Tempered holomorphic functions in analytic geometry

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Dedicated to the memory of Tim Lichtnau

Abstract. We introduce the theory of analytic rings of Clausen and Scholze and apply it to rediscover the classical notion of a tempered holomorphic function. More precisely, we show that the derived q-gaseous completion of $\mathbb{C}[q]$ is concentrated in degree 0, given by the ring of tempered holomorphic functions on the open unit disc.

This a revised online version of my thesis. Please contact me at lmv40 (at) cam (dot) ac (dot) uk for comments and corrections.

Introduction

The aim of this thesis is to rediscover the classical notion of a *tempered holomorphic function* within the analytic geometry of Clausen and Scholze. Our main result is as follows.

Theorem A. *The derived q-gaseous completion of $\mathbb{C}[q]$ is concentrated in degree 0, given by the ring of tempered holomorphic functions on the open unit disc.*

We will also obtain a new proof of the following arithmetic result from [CS24, Lecture 14].

Theorem B. *Let $\mathbb{Z}[\hat{q}]$ be the free Abelian group on a null sequence equipped with its natural ring structure. Then the derived q-gaseous completion of $\mathbb{Z}[\hat{q}]$ is concentrated in degree 0, given by the ring of arithmetic power series with coefficients of polynomial growth.*

Before giving definitions, let us motivate tempered holomorphic functions. They play a key role in a story that starts with Hilbert's 21st problem.

> Aus der Theorie der linearen Differentialgleichungen mit einer unabhängigen Veränderlichen z möchte ich auf ein wichtiges Problem hinweisen, welches wohl bereits Riemann im Sinne gehabt hat, und welches darin besteht, zu zeigen, daß es stets *eine lineare Differentialgleichung der Fuchsschen Klasse mit gegebenen singulären Stellen und einer gegebenen Monodromiegruppe giebt.* [Hil00]
>
> In the theory of linear differential equations in one variable z, I would like to point out an important problem, which likely already Riemann had in mind. The problem is to show that there always exists *a linear differential equation of Fuchsian class with given singular points and given monodromy group.*

This problem asks for the surjectivity of a map sending certain differential equations to certain topological invariants of their solutions. Ideally, we can hope for an equivalence of an "analytic" category of differential equations and a "topological" category of solutions. While the statement Hilbert likely had in mind is false by an example of Bolibrukh [Bol90], precise such equivalences, called *Riemann-Hilbert correspondences* nowadays, have been established. Deligne proved one in a complex-algebraic case [Del70], followed by Kaschiwara in a complex-analytic case [Kas84]. Recently, p-torsion versions have been proved by Bhatt-Lurie [BL19] and by Mann [Man22].

Example. Consider the differential operators $P := z^2\partial_z + z$ and $Q := z^2\partial_z + 1$ on $X = \mathbb{C}$. By Picard-Lindelöf, on a connected open $U \subseteq \mathbb{C}$, the equation $Pu = 0$ has holomorphic solutions $u(z) = a/z$ for $a \in \mathbb{C}$ if $0 \notin U$, and $u(z) = 0$ if $0 \in U$. Arguing analogously for Q, we find that the solutions of $Pu = 0$ and $Qu = 0$ are given by the sheaves of complex vector spaces

$$\mathbb{C}_{X\setminus\{0\}} \cdot \frac{1}{z} \quad \text{and} \quad \mathbb{C}_{X\setminus\{0\}} \cdot \exp(\frac{1}{z}).$$

Even though P and Q are distinct (precisely, their corresponding $\mathcal{D}_{\mathbb{C}}$-modules are not isomorphic), their holomorphic solutions are isomorphic as sheaves of complex vector spaces.

In this example, an obstruction to a Riemann-Hilbert correspondence is present. A close look suggests that the problem might have to do with the function $\exp(1/z)$, which grows enormously fast near its singularity. The notion of a tempered holomorphic function makes the sense in which $\exp(1/z)$ is not "tame" enough precise. On bounded opens in $\mathbb{C}$, it can be given as follows.

Definition. Let $U \subseteq \mathbb{C}$ be bounded and open with boundary ∂U. A holomorphic function $f\colon U \to \mathbb{C}$ is **tempered** if there exists an integer $N \geq 0$ such that $\sup_{z\in U} \mathrm{dist}(z, \partial U)^N |f(z)| < \infty$.

