Splitting an IMP score in his components

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Contents

1 Introduction

In [1] the author introduced the **Valet Score**, an improvement and a generalization of the Butler Score.

The general idea is to split the **Butler Score** achieved by a pair in a specific hand as the sum of two main components:

- the **Bid Score**, only depending on the contract played at the table;
- the **Play Score**, only depending on the number of tricks made by declarer.

The purpose of this document is to slightly modify the algorithm described in [1], introducing a third component of the Butler Score and to justify by a formal mathematical point of view the way in which the Butler Score is split.

Moreover, with respect to the original algorithm, also the point of view changes a bit. Instead of splitting the Butler Score, that is the average of the IMP's won or lost in comparison with all the other results, we'll try to split every single comparison between one result and another result.

The aim is, given two results r_j and r_k and defining the IMP function $\mathrm{IMP}(r_j, r_k)$ as the number of IMP's won or lost by table j against table k in a single hand, to obtain a formula like

$$\mathrm{IMP}(r_j, r_k) = \mathrm{Bid}(r_j, r_k) + \mathrm{Play}(r_j) + \mathrm{Play}(r_k). \qquad (1)$$

in which the first term only depends on the contracts bid at both tables, the second term only depends on the number or tricks made at the first table and the last one only depends on the number of tricks made at the second table.

We'll see that it's impossible to obtain such result, but that we can go "as near as possible" to it.

2 Definitions

First of all we need some definition.

Definition 1. *Let a **contract** be the datum of:*

- *a **level**, i.e a number* $\in \{1, 2, 3, 4, 5, 6, 7\}$*,*
- *a **denomination**, i.e. a suit* $\in \{N, S, H, D, C\}$*,*
- *a **declarer*** $\in \{N, E, S, W\}$*,*
- *a **doubled flag*** $\in \{undoubled, doubled, redoubled\}$*,*

plus the result all pass*, and let a **result** be the datum of a contract and the number of tricks made, i.e. a number* $n \in \mathbb{N} : 0 \leq n \leq 13$*,*

Definition 2. *Let* $\mathbf{R}$ *be the set of all the* 5881 *possible bridge results and let* $\mathbf{V}$ *be the free vector space over* $\mathbb{Q}$*, i.e. the set of all the formal sums*

$$\sum_i \lambda_i r_i \qquad \text{where } r_i \in R \text{ and } \lambda_i \in \mathbb{Q}.$$

Definition 3. *For every pair* $(r_j, r_k) \in R \times R$*, let* $\mathbf{i_{j,k}}$ *the number of IMP (possibly negative or null) obtained by the result* r_j *against the result* r_k*, and let* $\mathbf{M}$ *be the antisymmetric matrix of dimension equal to* $\dim V$*, whose coefficient in the* j*-th row and in the* k*-th column is equal to* $i_{j,k}$ *for all* $1 \leq j, k \leq 5881$*.*

Definition 4. *Let*

$$\mathrm{IMP} : V \times V \to \mathbb{Q}$$

be the bilinear form on V *defined by*

$$\mathrm{IMP}(v_j, v_k) = v_j^T M v_k \qquad \forall\, v_j, v_k \in V$$

The function IMP extends the definition of the usual IMP function

$$\mathrm{IMP} : R \times R \to \mathbb{Z},$$

defined only on real bridge results, to all the pairs of linear combination of results.

Definition 5. *Given a hand* H *played in* n *tables* $t_1, t_2, \cdots, t_n$*, let* $\mathbf{r_{H_i}}$ *the result obtained at table* i *and let* $\mathbf{R_H}$ *be the multiset (i.e. a set with possible multiple elements) of the results* $r_{H_i}\ \forall\, i : 1 \leq i \leq n$*.*

Definition 6. *Given a hand* H *and an element* $r_{H_i} \in R_H$*, let* $\mathbf{S_{H_i}} \subseteq R_H$ *be the multisubset of* R_H *given by all the results with the same denomination and the same declarer as* r_{H_i} *and let* $\overline{\mathbf{S_{H_i}}}$ *be the multiset of all the contracts with the level, denomination, declarer and flag of* r_{H_i} *and number of tricks of any element of* S_{H_i}*. Finally define*

$$\mathbf{e_{H_i}} = \frac{\sum_{x \in \overline{S_{H_i}}} x}{\left|\overline{S_{H_i}}\right|}.$$

So e_{H_i} is a linear combination of bridge results (not necessarily belonging to R_H) with the sum of all coefficients equal to 1. We'll call this element **expected result**. This definition is nothing more than the formal definition used in [1] to calculated the Bid Score.

