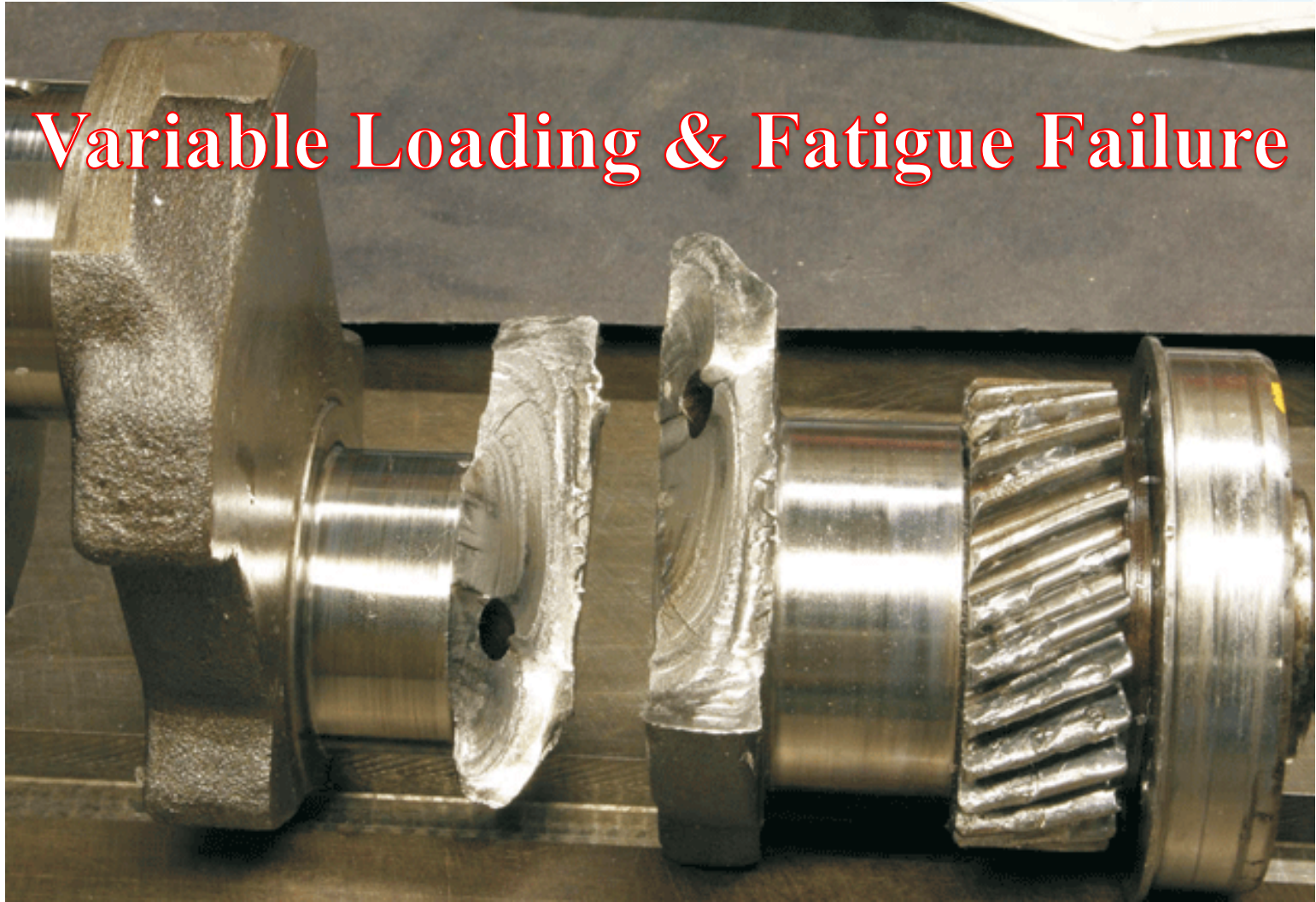


Introduction to **Mechanical Engineering Design**

Variable Loading & Fatigue Failure



Graphical Representation of Fatigue Data

For better visualization and analysis, among other, fatigue test data is graphically represented used the following graphs:

- ❑ **Wöhler curve (S-N diagram)**
gives the relation between amplitude stress σ_a and number of cycles to fracture N
- ❑ **Smith-diagram (Modified Goodman diagram)**
shows the max. $(\sigma_m + \sigma_a)$ and min. $(\sigma_m - \sigma_a)$ values of the fatigue strength as a function of the average (midrange) stress σ_m
- ❑ **Haigh-diagram**
shows the amplitude fatigue stress σ_a as a function of the midrange stress σ_m

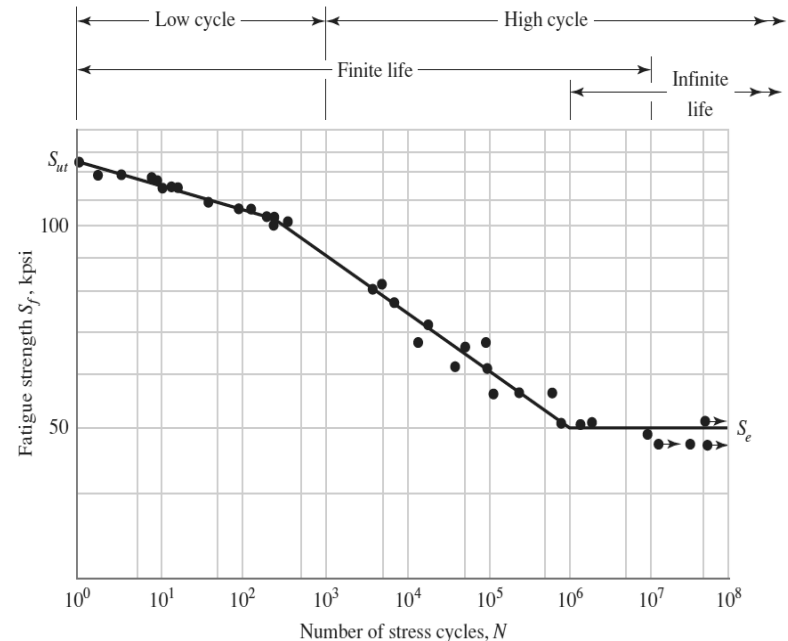
Graphical Representation of Fatigue Data

Relation between σ_a & N

- ❑ Determined through several tests (25-30 samples)
- ❑ Plotted on semilog or log-log scale, called **Wöhler diagram or S-N diagram**

