Exercise 7

Gormery K. Wanjiru

Problem 1

Given the impulse response of the filter $h(n) = 5, 4, 3, 2, 1$, and the input sequence $x(n) = 1, 2, 2, 0, 0, 0, 0, \ldots$

a) Find $H(z)$ for the filter.

The Z-transform of the impulse response is given by:

$$\begin{aligned} H(z) &= 5z^0 + 4z^{-1} + 3z^{-2} + 2z^{-3} + 1z^{-4} \\ &= 5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \end{aligned}$$

b) Find $X(z)$ for the input sequence $x(n)$.

The Z-transform of the input sequence is given by:

$$\begin{aligned} X(z) &= 1z^0 + 2z^{-1} + 2z^{-2} \\ &= 1 + 2z^{-1} + 2z^{-2} \end{aligned}$$

c) Find $Y(z)$ for the output sequence.

The output sequence in the Z-domain is obtained by multiplying $H(z)$ with $X(z)$:

$$\begin{aligned} Y(z) &= H(z) \cdot X(z) \\ &= (5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}) \cdot (1 + 2z^{-1} + 2z^{-2}) \\ &= 5 + 14z^{-1} + 20z^{-2} + 16z^{-3} + 9z^{-4} + 2z^{-5} + z^{-6} \end{aligned}$$

d) Find $y(n)$ from $Y(z)$.

By applying the inverse Z-transform, we find the output sequence $y(n)$. Given that $Y(z) = 5+14z^{-1}+20z^{-2}+16z^{-3}+9z^{-4}+2z^{-5}+z^{-6}$, the coefficients directly give us the values of $y(n)$ at each time step starting from $n = 0$. So, the sequence $y(n)$ can be written as:

$$y(n) = \begin{cases} 5 & \text{for } n = 0 \\ 14 & \text{for } n = 1 \\ 20 & \text{for } n = 2 \\ 16 & \text{for } n = 3 \\ 9 & \text{for } n = 4 \\ 2 & \text{for } n = 5 \\ 1 & \text{for } n = 6 \\ 0 & \text{for } n \geq 7 \end{cases}$$

$$y(n) = 5, 14, 20, 16, 9, 2, 1, 0, 0, \ldots$$

Problem 2

Given the difference equation $y(k) + 0.5y(k-1) = x(k) - x(k-1)$.

a) Find the pole and zero positions for the filter.

Using the Z-transform:

$$\begin{aligned} Y(z) + 0.5z^{-1}Y(z) &= X(z) - z^{-1}X(z) \\ \left(1 + 0.5z^{-1}\right) Y(z) &= (1 - z^{-1})X(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + 0.5z^{-1}} \end{aligned}$$

Zero at $z = 1$, pole at $z = -0.5$.

c) Find an expression $H(\omega T)$ for the filter.

Simply substituting $z = e^{j\omega T}$ into $H(z)$:

$$H(\omega T) = \frac{1 - e^{-j\omega T}}{1 + 0.5e^{-j\omega T}}$$

d) Sketch the magnitude frequency response based on the pole and zero positions.

For a filter with the transfer function $H(z) = \frac{1-z^{-1}}{1+0.5z^{-1}}$, converting to the frequency domain involves substituting $z = e^{j\omega T}$, yielding:

$$H(\omega T) = \frac{1 - e^{-j\omega T}}{1 + 0.5e^{-j\omega T}}$$

The magnitude frequency response, $|H(\omega T)|$, can be sketched by evaluating the magnitude of this expression over a range of frequencies. Given these points, the sketch would show a magnitude response starting at 0, peaking, and then gradually decreasing.

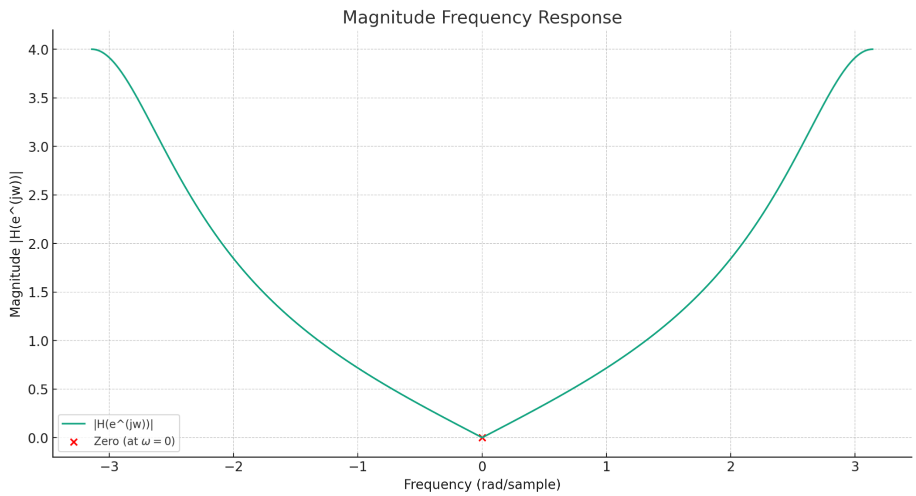


Figure 1: a og b

e) Calculate the magnitude response at the frequency where $|H(\omega T)|$ has its maximum.

