Exercise 3

Gormery K. Wanjiru

February 2, 2024

Problem 1

a) A signal $x_1(t) = \cos(2\pi f_0 t)$ where $f_0 = 125$ Hz, is sampled with spacing $T = 0.5$ ms between the samples, and the first sample is taken at $t = 0$. Suppose we calculate a 128-point DFT of the first 128 samples of $x_1(t)$. For what values of k, will $X(k)$ be different from 0?

The formula for the Discrete Fourier Transform (DFT) is given by:

$$X(k) = \sum_{n=0}^{127} \cos(2\pi f_0 nT) \cdot e^{-j\frac{2\pi}{128}kn}$$

Values of k for which $X(k)$ is different from 0 are those that satisfy $f_0 nT = \frac{k}{128}$.

Given parameters: $f_0 = 125$ Hz, $T = 0.5$ ms $= 0.0005$ s.

Substitute into the equation: $f_0 nT = \frac{k}{128}$.

$$125 \times n \times 0.0005 = \frac{k}{128}$$

Solve for k:

$$n = \frac{k}{(125 \times 0.0005) \times 128}$$

$$n = \frac{k}{0.016}$$

$$k = 0.016n$$

Since k must be an integer, let n take integer values. Therefore, to get this numbers this student used the following simple python code:

```python
# Given parameters
f0 = 125  # Hz
T = 0.0005  # s
N = 128

# Solve for k
k_values = [n * N // (f0 * T * 128) for n in range(N)]

print(k_values)
```

Listing 1: print the array of k values

These values represent the indices in the DFT where the corresponding $X(k)$ will be different from 0. k can take on integer values $0, 16, 32, \ldots, 2016$

b) What is the frequency resolution, i.e., the frequency distance between the calculated values in a 128-point DFT? Suppose the sequence is sampled with 2000 Hz.

The frequency resolution in a 128-point DFT is given by:

$$\text{Frequency Resolution} = \frac{\text{Sampling Frequency}}{\text{Number of Points in DFT}}$$

Suppose the sequence is sampled with 2000 Hz.

Calculate the frequency resolution:

$$\text{Frequency Resolution} = \frac{2000}{128}\ \text{Hz} = 15,625\ \text{Hz}$$

c) A signal is sampled with a 2000 Hz sampling frequency. How long a section of the signal (measured in time) must be analyzed to determine the frequency content with better than 1 Hz frequency resolution?

To determine the frequency content with better than 1 Hz resolution when sampled with a 2000 Hz frequency, the time duration T_{analyze} is given by:

$$T_{\text{analyze}} = \frac{1}{\text{Frequency Resolution}}$$

Calculate T_{analyze}:

$$T_{\text{analyze}} = \frac{1}{15,625} \approx 0.000064\ \text{s}$$

Problem 2

a) Find with an 8-sample DFT of the unit impulse, $x(n) = 1, 0, 0, 0, 0, 0, 0, 0, \ldots$.

$$X(k) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi}{8}kn}$$

Calculate the 8-sample DFT:

$$X(k) = 1 + 0 + 0 + \ldots = 1$$

b) Explain the result.

The result shows that the 8-sample DFT of the unit impulse sequence has a non-zero value only at $k = 0$. This is expected as the unit impulse has all its energy concentrated at $n = 0$, and the DFT is essentially calculating the sum of the sequence.

c) What result do you expect if you take a 128-sample DFT of the unit impulse sequence?

$$\text{Frequency Resolution} = \frac{\text{Sampling Frequency}}{128}$$

For a 128-sample DFT of the unit impulse sequence, the frequency resolution will be $\frac{2000}{128}$ Hz.

Problem 3

a) $X_1(k) = \textbf{DFT}\{x_1(n)\}$. Find $X_1(k)$ for the sequence $x_1(n) = \{2, 2, 0, 0, 2, 2, 0, 0\}$.

Calculate $X_1(k)$ for the given sequence:

$$X_1(k) = \sum_{n=0}^{7} x_1(n) \cdot e^{-j\frac{2\pi}{8}kn}$$

$$X_1(k) = 2 + 2 + 0 + 0 + 2 + 2 + 0 + 0 = 8$$

b) The sequence $x_1(n)$ is sampled with a sampling frequency of 1000 Hz. What frequency components in the range $-\frac{\text{fs}}{2}$ and $+\frac{\text{fs}}{2}$ are present in $x(n)$?

$$f = \frac{k \cdot \text{fs}}{N}$$

For the given sequence and sampling frequency:

$$f = \frac{k \cdot 1000}{8}$$

$$f = \frac{k \cdot 125}{1}$$

The frequency components present are at multiples of 125 Hz: -500 Hz, -375 Hz, -250 Hz, -125 Hz, 0 Hz, 125 Hz, 250 Hz, 375 Hz.

c) Suppose $x_1(n)$ is a sampled version of $x_1(t)$ and $x_1(t)$ is low pass limited to $\frac{\mathbf{fs}}{2}$. Write the expression for $x(t)$ from what you found in a) and b) above.

Since $x_1(t)$ is low pass limited to $\frac{\text{fs}}{2}$, the frequency components present in $x(t)$ are the same as in $x(n)$. Therefore, $x(t)$ can be expressed as a sum of sinusoids with frequencies at multiples of 125 Hz.

The expression for $x(t)$ is given by:

$$x(t) = \sum_{k=0}^{7} X_1(k) \cdot e^{j\frac{2\pi}{8}kt}$$

Substitute the calculated values of $X_1(k)$:

$$x(t) = 8 \cdot e^{j\frac{2\pi}{8}\cdot 0\cdot t} + 8 \cdot e^{j\frac{2\pi}{8}\cdot 1\cdot t} + \ldots + 8 \cdot e^{j\frac{2\pi}{8}\cdot 7\cdot t}$$

Simplify the expression based on the frequencies calculated in part b):

$$x(t) = 8 + 8 \cdot e^{j\frac{2\pi}{8}\cdot 1\cdot t} + \ldots + 8 \cdot e^{j\frac{2\pi}{8}\cdot 7\cdot t}$$

This expression represents the sampled signal $x_1(n)$ in the continuous-time domain.