1
- (b) 1
- (c) 73
- (d) 33

10. What is true for any two unitary (orthogonal) matrices U and V ?

- (a) $\det(UV)$ is equal to 1 or -1
- (b) $(UV)^T UV = U^T U$
- (c) $(UV)^T UV = V^T V$
- (d) $(UV)^T UV = I$

11. A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find:

- (a) the eigenvalues of $B^T B$
- (b) the determinant of B
- (c) the determinant of $B^T B$
- (d) the eigenvalues of $(B^2 + I)^{-1}$

12.

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 7 & -3 \\ -2 & 6 \end{pmatrix}$$

A^2 has the same _____ as A .

- (a) determinant
- (b) eigenvalues
- (c) eigenvectors
- (d) characteristic polynomial

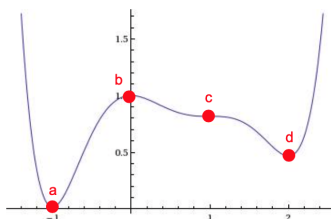
13.

$$y = \sqrt[3]{1 - 8z}$$

$$\frac{dy}{dz} = ?$$

- (a) $-\frac{1}{3}(1 - 8z)^{-\frac{1}{3}}$
- (b) $-\frac{8}{3}(1 - 8z)^{-\frac{2}{3}}$
- (c) $-\frac{8}{3}(1 - 8z)^{-\frac{1}{3}}$
- (d) $-\frac{1}{3}(1 - 8z)^{-\frac{2}{3}}$

14. Identify the local maximum and local minimum points in the below graph



- (a) b and c are local maxima points
- (b) c is a local maxima point
- (c) a and d are local minima points
- (d) a, b, c, d are local minima points

15. What is true about local maxima and minima points of the function $-x^3 + 6x$?

- (a) $\sqrt{2}$ and $-\sqrt{2}$ are local minima points
- (b) $\sqrt{2}$ and $-\sqrt{2}$ are local maxima points
- (c) $\sqrt{2}$ is local maxima and $-\sqrt{2}$ is local minima
- (d) $\sqrt{2}$ is local minima and $-\sqrt{2}$ is local maxima

2 Open-ended questions

1. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}, \quad A + 2I = \begin{pmatrix} -3 & 2 \\ -7 & 6 \end{pmatrix}$$

2. Calculate the Hessian matrix of

$$f(x, y) = x^3 + 2xy - y^6$$

3. (a) Write down any 3 by 3 matrix A . Split A into $B + C$ where $B = B^T$ and $C = -C^T$.
(b) Find formulas for B and C involving A and A^T . We want $A = B + C$ with $B = B^T$ and $C = -C^T$.