



Introduction to bandit problems and algorithms

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This lecture is a short introduction to **bandit problems and algorithms**.

For an in-depth treatment, we suggest the recent book *Bandit algorithms* by Lattimore and Szepesvári (2020). See also [this tutorial](#) or [this blog](#).

Outline:

- 1 The K -armed bandit problem
- 2 Various extensions for numerous applications
- 3 An example in ad auction optimization
- 4 Next: Reinforcement Learning

1 The K -armed bandit problem

- Setting
- Well-known (but suboptimal) bandit algorithms
- Better performances with refined algorithms

2 Various extensions for numerous applications

- K -armed bandits: loosening the i.i.d. assumption
- Bandit problems with more complex decision space
- Best-arm identification

3 An example in ad auction optimization

4 Next: Reinforcement Learning

The Multi-Armed Bandit problem (MAB) is a toy problem that models sequential decision tasks where the learner must simultaneously exploit their knowledge and explore unknown actions to gain knowledge for the future (**exploration-exploitation** tradeoff).

Toy example: playing in a casino.

- Imagine we are given 1000 USD that we can use on 10 different slot machines (or *one-armed bandits*), 1 USD each.
- The average reward may vary from one slot machine to another. We initially do not know which machine is optimal.
- What is the best strategy to optimize our cumulative reward after 1000 rounds?
- We should both try all machines (exploration) while playing an empirically good machine sufficiently often (exploitation).

Imagine you are a doctor:

- Patients visit you *one after another* for a given disease.
- You prescribe one of the (say) *5 treatments* available.
- The treatments are *not equally efficient*.
- You do not know which one is the best, you *observe the effect* of the prescribed treatment on each patient

⇒ What should you do?

- You must choose each prescription using only the *previous observations*.
- Your goal is not to estimate each treatment's efficiency precisely, but to *heal as many patients as possible* (\neq clinical trials).

We write $g_t(i)$ for the reward (gain) of arm $i \in \{1, \dots, K\}$ at round $t \geq 1$. We assume that the sequence of reward vectors $g_1, g_2, \dots \in \mathbb{R}^K$ is chosen at the beginning of the game, and is i.i.d. for the moment. We set:

$$\mu_i := \mathbb{E}[g_1(i)] \quad \text{and} \quad \mu^* := \max_{1 \leq i \leq K} \mu_i.$$

Online protocol: at each round $t \in \mathbb{N}^*$,

- 1 The learner chooses an action $I_t \in \{1, \dots, K\}$, possibly at random.
- 2 The learner receives and observes the reward $g_t(I_t)$, but does not observe the reward $g_t(i)$ they would have got had they played another action $i \neq I_t$.

Goal: minimize the (pseudo) regret

$$R_T := \max_{1 \leq i \leq K} \mathbb{E} \left[\sum_{t=1}^T g_t(i) \right] - \mathbb{E} \left[\sum_{t=1}^T g_t(I_t) \right] = T\mu^* - \mathbb{E} \left[\sum_{t=1}^T g_t(I_t) \right].$$

A low regret means that the learner played (in expectation) almost as good as the best action, which is unknown to the learner.

Explore-Then-Commit (ETC)

Parameter: number $m \in \mathbb{N}^*$ of initial draws for each arm.

- 1 At each round $t \in \{1, \dots, mK\}$, choose action $I_t = (t \bmod K) + 1$.
- 2 At each round $t \geq mK + 1$, choose the action that was empirically best after the first phase: $I_t = \operatorname{argmax}_{1 \leq i \leq K} \hat{\mu}_i(mK)$.

Theoretical guarantee: if the reward vectors $g_1, g_2, \dots \in \mathbb{R}^K$ are i.i.d. and each $g_1(i) - \mu_i$ is subgaussian with variance factor σ^2 , then ETC satisfies (see, e.g., Thm 6.1 by Lattimore and Szepesvári 2020)

$$R_T \leq m \sum_{i=1}^K \Delta_i + T \sum_{i=1}^K \Delta_i \exp\left(-\frac{m\Delta_i^2}{4\sigma^2}\right),$$

where $\Delta_i = \mu^* - \mu_i$ is the suboptimality gap of arm i .

Consequence: for $K = 2$ arms with gap $\Delta > 0$, tuning $m \approx \log(T\Delta^2)/\Delta^2$ yields $R_T \lesssim \log(T\Delta^2)/\Delta$.

