von Mises Stress

Spring 2023

What is von Mises stress theory?

The von Mises stress is an equivalent or effective stress at which yielding is predicted to occur in ductile materials!

- **Principal axes**

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

- **Non-principal axes**

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

Where: **Shear stress:** τ
Normal stress: σ
Equivalent stress: σ'

Distortion Energy

Yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.

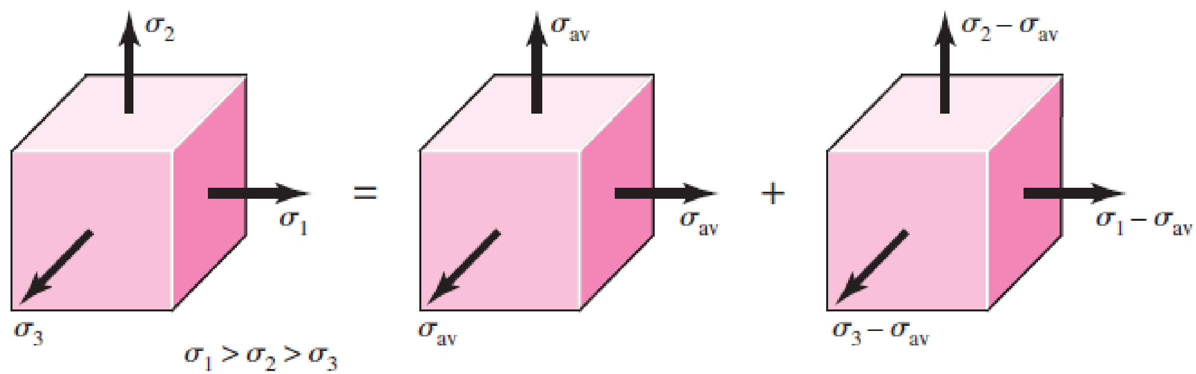
Using principal axes, we can first decompose the state of stress at a point that is given in terms of the principal stresses σ_1 , σ_2 , and σ_3 into the sum of two states:

(a) a state of hydrostatic stress due to the stresses σ_{av} acting in each of the principal directions and causing only volume change

(b) a state of deviatoric stress causing angular distortion without volume change

Where:

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$



(*a*) Triaxial stresses (*b*) Hydrostatic component (*c*) Distortional component

- **(a) Element with triaxial stresses; this element undergoes both volume change and angular distortion.**
- **(b) Element under hydrostatic tension undergoes only volume change.**
- **(c) Element has angular distortion without volume change.**

Applications: Plasticity, Thermo-elasticity, Fluid dynamics, Other fields of engineering science

In simple tension, we have (Hooke's law):

$$\sigma = E\varepsilon$$

where σ is the stress, ε is the strain, and *E* is the modulus of elasticity

Therefore, the strain energy per unit volume for simple tension is

$$u = \int_0^{\varepsilon} \sigma \, d\varepsilon = \int_0^{\varepsilon} E\varepsilon \, d\varepsilon = \frac{1}{2} E\varepsilon^2 = \frac{1}{2}\varepsilon\,\sigma$$

For the state of stress in Fig. 1(*a*), we readily write the total strain energy per unit volume as

$$u = \frac{1}{2}\left(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3\right)$$

where

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_3 + \sigma_1)\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]$$

in which ν is the Poison's ratio. Substituting above Eqs, we get the total strain energy as

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

By letting σ1 = σ2 = σ3 = σav in total strain energy Eq, we obtain the strain energy associated with hydrostatic loading, or only volume change, as

$$u_v = \frac{1-2\nu}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

Clearly, the distortion energy per unit volume is

$$u_d = u - u_v$$

$$u_d = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]$$

Note that ud=0 if σ1 =σ2 =σ3 ; i.e., *no distortion exists in hydrostatic state of stress*.