Wöhler diagram or S-N diagram is

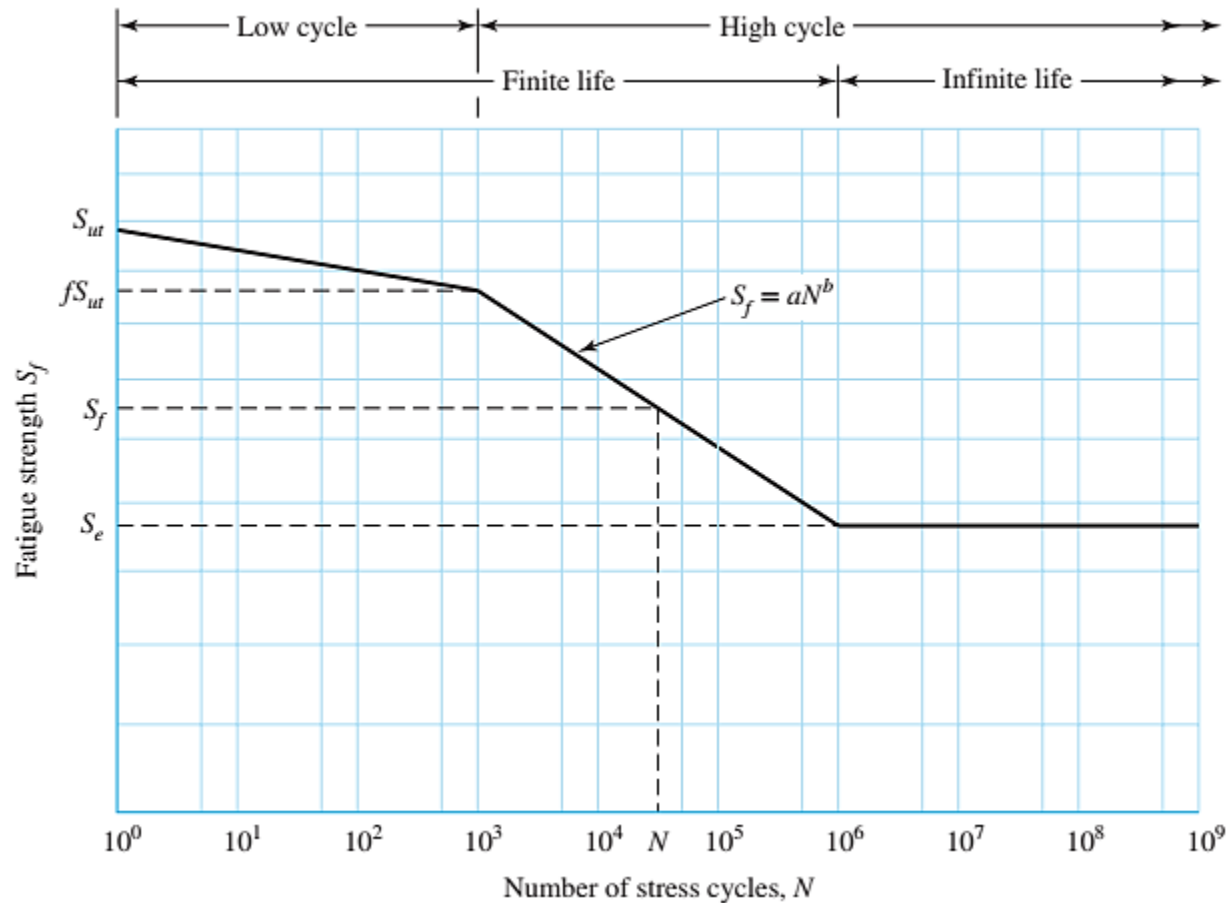
- ❑ Standard method of representing fatigue strength
- ❑ Simple and informative



Test results depend on

- Material, loading art and amplitude stress
- Material classification
 - Those with clearly defined fatigue limit
 - Those with no clear fatigue limit

The Idealized S-N Diagram for Steels



The Idealized S-N Diagram for Steels

$$S'_e = \begin{cases} 0.5S_{ut} & S_{ut} \leq 200 \text{ kpsi (1400 MPa)} \\ 100 \text{ kpsi} & S_{ut} > 200 \text{ kpsi} \\ 700 \text{ MPa} & S_{ut} > 1400 \text{ MPa} \end{cases}$$

$$f = 1.06 - 2.8(10^{-3})S_{ut} + 6.9(10^{-6})S_{ut}^2 \quad 70 < S_{ut} < 200 \text{ kpsi}$$

$$f = 1.06 - 4.1(10^{-4})S_{ut} + 1.5(10^{-7})S_{ut}^2 \quad 500 < S_{ut} < 1400 \text{ MPa}$$

$$S_f = aN^b$$

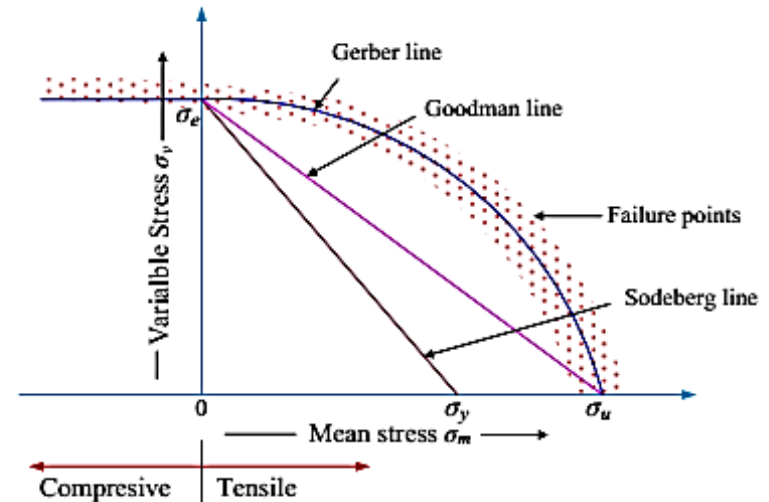
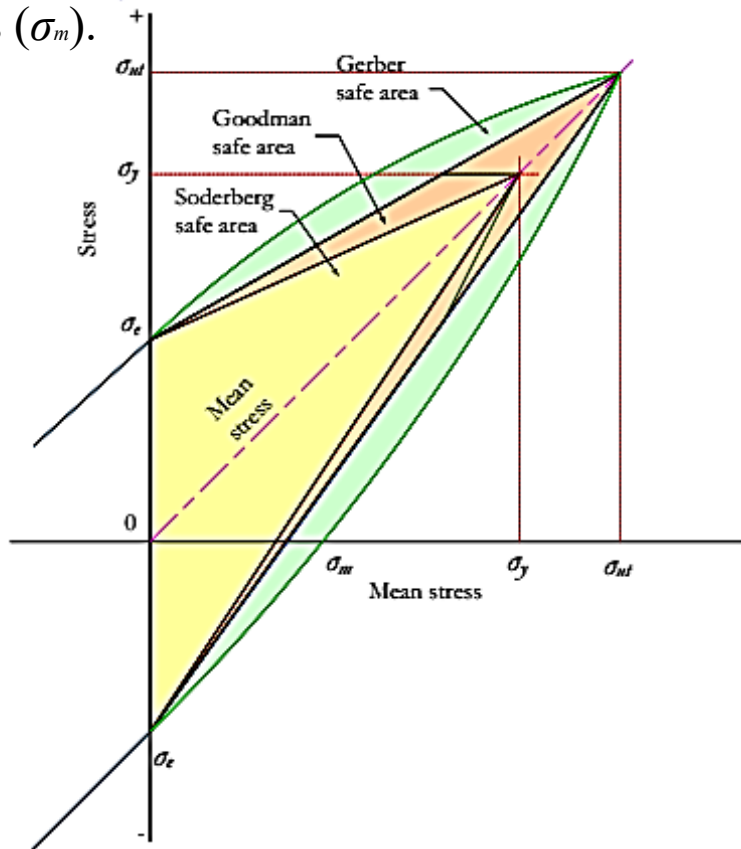
$$a = \frac{(fS_{ut})^2}{S_e}$$

