$$\frac{}{\psi_{\mathrm{CM}}}\frac{\partial\psi_{\mathrm{CM}}}{\partial t} = \frac{\hbar^2}{2\mu}\frac{1}{\psi_{\mathrm{rel}}}\frac{\partial^2\psi_{\mathrm{rel}}}{\partial x^2} - \frac{1}{2}\mu\omega$$
$$+\frac{\mu}{m_1}Fx + \frac{i\hbar}{\psi_{\mathrm{rel}}}\frac{\partial\psi}{\partial}$$

We now see that the left-hand side of the Schrödinger equation is onl
on R and t, while the right-hand side is only dependent on x and t.
any time, the two sides of the equation must be equal to each other fo
R. Thus, each side can be set equal to a separation constant that is d
time, which we will call $C(t) = C$.

3.1 Solving for the Center of Mass Wave Function

First, let us consider the left-hand side of (13), given by

$$-\frac{\hbar^2}{2M}\frac{1}{\psi_{\mathrm{CM}}}\frac{\partial^2\psi_{\mathrm{CM}}}{\partial R^2} - FR - \frac{i\hbar}{\psi_{\mathrm{CM}}}\frac{\partial\psi_{\mathrm{CM}}}{\partial t} = C.$$

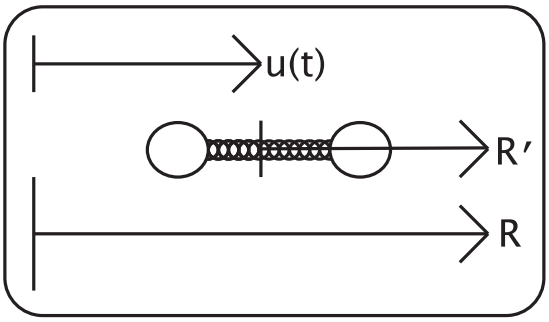

Figure 2: Diagram of the Transformation from the Stationary Frame to the Center of Mass Frame. The center of mass frame, given by R', moves with the system. The function $u(t)$ shows the difference between the frames.

At this point, separation of variables must be applied once more in order obtain the form of ψ_{CM}. However, the pulse, $F(t)$, is time dependent. Thus, the time and spatial dependencies of ψ_{CM} are still entangled in the second term of (15). In order to remedy this, the problem must be moved from the lab frame to a reference frame that moves with the center of mass of the system. We accomplish this with the extended Galilean transformation [5]

$$R' = R - u(t) \quad \text{and} \quad t' = t, \tag{16}$$

where $u(t)$ is a function that describes the difference between the two frames (see Figure 2) and is defined such that the terms with time and spatial dependencies in (15) drop out. It follows from the definitions of R' and t' that

$$\frac{\partial^2}{\partial R'^2} = \frac{\partial^2}{\partial R^2} \quad \text{and} \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \dot{u}\frac{\partial}{\partial R'}. \tag{17}$$

Following the procedure in Kerner [6], we assume the following form for ψ_{CM}

$$\psi_{CM}(R,t) = \Phi(R',t)e^{Rg(t)}, \tag{18}$$

where $g(t)$ is a function we will choose later. Now, (16)-(18) are substituted into

(15). Once simplified, the new form of (15) is

$$-\frac{\hbar^2}{2M}\frac{\partial^2\Phi}{\partial R'^2} + (i\hbar\dot{u} - \frac{\hbar^2}{M}g)\frac{\partial\Phi}{\partial R'} + \left[R'(-F - i\hbar\dot{g}) - Fu - i\hbar u\dot{g} - \frac{\hbar^2}{2M}g^2\right]\Phi$$
$$= i\hbar\frac{\partial\Phi}{\partial t} + C\Phi. \quad (19)$$

At this point, we consider each term separately and attempt to rid the equation of those that contain both R' and t dependencies. In order to do so, we make careful choices of the functions $u(t)$ and $g(t)$. Thus, we eliminate the terms in $\frac{\partial\Phi}{\partial R'}$ by writing

$$g(t) = i\frac{M}{\hbar}\dot{u} \quad (20)$$

and similarly set the terms in R' to zero by assuming

$$-i\hbar\dot{g} - F(t) = 0. \quad (21)$$

We also note that if (20) and (21) are combined, we arrive at an expression of Newton's Second Law

$$F = M\ddot{u}, \quad (22)$$

which shows that the external force on the system is equal to the total mass of the system multiplied by the acceleration of the center of mass.

Equations (20) and (21) allow us to arrive at a greatly simplified form of (19)

$$-\frac{\hbar^2}{2M}\frac{\partial^2\Phi}{\partial R'^2} + \frac{1}{2}M\dot{u}^2\Phi = i\hbar\frac{\partial\Phi}{\partial t} + C\Phi. \quad (23)$$

To solve for $\Phi(R', t)$, we will assume that it is separable

$$\Phi(R', t) = \mathcal{R}(R')\mathcal{T}(t), \quad (24)$$

and separate variables by dividing (23) by (24) such that

$$-\frac{\hbar^2}{2M}\frac{1}{\mathcal{R}}\frac{d^2\mathcal{R}}{dR'^2} + \frac{1}{2}M\dot{u}^2 = \frac{i\hbar}{\mathcal{T}}\frac{d\mathcal{T}}{dt} + C. \quad (25)$$

Rearranging terms yields

$$-\frac{\hbar^2}{2M}\frac{1}{\mathcal{R}}\frac{d^2\mathcal{R}}{dR'^2} = \frac{i\hbar}{\mathcal{T}}\frac{d\mathcal{T}}{dt} - \frac{1}{2}M\dot{u}^2 + C. \tag{26}$$

We notice that the left half of this expression is dependent only on R, while the right half is dependent only on t. Thus, we can set each side equal to a separation constant, E. First, let us consider the left-hand side of the expression

$$-\frac{\hbar^2}{2M}\frac{1}{\mathcal{R}}\frac{d^2\mathcal{R}}{dR'^2} = E. \tag{27}$$

This can be rearranged to look like

$$\frac{d^2\mathcal{R}}{dR'^2} = -c^2\mathcal{R}, \tag{28}$$

where $c = \pm\frac{\sqrt{2ME}}{\hbar}$. This differential equation is the time-independent Schrödinger equation for a free particle. The solutions to it are known to be

$$\mathcal{R} = Ae^{icR'}, \tag{29}$$

where A is a normalization constant which cannot be explicitly determined due to the nature of the free particle system [4]. The separation constant, E, thus translates into the energy the system would have as a free particle of mass M in the R' frame.

Now let us consider the right-hand side of (26)

$$\frac{i\hbar}{\mathcal{T}}\frac{d\mathcal{T}}{dt} - \frac{1}{2}M\dot{u}^2 + C = E, \tag{30}$$

and rewrite it to look like

$$\frac{i\hbar}{\mathcal{T}}\frac{d\mathcal{T}}{dt} = \frac{1}{2}M\dot{u}^2 - C + E. \tag{31}$$

Rewriting once more to separate variables gives

$$\frac{d\mathcal{T}}{\mathcal{T}} = -\frac{i}{\hbar}(\frac{1}{2}M\dot{u}^2 - C + E)dt. \tag{32}$$

Integrating both sides, we find the form of $\mathcal{T}$ to be

$$\mathcal{T} = e^{-\frac{i}{\hbar}\left[\int\left(\frac{1}{2}M\dot{u}^2 - C\right)dt + Et)\right]}. \tag{33}$$

Finally, we can write the expression for $\psi_{CM}(R,t)$ by combining the results in (18), (24), (29), and (33) to get

$$\psi_{CM}(R,t) = Ae^{ic(R-u)}e^{-\frac{i}{\hbar}\left[\int\left(\frac{1}{2}M\dot{u}^2 - C\right)dt + Et\right]}e^{iR\frac{M}{\hbar}\dot{u}}. \tag{34}$$

3.2 Solving for the Relative Wave Function

The expression for $\psi_{rel}(x,t)$ in (11) is needed before the full-blown wave function, $\Psi(x,R,t)$, can be determined. In the previous section, we found that the portion of the wave function dependent on the position of the center of mass (see equation (34)) describes the translational motion of our system and is analogous to the wave function of a free particle. We hope that the relative portion of the wave function will provide a similarly enlightening form. In order to determine $\psi_{rel}(x,t)$, we will take an approach similar to that used to find $\psi_{CM}(R,t)$. First, we consider the right-hand side of (13)

$$\frac{\hbar^2}{2\mu}\frac{1}{\psi_{rel}}\frac{\partial^2\psi_{rel}}{\partial x^2} - \frac{1}{2}\mu\omega^2x^2 + \frac{\mu}{m_1}Fx + \frac{i\hbar}{\psi_{rel}}\frac{\partial\psi_{rel}}{\partial t} = C. \tag{35}$$