Roughly, a holomorphic function is tempered if it does not grow too fast near the boundary of its domain. For example, $1/z$ is tempered on the punctured open unit disc, while $\exp(1/z)$ is not.

In this thesis, we rediscover tempered holomorphic functions through "analysis-free formal nonsense". In [CS19, CS20, CS22, CS24], Clausen and Scholze introduced a theory of analytic geometry that resolves two shortcomings of previous approaches. It unifies Archimedean and non-Archimedean geometry, and it allows for well-behaved categories of quasi-coherent sheaves.

The basic building blocks in this theory are *analytic rings*, which play a role analogous to that of rings in algebraic geometry. An analytic ring combines algebra, topology, and analysis within one object. To mix algebra and topology, we consider *light condensed rings*, which are certain sheaves of rings on the category of metrisable profinite sets. They are close to topological rings and enjoy the property that the category of modules over a light condensed ring is closed symmetric monoidal Grothendieck Abelian. To do analysis, we then single out the modules we consider complete. A *pre-analytic ring structure* on a light condensed ring $A^\triangleright$ is a full subcategory $\mathrm{Mod}_A \subseteq \mathrm{Mod}_{A^\triangleright}$ subject to certain closure properties. It is an *analytic ring structure* if moreover $A^\triangleright \in \mathrm{Mod}_A$. The closure properties guarantee that the inclusion $\mathrm{Mod}_A \hookrightarrow \mathrm{Mod}_{A^\triangleright}$ has a left adjoint by the adjoint functor theorem, which we regard as a completion functor.

The most recent iteration of the theory in [CS24] (switching to the *light* setting) provides a key tool to construct examples. The *free Abelian group on a null sequence* $P := \mathbb{Z}[\mathbb{N} \cup \infty]/\mathbb{Z}[\infty]$ is an internally projective light condensed Abelian group, and we can use it to capture completeness in terms of the summability of null sequences. Let $\mathrm{shift}\colon P \to P$ be the map induced by $n \mapsto n+1$.

Definition. Let $A^\triangleright$ be a light condensed ring and let $f \in A^\triangleright(*)$. An $A^\triangleright$-module M is ***f*-gaseous** if $\mathrm{id} - f\mathrm{shift}$ induces an isomorphism on $\underline{\mathrm{Hom}}_{A^\triangleright}(A^\triangleright \otimes P, M)$.

Intuitively, $\underline{\mathrm{Hom}}_{A^\triangleright}(A^\triangleright \otimes P, M)$ consists of the null sequences in M and $\mathrm{id} - f\mathrm{shift}$ acts by

$$(m_0, m_1, \ldots) \mapsto (m_0 - fm_1, m_1 - fm_2, \ldots).$$

If this map is an isomorphism, then $(m'_0, m'_1, \ldots) \mapsto (\sum_{i=0}^\infty m'_i f^i, \sum_{i=1}^\infty m'_i f^{i-1}, \ldots)$ should be its inverse, witnessing the summability of null sequences against f. For example, $\mathbb{R}$ will be f-gaseous for any $f \in (0,1)$ because null sequences in $\mathbb{R}$ are summable against such f. The category of f-gaseous modules is a pre-analytic ring structure on $A^\triangleright$, and we call its completion functor *f-gaseous completion*. It has a derived analogue, called *derived f-gaseous completion*. The tension in Theorem A lies in between the very concrete notion of a tempered holomorphic function and the formal means we choose to recover it.

Outline

In Section 1, we introduce condensed mathematics in its recent light formulation. Section 2 is devoted to analytic rings with an emphasis on the gaseous theory. The final Section 3 is concerned with our results. There, we also provide more background on tempered holomorphic functions and give an outline of the proofs. The author hopes that a reader familiar with analytic rings will be able to follow Section 3 right away, and that a reader looking for an introduction to Clausen's and Scholze's theory will find Sections 1 and 2 helpful on their own.

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1 Light condensed mathematics

The starting point of condensed mathematics is the assessment that topological spaces mix so poorly with algebra that they should be replaced by a different notion. The main issue with topology is that categories of topological modules are usually not Abelian. For example, in the category of topological Abelian groups, the set-theoretic identity $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{eucl}}$ from the discrete to the Euclidean reals is monic and epic but not an isomorphism. Condensed mathematics resolves this problem in the spirit of Grothendieck. Viewing morphisms as more fundamental than objects, it replaces the category of topological spaces by a suitable category of sheaves, so that a topological space X can be replaced by its functor of points $\mathrm{Cont}(-, X)$.