Note that the expected result only depends on the contract played in a table, but not on the number of tricks really made (or better: it depends on the tricks really made at the table but in the same measure in which it depends on the number of tricks made on every other table that played in the same denomination with the same declarer).

3 Splitting the Butler Score

Definition 7. *Given a hand* H *let*

$$f_H : R_H \to V \qquad f_H(r_i) = e_i.$$

be the function that associate to a result r_i *his expected result* e_i*.*

We have now that, given a hand H, $\forall\, r_j, r_k \in R_H$

$$
\begin{aligned}
\mathrm{IMP}(r_j, r_k) &= \mathrm{IMP}\left(f_H(r_j) + (r_j - f_H(r_j)), f_H(r_k) + (r_k - f_H(r_k))\right) = \\
&= \mathrm{IMP}(e_j + (r_j - e_j), e_k + (r_k - e_k)) = \\
&= \mathrm{IMP}(e_j, e_k) + \mathrm{IMP}(e_j, r_k - e_k) + \mathrm{IMP}(r_j - e_j, e_k) + \mathrm{IMP}(r_j - e_j, r_k - e_k).
\end{aligned}
\tag{2}
$$

The first of the 4 terms of (2) doesn't depend on the tricks made on both the tables, but only on the contracts played. This is a very good definition for what can be called Bid Score.

The second term of (2) only depends on the result obtained at table k and the third one only depends on the result obtained at table j. These could be a good definition too, respectively, for Play(r_k) and Play(r_j) Score, if it didn't exist the last term. We could define it as **Cross Play Score**, since it depends, in an linked way, on both the real results.

Note that when you calculated the Butler Score for the table j you have to do the average of all the scores obtained and obviously the split of the Butler score reflects the split of every single result. In other words, in the Global Cross Play Score, i.e.

$$
\frac{\sum_{k \neq j} \mathrm{IMP}(r_j - e_j, r_k - e_k)}{n-1} = \mathrm{IMP}\left((r_j - e_j, \frac{\sum_{k \neq j} r_k - \sum_{k \neq j} e_k}{n-1}\right) \tag{3}
$$

the difference between the sum of all the real results and their associated expected score is, in some sense, "small"[1]. In fact this is equal to zero if any denomination is played at the same level with the same flag in every table, and is equal to zero if the distribution of the tricks made is the same at different levels of the same denomination too. So, in general, the contribution of this term to the Global Butler Score is very small.

Anyway it shouldn't be right to exclude this term by the Play Score, because in fact it depends on the result obtained at the table, and also it shouldn't be right that the Play Score is completely independent on what is happening at the other table. For example an overtrick in 3NT can have a different value either at the other table 3NT was made or if at the other table they went down. It weights 1 IMP in the first case and probably nothing in the second.

So we need to change a bit the formula (1) and define the Play Score of a table not as the part of the Butler score only depending on the result obtained, but as the part of the Butler Score depending on the difference between the real result and the expected result.

To do this we can write the Cross Play Score in this way:

$$
\begin{aligned}
&\mathrm{IMP}(r_j - e_j, r_k - e_k) = \tfrac{1}{2}\,\mathrm{IMP}(r_j - e_j, r_k - e_k) + \tfrac{1}{2}\,\mathrm{IMP}(r_j - e_j, r_k - e_k) = \\
&= \tfrac{1}{2}\,\mathrm{IMP}(r_j, r_k - e_k) - \tfrac{1}{2}\,\mathrm{IMP}(e_j, r_k - e_k) + \tfrac{1}{2}\,\mathrm{IMP}(r_j - e_j, r_k) - \tfrac{1}{2}\,\mathrm{IMP}(r_j - e_j, e_k).
\end{aligned}
\tag{4}
$$

The first and the second term of (4) are depending on the difference between the real result and the expected result obtained at table k and the other two terms are depending on the difference between the real result and the expected result of table j.

Moreover we cannot say that the term $\mathrm{IMP}(r_j, (r_k - e_k))$ is completely independent on the result obtained at table j, but surely, in some sense, it's more dependent on the result of table k. For example if at table k the real result is much worse that the expected result,

[1] This could be formally proven introducing a norm on V

$\mathrm{IMP}(r_j,(r_k-e_k))$ will be anyway positive, even if the value of this expression can change a bit according to the result achieved on table j.