$$b = -\frac{1}{3} \log \left(\frac{fS_{ut}}{S_e} \right)$$

$$N = \left(\frac{\sigma_{ar}}{a} \right)^{1/b}$$

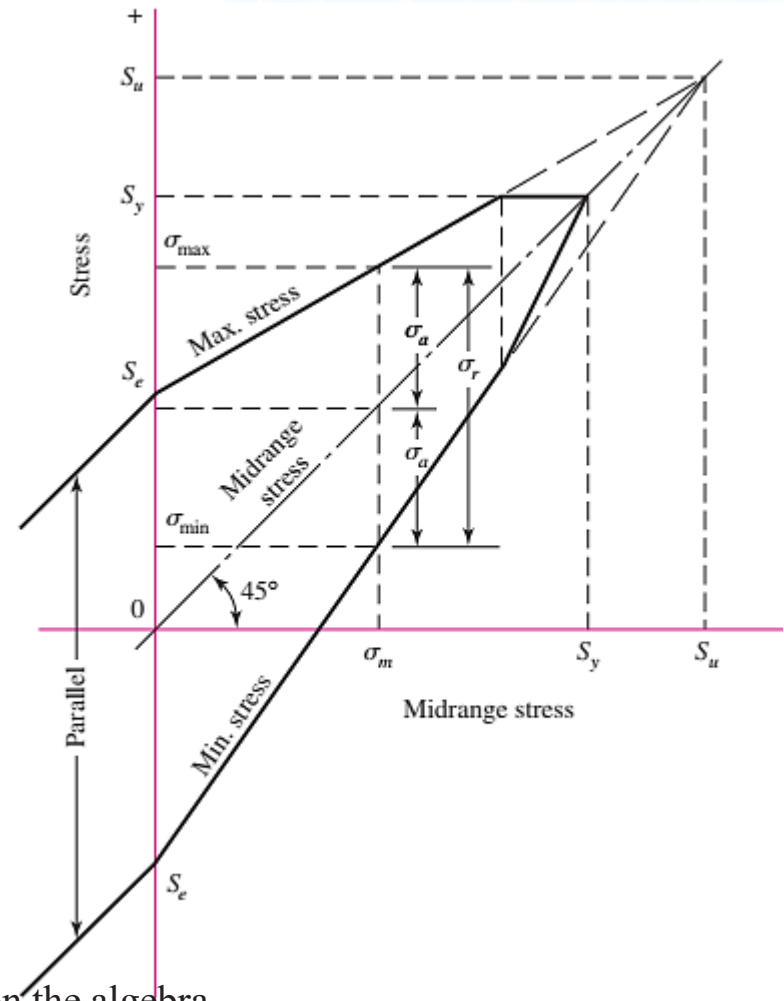
Combined Steady and Variable Stress

The failure points from fatigue tests made with different steels and combinations of mean and variable stresses are plotted in the figure as functions of variable stress (σ_v) and mean stress (σ_m).



Modified Goodman Diagram

- Modified Goodman diagram showing all the strengths and the limiting values of all the stress components for a particular mean stress.
 - The midrange stress line is a 45° line from the origin to the tensile strength of the part.
 - The yield strength is also plotted on both axes, because yielding would be the criterion of failure if σ_{\max} exceeded S_y !
-
- It is a straight line and the algebra is linear and easy.
 - It is easily graphed, every time for every problem.
 - It reveals subtleties of insight into fatigue problems.
 - Answers can be scaled from the diagrams as a check on the algebra.



M. Goodman Method for Combination of Stresses

A straight line connecting the endurance limit (σ_e) and the ultimate strength (σ_u) as shown by line AB , follows the suggestion of Goodman. A Goodman line is used when the design is based on ultimate strength and may be used for ductile or brittle materials.

Line AB connecting σ_e and σ_u is called Goodman failure stress line. If a suitable factor of safety (F.S.) is applied to endurance limit and ultimate strength, a safe line CD may be drawn parallel to the line AB . Now let us consider a design point P on the line CD

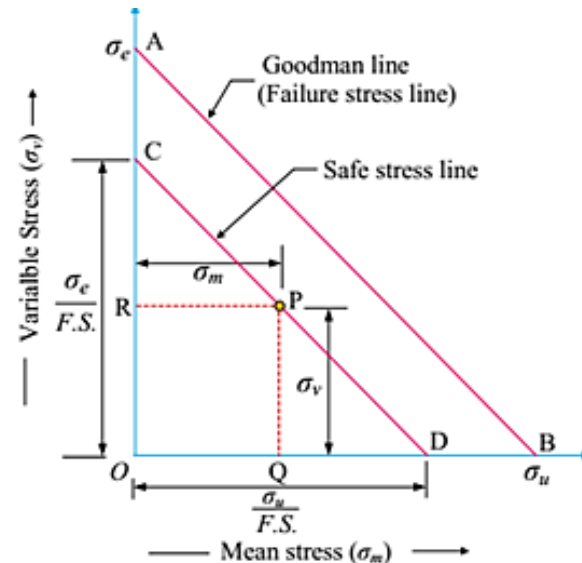
Now from similar triangle COD and PQD ,

$$\frac{PQ}{CO} = \frac{QD}{OD} = \frac{OD - OQ}{OD} = 1 - \frac{OQ}{OD}$$

$$\therefore \frac{* \sigma_v}{\sigma_e / F.S.} = 1 - \frac{\sigma_m}{\sigma_u / F.S.}$$

or

$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v}{\sigma_e}$$



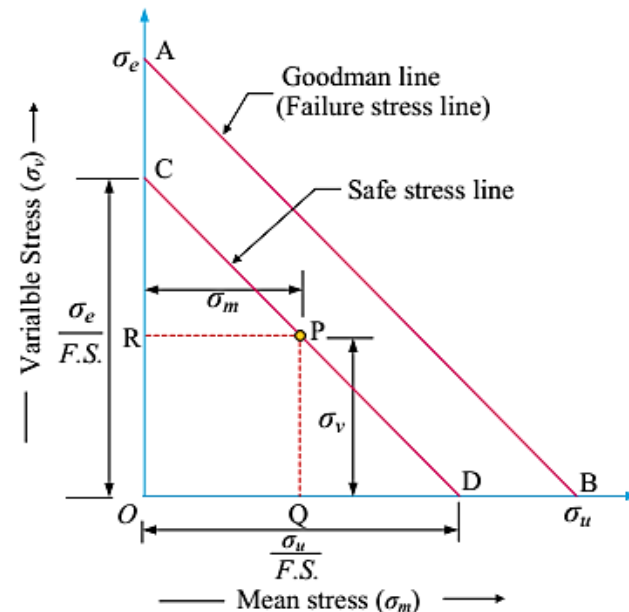
M. Goodman Method for Combination of Stresses

Since many machine and structural parts that are subjected to fatigue loads contain regions of high stress concentration, therefore, the equation must be altered to include this effect. In such cases, the fatigue stress concentration factor (k_f) is used to multiply variable stress (σ_v). The equation may now be written as:

$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_e}$$

Where

- $F.S.$ = Factor of safety
- σ_m = Mean stress
- σ_u = Ultimate stress
- σ_v = Variable stress
- σ_e = Endurance limit for reversed loading.
- k_f = Fatigue stress concentration factor



M. Goodman Method for Combination of Stresses

Now, consider the load factor, surface finish factor and size factor.

The equation
$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_e}$$

may be written as:
$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_{eb} \times K_{sur} \times K_{sz}} = \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_e \times K_b \times K_{sur} \times K_{sz}}$$

where K_b = Load factor for reversed bending load.

K_{sur} = Surface finishing factor.

and K_{sz} = Size factor.