In Subsection 1.1, we introduce *light profinite sets*, providing a reasonable category of test spaces. Afterwards, in Subsection 1.2, we choose a Grothendieck topology on it and study the resulting sheaves, called *light condensed sets*. In Subsection 1.3, we introduce and study the corresponding Abelian sheaves, the *light condensed Abelian groups*. A final Subsection 1.4 is concerned with the study of one particular light condensed Abelian group, the *free light condensed Abelian group on a null sequence*. We will follow [CS24, Lectures 1 – 4].

1.1 Light profinite sets

Let us begin with the equivalence of a "combinatorial" and a "topological" category.

Proposition 1.1. *Let* $\mathcal{F}\mathrm{in}$ *denote the category of finite sets and let* $\mathcal{T}\mathrm{op}$ *denote the category of topological spaces. Then the following categories are equivalent:*

(1) $\mathrm{Pro}(\mathcal{F}\mathrm{in})$*, the pro-category of* $\mathcal{F}\mathrm{in}$*. Its objects are diagrams* $I^{\mathrm{op}} \to \mathcal{F}\mathrm{in}$ *for filtered posets* I*, written "*$\varprojlim_i S_i$*", and* $\mathrm{Mor}_{\mathrm{Pro}(\mathcal{F}\mathrm{in})}(\text{“}\varprojlim_i S_i\text{”}, \text{“}\varprojlim_j T_j\text{”}) \coloneqq \varprojlim_j \varinjlim_i \mathrm{Map}(S_i, T_j)$*.*

(2) The full subcategory of $\mathcal{T}\mathrm{op}$ *spanned by the totally disconnected compact Hausdorff spaces.*

Explicitly, "$\varprojlim_i S_i$*"* $\mapsto \varprojlim_i S_i$ *defines an equivalence*

Totally disconnected compact Hausdorff spaces are closed under limits, so "$\varprojlim_i S_i$" $\mapsto \varprojlim_i S_i$ indeed defines a functor between the two categories. The following is the key to full faithfulness.

Lemma 1.2. *Let* $S = \varprojlim_i S_i$ *be a cofiltered limit of finite discrete spaces and let* T *be any discrete space. Then the map* $\varinjlim_i \mathrm{Map}(S_i, T) \to \mathrm{Cont}(\varprojlim_i S_i, T), [f \colon S_i \to T] \mapsto f \circ \mathrm{pr}_i$ *is bijective.*

Proof. We may assume that all S_i are non-empty. Let us first prove the lemma in the case when all projections $\mathrm{pr}_i \colon \varprojlim_i S_i \to S_i$ are surjective. Let $\varphi_{k\ell}$ denote the transition map $S_k \to S_\ell$.

For injectivity, take $[f \colon S_i \to T], [g \colon S_j \to T] \in \varinjlim_i \mathrm{Cont}(S_i, T)$ with $f \circ \mathrm{pr}_i = g \circ \mathrm{pr}_j$. Take i_0 with $i_0 \geq i, j$. Then $f \circ \varphi_{i_0,i} \circ \mathrm{pr}_{i_0} = f \circ \mathrm{pr}_i = g \circ \mathrm{pr}_j = g \circ \varphi_{i_0,j} \circ \mathrm{pr}_{i_0}$. As pr_{i_0} is surjective, we get $f \circ \varphi_{i_0,i} = g \circ \varphi_{i_0,j}$, and thus $[f] = [g]$ in the colimit.

For surjectivity, let $f \colon \varprojlim_i S_i \to T$ be continuous. We show that f factors through some S_{i_0}. As $\varprojlim_i S_i$ is compact, we may assume that T is finite, say $T = \{t_0, \ldots, t_n\}$. Set $U_j \coloneqq f^{-1}(t_j)$, then

$$S = \textstyle\bigsqcup_{j=0}^n U_j.$$

Since $\{\mathrm{pr}_i^{-1}(x) : i \in I, x \in S_i\}$ is a basis of $\varprojlim_i S_i$ and each U_j is compact, we have

$$U_j = \textstyle\bigsqcup_{k=0}^{n_j} U_{j,k},$$

where $U_{j,k} \coloneqq \mathrm{pr}_{i(j,k)}^{-1}(s_{i(j,k)})$ for some $s_{i(j,k)} \in S_{i(j,k)}$. Pick i_0 with $i_0 \geq i(j,k)$ for all j, k. Then, for all $x \in S_{i_0}$, there is a unique $j \in \{0, \ldots, n\}$ with $x \in \mathrm{pr}_{i_0}(U_j)$. We can thus define a map $\overline{f} \colon S_{i_0} \to T$ sending x to the unique t_j with $x \in \mathrm{pr}_{i_0}(U_j)$ and have $f = \overline{f} \circ \mathrm{pr}_{i_0}$ as wanted.