So we are going to add the first two terms of (4) to the one already obtained for $\mathrm{Play}(r_k)$ Score and the other two terms to the $\mathrm{Play}(r_j)$ Score obtaining that

$$\begin{aligned}\mathrm{Play}(r_j) &= \mathrm{IMP}(r_j-e_j,r_k)+\left(\tfrac{1}{2}\,\mathrm{IMP}(r_j-e_j,e_k)-\tfrac{1}{2}\,\mathrm{IMP}(r_j-e_j,r_k)\right)=\\ &=\tfrac{1}{2}\,\mathrm{IMP}(r_j-e_j,e_k)+\tfrac{1}{2}\,\mathrm{IMP}(r_j-e_j,r_k).\end{aligned} \tag{5}$$

and in the same way

$$\begin{aligned}\mathrm{Play}(r_k) &= \mathrm{IMP}(r_j,r_k-e_k)+\left(\tfrac{1}{2}\,\mathrm{IMP}(e_j,r_k-e_k)-\tfrac{1}{2}\,\mathrm{IMP}(r_j,r_k-e_k)\right)=\\ &=\tfrac{1}{2}\,\mathrm{IMP}(r_j,r_k-e_k)+\tfrac{1}{2}\,\mathrm{IMP}(e_j,r_k-e_k).\end{aligned} \tag{6}$$

Roughly speaking we can say that the Play Score of a table is obtained comparing the difference between the real result and the expected result at this table with two different objects: the real result and the expected result at the other table. These two terms has the same weight.

4 Conclusions

We can summarize the results obtained in the

Theorem 1. *Given a hand H, let $r_j,r_k\in R_H$ be two results obtained at the tables j and k and e_j,e_k the expected results associated to r_j,r_k respectively. Let*

- $\mathrm{Butler}_{jk}=\mathrm{IMP}(r_j,r_k)$,
- $\mathrm{Bid}_{jk}=\mathrm{IMP}(e_j,e_k)$,
- $\mathrm{Play}_j=\frac{1}{2}\left(\mathrm{Butler}_{jk}-\mathrm{Bid}_{jk}+\mathrm{IMP}(r_j,e_k)-\mathrm{IMP}(e_j,r_k)\right)$,
- $\mathrm{Play}_k=\frac{1}{2}\left(\mathrm{Butler}_{jk}-\mathrm{Bid}_{jk}+\mathrm{IMP}(e_j,r_k)-\mathrm{IMP}(r_j,e_k)\right)$.

Then

$$\mathrm{Butler}_{jk}=\mathrm{Bid}_{jk}+\mathrm{Play}_j+\mathrm{Play}_k\,. \tag{7}$$

Proof. The proof is obvious by the definitions. The term Play_j is exactly the one defined in (5). In fact

$$\begin{aligned}&\tfrac{1}{2}\,\mathrm{IMP}(r_j-e_j,e_k)+\tfrac{1}{2}\,\mathrm{IMP}(r_j-e_j,r_k)=\\ &=\tfrac{1}{2}\left(\mathrm{IMP}(r_j,e_k)-\mathrm{IMP}(e_j,e_k)+\mathrm{IMP}(r_j,r_k)-\mathrm{IMP}(e_j,r_k)\right)=\\ &=\tfrac{1}{2}\left(\mathrm{Butler}_{jk}-\mathrm{Bid}_{jk}+\mathrm{IMP}(r_j,e_k)-\mathrm{IMP}(e_j,r_k)\right)=\mathrm{Play}_j\,.\end{aligned} \tag{8}$$

and in the same way the term Play_k is the same defined in (6). $\square$

Note that, considering a table j, the absolute value of the Play Score at the other table in a single comparison can be obviously very high, when the result at the other table is much better or much worse than the expected result. But when calculating the Global Butler of single result i.e. when you calculate the average for every $k \neq j$, the value of the Play at the other tables is very small, because of the way in which it is calculated. If the Play Score at another table is high, it means that the real result is worse than the expected one, but this also means that there will be other tables in which the contrary is true.

The average of the Play Score at the other tables measures, in some sense, the difference between the multiset of expected results and the multiset of real results. In a ”perfectly random” hand this value is equal to zero. But it can happen, for example, than all the tables playing 3NT made the contract and all the tables playing 2NT made exactly 8 tricks. This differs from a perfectly random situation in which the percentage of the tables the made 9 tricks is the same for the tables playing 3NT and the tables playing 2NT. In the first situation table j lose more IMP's than in the second, due to a different distribution of the tables that made 9 tricks instead of 8. This difference is exactly the value of other tables' play, and it measure what that can be called **luck**.

References

[1] [Hein] Søren Hein. *The Valet Score: Principles.* November 2015.