Once the expression has been multiplied through by ψ_{rel} and the terms have been rearranged, we obtain

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\psi_{rel}}{\partial x^2} + (\frac{1}{2}\mu\omega^2x^2 - \frac{\mu}{m_1}Fx)\psi_{rel} = i\hbar\frac{\partial\psi_{rel}}{\partial t} - C\psi_{rel}. \tag{36}$$

As before, there are still some terms containing both spatial and time dependencies. In order to fix this problem, we will perform another extended Galilean transformation into a new frame that moves with the oscillation of the masses after

the force has been applied [5]. The second extended Galilean transformation is

$$x' = x - \xi(t) \quad \text{and} \quad t' = t, \tag{37}$$

where $\xi(t)$ is a function defined such that the terms in (36) that are dependent on both time and position will drop. It follows from these definitions and the multivariable chain rule that

$$\frac{\partial^2}{\partial x'^2} = \frac{\partial^2}{\partial x^2} \quad \text{and} \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \dot{\xi}\frac{\partial}{\partial x'}. \tag{38}$$

We assume the following form for $\psi_{rel}(x,t)$ according to the technique in Kerner [6], as we did for $\psi_{CM}(R,t)$ in the previous section, such that

$$\psi_{rel}(x,t) = \varphi(x',t)e^{xf(t)}, \tag{39}$$

where $f(t)$ is an unspecified imaginary function. If we substitute (37)-(39) into (36), the expression can be rewritten as

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\varphi}{\partial x'^2} + (i\hbar\dot{\xi} - \frac{\hbar^2}{\mu}f)\frac{\partial\varphi}{\partial x'} + \Big[\frac{1}{2}kx'^2 + x'(k\xi - \frac{\mu}{m_1}F - i\hbar\dot{f}) + (\frac{1}{2}k\xi^2 - \frac{\mu}{m_1}F\xi - i\hbar\xi\dot{f} - \frac{\hbar^2}{2\mu}f^2)\Big]\varphi = i\hbar\frac{\partial\varphi}{\partial t} - C\varphi. \tag{40}$$

Again, we want to choose $\xi(t)$ and $f(t)$ so that the terms containing both time and spatial dependencies cancel out. This can be done by letting

$$f(t) = i\frac{\mu}{\hbar}\dot{\xi}, \tag{41}$$

so that the terms in $\frac{\partial\varphi}{\partial x'}$ go to zero, and

$$k\xi - \frac{\mu}{m_1}F - i\hbar\dot{f} = 0, \tag{42}$$

to eliminate terms with both x' and t. The previous two equations combine to form the following expression for the force

$$\frac{\mu}{m_1}F = \mu\ddot{\xi} + k\xi. \tag{43}$$

Equations (41) and (42) further allow us to simplify the form of (40), such that

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\varphi}{\partial x'^2} + (\frac{1}{2}kx'^2 + \frac{1}{2}\mu\dot{\xi}^2 - \frac{1}{2}k\xi^2)\varphi = i\hbar\frac{\partial\varphi}{\partial t} - C\varphi. \tag{44}$$

We can write (44) in a compacted form when we recognize that $\mathcal{L}(t) = \frac{1}{2}\mu\dot{\xi}^2 - \frac{1}{2}k\xi^2$ is the Lagrangian. Thus,

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\varphi}{\partial x'^2} + (\frac{1}{2}kx'^2 + \mathcal{L})\varphi = i\hbar\frac{\partial\varphi}{\partial t} - C\varphi. \tag{45}$$

All of our terms are now either dependent on x' or t, but not both. Therefore, we can assume $\varphi(x', t)$ has a separable form given by

$$\varphi(x', t) = \mathcal{X}(x')\mathcal{T}(t) \tag{46}$$

and divide (45) by it. After rearranging terms, (45) becomes

$$-\frac{\hbar^2}{2\mu}\frac{1}{\mathcal{X}}\frac{d^2\mathcal{X}}{dx'^2} + \frac{1}{2}kx'^2 = \frac{i\hbar}{\mathcal{T}}\frac{d\mathcal{T}}{dt} - \mathcal{L} - C. \tag{47}$$

Setting each side equal to a separation constant, E_n, we consider the left side of the equation

$$-\frac{\hbar^2}{2\mu}\frac{1}{\mathcal{X}}\frac{d^2\mathcal{X}}{dx'^2} + \frac{1}{2}kx'^2 = E_n. \tag{48}$$

This is the time-independent Schrödinger equation for the unperturbed quantum harmonic oscillator. The solution is known to be

$$\mathcal{X} = (\frac{\beta^2}{\pi})^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x')e^{-\frac{1}{2}(\beta x')^2}, \tag{49}$$

where $\beta = \sqrt{\frac{\mu\omega}{\hbar}}$, ω is the angular frequency of the oscillator, and H_n is a Hermite polynomial of order n [4]. Thus, the separation constant, which we so fortuitously chose to be a function of n, is the energy of an unperturbed quantum harmonic

oscillator in a state $n = 0, 1, 2...$, where $n = 0$ is the ground state, $n = 1$ is the first excited state, and so on. These energies are given by

$$E_n = (n + \frac{1}{2})\hbar\omega. \tag{50}$$

Now we will consider the right hand side of (47)

$$\frac{i\hbar}{\mathcal{T}}\frac{d\mathcal{T}}{dt} - \mathcal{L} - C = E_n. \tag{51}$$

Rearranging terms to separate variables, we obtain

$$\frac{d\mathcal{T}}{\mathcal{T}} = -\frac{i}{\hbar}(\mathcal{L} + C + E_n)dt. \tag{52}$$

Integrating both sides and solving for $\mathcal{T}$, we find

$$\mathcal{T} = e^{-\frac{i}{\hbar}\left[\int(\mathcal{L}+C)dt + E_n t\right]}. \tag{53}$$

We can now write the form of φ from (46), (49), and (53)

$$\varphi(x', t) = (\frac{\beta^2}{\pi})^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x')e^{-\frac{1}{2}(\beta x')^2}e^{-\frac{i}{\hbar}\left[\int(\mathcal{L}+C)dt + E_n t\right]}. \tag{54}$$

(39), (41), and (54) combine as

$$\psi_{rel_n}(x, t) = (\frac{\beta^2}{\pi})^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n\left[\beta(x - \xi)\right]e^{-\frac{1}{2}[\beta(x-\xi)]^2}e^{-\frac{i}{\hbar}\left[\int(\mathcal{L}+C)dt + E_n t\right]}e^{ix\frac{\mu}{\hbar}\dot{\xi}}. \tag{55}$$

Finally, the complete form of the wave function for the entire system in a given state $n = 0, 1, 2...$ can be written from (11), (34), and (55) as

$$\Psi_n(x, R, t) = A(\frac{\beta^2}{\pi})^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n\left[\beta(x - \xi)\right]e^{-\frac{1}{2}[\beta(x-\xi)]^2}e^{ix\frac{\mu}{\hbar}\dot{\xi}}e^{ic(R-u)}e^{iR\frac{M}{\hbar}\dot{u}}$$
$$\times\ e^{-\frac{i}{\hbar}\left[\int\mathcal{L}dt + \int\frac{1}{2}M\dot{u}^2dt + (E+E_n)t\right]}. \tag{56}$$

Thus, the wave function of two particles connected by a harmonic potential, $V(x)$, and subject to a time-dependent force, $F(t)$, will be represented by the product of a center of mass wave function akin to that of a free particle and a relative motion wave function akin to that of the quantum harmonic oscillator.