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- Issue 1: if T is unknown, the choice of m is impractical. Besides, the regret R_T usually grows **linearly with T** if m is fixed and $T \rightarrow +\infty$.
Completely stopping exploring if we do not know T is a bad idea!
- Issue 2: **Δ is usually unknown.**

Definition

Let $v \in \mathbb{R}_+$. A real random variable X is said to be **subgaussian with variance factor v** iff

$$\forall \lambda \in \mathbb{R}, \quad \mathbb{E} [e^{\lambda X}] \leq \exp \left(\frac{\lambda^2 v}{2} \right). \quad (1)$$

It can be shown that a subgaussian r.v. has finite moments at all orders, and has mean 0 and variance at most v .

Examples:

- if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $X - \mu$ satisfies (1) with equality for $v = \sigma^2$;
- if $X \in [a, b]$ is a bounded random variable, then $X - \mathbb{E}[X]$ satisfies (1) with $v = (b - a)^2/4$.

Let $v > 0$. If X is subgaussian with variance factor v , then by Markov's inequality, for all $x > 0$ and all $\lambda > 0$,

$$\mathbb{P}(X \geq x) = \mathbb{P}(e^{\lambda X} > e^{\lambda x}) \leq e^{-\lambda x} \mathbb{E}[e^{\lambda X}] \leq e^{-\lambda x + \lambda^2 v / 2}.$$

Optimizing in λ yields $\mathbb{P}(X \geq x) \leq e^{-x^2/(2v)}$ for all $x > 0$, and $\mathbb{P}(X \leq -x) \leq e^{-x^2/(2v)}$ as well. For n independent r.v., we have:

Lemma (Subgaussian deviation inequality for the empirical mean)

Let X_1, X_2, \dots be i.i.d. real random variables such that $X_1 - \mu$ is subgaussian with variance factor σ^2 . Then, the empirical mean $\hat{\mu}_n = \frac{1}{n} \sum_{k=1}^n X_k$ satisfies, for all $n \in \mathbb{N}^*$ and $x > 0$,

$$\mathbb{P}(\hat{\mu}_n \geq \mu + x) \leq e^{-nx^2/(2\sigma^2)}$$

$$\mathbb{P}(\hat{\mu}_n \leq \mu - x) \leq e^{-nx^2/(2\sigma^2)}$$

The deviation probability bounds decrease exponentially fast with n and x^2 , but increase with σ^2 .

ε -Greedy

Parameters: $\varepsilon_1, \varepsilon_2, \dots \in (0, 1]$.

At each round $t \geq 1$,

- 1 let J_t be the best arm so far (highest empirical average);
- 2 play J_t with probability $1 - \varepsilon_t$ or a random uniform arm with probability ε_t .

Theoretical guarantee: Auer et al. (2002a) proved that if the reward vectors $g_1, g_2, \dots \in [0, 1]^K$ are i.i.d. and if $\varepsilon_t \approx K/(\Delta^2 t)$, then ε -Greedy satisfies

$$R_T \lesssim \frac{K \log T}{\Delta^2},$$

where the gap Δ is the difference between the reward expectations of the best arm and the next best arm.

Now, T is not required to tune the algorithm, but Δ still is.

This algorithm follows the 'Optimism in face of uncertainty' principle.

UCB1 (Upper Confidence Bound)

UCB.avi

Initialization: play each arm once.

At each round $t \geq K + 1$,

- 1 play arm $I_t \in \operatorname{argmax}_{1 \leq i \leq K} \left\{ \hat{\mu}_{t-1}(i) + \sqrt{\frac{2 \log t}{T_i(t-1)}} \right\}$, where
 $\hat{\mu}_{t-1}(i)$ is the average reward of arm i up to round $t - 1$, and
 $T_i(t - 1)$ is the number of times arm i was played.

Theoretical guarantee: Auer et al. (2002a) proved that if the reward vectors $g_1, g_2, \dots \in [0, 1]^K$ are i.i.d., then UCB1 satisfies

$$R_T \leq \sum_{i: \Delta_i > 0} \frac{8 \log T}{\Delta_i} + \left(1 + \frac{\pi^2}{3}\right) \sum_{i=1}^K \Delta_i,$$

where $\Delta_i = \mu^* - \mu_i$ is the difference between the reward expectations of the best arm and the i -th arm. (Now, the algorithm does not use the Δ_i .)