Now let $\varprojlim_i S_i$ be any cofiltered limit of finite discrete spaces, possibly with non-surjective projections. Set $S'_j := \bigcap_{i \geq j} \operatorname{im}(\varphi_{ij} \colon S_i \to S_j) \subseteq S_j$. Then the φ_{ij} restrict to surjections $\varphi'_{ij} \colon S'_i \to S'_j$ and we can consider $\varprojlim_i S'_i$. Since all tranistion maps are surjective, so are all projections $\mathrm{pr}'_j \colon \varprojlim_i S'_i \to S'_j$ by a compactness argument.

As shown above, the natural map $\varinjlim_i \operatorname{Map}(S'_i, T) \to \operatorname{Cont}(\varprojlim_i S'_i, T)$ is bijective.

The composites $\varprojlim_i S'_i \to S'_i \hookrightarrow S_i$ induce an injective continuous map $\varprojlim_i S'_i \to \varprojlim_i S_i$. It is also surjective, since for all $(s_i) \in \varprojlim_i S_i$ we have $s_j \in S'_j = \bigcap_{i \geq j} \operatorname{im} \varphi_{ij}$ for all j. Hence, it is a homeomorphism as a continuous bijection of compact Hausdorff spaces.

On the other hand, the surjection $\varinjlim_i \operatorname{Map}(S_i, T) \to \varinjlim_i \operatorname{Map}(S'_i, T)$ is also injective. Take $[f \colon S_i \to T]$, $[g \colon S_j \to T]$ in the left colimit with $[f|_{S'_i}] = [g|_{S'_j}]$. This means that we find $k \geq i, j$ such that $f|_{S'_i} \circ \varphi'_{ki} = g|_{S'_j} \circ \varphi'_{kj}$. If $S'_k = S_k$, then $[f] = [g]$ follows immediately. If not, say if $S_k \setminus S'_k = \{x_1, \ldots, x_n\}$, then $x_m \notin \bigcap_{\ell \geq k} \operatorname{im} \varphi_{\ell k}$ for all $m \in \{1, \ldots, n\}$, and we find $\ell_1, \ldots, \ell_n$ with $x_m \notin \operatorname{im} \varphi_{\ell_m k}$. Take ℓ with $\ell \geq \ell_1, \ldots \ell_n$, then $f \circ \varphi_{\ell i} = g \circ \varphi_{\ell j}$ shows $[f] = [g]$.

We conclude that $\varinjlim_i \operatorname{Map}(S_i, T) \to \operatorname{Cont}(\varprojlim_i S_i, T)$ is bijective. □

It is worth recording the following ingredient in the proof of Lemma 1.2 as an auxiliary result.

Lemma 1.3. *The category* $\operatorname{Pro}(\mathcal{F}\mathrm{in})$ *is equivalent to its full subcategory* $\operatorname{Pro}^{\mathrm{surj}}(\mathcal{F}\mathrm{in})$ *spanned by the diagrams with surjective transition maps.*

Proof of Proposition 1.1. Full faithfulness holds by Lemma 1.2. For essential surjectivity, let S be a totally disconnected compact Hausdorff space. Define I as the set of finite disjoint clopen covers of S,

$$I := \{\{U_j\}_{j \in J} : J \text{ is finite and } U_j \subseteq S \text{ is clopen for all } j \in J \text{ such that } X = \textstyle\bigsqcup_{j \in J} U_j\},$$

and declare that we have $\mathfrak{V} \leq \mathfrak{U}$ for $\mathfrak{V}, \mathfrak{U} \in I$ if and only if $\mathfrak{U}$ refines $\mathfrak{V}$. The cover $\{U \cap V\}_{U \in \mathfrak{U}, V \in \mathfrak{V}}$ refines $\mathfrak{U}$ and $\mathfrak{V}$, so I is filtered. Equipping each $\mathfrak{U} \in I$ with the discrete topology, we have maps $\mathfrak{V} \to \mathfrak{U}$ whenever $\mathfrak{U} \leq \mathfrak{V}$, and, for $\mathfrak{U} \in I$, the continuous maps $S \to \mathfrak{U}$, $(x \in U_i) \mapsto U_i$ induce a continuous map of compact Hausdorff spaces $S \to \varprojlim_{\mathfrak{U} \in I} \mathfrak{U}$. We show that it is bijective.