4 The Gaussian Pulse

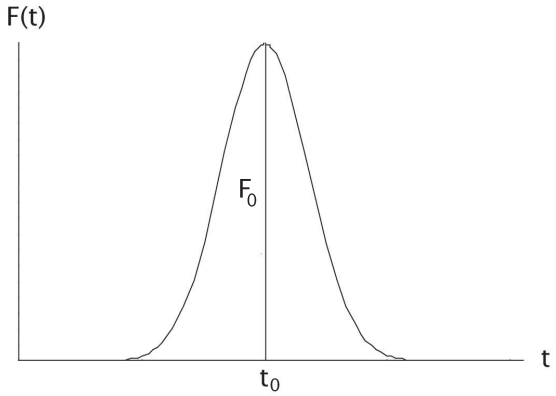

Figure 3: Graph of the Gaussian Pulse.

The force applied to the system will now be specified as a Gaussian pulse given by

$$F(t) = F_0 e^{-\gamma(t-t_0)^2}, \tag{57}$$

where F_0 and γ are constants dependent on the physical attributes of the specific force applied, and t_0 is the time at which the force applied to m_1 is maximized (See Figure 3). Specifying the force will allow us to further analyze our system and calculate such quantities as the probability distribution and the expectation value for the energy. This particular form for the force is of interest because it is a reasonable approximation of a real force that could be applied in a laboratory experiment. That is, it is a force that contains no discontinuities and only acts upon the system for a finite amount of time. Furthermore, the function itself is easily differentiated and integrated, which simplifies our math considerably.

4.1 Calculating $u(t)$ and $\xi(t)$

Once the force is specified, we can determine the functions $u(t)$ and $\xi(t)$ from

our Galilean transformations. From (22) and (57), we write the differential equation

$$M\ddot{u} = F_0 e^{-\gamma(t-t_0)^2}. \tag{58}$$

In order to find the correct solution to this equation, we must stipulate that the center of mass frame and the lab frame agree until the force is applied to the system. Therefore, we can assume that both $u(t) = 0$ and $\dot{u}(t) = 0$ when $t = 0$ and proceed to solve by integration. The form is then

$$u(t) = \frac{F_0}{2M\gamma}\Big[e^{-\gamma(t-t_0)^2} - e^{-\gamma t_0{}^2} + \sqrt{\pi\gamma}(t-t_0)\left(erf[\sqrt{\gamma}(t-t_0)] + erf[\sqrt{\gamma}t_0]\right)\Big], \tag{59}$$

where $erf[z]$ is the error function, which is obtained by the integration of the normalized Gaussian function.

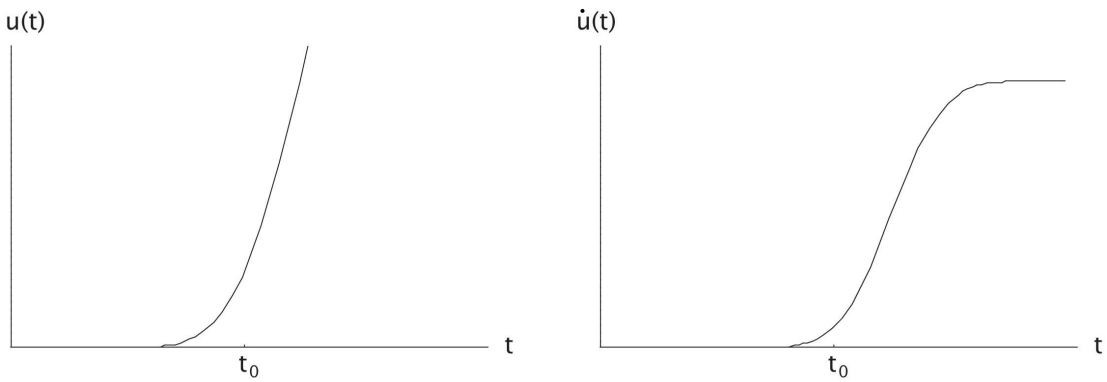

Figure 4: Graphs of $u(t)$ and $\dot{u}(t)$.

The graphs of $u(t)$ and $\dot{u}(t)$ show us that the center of mass accelerates during the time when the force is applied and continues at a constant velocity afterwards (see Figure 4).

The expression for $\xi(t)$ can be found by solving the differential equation suggested by combining (43) and (57)

$$\frac{\mu}{m_1} F_0 e^{-\gamma(t-t_0)^2} = \mu\ddot{\xi} + \mu\omega^2\xi(t). \tag{60}$$

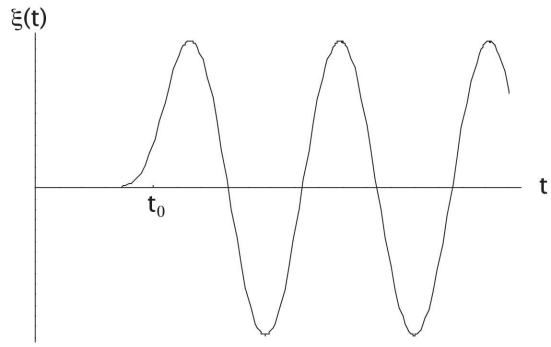

Figure 5: Graph of $\xi(t)$. The oscillation begins when the pulse acts on the system at t_0.

This is a nonhomogenous, second-order linear differential equation. Its solution is the general solution of its homogenous analogue plus a particular solution found using the method of variation of parameters [14], where the Wronskian for our equation is

$$W = c_1\omega\cos^2(\omega t) + c_2\omega\sin^2(\omega t). \tag{61}$$

Therefore, the general solution plus the particular solution of $\xi(t)$ is given by

$$\begin{aligned}\xi(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + c_1\cos(\omega t)\int\frac{-c_2\sin(\omega t)F_0e^{-\gamma(t-t_0)^2}}{W\mu}dt \\ + c_2\sin(\omega t)\int\frac{c_1\cos(\omega t)F_0e^{-\gamma(t-t_0)^2}}{W\mu}dt.\end{aligned} \tag{62}$$

The two constants, c_1 and c_2, are found by requiring that the new frame and the lab frame agree until the force has acted upon the system. This means that our initial conditions require that $\xi(t) = 0$ whenever $t < t_0$. We accomplish this by requiring that $\xi(t) = 0$ when $t = \frac{-\pi}{\omega}$, which allows us to isolate and solve for c_1. Then we set $\xi(t) = 0$ when $t = \frac{-\pi}{2\omega}$, which allows us to find c_2. Because our force only acts in the neighborhood of t_0, forcing $\xi(t) = 0$ for a pair of arbitrary points before $t = 0$ means that $\xi(t)$ must be zero everywhere before the force acts. The equation for

$\xi(t)$, given by (62), thus yields

$$\xi(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{4m_1\omega}\sqrt{\frac{\pi}{\gamma}} e^{-\left(\frac{\omega^2}{4\gamma} + i\omega(t-t_0)\right)} \times \left(e^{2i\omega(t-t_0)} erfi[\frac{\omega - 2i\gamma(t-t_0)}{2\sqrt{\gamma}}] + erfi[\frac{\omega + 2i\gamma(t-t_0)}{2\sqrt{\gamma}}] \right), \quad (63)$$

where $erfi[z]$ is the imaginary error function, given by $erfi[z] = -ierf[iz]$. The constants, c_1 and c_2, are given by

$$c_1 = -\frac{iF_0}{4m_1\omega}\sqrt{\frac{\pi}{\gamma}} e^{-\left(\frac{\omega^2}{4\gamma} + i\omega t_0\right)} \times \left(e^{2i\omega t_0} erf[\frac{-i\omega - 2\gamma(\pi + \omega t_0)}{2\omega\sqrt{\gamma}}] - erf[\frac{i\omega - 2\gamma(\pi + \omega t_0)}{2\omega\sqrt{\gamma}}] \right), \quad (64)$$

and

$$c_2 = -\frac{F_0}{4m_1\omega}\sqrt{\frac{\pi}{\gamma}} e^{-\left(\frac{\omega^2}{4\gamma} + i\omega t_0\right)} \times \left(erf[\frac{i\omega^2 - \gamma(\pi + 2\omega t_0)}{2\omega\sqrt{\gamma}}] - ie^{2i\omega t_0} erfi[\frac{\omega - i\gamma(\pi + 2\omega t_0)}{2\omega\sqrt{\gamma}}] \right). \quad (65)$$

The graph of $\xi(t)$ shows us that the function is zero until the force acts upon the system, at which time it begins to oscillate. That is, once the force is applied, the difference between the relative frame and the lab frame is oscillatory (see Figure 5). We also note that at large t, the force goes to zero, and the motion given by $\xi(t)$ is reduced to the cosine and sine terms and is simply that of a free oscillation. In fact, one can see from (43) that *any* force that eventually goes to zero for large t will produce this behavior. This is an important result that will be revisited in the next section.