Warning: UCB should not be used in practice!

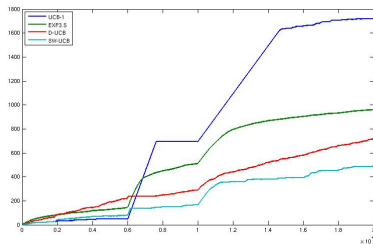
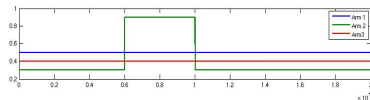
UCB uses unnecessarily large confidence intervals (based on Hoeffding's inequality), which results in a suboptimal multiplicative constant before $\log(T)$, and slow convergence in practice.

Refined algorithms with better (and sometimes optimal) performances include:

- algorithms using tighter confidence intervals (e.g., KL-UCB or KL-UCB-switch);
- Bayesian algorithms (e.g., Thompson sampling).

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- Changepoint: reward distributions change *abruptly*
- Goal: *follow the best arm*
- Application: scanning tunnelling microscope



- Variants D-UCB et SW-UCB including a progressive *discount* of the past
- Bounds $O(\sqrt{n \log n})$ are proved, which is (almost) optimal

We now consider arbitrary reward vectors $g_1, g_2, \dots \in [0, 1]^K$ (not necessarily drawn i.i.d. from a given distribution).

Exp3 algorithm

Parameters: $\eta_1, \eta_2, \dots > 0$.

At each round $t \geq 1$,

- 1 compute the weight vector $w_t = (w_t(1), \dots, w_t(K))$ given by

$$w_t(i) = \frac{\exp\left(\eta_t \sum_{s=1}^{t-1} \tilde{g}_s(i)\right)}{\sum_{j=1}^K \exp\left(\eta_t \sum_{s=1}^{t-1} \tilde{g}_s(j)\right)}, \quad 1 \leq i \leq K;$$

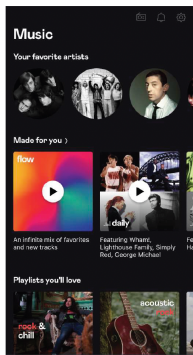
where $\tilde{g}_s(i) = 1 - \frac{1-g_s(i)}{w_s(i)} \mathbb{1}_{I_s=i}$ is an unbiased estimator of $g_s(i)$;

- 2 draw I_t at random such that $\mathbb{P}(I_t = i) = w_t(i)$.

Theoretical guarantee: Auer et al. (2002b) proved $R_T \leq 2\sqrt{TK \ln K}$ with $\eta_t = \sqrt{\ln(K)/(tK)}$, for **arbitrary** reward vectors $g_1, g_2, \dots \in [0, 1]^K$. (Worst guarantee than UCB1, but more robust.)

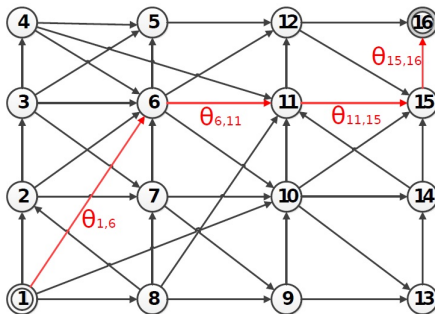
Sequentially choose an (ordered) subset of arms from a huge set.

Example: a recommender system suggests a list of products from a huge catalogue to any new customer on the website, with the goal of maximizing the total number of clicks or the total revenue.



Source: <https://www.deezer.com/>

- Sequentially choose a path in a graph (with costs on edges).

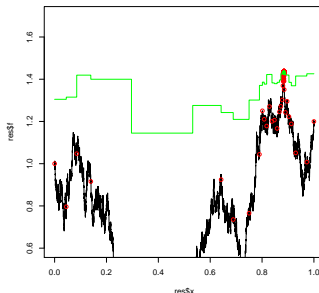


Source: path routing example of Combes and Proutière in

https://www.sigmetrics.org/sigmetrics2015/tutorial_sigmetrics.pdf

- Sequentially choose a perfect matching in a complete bipartite graph (assignment problem).