For injectivity, take $x, y \in S$ with $x \neq y$. As S is compact Hausdorff and totally disconnected, there exists a clopen neighbourhood U of x that does not contain y. Then $\{U, (S \setminus U)\}$ witnesses that x and y have different images in $\varprojlim_{\mathfrak{U} \in I} \mathfrak{U}$. For surjectivity, take $(U_{\mathfrak{U}})_{\mathfrak{U} \in I} \in \varprojlim_{\mathfrak{U} \in I} \mathfrak{U}$. Given finitely many $\mathfrak{U}_1, \ldots \mathfrak{U}_n \in I$ we find a common refinement $\mathfrak{U}_i \leq \mathfrak{V}$, and then have $U_{\mathfrak{V}} \subseteq U_{\mathfrak{U}_i}$ for all $i \in \{1, \ldots, n\}$. Hence the intersection of any finite subcollection of $(U_{\mathfrak{U}})_{\mathfrak{U} \in I}$ is non-empty. Then also $\bigcap_{\mathfrak{U} \in I} U_{\mathfrak{U}}$ is non-empty by compactness. Any element of $\bigcap_{\mathfrak{U} \in I} U_{\mathfrak{U}}$ is a preimage of $(U_{\mathfrak{U}})_{\mathfrak{U} \in I}$. □

With Proposition 1.1 in mind, we commit to the following definition.

Definition 1.4. A **profinite set** is a totally disconnected compact Hausdorff space.

Examples 1.5.

(1) The one-point-compactification of the natural numbers $\mathbb{N} \cup \infty$ is a profinite set. Indeed, $\mathbb{N} \cup \infty = \varprojlim_n \{0, 1, \ldots, n, \infty\}$ where the transition map $\{0, 1, \ldots, n, \infty\} \to \{0, 1, \ldots, m, \infty\}$ for $m \leq n$ fixes $\{0, \ldots, m\}$ and sends all $i \in \{m+1, \ldots, n, \infty\}$ to ∞.

(2) The Cantor set $\mathfrak{C} = \{0, 1\}^{\mathbb{N}} \cong \varprojlim_n \{0, 1\}^n$ is a profinite set.

(3) The Stone-Čech compactification of the natural numbers $\beta\mathbb{N}$ is a profinite set.

We record an important structural property, which already showed in the proof of Lemma 1.2.

Lemma 1.6. *Let S be a profinite set. Then every open cover of S has a refinement by a cover consisting of finitely many disjoint clopens.*

Proof. Write $S = \varprojlim_{i \in I} S_i$ with S_i finite and discrete. A basis of S is $\{\mathrm{pr}_i^{-1}(s_i) : i \in I, s_i \in S_i\}$, so upon refining and using compactness, we may assume that the cover is given by $\bigcup_{j=1}^n \mathrm{pr}_{i(j)}^{-1}(s_{i(j)})$. Take $i \in I$ with $i \geq i(j)$ for all $j \in \{1, \ldots, n\}$. Then $\bigsqcup_{t \in S_i} \mathrm{pr}_i^{-1}(t)$ is a refinement as wanted. □

The light profinite set $\mathbb{N} \cup \infty$ classifies convergent sequences with a specified limit point, so it is certainly a reasonable test space. The Cantor set will prove useful because every metrisable compact Hausdorff space admits a continuous surjection from it. In a sense we make precise next, $\beta\mathbb{N}$ is "too big" to be strictly relevant.

Proposition 1.7. *The equivalence of Proposition 1.1 restricts to an equivalence of the following full subcategories:*

(1) The sequential pro-category of finite sets $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ *(spanned by diagrams* $\mathbb{N}^{\mathrm{op}} \to \mathcal{F}\mathrm{in}$*).*

(2) The category of metrisable totally disconnected compact Hausdorff spaces.

Proof. By Urysohn's metrisation theorem, a compact Hausdorff space is metrisable if and only if it has a countable basis. Therefore, it suffices to show that a profinite set S can be written as a sequential limit of finite discrete spaces if and only if it has a countable basis.