4.2 The Probability Distribution and Coherent States

It is now possible to consider the probability distribution of the system. An

Figure 6: Graph of the Probability Distribution. The oscillation in the wave packet begins when the pulse acts on the system at t_0.

inspection of (56) yields the form

$$\Psi_n^* \Psi_n = \frac{A^2}{2^n n!} \left(\frac{\beta^2}{\pi} \right)^{1/2} {H_n}^2(\beta x') e^{-(\beta x')^2}, \tag{66}$$

as all of the imaginary exponential terms cancel with their complex conjugates. We note that this expression is not dependent on the spatial variable, R. Therefore, the behavior of the probability density does not depend on the overall motion of the center of mass of the system. If the probability distribution is graphed in the lab frame, using (37) and (63) to write everything in terms of x and t, the result is a wave packet that keeps the shape of that of the unperturbed quantum harmonic oscillator but oscillates in time with respect to x after the force is applied (See Figure 6). This behavior indicates that the system is in a displacement-operator coherent state after being subjected to the force.

Before we go further, let us consider the effect the form of $F(t)$ has on (66). Examination of the expression shows that the force only affects the probability distribution through $\xi(t)$. Interestingly, as discussed in the previous section, $\xi(t)$ shows the same type of behavior *regardless of the force* after the force has died off.

Thus, for any general force that eventually goes to zero, the probability distribution for our model will exhibit the behavior shown in Figure 6 after sufficient time. The only exception to this rule is when the force is turned off at the exact instant when $\xi(t) = 0$ and $\dot{\xi}(t) = 0$. That is, when the oscillator is in its original position, $x' = x$, and has no velocity away from this position. In this case, we see that $\xi(t)$ will continue to be zero for any subsequent t as long as the system is not acted upon by another force. Therefore, the relative wave function, given by (55), will return to its original stationary state for the unperturbed quantum harmonic oscillator. We note that this behavior is impossible for the Gaussian pulse, since inspection of Figure 5 shows that there is never a point at which both $\xi(t)$ and its first derivative equal zero. However, there are forces that do produce this behavior, such as a sinusoidal force of frequency not equal to the natural frequency of the oscillator.

We have shown that the displacement-operator definition of coherent states holds for our model (see Figure 6). Now, we will examine another definition of coherent states. The annihilation-operator coherent states for the harmonic oscillator occur when the probability, $|C_m|^2$, that the new system is in the unperturbed state, $\psi_m(x,t)$, is given by

$$|C_m|^2 \propto e^{-|z|^2} \frac{|z|^{2m}}{m!}, \tag{67}$$

where m is the state of the unperturbed harmonic oscillator and z is a complex number [15]. We recall that the states of the unperturbed quantum harmonic oscillator are given by

$$\varphi_m(x,t) = (\frac{\beta^2}{\pi})^{\frac{1}{4}} \frac{1}{\sqrt{2^m m!}} H_m(\beta x) e^{-\frac{1}{2}(\beta x)^2} e^{-\frac{i}{\hbar} E_m t} \tag{68}$$

and are well-known [4]. We are now able to evaluate the expression

$$C_m =< \varphi_m(x,t) | \psi_{rel_n}(x,t) >, \tag{69}$$

where we are comparing the relative portion of our wave function to the unperturbed wave function. We will use the ground state of the kicked oscillator, $n = 0$, in order to simplify our math. Equation (69) eventually generates a very complicated expression, which is reduced when we multiply it by its complex conjugate to obtain

$$|C_m|^2 = \beta^2 e^{-\frac{1}{2}\left(\beta^2\xi^2 + \frac{\mu^2\dot{\xi}^2}{\hbar^2\beta^2}\right)} \frac{1}{m!} \left[\frac{1}{2}\left(\beta^2\xi^2 + \frac{\mu^2\dot{\xi}^2}{\hbar^2\beta^2}\right)\right]^m, \tag{70}$$

which has the form given in (67) with $|z|^2 = \frac{1}{2}\left(\beta^2\xi^2 + \frac{\mu^2\dot{\xi}^2}{\hbar^2\beta^2}\right)$. Therefore, our model does indeed yield annihilation-operator coherent states [8].

If we consider the previously mentioned case, in which the force dies off at the exact moment when $\xi(t) = 0$ and $\dot{\xi}(t) = 0$, then we see from (70) that the probability of being in any stationary state except the ground state is zero. This agrees with our previous prediction that the wave function would return to the original stationary state under these conditions. That is, in order to find (70), we set $n = 0$, which means that the kicked oscillator is in its ground state and will return to the ground state of the unperturbed oscillator if $F(t)$ is turned off when $\xi(t) = 0$ and $\dot{\xi}(t) = 0$.

4.3 The Average Energy

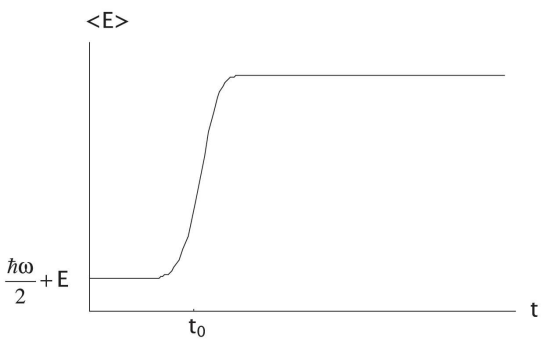

Figure 7: Graph of the Average Energy of the System Given by Equation (73)

Classically, the energy of our system is constant before and after the force has

acted on it. According to Ehrenfest's theorem [4], the expectation value of the quantum mechanic energy operator should show the same behavior. We will show this to be true by calculating the expectation value of the energy of our model when it is subjected to a Gaussian pulse. This calculation looks like

$$< E >=< H >=< \Psi|H|\Psi >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\Psi^* H\Psi)dxdR. \tag{71}$$

The integrand of this expression is very complicated. However, simplification and the stipulation that the oscillator was initially in the ground state, $n = 0$, before the pulse eventually provide the expression

$$\Psi^* H\Psi = \frac{A^2}{2m_1}\left(\frac{\beta^2}{\pi}\right)^{1/2} e^{(\beta x')^2}\Big[-2F(m_1R+\mu x)+m_1\Big(\hbar\omega+2E+2\hbar c\dot{u} \\ +M\dot{u}^2+\mu(\omega\xi-i\dot{\xi})(2\omega x-\omega\xi+i\dot{\xi})\Big)\Big]. \tag{72}$$

This function must be integrated over all space with respect to both x and R to yield the expectation value for the energy. The integral over x is simple, since the relative part of the wave function is already normalized. The integration over R requires that we divide each term by $< \Psi|\Psi >$ to normalize, since it is impossible to explicitly normalize the free particle part of our center of mass wave function. After integrating over all space and normalizing, the following expression for the average energy is obtained

$$< E >= \frac{\hbar\omega}{2}+E-\frac{\mu}{m_1}F(t)\xi(t)+\frac{1}{2}\mu\omega^2\xi(t)^2+\sqrt{2ME}\dot{u}(t)+\frac{1}{2}M\dot{u}(t)^2+\frac{1}{2}\mu\dot{\xi}(t)^2, \tag{73}$$

where the definitions $c = \pm\frac{\sqrt{2ME}}{\hbar}$ and $\beta = \sqrt{\frac{\mu\omega}{\hbar}}$ have been substituted into the equation. Note that all the terms except the first two are zero until the force is applied; these two terms are the ground state energy of the harmonic oscillator given by (50) and the kinetic energy of the center of mass, respectively. The force

then does work on the system, bringing it to a new constant energy (see Figure 7). Many of the other terms are easily recognizable quantities. For instance, the fourth term is the potential due to the spring, and the last two terms are the kinetic energies of the center of mass and the oscillation, respectively. Finally, the third term can be understood as a potential energy of the spring due to the external force.

5 The General Potential

In the previous section, we demonstrated that our model produces coherent states for a general force, $F(t)$, that dies off with time. The next step is to determine if any general potential, $V(x)$, acting between the two masses will do the same. For the the harmonic potential case, we were able to use extended Galilean tranformations [5] and the techniques developed by Kerner [6] to solve for the wave function directly from the Schrödinger equation. However, these techniques are specific to the harmonic potential and therefore useless for a general $V(x)$. Thus, we will have to develop a new method for solving for the wave function of our system.