When a machine component is subjected to reversed axial loading:

$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_{ea} \times K_{sur} \times K_{sz}}$$

...(For ductile materials)

$$\frac{1}{F.S.} = \frac{\sigma_m \times K_t}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_{ea} \times K_{sur} \times K_{sz}}$$

...(For brittle materials)

When a machine component is subjected to reversed shear loading:

$$\frac{1}{F.S.} = \frac{\tau_m}{\tau_u} + \frac{\tau_v \times K_{fs}}{\tau_e \times K_{sur} \times K_{sz}}$$

Where $\tau_u = 0.8 \sigma_u$; and $\tau_e = 0.8 \sigma_e$

Soderberg Method

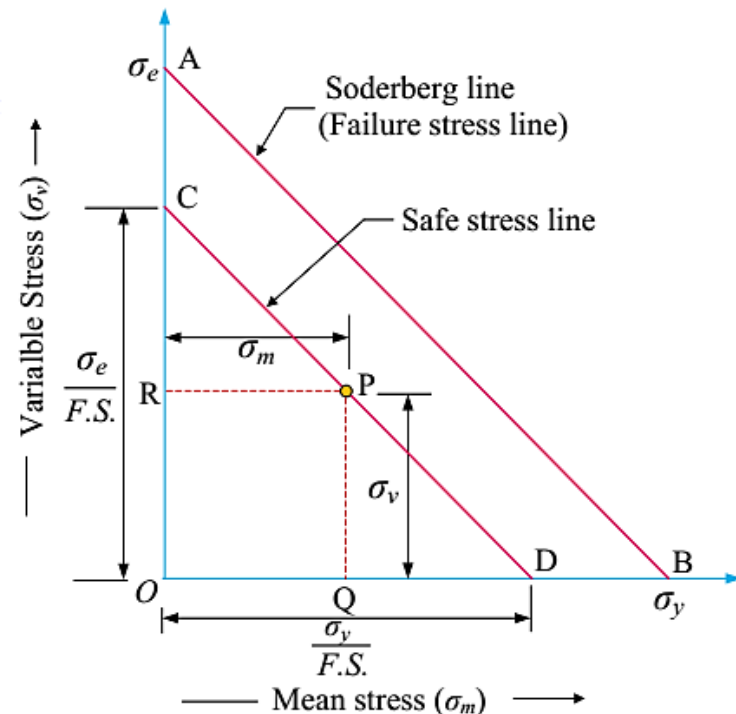
A straight line connecting the endurance limit (σ_e) and the yield strength (σ_y), (the line AB in the figure), follows the suggestion of Soderberg line. This line is used when the design is based on *yield strength* (σ_y).

The equation looks like the Goodman equation except that we use (σ_y) and may be written as:

$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_y} + \frac{\sigma_v \times K_f}{\sigma_e}$$

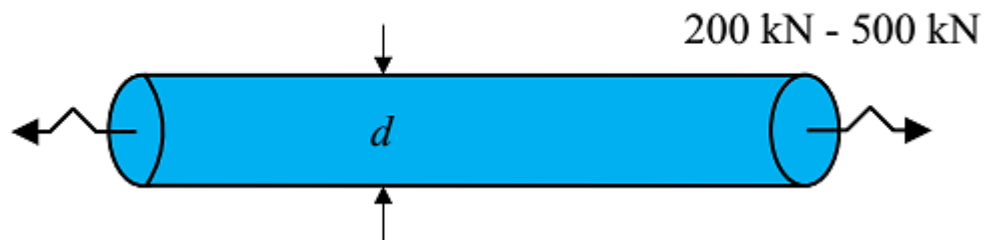
Consider the load factor, surface finish factor and size factor. The equation

$$\frac{1}{F.S.} = \frac{\sigma_m}{\sigma_y} + \frac{\sigma_v \times K_f}{\sigma_{eb} \times K_{sur} \times K_{sz}}$$



Example 1

A bar of circular cross-section is subjected to alternating tensile forces varying from a minimum of 200 kN to a maximum of 500 kN. It is to be manufactured of a material with an ultimate tensile strength of 900 MPa and an endurance limit of 700 MPa. Determine the diameter of bar using safety factors of 3.5 related to ultimate tensile strength and 4 related to endurance limit and a stress concentration factor of 1.65 for fatigue load. Use Goodman straight line as basis for design.



Solution

Solution. Given : $W_{min} = 200 \text{ kN}$; $W_{max} = 500 \text{ kN}$; $\sigma_u = 900 \text{ MPa} = 900 \text{ N/mm}^2$; $\sigma_e = 700 \text{ MPa} = 700 \text{ N/mm}^2$; $(F.S.)_u = 3.5$; $(F.S.)_e = 4$; $K_f = 1.65$

Let d = Diameter of bar in mm.

$$\therefore \text{Area, } A = \frac{\pi}{4} \times d^2 = 0.7854 d^2 \text{ mm}^2$$

We know that mean or average force,

$$W_m = \frac{W_{max} + W_{min}}{2} = \frac{500 + 200}{2} = 350 \text{ kN} = 350 \times 10^3 \text{ N}$$

$$\therefore \text{Mean stress, } \sigma_m = \frac{W_m}{A} = \frac{350 \times 10^3}{0.7854 d^2} = \frac{446 \times 10^3}{d^2} \text{ N/mm}^2$$

$$\text{Variable force, } W_v = \frac{W_{max} - W_{min}}{2} = \frac{500 - 200}{2} = 150 \text{ kN} = 150 \times 10^3 \text{ N}$$

$$\therefore \text{Variable stress, } \sigma_v = \frac{W_v}{A} = \frac{150 \times 10^3}{0.7854 d^2} = \frac{191 \times 10^3}{d^2} \text{ N/mm}^2$$

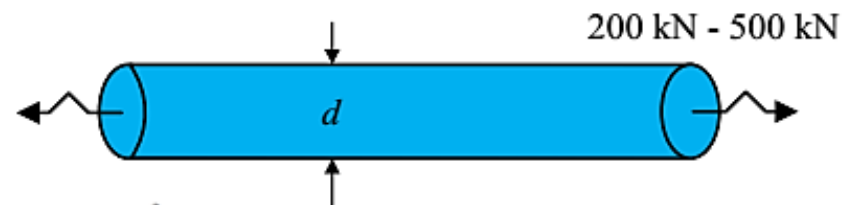
We know that according to Goodman's formula,

$$\frac{\sigma_v}{\sigma_e / (F.S.)_e} = 1 - \frac{\sigma_m \cdot K_f}{\sigma_u / (F.S.)_u}$$

$$\frac{\frac{191 \times 10^3}{d^2}}{700/4} = 1 - \frac{\frac{446 \times 10^3}{d^2} \times 1.65}{900/3.5}$$

$$\frac{1100}{d^2} = 1 - \frac{2860}{d^2} \quad \text{or} \quad \frac{1100 + 2860}{d^2} = 1$$

$$\therefore d^2 = 3960 \quad \text{or} \quad d = 62.9 \text{ say } 63 \text{ mm} \quad \text{Ans.}$$



Example 2

A cantilever beam made of cold drawn carbon steel of circular cross-section as shown in Figure below is subjected to a load which varies from $-F$ to $3F$. Determine the maximum load that this member can withstand for an indefinite life using a factor of safety as 2. The theoretical stress concentration factor is 1.42 and the notch sensitivity is 0.9. Assume the following values :