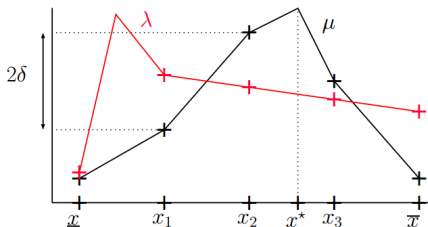
- Goal: sequentially play almost as good as the maximum of a function $f : \mathcal{C} \subset \mathbb{R}^d \rightarrow \mathbb{R}$ that we observe (possibly) with noise.



- Various possible models : f has a certain regularity (e.g., Lipschitz or gradient-Lipschitz), f is the realization of a Gaussian Process, etc.
- Several algorithms: zooming algorithm, HOO, GP-UCB, etc (and other algorithms for the simple regret).

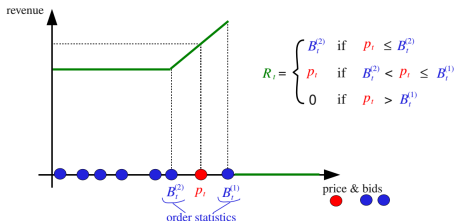
Unimodal bandits without smoothness: trisection algorithms, and better (Combes and Proutiere, 2014).

Application to internet network traffic optimization.



Reserve Price Optimization in Second-price Auctions (Cesa-Bianchi et al., 2015).

Application to ad placement.



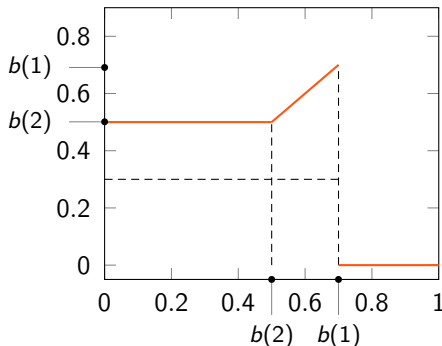
Ad auction:

- Online advertising: consider a publisher (seller) who want to sell an ad space to advertisers (buyers) through second-price auctions managed by an ad exchange.
- For each impression (ad display) created on the publisher's website, the ad exchange runs an auction on the fly.

Second-price auction:

- Simultaneously, all buyers propose a price (bid) to the ad exchange.
- The buyer with the highest bid wins the auction but pays the second highest price.
- This is a truthful mechanism.

- The seller has an additional degree of freedom: the **reserve price**, which corresponds to the minimal revenue they are willing to get.
- Before the auction, the seller communicates a reserve price y to the ad exchange (the reserve price is unknown to the buyers).
- If the reserve price y is larger than the highest bid $b(1)$, the auction is lost. Otherwise, the buyer with the highest bid wins the auction.
- The winner pays the maximum of the second-highest bid $b(2)$ and the reserve price y . The seller's revenue is $g(y) = \max\{b(2), y\} \mathbb{1}_{b(1) \geq y}$.



Assume now that the publisher participates to a series of auctions. The task of sequentially optimizing the reserve price can be phrased as a continuum-armed bandit problem: at each round $t \geq 1$,

- the seller sets a reserve price $\hat{y}_t \in [0, 1]$;
- simultaneously, a set of buyers propose bids $b_t(1) \geq b_t(2) \geq \dots \in [0, 1]$ (sorted in decreasing order);
- the seller receives and observes the revenue $g_t(\hat{y}_t) = \max\{b_t(2), \hat{y}_t\} \mathbb{1}_{b_t(1) \geq \hat{y}_t}$.

Cesa-Bianchi et al. (2015) proposed an algorithm for the case when the bids are i.i.d. accross the buyers and time. They proved a $\tilde{O}(\sqrt{T})$ upper bound on the (pseudo) regret

$$R_T := \sup_{y \in [0,1]} \mathbb{E} \left[\sum_{t=1}^T g_t(y) \right] - \mathbb{E} \left[\sum_{t=1}^T g_t(\hat{y}_t) \right] .$$

Before choosing the arm $I_t \in \{1, \dots, K\}$ or (more generally) the action $\hat{y}_t \in \mathcal{Y}$, the learner has access to a context $x_t \in \mathcal{X}$.

Example: in ad auctions, the context may contain different properties of the customer or of the ad space.

General setting = contextual bandits: at each round $t \in \mathbb{N}^*$,

- 1 The environment reveals a context $x_t \in \mathcal{X}$.
- 2 The learner chooses an action $\hat{y}_t \in \mathcal{Y}$, possibly at random.
- 3 The learner receives and observes a reward $g_t(\hat{y}_t)$.