The first step in developing the new method is to assume a delta function pulse such that

$$F(t) = F_0\delta(t), \tag{74}$$

where F_0 scales the force and $\delta(t)$ is a Dirac delta function centered on $t = 0$. Although the Gaussian model for the force is ultimately more realistic, modeling the pulse as a Dirac delta function at $t = 0$ simplifies the problem and allows us to more easily expand our analysis of the system's behavior.

We now consider the Schrödinger equation for a system of two masses interacting in a general potential, $V(x)$, and subject to an external force of the form given in (74). If we substitute the harmonic potential in (10) for a general form, $V(x)$, we obtain the following Schrödinger equation

$$-\frac{\hbar^2}{2M}\frac{\partial^2\Psi}{\partial R^2} - F_0\delta(t)R\Psi - \frac{\hbar^2}{2\mu}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi - \frac{\mu}{m_1}F_0\delta(t)x\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{75}$$

As before, we can perform separation of variables by assuming that the total wave function is the product of a relative motion wave function, $\psi_{rel}(x,t)$, and a center

of mass wave function, $\psi_{CM}(R,t)$. However, as we are only interested in $\psi_{rel}(x,t)$ and whether it meets the requirements for displacement-operator coherent states, we may ignore the center of mass wave function in this section. Thus, we write the Schrödinger equation for $\psi_{rel}(x,t)$, as given by (36) with $V(x)$ substituted for the harmonic potential and our delta function kick substituted for the force such that

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\psi_{rel}}{\partial x^2} + \left(V(x) - \frac{\mu}{m_1}F_0\delta(t)x + C\right)\psi_{rel} = i\hbar\frac{\partial\psi_{rel}}{\partial t}, \tag{76}$$

where we recall that C is the separation constant and is dependent on time. Then, we pull all of the terms of the Hamiltonian that are not dependent on t together into one time-independent operator, given by

$$H_0 = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + V(x). \tag{77}$$

Using this substitution and rearranging terms, we may rewrite (76) in a more manageable form

$$i\hbar\frac{\partial\psi_{rel}}{\partial t} = \left(H_0 - \frac{\mu}{m_1}F_0\delta(t)x + C(t)\right)\psi_{rel}(x,t). \tag{78}$$

Now, we consider an infinitesimally small amount of time, $-\epsilon \leq t \leq \epsilon$, such that $\psi_{rel}(x,0)$ is approximately the average of the wave functions right before and right after $t = 0$

$$\psi_{rel}(x,0) = \frac{1}{2}\left[\psi_{rel}(x,\epsilon) + \psi_{rel}(x,-\epsilon)\right], \tag{79}$$

and the first time derivative of the wave function at $t = 0$ may be approximated as simply the increase in the wave function divided by the increase in time

$$\frac{\partial\psi_{rel}}{\partial t} = \frac{1}{2\epsilon}\left[\psi_{rel}(x,\epsilon) - \psi_{rel}(x,-\epsilon)\right]. \tag{80}$$

We rewrite the Schrödinger equation given by (78) with the assumption that $-\epsilon \leq t \leq \epsilon$, where ϵ is an infinitesimally small displacement from $t = 0$. Recalling

that, by definition, $\delta(t)$ peaks at $t = 0$ and has a width of 2ϵ, we determine that our time period encompasses the exact moment during which the force acts on the system. Applying (80) to (78) yields

$$\begin{aligned} &i\hbar\left[\psi_{rel}(x,\epsilon) - \psi_{rel}(x,-\epsilon)\right] \\ &\quad = 2\epsilon H_0\psi_{rel}(x,t) - 2\epsilon\frac{\mu}{m_1}F_0\delta(t)x\psi_{rel}(x,t) + 2\epsilon C(t)\psi_{rel}(x,t). \end{aligned} \tag{81}$$

As mentioned previously, the width of the delta function is equal to 2ϵ. Other curious properties of this function are that its peak is infinitely tall and its area is equal to one unit. That is,

$$\int_{-\epsilon}^{\epsilon}\delta(t)dt = 2\epsilon\delta(t) = 1. \tag{82}$$

As for the other terms in (81), we recognize that any terms containing ϵ but not $\delta(t)$ must be vanishingly small. Therefore, by combining (79) and (81) we obtain

$$i\hbar\left[\psi_{rel}(x,\epsilon) - \psi_{rel}(x,-\epsilon)\right] = -\frac{\mu}{2m_1}F_0x\left[\psi_{rel}(x,\epsilon) + \psi_{rel}(x,-\epsilon)\right], \tag{83}$$

which can be rewritten in a more illuminating form

$$\psi_{rel}(x,\epsilon) = \left(\frac{1 + \frac{i\mu}{2\hbar m_1}F_0x}{1 - \frac{i\mu}{2\hbar m_1}F_0x}\right)\psi_{rel}(x,-\epsilon). \tag{84}$$

which is equal to

$$\psi_{rel}(x,\epsilon) = e^{\frac{i\mu}{\hbar m_1}F_0x}\psi_{rel}(x,-\epsilon), \tag{85}$$

for small ϵ.

We will now consider what $\psi_{rel}(x,\epsilon)$ and $\psi_{rel}(x,-\epsilon)$ mean. Both should be approximately equal to some wave function at $t = 0$, as ϵ approaches zero. However, for $\psi_{rel}(x,-\epsilon)$, we are approaching $t = 0$ before the force has acted. This means that the system's wave function must still be that of the stationary state, as it has

not been disturbed yet. Similarly, $\psi_{rel}(x,\epsilon)$ is the wave function right after the force has been applied. Therefore, (85) becomes

$$\psi_{rel}(x,0^+) = e^{\frac{i\mu}{\hbar m_1}F_0 x}\varphi(x,0^-), \tag{86}$$

where $\psi_{rel}(x,0^+)$ is the resultant wave function at $t = 0$ after the force has been applied and $\varphi(x,0^-)$ is the stationary state at $t = 0$ before the system is kicked.

If (86) gives the resultant wave function at $t = 0^+$, we must let the following operator act on it to find the wave function for a later time

$$\psi_{rel}(x,t) = e^{-\frac{i}{\hbar}H_0 t}\psi_{rel}(x,0^+) = e^{-\frac{i}{\hbar}H_0 t}e^{\frac{i\mu}{\hbar m_1}F_0 x}\varphi(x,0^-). \tag{87}$$

Unfortunately, the operators H_0 and x do not commute, and this equation is difficult to evaluate in its current form. Although we know how $e^{-\frac{i}{\hbar}H_0 t}$ acts on one of its eigenstates, we can not easily determine its effect on other wave functions. The rest of this section will deal with generating an analyzable form of (87).

First, we shall multiply the right-hand side of (87) by a quantity equal to one

$$\psi_{rel}(x,t) = e^{-\frac{i}{\hbar}H_0 t}e^{\frac{i\mu}{\hbar m_1}F_0 x}\left(e^{\frac{i}{\hbar}H_0 t}e^{-\frac{i}{\hbar}H_0 t}\right)\varphi(x,0^-). \tag{88}$$

Next, we use the identity given in Parker [13]

$$e^{-A}e^{B}e^{A} = exp\left(e^{-A}Be^{A}\right) \tag{89}$$

and the following substitutions

$$\begin{aligned} A &= \frac{it}{\hbar}H_0 = -\lambda H_0 = -\lambda a \\ B &= x \end{aligned} \tag{90}$$

to rewrite (88) as

$$\begin{aligned}\psi_{rel}(x,t) &= exp\left(\frac{i\mu F_0}{\hbar m_1}e^{-\frac{i}{\hbar}H_0 t}xe^{\frac{i}{\hbar}H_0 t}\right)e^{-\frac{i}{\hbar}H_0 t}\varphi(x,0^-)\\ &= exp\left(\frac{i\mu F_0}{\hbar m_1}e^{\lambda a}xe^{-\lambda a}\right)e^{-\frac{i}{\hbar}H_0 t}\varphi(x,0^-). \qquad (91)\end{aligned}$$

This form for the wave function can be further evaluated in two different ways, which we will call the Heisenberg picture method and the infinite series method. Both methods deal primarily with finding a better form for $e^{\lambda a}xe^{-\lambda a}$.