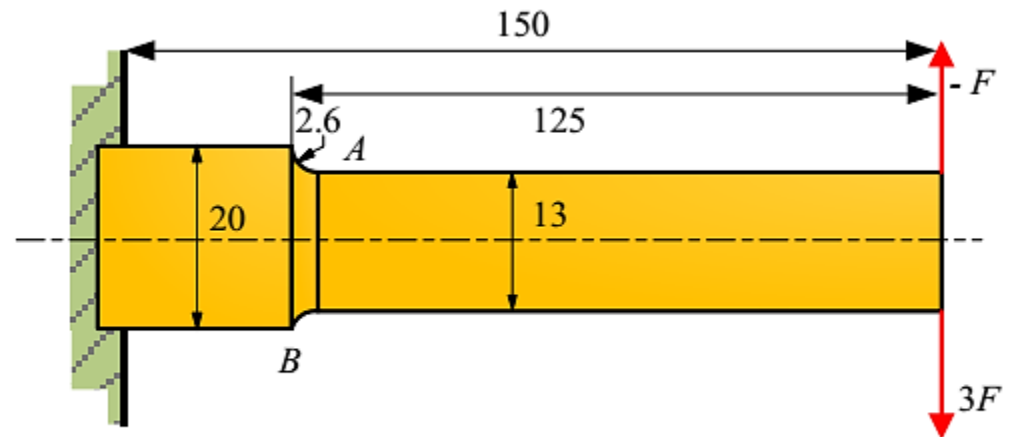
Ultimate stress = 550 MPa

Yield stress = 470 MPa

Endurance limit = 275 MPa

Size factor = 0.85

Surface finish factor = 0.89



Solution

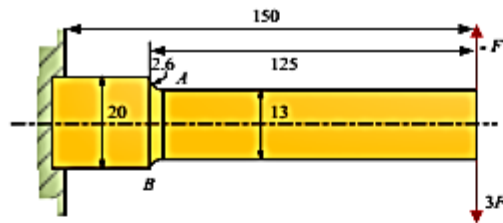
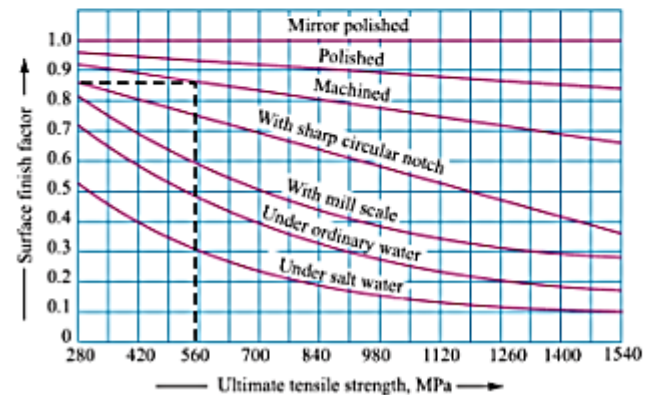
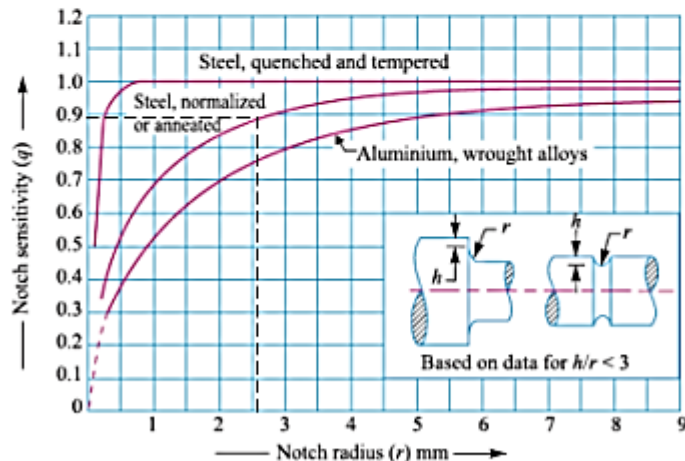


Table 6.4. Theoretical stress concentration factor (K_t) for a stepped shaft with a shoulder fillet (of radius r) in bending.

$\frac{D}{d}$	Theoretical stress concentration factor (K_t)									
	r/d									
	0.02	0.04	0.08	0.10	0.12	0.16	0.20	0.24	0.28	0.30
1.01	1.85	1.61	1.42	1.36	1.32	1.24	1.20	1.17	1.15	1.14
1.02	1.97	1.72	1.50	1.44	1.40	1.32	1.27	1.23	1.21	1.20
1.05	2.20	1.88	1.60	1.53	1.48	1.40	1.34	1.30	1.27	1.25
1.10	2.36	1.99	1.66	1.58	1.53	1.44	1.38	1.33	1.28	1.27
1.20	2.52	2.10	1.72	1.62	1.56	1.46	1.39	1.34	1.29	1.28
1.50	2.75	2.20	1.78	1.68	1.60	1.50	1.42	1.36	1.31	1.29
2.00	2.86	2.32	1.87	1.74	1.64	1.53	1.43	1.37	1.32	1.30
3.00	3.00	2.45	1.95	1.80	1.69	1.56	1.46	1.38	1.34	1.32
6.00	3.04	2.58	2.04	1.87	1.76	1.60	1.49	1.41	1.35	1.33



Solution (Continued)

We know that maximum bending moment at point A,

$$M_{max} = W_{max} \times 125 = 3F \times 125 = 375 F \text{ N-mm}$$

and minimum bending moment at point A,

$$M_{min} = W_{min} \times 125 = -F \times 125 = -125 F \text{ N-mm}$$

\therefore Mean or average bending moment,

$$M_m = \frac{M_{max} + M_{min}}{2} = \frac{375 F + (-125 F)}{2} = 125 F \text{ N-mm}$$

and variable bending moment,

$$M_v = \frac{M_{max} - M_{min}}{2} = \frac{375 F - (-125 F)}{2} = 250 F \text{ N-mm}$$

Section modulus,

$$Z = \frac{\pi}{32} \times d^3 = \frac{\pi}{32} (13)^3 = 215.7 \text{ mm}^3 \quad \dots (\because d = 13 \text{ mm})$$

\therefore Mean bending stress,

$$\sigma_m = \frac{M_m}{Z} = \frac{125 F}{215.7} = 0.58 F \text{ N/mm}^2$$

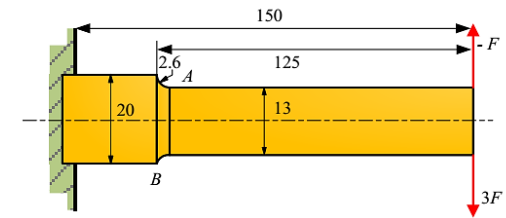
and variable bending stress,

$$\sigma_v = \frac{M_v}{Z} = \frac{250 F}{215.7} = 1.16 F \text{ N/mm}^2$$

Fatigue stress concentration factor, $K_f = 1 + q (K_t - 1) = 1 + 0.9 (1.42 - 1) = 1.378$

We know that according to Goodman's formula

$$\begin{aligned} \frac{1}{F.S.} &= \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_e \times K_{sur} \times K_{sz}} \\ \frac{1}{2} &= \frac{0.58 F}{550} + \frac{1.16 F \times 1.378}{275 \times 0.89 \times 0.85} \\ &= 0.00105 F + 0.00768 F = 0.00873 F \\ \therefore F &= \frac{1}{2 \times 0.00873} = 57.3 \text{ N} \end{aligned}$$



The beam as shown in Figure is subjected to a reversed bending load only. Since the point A at the change of cross section is critical, therefore we shall find the bending moment at point A.