Any sequential limit of finite discrete spaces $\varprojlim_n S_n$ has the countable basis $\{\mathrm{pr}_n^{-1}(s_n) : n \in \mathbb{N}, s_n \in S_n\}$. Conversely, let S be a profinite set admitting a countable basis $\mathcal{B}$. Then S has only countably many clopens. Indeed, any clopen of S is a finite union of sets in $\mathcal{B}$, so

$$|\{\text{clopens of } S\}| \leq \textstyle\sum_{n=0}^{\infty} |\{\text{unions of } n \text{ sets in } \mathcal{B}\}| \leq \sum_{n=0}^{\infty} \omega^n = \omega.$$

As in the proof of Proposition 1.1, we can write S as a cofiltered limit over the poset I of finite partitions of S into non-empty clopens. For the cardinality of I we obtain

$$|I| \leq \textstyle\sum_{n=1}^{\infty} |\{\text{clopens of } S\}|^n \leq \sum_{n=1}^{\infty} \omega^n = \omega.$$

It follows that S is a countable cofiltered limit of finite discrete spaces. Taking an increasing subsequence of an enumeration, we see that S is a sequential such limit. □

Definition 1.8. A profinite set is **light** if it is metrisable.

Examples 1.9.

(1) The one-point compactification of the natural numbers $\mathbb{N} \cup \infty$ is light.

(2) The Cantor set $\mathfrak{C}$ is light.

(3) The Stone-Čech compactification of the natural numbers $\beta\mathbb{N}$ is not light. Indeed, the universal property $\mathrm{Cont}(\beta\mathbb{N}, \mathbb{F}_2) = \mathrm{Cont}(\mathbb{N}, \mathbb{F}_2)$ shows that $\beta\mathbb{N}$ has uncountably many clopens, hence it cannot have a countable basis, hence it cannot be metrisable.

Light profinite sets are our test spaces of choice. They are intuitive spaces and have the technical advantage that their category is essentially small, allowing us to define sheaves without worrying about set theory. Before introducing them, we highlight one last structural property.

Lemma 1.10. *Non-empty light profinite sets are injective in the category of profinite sets. In particular, any injection of non-empty light profinite sets has a retraction.*

Proof. Let S be a non-empty light profinite set. We have to show that for any injection $Z \hookrightarrow X$ of profinite sets, any continuous map $Z \to S$ can be extended to a continuous map $X \to S$.

Proving this for $S = \{0, 1\}$ amounts to proving that any clopen subset of Z extends to a clopen subset of X. Let $U \subseteq Z$ be clopen. As X is totally disconnected compact Hausdorff, for each $x \in U$ and $z \in Z \setminus U$ we find a clopen $V_{x,z} \subseteq X$ such that $x \in V_{x,z}$ and $z \notin V_{x,z}$. Set $V_x \coloneqq \bigcap_{z \in Z \setminus U} V_{x,z}$. Then $V \coloneqq \bigcup_{x \in U} V_x$ is clopen in X with $V \cap Z = U$.

The assertion now also follows by induction when S is any finite set, because $\mathrm{Cont}(X, \{0, \dots, n\})$ is in bijection with tuples of clopens $(U_0, \dots, U_n)$ satisfying $X = \bigsqcup_{i=0}^{n} U_i$.

For the general case, let $S = \varprojlim_n S_n$ be light profinite with surjective transition maps, and let $g \colon Z \to S$ be continuous. To extend g to X, it is enough to find, for all n, compatible maps $f_n \colon X \to S_n$ extending g_n. We find f_0 since S_0 is finite. Now assume we have already found f_n. We aim to find a lift f_{n+1} making the diagram

$$\begin{tikzcd}
Z \arrow[r, "g_{n+1}"] \arrow[d, hook, "i"] & S_{n+1} \arrow[d, two heads, "\varphi_n"] \\
X \arrow[r, "f_n"] \arrow[ur, dashed, "f_{n+1}"] & S_n
\end{tikzcd}$$

commute. It suffices to find lifts over the fibre of each $s \in S_n$, which exist by the finite case. $\square$

1.2 Light condensed sets and condensification

With the category of test spaces $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ at hand, we now define suitable sheaves on it.

Definition 1.11. A **light condensed set** is a functor $X \colon \mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ satisfying the following conditions:

(1) We have $X(\emptyset) = *$.

(2) If S_1 and S_2 are light profinite sets, then $X(S_1 \amalg S_2) \to X(S_1) \times X(S_2)$ is a bijection.