5.1 The Heisenberg Picture Method

We will begin analyzing (91) by taking our calculations into the Heisenberg picture [12]. Until now, we have been conducting our calculations in the Schrödinger picture, where the wave functions carry all the time dependencies and the operators are time-independent. For instance, each time we used the operator x in the previous sections, what we really meant was x_S, the x-operator in the Schrödinger picture. For simplicity's sake, we shall continue the convention of writing O for O_S, where O is an arbitrary operator.

In the Heisenberg picture, the wave functions are time-independent and the operators are time-dependent. An arbitrary operator, O_H, in the Heisenberg picture is defined in terms of its Schrödinger picture counterpart, O, by the expression

$$O_H = e^{\frac{i}{\hbar}Ht}Oe^{-\frac{i}{\hbar}Ht}. \qquad (92)$$

For our system, it is useful to set the H in (92) equal to $-H_0$. This stipulation changes the picture slightly; however, our picture retains all of the mathematical qualities of the Heisenberg picture. Therefore,

$$O_H = e^{-\frac{i}{\hbar}H_0 t}Oe^{\frac{i}{\hbar}H_0 t}, \qquad (93)$$

where O_H is an operator *only* dependent on time. Then, for our system, the x-operator in the Heisenberg picture is given by

$$x_H = e^{-\frac{i}{\hbar}H_0 t} x e^{\frac{i}{\hbar}H_0 t}. \tag{94}$$

By combining this new expression with (91), we are able to write the wave function as

$$\psi_{rel}(x,t) = e^{\frac{i\mu F_0}{\hbar m_1} x_H} e^{-\frac{i}{\hbar}H_0 t} \varphi(x, 0^-). \tag{95}$$

Therefore, an alternate expression for x_H may allow us to further analyze the wave function. In order to find this expression, we shall continue our analysis of operators in the Heisenberg picture.

The first time derivative of O_H in terms of O is given by

$$\frac{dO_H}{dt} = \frac{i}{\hbar}\left(e^{\frac{i}{\hbar}Ht}[H,O]e^{-\frac{i}{\hbar}Ht}\right), \tag{96}$$

where $[H,O]$ is the commutator of the arbitrary Hamiltonian with the arbitrary operator. For our system, then,

$$\frac{dO_H}{dt} = \frac{-i}{\hbar}\left(e^{-\frac{i}{\hbar}H_0 t}[H_0,O]e^{\frac{i}{\hbar}H_0 t}\right). \tag{97}$$

Thus, the first time derivative of the x-operator in the Heisenberg picture is given by

$$\frac{dx_H}{dt} = \frac{-i}{\hbar}\left(e^{-\frac{i}{\hbar}H_0 t}[H_0,x]e^{\frac{i}{\hbar}H_0 t}\right) \tag{98}$$

for our system. If we recognize that our time-independent Hamiltonian from (77) can be written in terms of the momentum operator as

$$H_0 = \frac{p^2}{2\mu} + V(x), \tag{99}$$

then the commutator $[H_0, x]$ is

$$[H_0, x] = \left[\frac{p^2}{2\mu} + V(x), x\right] = \frac{1}{2\mu}[p^2, x] = \frac{-i\hbar}{\mu}p, \tag{100}$$

since x commutes with any function of x. By substituting (100) into (98) we find that

$$\frac{dx_H}{dt} = \frac{-1}{\mu}\left(e^{-\frac{i}{\hbar}H_0 t} p e^{\frac{i}{\hbar}H_0 t}\right). \tag{101}$$

The operators in the Heisenberg picture obey equations analogous to the classical equations of motion. To show this, we shall consider the relationships between our operators. We already have the expression for $\frac{dx_H}{dt}$ given in (101). According to (93), the momentum operator in the Heisenberg picture is given by

$$p_H = e^{-\frac{i}{\hbar}H_0 t} p e^{\frac{i}{\hbar}H_0 t}. \tag{102}$$

Thus, from (101) and (102), we have shown that

$$p_H = -\mu \frac{dx_H}{dt}, \tag{103}$$

which is analogous to the classical equation of motion $p = mv$.

If a time derivative of each side of (103) is taken, we see that

$$\frac{dp_H}{dt} = -\mu \frac{d^2 x_H}{dt^2}, \tag{104}$$

which is analogous to Newton's Second Law. The minus signs in (103) and (104) are due to the fact that we have defined $H = -H_0$ and are thus not strictly in the Heisenberg picture. We can also write $\frac{dp_H}{dt}$ according to the general formula given in (97)

$$\frac{dp_H}{dt} = \frac{-i}{\hbar}\left(e^{-\frac{i}{\hbar}H_0 t}[H_0, p] e^{\frac{i}{\hbar}H_0 t}\right). \tag{105}$$

The commutator of the time-independent portion of our Hamiltonian and the momentum operator is equal to

$$[H_0, p] = \left[\frac{p^2}{2\mu} + V(x), p\right] = [V(x), p] = i\hbar \frac{dV(x)}{dx}, \tag{106}$$

since p commutes with powers of itself. Therefore, (105) becomes

$$\frac{dp_H}{dt} = \left(e^{-\frac{i}{\hbar}H_0 t} \frac{dV(x)}{dx} e^{\frac{i}{\hbar}H_0 t} \right). \tag{107}$$

Inspection of (93) and (107) yields the following expression

$$\frac{dp_H}{dt} = \frac{dV_H}{dx_H}, \tag{108}$$

both sides of which are forms for a force. Altogether, then, from (104) and (108) we obtain

$$F = \mu \frac{d^2 x_H}{dt^2} = -\frac{dV_H}{dx_H}. \tag{109}$$

Now that we have shown in (103), (104), and (109) that x_H obeys the same equations of motion as the corresponding classical x, we will attempt to use these equations to obtain an expression for x_H. Equation (109) is a second-order differential equation. A form for x_H in terms of t can be obtained by taking two integrals of it and setting the initial conditions such that

$$x_H(t = 0) = x \tag{110}$$

and

$$\left.\frac{dx_H}{dt}\right|_{t=0} = \frac{-1}{\mu} p_H(t = 0) = \frac{-1}{\mu} p, \tag{111}$$

which are seen from inspection of (94) and (103), respectively. Thus, we see that the solution for x_H will have terms in t, x, and p. However, we cannot find an explicit equation for x_H unless the potential, V(x), is specified. Ultimately, then, this method tells us little about a general potential but is useful for specific $V(x)$ that allow (109) to be solved.

5.2 The Infinite Series Method

As we were unable to get an analyzable form of (91) from the Heisenberg picture, we will try another method. Another identity from Parker [13] tells us that

$$e^{\lambda a} x e^{-\lambda a} = x + \lambda[a, x] + \frac{\lambda^2}{2!}[a, [a, x]] + ... = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} C_j, \tag{112}$$

where C_j can be defined recursively. From (112) it is apparent that

$$C_0 = x, \tag{113}$$

$$C_1 = [a, x] = [a, C_0], \tag{114}$$

$$C_2 = [a, [a, x]] = [a, C_1]. \tag{115}$$

It follows, then, that C_j is of the form

$$C_{j+1} = [a, C_j], \tag{116}$$

where $j = 0, 1, 2...$ is an integer.

Now that we have the infinite series for which this method is named, we may begin calculating values of C_j. These quickly become very complicated and unwieldy, but we will show the first few of them and then conjecture on the behavior of the rest. With that aim and the knowledge that $a = H_0$, where H_0 is given by (99), the value of C_1 is

$$C_1 = [a, x] = \frac{1}{2\mu}[p^2, x] = \frac{-i\hbar}{\mu} p, \tag{117}$$

the expression for C_2 is

$$C_2 = [a, C_1] = \left[\frac{1}{2\mu}p^2 + V(x), \frac{-i\hbar}{\mu}p\right] = \frac{-i\hbar}{\mu}[V(x), p] = \frac{\hbar^2}{\mu}\frac{dV(x)}{dx}, \tag{118}$$

and, finally, the expression for C_3 is

$$C_3 = [a, C_2] = \left[\frac{1}{2\mu}p^2 + V(x), \frac{\hbar^2}{\mu}\frac{dV(x)}{dx}\right] = \frac{-i\hbar^3}{2\mu^2}\frac{d^2V(x)}{dx^2}p - \frac{\hbar^4}{2\mu^2}\frac{d^3V(x)}{dx^3}. \tag{119}$$