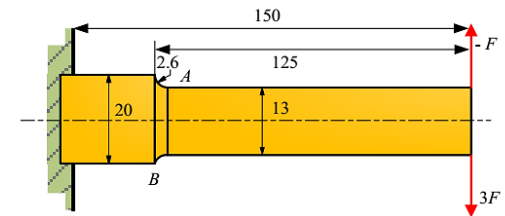
Solution (Continued)

and according to Soderberg's formula,

$$\begin{aligned} \frac{1}{F.S.} &= \frac{\sigma_m}{\sigma_y} + \frac{\sigma_v \times K_f}{\sigma_e \times K_{sur} \times K_{sz}} \\ \frac{1}{2} &= \frac{0.58 F}{470} + \frac{1.16 F \times 1.378}{275 \times 0.89 \times 0.85} \\ &= 0.00123 F + 0.00768 F = 0.00891 F \end{aligned}$$

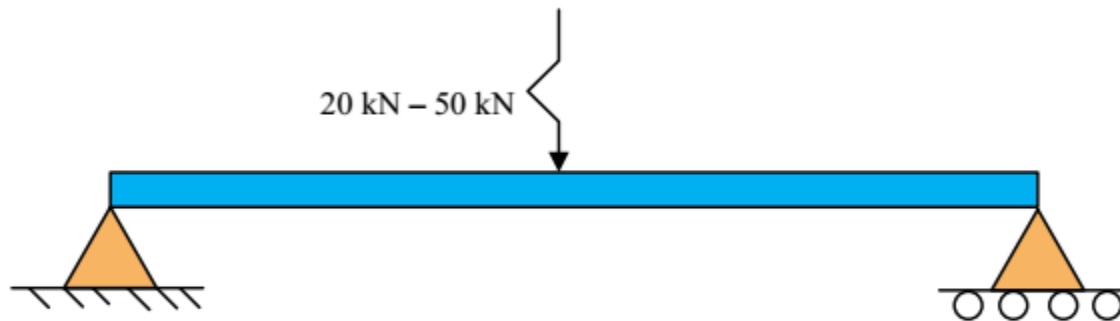
$$\therefore F = \frac{1}{2 \times 0.00891} = 56 \text{ N}$$

Taking larger of the two values, we have $F = 57.3 \text{ N}$ **Ans.**



Example 3

A circular bar of 500 mm length is supported freely at its two ends. It is acted upon by a central concentrated cyclic load having a minimum value of 20 kN and a maximum value of 50 kN. Determine the diameter of bar by taking a factor of safety of 1.5, size effect of 0.85, surface finish factor of 0.9. The material properties of bar are given by : ultimate strength of 650 MPa, yield strength of 500 MPa and endurance strength of 350 MPa.



Solution

Let d = Diameter of the bar in mm.

We know that the maximum bending moment,

$$M_{max} = \frac{W_{max} \times l}{4} = \frac{50 \times 10^3 \times 500}{4} = 6250 \times 10^3 \text{ N-mm}$$

and minimum bending moment,

$$M_{min} = \frac{W_{min} \times l}{4} = \frac{20 \times 10^3 \times 500}{4} = 2500 \times 10^3 \text{ N-mm}$$

∴ Mean or average bending moment,

$$M_m = \frac{M_{max} + M_{min}}{2} = \frac{6250 \times 10^3 + 2500 \times 10^3}{2} = 4375 \times 10^3 \text{ N-mm}$$

and variable bending moment,

$$M_v = \frac{M_{max} - M_{min}}{2} = \frac{6250 \times 10^3 - 2500 \times 10^3}{2} = 1875 \times 10^3 \text{ N-mm}$$

Section modulus of the bar,

$$Z = \frac{\pi}{32} \times d^3 = 0.0982 d^3 \text{ mm}^3$$

∴ Mean or average bending stress,

$$\sigma_m = \frac{M_m}{Z} = \frac{4375 \times 10^3}{0.0982 d^3} = \frac{44.5 \times 10^6}{d^3} \text{ N/mm}^2$$

Solution (Continued)

and variable bending stress,

$$\sigma_v = \frac{M_v}{Z} = \frac{1875 \times 10^3}{0.0982 d^3} = \frac{19.1 \times 10^6}{d^3} \text{ N/mm}^2$$

We know that according to Goodman's formula,

$$\begin{aligned} \frac{1}{F.S.} &= \frac{\sigma_m}{\sigma_u} + \frac{\sigma_v \times K_f}{\sigma_e \times K_{sur} \times K_{sz}} \\ \frac{1}{1.5} &= \frac{44.5 \times 10^6}{d^3 \times 650} + \frac{19.1 \times 10^6 \times 1}{d^3 \times 350 \times 0.9 \times 0.85} \quad \dots(\text{Taking } K_f = 1) \\ &= \frac{68 \times 10^3}{d^3} + \frac{71 \times 10^3}{d^3} = \frac{139 \times 10^3}{d^3} \end{aligned}$$

$$\therefore d^3 = 139 \times 10^3 \times 1.5 = 209 \times 10^3 \text{ or } d = 59.3 \text{ mm}$$