(3) If $T \to S$ is a surjection of light profinite sets, then $X(S) \to \mathrm{eq}(X(T) \rightrightarrows X(T \times_S T))$ is a bijection, where the right side is the equaliser of the maps induced by the two projections.

Equivalently, a light condensed set is a sheaf on the site of light profinite sets with coverings given by finite families of jointly surjective maps (that is, $\{S_i \to S\}_{i \in I}$ is a covering if I is finite and $\coprod_{i \in I} S_i \to S$ is surjective). We denote this category of sheaves by $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$.

Lemma 1.12. *If A is a topological space, then $\underline{A} \coloneqq \mathrm{Cont}(-, A)$ is a light condensed set.*

Proof. We have $\underline{A}(\emptyset) = *$ and $\underline{A}(S_1 \amalg S_2) = \underline{A}(S_1) \times \underline{A}(S_2)$. Let $q \colon T \to S$ be surjective. Then

$$\mathrm{eq}(\underline{A}(T) \rightrightarrows \underline{A}(T \times_S T)) = \{f \in \mathrm{Cont}(T, A) : \text{for all } x, y \in T,\ q(x) = q(y) \text{ implies } f(x) = f(y)\}.$$

Hence bijectivity of $\underline{A}(S) \to \mathrm{eq}(\underline{A}(T) \rightrightarrows \underline{A}(T \times_S T))$ holds if q is a quotient map, which is true since q is a continuous surjection of compact Hausdorff spaces. $\square$

Definition 1.13. For a topological space A, we call $\underline{A}$ the **condensification** of A.

In particular, Lemma 1.12 applies when A is a light profinite set, so the Yoneda embedding includes $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ fully faithfully into $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$.

Let us study how tight the relation between topological spaces and light condensed sets is. For any light condensed set X, we view $X(*)$ as the "underling set" of X. For a light profinite set S, the Yoneda Lemma asserts $\mathrm{Mor}_{\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(\underline{S}, X) \cong X(S)$, so any $\alpha \in X(S)$ induces a map $S = \underline{S}(*) \to X(*)$. If $S \neq \emptyset$, then this gives a surjection $\coprod_{\alpha \in X(S)} S \to X(*)$.

Proposition 1.14. *Condensification $\mathcal{T}\mathrm{op} \to \mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ has a left adjoint. It sends X to $X(*)_{\mathrm{top}}$, the set $X(*)$ equipped with the quotient topology from $\coprod_{S \in \mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})} \coprod_{\alpha \in X(S)} S \to X(*)$.*

Proof. Let A be a topological space and let X be a light condensed set. We show that taking sections over the point defines a bijection

$$\mathrm{Mor}_{\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(X, \underline{A}) \xrightarrow{\cong} \mathrm{Cont}(X(*)_{\mathrm{top}}, A).$$

For a light condensed set S, the natural map $\underline{A}(S) \to \prod_{s \in S} \underline{A}(*) \cong \prod_{s \in S} A$ is injective. It is the inclusion of the set of continuous maps $S \to A$ into the set of all maps $S \to A$. For any map of light condensed sets $f \colon X \to \underline{A}$ we get a commutative diagram of sets

$$\begin{tikzcd} X(S) \arrow[r] \arrow[d, "f_S"'] & \prod_{s \in S} X(*) \arrow[d, "\prod f_{\{*\}}"] \\ \underline{A}(S) \arrow[r] & \prod_{s \in S} A \end{tikzcd}$$

in which $\underline{A}(S) \to \prod_{s \in S} A$ is injective. Thus $\mathrm{Mor}_{\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(X, \underline{A}) \to \mathrm{Map}(X(*), A)$ is injective. It remains to show that the image consists of the maps that are continuous with respect to $X(*)_{\mathrm{top}}$. Let $g \colon X(*) \to A$ be a map of sets. Then g lies in the image of $\mathrm{Mor}_{\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(X, \underline{A}) \hookrightarrow \mathrm{Map}(X(*), A)$ if and only if for all light profinite sets S, the composite

$$X(S) \to \prod_{s \in S} X(*) \xrightarrow{\prod g} \prod_{s \in S} A$$

has image in $\underline{A}(S) \subseteq \prod_{s \in S} A$. This holds if and only if for all $S \in \mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ and all $\alpha \in X(S)$, $S \to X(*) \to A$ is continuous, which means precisely continuity of $g \colon X(*)_{\mathrm{top}} \to A$. $\square$

By general sheaf theory, $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ has all limits, given by limits of presheaves, and all colimits, given by sheafifications of colimits of presheaves. For filtered colimits, the sheafification is not necessary because the Grothendieck topology is finitary.