Beyond C_3, the C_j values get significantly more complicated. Therefore, we will pull all of the constants together and give them for each term as α_{jl}, where l is the number of the term, starting at $l = 0$. The expression for C_4 is then

$$\begin{aligned} C_4 = [a, C_3] &= \left[\frac{1}{2\mu}p^2 + V(x), \frac{-i\hbar^3}{2\mu^2}\frac{d^2V(x)}{dx^2}p - \frac{\hbar^4}{2\mu^2}\frac{d^3V(x)}{dx^3}\right] \\ &= \alpha_{30}\left[\frac{1}{2\mu}p^2 + V(x), \frac{d^2V(x)}{dx^2}p\right] + \alpha_{31}\left[\frac{1}{2\mu}p^2 + V(x), \frac{d^3V(x)}{dx^3}\right] \\ &= \alpha_{30}\left(i\hbar\frac{d^2V(x)}{dx^2} - \frac{\hbar^2}{2\mu}\frac{d^4V(x)}{dx^4}p - \frac{i\hbar}{\mu}\frac{d^3V(x)}{dx^3}p^2\right) \\ &\quad + \alpha_{31}\left(\frac{-i\hbar}{\mu}\frac{d^4V(x)}{dx^4}p - \frac{\hbar^2}{2\mu}\frac{d^5V(x)}{dx^5}\right), \end{aligned} \tag{120}$$

which can be rewritten with time-independent factors, $\sigma_{jl}(x)$, as

$$C_4 = \sigma_{40}(x) + \sigma_{41}(x)p + \sigma_{42}(x)p^2. \tag{121}$$

Note that the term number, l, corresponds to the power of p in the term and increases with larger j values. Thus, as j increases, the power of p will also increase, such that the identity given in (112) becomes an infinite series in p unless the factors, $\sigma_{jl}(x)$, are zero for higher order terms. Inspection of (120) and (121) show that the factors are dependent on spatial derivatives of the potential, $V(x)$, so it is possible that a specific $V(x)$ will cause them to go to zero. In general, however, (91) is of the form

$$\psi_{rel}(x,t) = exp\left(\sum_{h=0}^{\infty} C_h p^h\right) e^{-\frac{i}{\hbar}H_0 t}\varphi(x, 0^-), \tag{122}$$

where C_h is a coefficient related to $\sigma_{jl}(x)$. Thus, the general wave function can not be written in the form

$$\psi_{rel}(x,t) = e^D e^{d_0 x} e^{d_1 p} e^{-\frac{i}{\hbar}H_0 t}\varphi(x, 0^-), \tag{123}$$

where D, d_0, and d_1 do not depend on x, and both d_0 and d_1 are imaginary and time-dependent. Consideration of (123) yields the observation that it is a displacement-

operator coherent state, as $e^{d_1 p}$ is a shift-operator that causes the wave function $\varphi(x, 0^-)$ to shift spatially by some time-dependent amount.

As its wave function does not have the form in (123), the system of two masses interacting in a general potential and subject to an external pulse of force does not produce a wave packet that moves classically and keeps its shape with time. However, this does not conclusively prove that the system does not produce coherent states in general, only that it does not meet the criteria for displacement-operator coherent states [8].

6 The Delta-Kicked Harmonic Oscillator

In the previous section, we developed techniques for determining the wave function of a system of two masses in a general potential subjected to a delta function pulse. In this section, we will test these techniques on our original case of two masses interacting in a harmonic potential. The harmonic potential case is a good check of the Heisenberg picture and infinite series methods since we already know it produces coherent states.

Before we employ the new methods, however, we will generate the relative wave function, $\psi_{rel_n}(x,t)$, of a delta-kicked harmonic oscillator using our first method of directly solving the Schrödinger equation. This will allow us to compare the results of the three techniques.

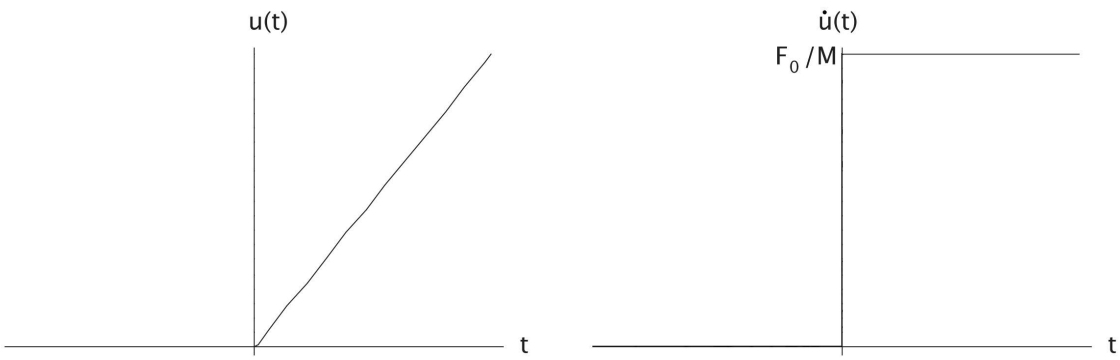

Figure 8: Graphs of $u(t)$ and $\dot{u}(t)$.

The general form of $\psi_{rel_n}(x,t)$ is given by (55). In order make this equation specific to the delta function pulse, we must simply calculate expressions for $u(t)$ and $\xi(t)$. It is important to remember throughout these calculations that the improper integral of the delta function is the step function, $\theta(t)$. If the force is given by (74), then the equation for $u(t)$ as given by (22) yields the differential equation

$$F_0\delta(t) = M\ddot{u}. \tag{124}$$

This expression may be integrated twice with the initial conditions that the lab frame and the center of mass frame agree at $t = 0$ and that the distance between them is not increasing at $t = 0$. That is, $u(t) = 0$ and $\dot{u}(t) = 0$ at $t = 0$. In this case, the solution to (124) is given by

$$u(t) = \frac{F_0 t}{M}\theta(t). \tag{125}$$

The graphs of this function and its first derivative show us that the center of mass frame is instantaneously accelerated away from the lab frame when the force acts at $t = 0$ and then continues at a constant speed relative to the lab frame thereafter (see Figure 8). Again, this is not as realistic as the Gaussian pulse model of the force, but it does simplify the math immensely, as can be seen from the relative complexities of (59) and (125).

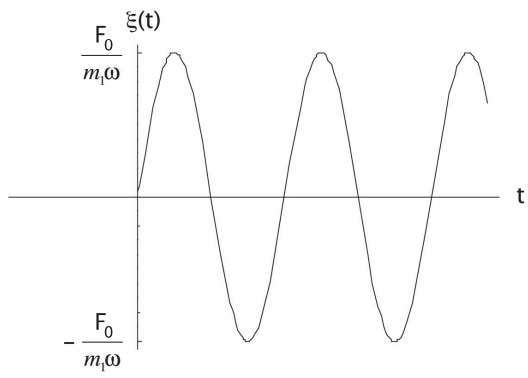

Figure 9: Graph of $\xi(t)$.

Next, substituting our new force into (43) yields

$$\frac{\mu}{m_1} F_0 \delta(t) = \mu\ddot{\xi} + k\xi. \tag{126}$$

We solve this nonhomogenous, second-order linear differential equation by the same method used for the Gaussian pulse case, where the particular solution is a function

of the Wronskian [14]. Once again stipulating that the lab frame and the new frame agree before the pulse has acted on the system, such that $\xi(t) = 0$ when $t < 0$, we obtain

$$\xi(t) = \frac{F_0}{m_1\omega}\sin(\omega t)\theta(t), \tag{127}$$

where $\omega = \sqrt{\frac{k}{\mu}}$ is the angular frequency of the oscillator. The graph of $\xi(t)$ shows that the relative frame begins oscillating with respect to the lab frame after the force acts on the system at $t = 0$ (see Figure 9).

If we substitute (125) and (127) into the relative wave function from (55), we can obtain the following form for for the relative wave function of the system when it is hit by a Dirac delta function pulse

$$\psi_{rel_n}(x,t) = (\frac{\beta^2}{\pi})^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta(x-\xi))e^{-\frac{1}{2}(\beta(x-\xi))^2}e^{-\frac{i}{\hbar}(C+E_n)t} \times e^{-\frac{i}{\hbar}\frac{\mu F_0}{2{m_1}^2\omega}\theta(t)e^{-i\omega t}(F_0\sin(\omega t)-2m_1\omega x)}. \tag{128}$$

This wave function, which was found by the old method, will be used as a check of our two new techniques as they are applied to the harmonic potential case.