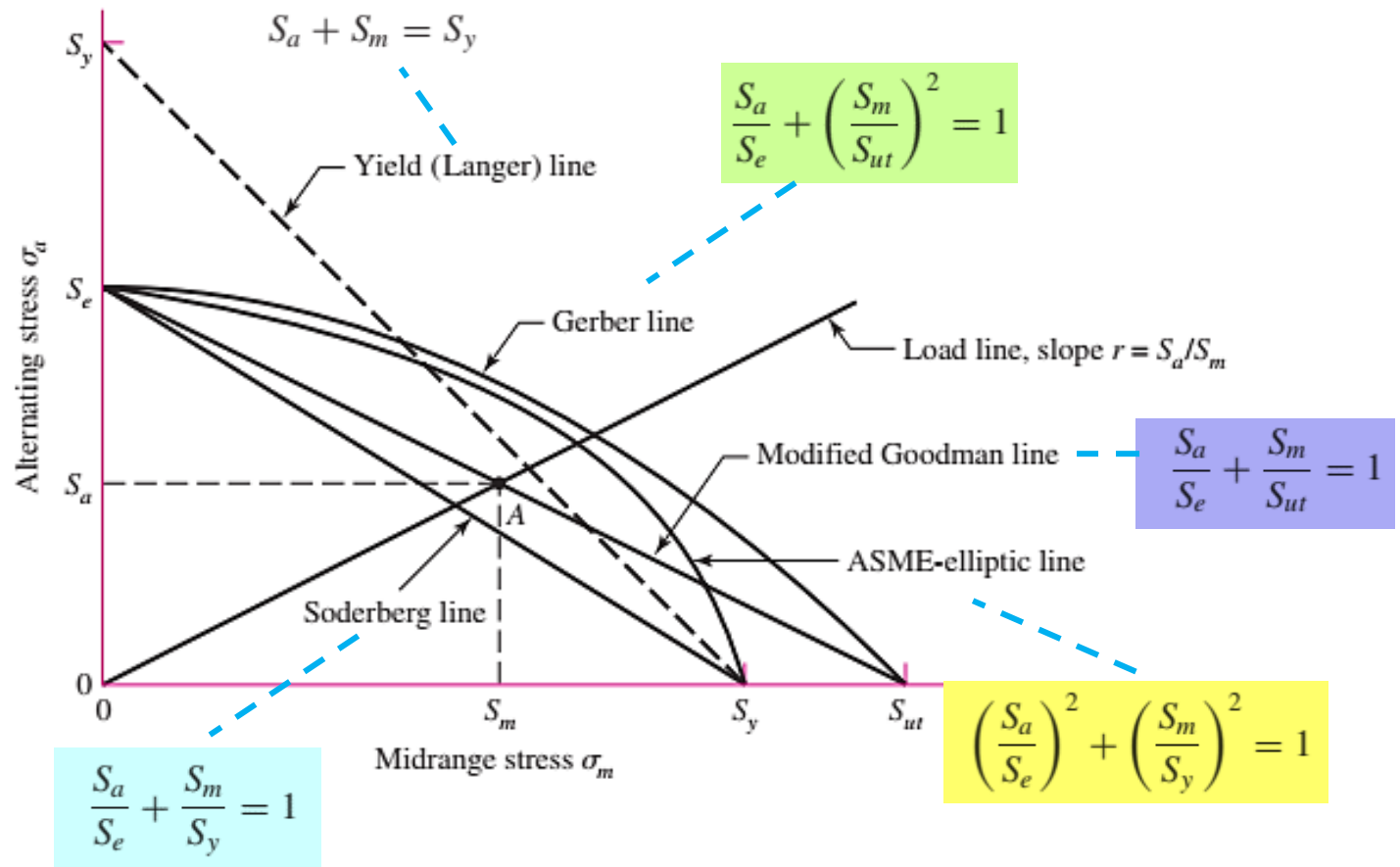
and according to Soderberg's formula,

$$\begin{aligned} \frac{1}{F.S.} &= \frac{\sigma_m}{\sigma_y} + \frac{\sigma_v \times K_f}{\sigma_e \times K_{sur} \times K_{sz}} \\ \frac{1}{1.5} &= \frac{44.5 \times 10^6}{d^3 \times 500} + \frac{19.1 \times 10^6 \times 1}{d^3 \times 350 \times 0.9 \times 0.85} \quad \dots(\text{Taking } K_f = 1) \\ &= \frac{89 \times 10^3}{d^3} + \frac{71 \times 10^3}{d^3} = \frac{160 \times 10^3}{d^3} \end{aligned}$$

$$\therefore d^3 = 160 \times 10^3 \times 1.5 = 240 \times 10^3 \text{ or } d = 62.1 \text{ mm}$$

Taking larger of the two values, we have $d = 62.1 \text{ mm}$ **Ans.**

General Form of Fatigue Diagram



M. Goodman Method for Combination of Stresses

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for **Modified Goodman** and Langer Failure Criteria

Intersecting Equations	Intersection Coordinates
$\frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1$ $\text{Load line } r = \frac{S_a}{S_m}$	$S_a = \frac{r S_e S_{ut}}{r S_{ut} + S_e}$ $S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$ $\text{Load line } r = \frac{S_a}{S_m}$	$S_a = \frac{r S_y}{1 + r}$ $S_m = \frac{S_y}{1 + r}$
$\frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1$ $\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_m = \frac{(S_y - S_e) S_{ut}}{S_{ut} - S_e}$ $S_a = S_y - S_m, r_{\text{crit}} = S_a / S_m$

Fatigue factor of safety

$$n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}$$

Gerber Method for Combination of Stresses

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for **Gerber** and Langer Failure Criteria

Intersecting Equations	Intersection Coordinates
$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = 1$	$S_a = \frac{r^2 S_{ut}^2}{2S_e} \left[-1 + \sqrt{1 + \left(\frac{2S_e}{r S_{ut}}\right)^2} \right]$
Load line $r = \frac{S_a}{S_m}$	$S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = \frac{r S_y}{1 + r}$
Load line $r = \frac{S_a}{S_m}$	$S_m = \frac{S_y}{1 + r}$
$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = 1$	$S_m = \frac{S_{ut}^2}{2S_e} \left[1 - \sqrt{1 + \left(\frac{2S_e}{S_{ut}}\right)^2 \left(1 - \frac{S_y}{S_e}\right)} \right]$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = S_y - S_m, r_{crit} = S_a/S_m$

Fatigue factor of safety

$$n_f = \frac{1}{2} \left(\frac{S_{ut}}{\sigma_m}\right)^2 \frac{\sigma_a}{S_e} \left[-1 + \sqrt{1 + \left(\frac{2\sigma_m S_e}{S_{ut} \sigma_a}\right)^2} \right] \quad \sigma_m > 0$$

ASME Elliptic Method for Combination of Stresses

Amplitude and Steady
Coordinates of Strength
and Important Intersections
in First Quadrant for **ASME
Elliptic** and Langer
Failure Criteria

Intersecting Equations	Intersection Coordinates
$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = 1$ <p>Load line $r = S_a/S_m$</p>	$S_a = \sqrt{\frac{r^2 S_e^2 S_y^2}{S_e^2 + r^2 S_y^2}}$ $S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$ <p>Load line $r = S_a/S_m$</p>	$S_a = \frac{r S_y}{1 + r}$ $S_m = \frac{S_y}{1 + r}$
$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = 1$ $\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = 0, \frac{2 S_y S_e^2}{S_e^2 + S_y^2}$ $S_m = S_y - S_a, r_{\text{crit}} = S_a/S_m$

Fatigue factor of safety

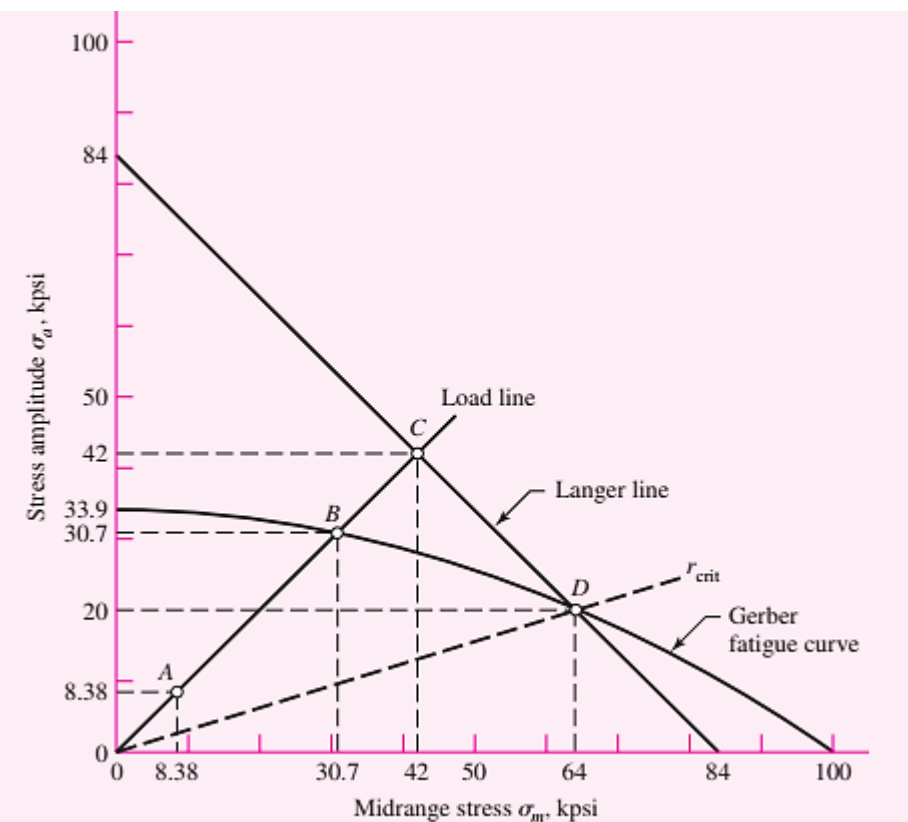
$$n_f = \sqrt{\frac{1}{(\sigma_a/S_e)^2 + (\sigma_m/S_y)^2}}$$