For simple tensile test of a ductile material, we have σ1 = Sy, σ2 = σ3 = 0, therefore:

$$u_d = \frac{1+\nu}{3E} S_y^2$$

where Sy is the yield stress in tension for the material. By two previous Eqs, yield occurs whenever

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \geq S_y$$

Therefore, the left side this Eq. represents:

$$\boxed{\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2}} \geq S_y$$

An equivalent or effective stress at which yielding of any ductile material is predicted to occur.

$$\sigma' \geq S_y$$

This stress is usually denoted as σ′ and is known as the von Mises stress:

$$\boxed{\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}}$$

The general form of von Mises stress, σ′

1- Eigenvalues of the Stress Matrix:

If $\tau_{yx} = \tau_{xy}$ $\tau_{zy} = \tau_{yz}$ $\tau_{xz} = \tau_{zx}$

The state of stress at point *O* of the coordinate system *O*xyz in the material is given by this symmetric stress matrix (where the *xyz* axes are generally NOT principal axes):

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix}$$

The principal stresses σ1, σ2, and σ3 at *O* of the material are given by the eigenvalues of the stress matrix, which are simply the roots of this characteristic equation:

$$\begin{vmatrix} \sigma_x - \lambda & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \lambda & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \lambda \end{vmatrix} = 0$$

Upon expansion, the latest Eq. becomes this cubic equation:

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{zx} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix}$$
$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix}$$

The roots this cubic Eq. are the principal stresses σ1, σ2, and σ3 at *O* of the material. It means that ….

If the *xyz* coordinate axes at point *O* coincide with the principal axes at point *O*, the "cross shears" vanish and the values of I_1, I_2, I_3 can be expressed in terms of just the principal stresses as:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

Based on the two ★ Eqs. we write:

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z$$

Based on the two ★ Eqs. we write:

$$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \quad ★$$

Squaring both sides of these two Eqs. we write

$$\begin{aligned}
&\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \\
&= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) \\
&= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) + 2\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)
\end{aligned}$$

So: $$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \quad ★$$

Therefore based on von Mises stress formula and using ★ and ★ :

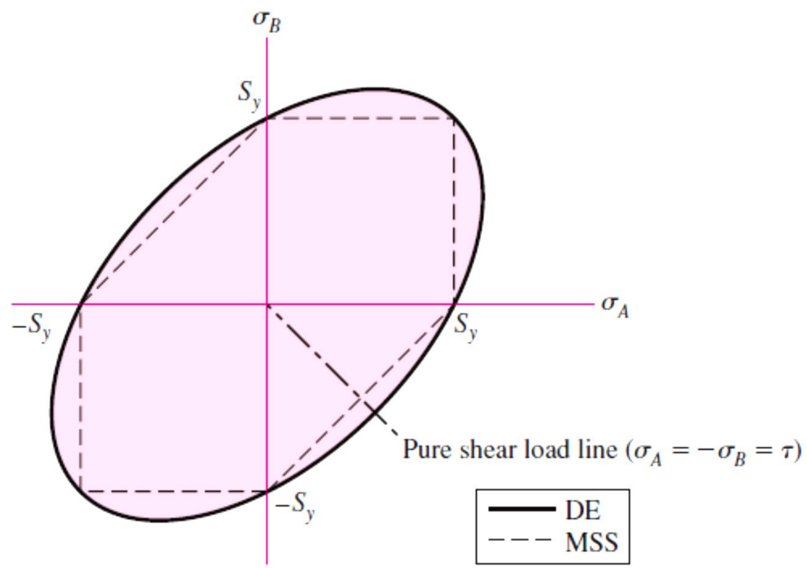
$$
\left\{
\begin{aligned}
&\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2} \\
&\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 = \sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2 \\
&\sigma_1^2+\sigma_2^2+\sigma_3^2 = \sigma_x^2+\sigma_y^2+\sigma_z^2+2\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)
\end{aligned}
\right.
$$

We write:

$$
\begin{aligned}
2(\sigma')^2 &= (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 \\
&= 2(\sigma_1^2+\sigma_2^2+\sigma_3^2) - 2\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right) \\
&= 2(\sigma_x^2+\sigma_y^2+\sigma_z^2) + 4(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2) \\
&\quad -2(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x) + 2(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2) \\
&= 2(\sigma_x^2+\sigma_y^2+\sigma_z^2) - 2(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x) + 6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2) \\
&= (\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)
\end{aligned}
$$

Therefore, we conclude that the following general form of the von Mises stress has been proved and is true:

$$\boxed{\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}}$$



SUMMARY

- **At the institution where the authors teach in the Department of Mechanical Engineering, the course Machine Element Design is a course for mainly seniors in the curriculum.**

- **Today the generally accepted failure theories (or yield criteria) used to evaluate machine components manufactured from ductile materials are:**
  - **Maximum shear stress theory,**
  - **Distortion energy theory,**
  - **Coulomb-Mohr theory**

- **In the *distortion energy theory*, yielding occurs when the von Mises stress σ′ is reached, or exceeded, by a state of stress in the machine component.**