6.1 The Heisenberg Picture Method

If we consider the form of the resultant wave function given in (95) for the harmonic potential case, we see that the stationary state, $\varphi(x, 0^-)$, must be the unperturbed quantum harmonic oscillator wave function, given by (68). We will write this wave function at $t = 0$ as $\varphi_n(x, 0^-)$. Thus, (95) becomes

$$\psi_{rel_n}(x,t) = e^{\frac{i\mu F_0}{\hbar m_1}x_H}e^{-\frac{i}{\hbar}H_0 t}\varphi_n(x,0^-). \tag{129}$$

Next we must recognize that

$$V(x) = \frac{1}{2}\mu\omega^2 x^2, \tag{130}$$

and, therefore,

$$H_0 = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\mu\omega^2 x^2. \tag{131}$$

Since $\varphi_n(x, 0^-)$ is an eigenstate of this Hamiltonian, (129) becomes

$$\psi_{rel_n}(x,t) = e^{\frac{i\mu F_0}{\hbar m_1}x_H} e^{-\frac{i}{\hbar}E_n t}\varphi_n(x, 0^-), \tag{132}$$

where $e^{-\frac{i}{\hbar}E_n t}$ is the time-dependent portion of the unperturbed quantum harmonic oscillator wave function, so (132) is

$$\psi_{rel_n}(x,t) = e^{\frac{i\mu F_0}{\hbar m_1}x_H}\varphi_n(x,t). \tag{133}$$

The final step in finding the form of $\psi_{rel_n}(x,t)$ is to determine x_H from its equations of motions. If we combine the harmonic potential (130) with equation (109), we find that

$$\frac{d^2 x_H}{dt^2} = -\omega^2 x_H. \tag{134}$$

The general solution to this equation is then

$$x_H = k_1\cos(\omega t) + k_2\sin(\omega t), \tag{135}$$

where k_1 and k_2 are constants to be determined by the initial conditions. When we apply the initial conditions given in (110) and (111), we find that

$$x_H = x\cos(\omega t) + \frac{-p}{\mu\omega}\sin(\omega t). \tag{136}$$

Therefore, we can rewrite (133) as

$$\psi_{rel_n}(x,t) = e^{\frac{i\mu F_0}{\hbar m_1}x\cos(\omega t) - \frac{iF_0}{\hbar m_1\omega}p\sin(\omega t)}\varphi_n(x,t), \tag{137}$$

where we will define x- and p-independent coefficients d_0 and d_1 such that (137) becomes

$$\psi_{rel_n}(x,t) = e^{d_0 x + d_1 p}\varphi_n(x,t). \tag{138}$$

The Baker-Campbell-Hausdorff formula [11] for the case where the commutator $[A, B]$ is constant is given by

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{139}$$

Thus, since $[p, x] = i\hbar$, (138) becomes

$$\psi_{rel_n}(x,t) = e^{d_0 x} e^{d_1 p} e^{-\frac{i\hbar}{2} d_0 d_1} \varphi_n(x,t) = e^{-\frac{i\hbar}{2} d_0 d_1} e^{d_0 x} e^{d_1 p} \varphi_n(x,t), \tag{140}$$

where we recall that d_0 and d_1 are not operators, so they may be moved throughout the equation. The expression $e^{d_1 p}$ is a shift operator that causes the wave function to be shifted spatially by an amount d_1. That is,

$$\psi_{rel_n}(x,t) = e^{-\frac{i\hbar}{2} d_0 d_1} e^{d_0 x} \varphi_n(x - d_1, t), \tag{141}$$

which shows us that the resultant wave function meets the displacement-operator definition of coherent states [8]. That is, the probability distribution, given by

$$\psi^*_{rel_n} \psi_{rel_n} = \varphi^*_n(x - d_1, t) \varphi_n(x - d_1, t), \tag{142}$$

is going to keep its shape and have classical motion over time. If we compare (141) with (128), we see that the functions have the same form. Thus, the Heisenberg method has delivered the correct form of the wave function when $V(x)$ is specified to be the harmonic potential.

6.2 The Infinite Series Method

The wave function we are attempting to evaluate is given by (91). As in the previous section, since the wave function of the unperturbed quantum harmonic oscillator is an eigenstate of H_0, we can rewrite (91) as

$$\psi_{rel}(x,t) = exp\left(\frac{i\mu F_0}{\hbar m_1} e^{\lambda a} x e^{-\lambda a}\right) e^{-\frac{i}{\hbar} E_n t} \varphi_n(x, 0^-), \tag{143}$$

where $\varphi_n(x, 0^-)$ is the time-independent part of the wave function of the unperturbed oscillator and the time-dependent part is such that

$$\psi_{rel}(x,t) = exp\left(\frac{i\mu F_0}{\hbar m_1} e^{\lambda a} x e^{-\lambda a}\right) \varphi_n(x,t). \tag{144}$$

We now consider the quantity $e^{\lambda a} x e^{-\lambda a}$ as given by (112). For the harmonic oscillator case, the C_j values given by (113) and (117)-C33 are

$$C_0 = x, \tag{145}$$

$$C_1 = \frac{-i\hbar}{\mu} p, \tag{146}$$

$$C_2 = \hbar^2 \omega^2 x = \sigma_{20}(x), \tag{147}$$

$$C_3 = \frac{-i\hbar^3 \omega^2}{2\mu} p = \sigma_{31}(x). \tag{148}$$

Thus, we see that, in general,

$$C_{2k} = (\hbar\omega)^{2k} C_0 \tag{149}$$

$$C_{2k+1} = (\hbar\omega)^{2k} C_1. \tag{150}$$

Combining these recursive formulae with (112) and remembering that we defined $\lambda = \frac{it}{\hbar}$, we obtain

$$e^{\lambda a} x e^{-\lambda a} = C_0 \sum_{k=0}^{\infty} (-1)^k \frac{(\omega t)^{2k}}{2k!} + \frac{iC_1}{\hbar\omega} \sum_{k=0}^{\infty} (-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}. \tag{151}$$

The series in (151) may be rewritten such that

$$e^{\lambda a} x e^{-\lambda a} = C_0 \cos(\omega t) + \frac{iC_1}{\hbar\omega} \sin \omega t. \tag{152}$$

Therefore, substituting (145) and (146) into (152), we may rewrite (144) as

$$\psi_{rel}(x,t) = e^{d_0 x + d_1 p} \varphi_n(x,t) \tag{153}$$

where d_0 and d_1 are not operators. This form is identical to the form found in the previous section using the Heisenberg method, given by (138). Thus, the infinite series method also agrees with (128) and shows that the harmonic potential produces displacement-operator coherent states.

7 Conclusion

The classical system of two masses connected by a spring and subject to a time-dependent force can be used to model a quantum harmonic oscillator system. The wave function of this system is equal to the product of its center of mass wave function and its relative motion wave function. We have shown that the relative wave function for this model produces coherent states by both the annihilation-operator definition and the displacement-operator definition for any general time-limited force [8]. The exception to this rule is when the forcing term is turned off just as the oscillator has reached its equilibrium position and has zero speed. In this case, the oscillator returns to its original stationary state.

Our results are related to the Mössbauer effect for the case when the force is cause by the emission of a photon. The Mössbauer effect occurs in solids with a sufficiently rigid lattice structure, where the emission of the photon is not energetic enough to excite the modes of the lattice, so the recoil is absorbed by the bulk of the solid. Thus, the Mössbauer effect is not truly a "recoilless emission"; rather, the solid is so large compared to the force exerted by the photon that the emission merely *seems* recoilless [1]-[3]. The annihilation-operator coherent states of our model show that for any time-limited force, there is a probability that the wave function remains in its original state. That is, if our model represents a diatomic molecule, and one atom emits a photon, there is a chance that the oscillations between them are not excited. The difference between this case and the Mössbauer effect is that for the diatomic molecule we see a recoil of the molecule whether its oscillations are excited or not.

We have also developed two different techniques, the Heisenberg picture method

and the infinite series method, for determining whether a delta-kicked system of two masses in a general potential produces coherent states. We have further found that only special forms of the potential will produce displacement-operator coherent states. Future research on this topic should include the development of techniques to determine whether the general potential will produce either minimum-uncertainty coherent states or annihilation-operator coherent states.

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