Example 4

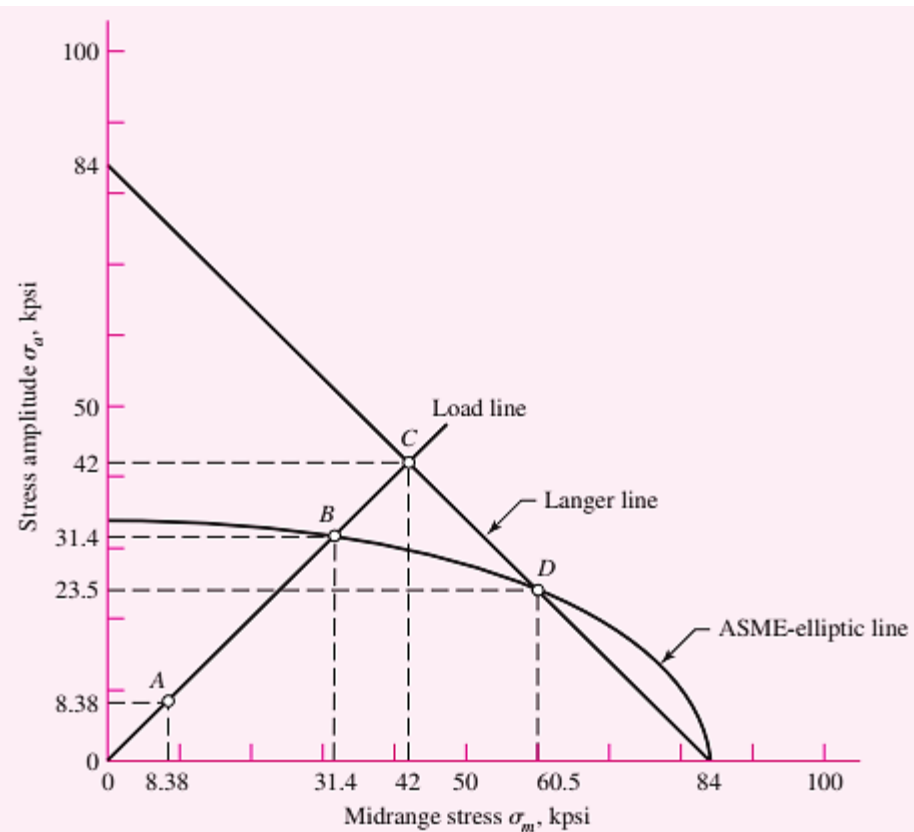
A 1.5-in-diameter bar has been machined from an AISI 1050 cold-drawn bar. This part is to withstand a fluctuating tensile load varying from 0 to 16 kip. Because of the ends, and the fillet radius, a fatigue stress-concentration factor K_f is 1.85 for 10^6 or larger life. Find S_a and S_m and the factor of safety guarding against fatigue and first-cycle yielding, using: (a) the Gerber fatigue line and (b) the ASME-elliptic fatigue line.

(Example 6-10, Shigley-9th edition)

Solution



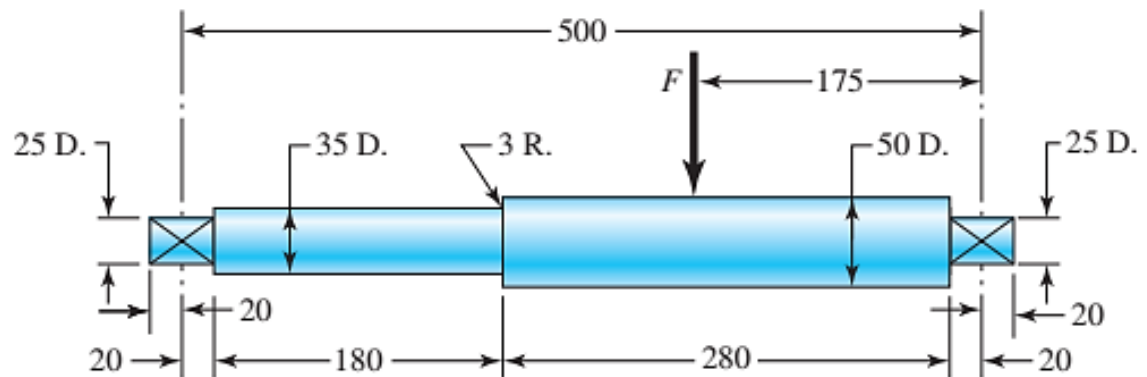
Solution (Continued)



Sample problem 1

The rotating shaft shown in the figure is machined from AISI 1020 CD steel. It is subjected to a force of $F = 6$ kN. Find the minimum factor of safety for fatigue based on infinite life. If the life is not infinite, estimate the number of cycles. Be sure to check for yielding. (all dimensions in mm)

(Ans. $n = 1.03$)

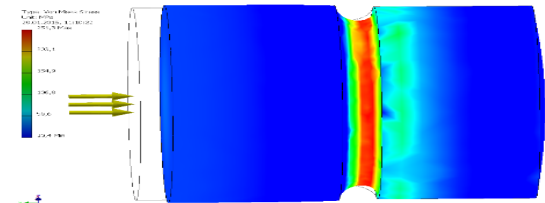
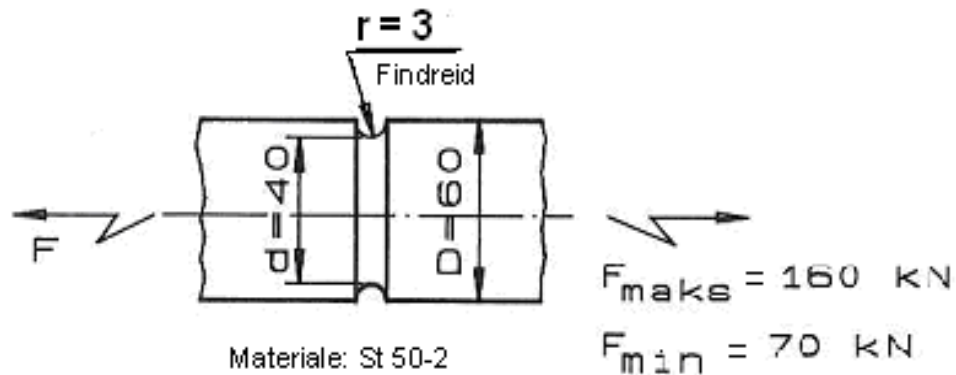


Sample problem 2

The figure shows part of a shaft subjected to a variable axial load (tensile) that varies between $F_{\min} = 70 \text{ kN}$ and $F_{\max} = 160 \text{ kN}$.

Check if the safety factor at the groove is satisfactory against fatigue failure.

(Ans. $n = 2.1$)





End of today's lecture!