The goal is now to minimize the pseudo regret w.r.t. a (nonparametric) set of functions $\mathcal{F} \subset \mathcal{Y}^{\mathcal{X}}$ (e.g., Cesa-Bianchi et al. 2017):

$$R_T := \sup_{f \in \mathcal{F}} \mathbb{E} \left[\sum_{t=1}^T g_t(f(x_t)) \right] - \mathbb{E} \left[\sum_{t=1}^T g_t(\hat{y}_t) \right] .$$

Also sometimes called **pure exploration**.

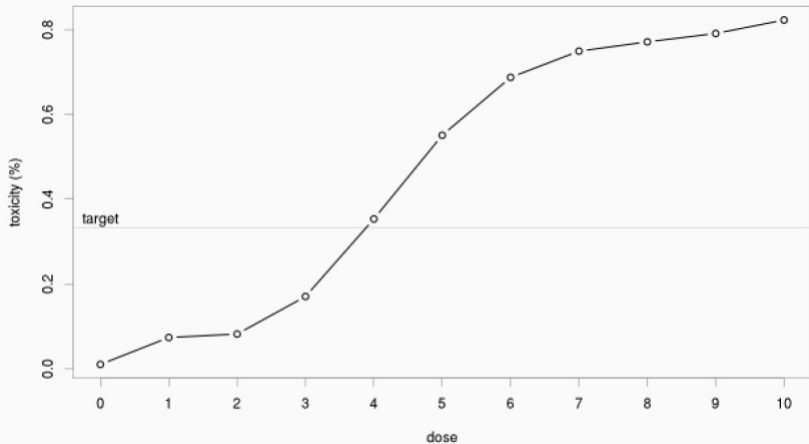
- Previous goal: maximize the cumulative reward.
- Now: identify arm with maximal expectation: $i^* \in \operatorname{argmax}_{1 \leq i \leq K} \mu_i$.
For example, given δ , minimize the expected number of trials $\mathbb{E}[\tau_\delta]$ while ensuring the final recommendation \hat{i} is most probably correct:

$$\mathbb{P}(\hat{i} \neq i^*) \leq \delta.$$

Applications:

- clinical trials
- A/B testing (for, e.g., website design)
- continuous action space: zero-order stochastic optimization

See, e.g., Garivier and Kaufmann (2016).



And much more!

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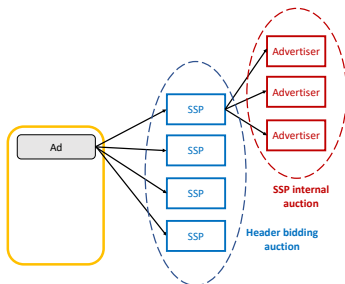
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An example of a continuum-armed bandit problem in ad auction optimization.
Joint work: Jauvion et al. (2018).



- A supply-side platform (SSP) runs repeated second-price auctions where several advertisers compete to display ads on a publisher's ad space.
- After running any second-price auction, the SSP wants to minimize the price they offer to the publisher (the SSP gains the difference).
- However, the publisher might not accept this price since they are asking to several SSPs simultaneously for a given price (first-price auction).

More details on https://alephd.github.io/assets/header_bidding/slides/.

Take-home message: bandits = exploration-exploitation tradeoff.

Bandit problems are sequential decision models where the learner must simultaneously:

- exploit their current knowledge;
- explore unknown actions to gain knowledge for the future.

Not thinking about the future can be terribly bad!

There are multiple variants of the simple K -armed bandit problem that have been designed for numerous applications.

There are also pure-exploration bandit problems.

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Bandits

- Bandit models are simple models that stress the importance to combine exploitation with exploration.
- Yet, making an action does not change the state of the environment.

Reinforcement Learning

- RL studies "learning from interaction to achieve a goal".
- Markov Decision Processes are more general models that include a **state** that can evolve over time, based on the actions of the learner.
- Example: inverted pendulum <https://www.youtube.com/watch?v=Lt-KLtkDlh8>
- See the book by Sutton and Barto (2018), and Erwan Le Pennec's reading notes:
<http://www.cmap.polytechnique.fr/~lepennec/files/RL/Sutton.pdf>

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