To determine the frequency at which the magnitude response $|H(\omega T)|$ of our filter is maximized, we need to find the maximum of the function. Since the filter has a pole at $z = -0.5$ and a zero at $z = 1$, we analyze the magnitude response over the frequency range to locate this maximum.

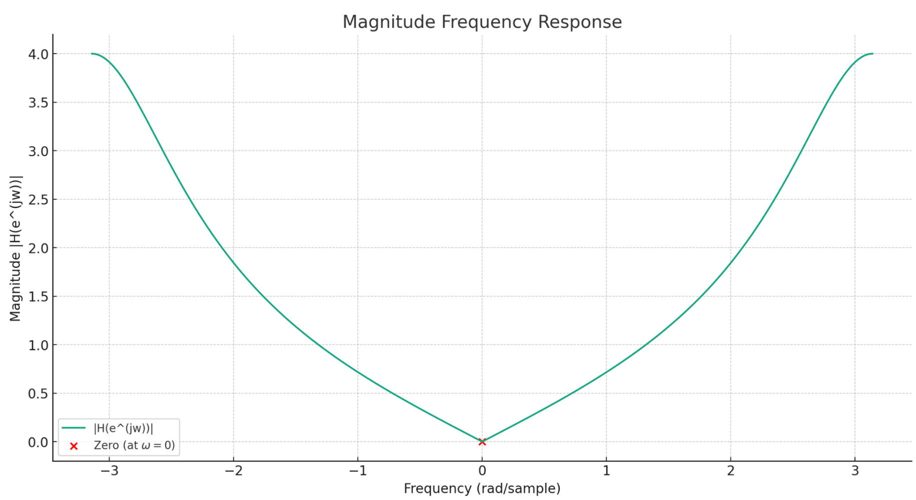


Figure 2: cause its the same plot as before

As the plot suggests, the magnitude response has a symmetrical shape with respect to the y-axis, due to the filter's real coefficients. The maximum magnitude occurs at both ends of the frequency spectrum, which are $\omega = \pm\pi$, (Nyquist frequency).

This result is expected due to the pole being close to the unit circle. This is typical for filters that are close to instability and have a sharp resonance peak.

Problem 3

Given the transfer function $H(z) = \frac{0.1\cdot(z^2+2z+1)}{z^2-1.6z+0.89}$.

a) Find the difference equation of the filter.

To find the difference equation, we multiply both sides of the transfer function by the denominator to get:

$$H(z)\cdot(z^2 - 1.6z + 0.89) = 0.1\cdot(z^2 + 2z + 1)$$
$$Y(z)\cdot(z^2 - 1.6z + 0.89) = X(z)\cdot 0.1\cdot(z^2 + 2z + 1)$$

Taking the inverse Z-transform, we get the difference equation in terms of $y[n]$ and $x[n]$:

$$y[n] - 1.6y[n-1] + 0.89y[n-2] = 0.1x[n] + 0.2x[n-1] + 0.1x[n-2]$$

This is the difference equation that describes the filter in the time domain.

b) Draw the pole-zero diagram.

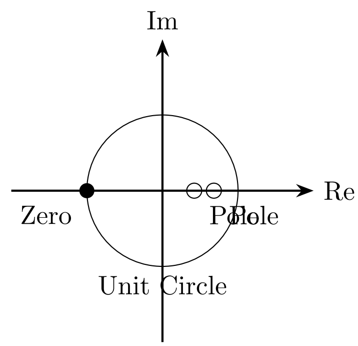
The zeros of the transfer function are given by the roots of the numerator $z^2 + 2z + 1 = 0$. This is a square and can be factored as $(z+1)^2 = 0$, so both zeros are at $z = -1$.

The poles are the roots of the denominator $z^2 - 1.6z + 0.89 = 0$, which we can find using the quadratic formula:

$$\begin{aligned} z &= \frac{1.6 \pm \sqrt{1.6^2 - 4\cdot 0.89}}{2} \\ &= \frac{1.6 \pm \sqrt{0.576}}{2} \\ &= \frac{1.6 \pm 0.76}{2} \end{aligned}$$

This gives us two poles, one at $z = \frac{1.6+0.76}{2}$ and the other at $z = \frac{1.6-0.76}{2}$.

The pole-zero diagram:



The unit circle is drawn to indicate the stability region in the z-plane, poles are indicated with $\circ$, and zeros with $\bullet$. Poles inside the unit circle indicate that the filter is stable.