As a right adjoint, condensification preserves limits. We will see that it also preserves many colimits of interest. As a start, note that the sheaf conditions in Definition 1.11 can be restated in terms of preservation of colimits. For example, if S_1 and S_2 are light profinite sets, then the natural map $\underline{S_1} \amalg \underline{S_2} \to \underline{S_1 \amalg S_2}$ is an isomorphism.

We identify a rich class of topological spaces on which condensification is fully faithful.

Definition 1.15. A topological space A is **sequential** if for all topological spaces B, a map of sets $f \colon A \to B$ is continuous if $f \circ \alpha \colon \mathbb{N} \cup \infty \to B$ is continuous for all continuous $\alpha \colon \mathbb{N} \cup \infty \to A$. We denote the full subcategory of $\mathcal{T}\mathrm{op}$ spanned by the sequential topological spaces by $\mathrm{s}\mathcal{T}\mathrm{op}$.

For example, all metrisable spaces are sequential. The inclusion $\mathrm{s}\mathcal{T}\mathrm{op} \hookrightarrow \mathcal{T}\mathrm{op}$ has a right adjoint $(-)^s$ that sends a topological space A to the set A equipped with the quotient topology from $\coprod_{\alpha \in \mathrm{Cont}(\mathbb{N} \cup \infty, A)} \mathbb{N} \cup \infty \to A$. Equivalently, as all light profinite sets are sequential, A^s carries the quotient topology from $\coprod_{S \in \mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})} \coprod_{\alpha \in \mathrm{Cont}(S, A)} S \to A$.

Proposition 1.16. *If X is a light condensed set, then the topological space $X(*)_{\mathrm{top}}$ is sequential. Moreover, condensification restricts to a fully faithful embedding $\mathrm{s}\mathcal{T}\mathrm{op} \hookrightarrow \mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$.*

Proof. As a quotient of a disjoint union of metrisable compact spaces, $X(*)_{\mathrm{top}}$ is sequential. To show that $\underline{(-)} \colon \mathrm{s}\mathcal{T}\mathrm{op} \hookrightarrow \mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ is fully faithful, we show that if A is sequential, then the counit $\varepsilon_A \colon \underline{A}(*)_{\mathrm{top}} \to A$ is an isomorphism. But $\underline{A}(*)_{\mathrm{top}}$ agrees with A^s by definition. $\square$

Our discussion so far emphasises the importance of $\mathbb{N} \cup \infty$ as a test space. We have not yet explained the role of the Cantor set $\mathfrak{C}$. Its importance stems from the fact that any metrisable compact Hausdorff space admits a continuous surjection from it. We first specialise the notions of injections and surjections of sheaves to the case of light condensed sets.

Definition 1.17. A map of light condensed sets $X \to Y$ is **injective** if for all light profinite S the map $X(S) \to Y(S)$ is injective, and **surjective** if for all light profinite S and all $\alpha \in Y(S)$ there exists a surjection of light profinite sets $T \to S$ such that $X(T) \to Y(T)$ hits $\alpha|_T$.

Equivalently, by the Yoneda Lemma, $X \to Y$ is surjective if and only if for all light profinite sets S and maps $\underline{S} \to Y$ there exists a surjection of light profinite sets $T \to S$ and a map $\underline{T} \to X$ making the following square commute

$$\begin{tikzcd} \underline{T} \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\ \underline{S} \arrow[r] & Y. \end{tikzcd}$$

As a fully faithful right adjoint, condensification $\mathrm{s}\mathcal{T}\mathrm{op} \to \mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ creates limits. Thus a map of sequential spaces $f\colon A \to B$ is injective if and only if $\underline{f}\colon \underline{A} \to \underline{B}$ is injective. On the other hand, surjectivity of $\underline{A} \to \underline{B}$ implies surjectivity of $A \to B$, and the sheaf axioms show the converse when A and B are light profinite. We can generalise this as follows.

Lemma 1.18. *A map of metrisable compact Hausdorff spaces $f\colon A \to B$ is surjective if and only if the map of light condensed sets $\underline{f}\colon \underline{A} \to \underline{B}$ is surjective.*

Proof. The reverse implication holds by adjunction. For the direct implication, let $f\colon A \to B$ be surjective and let $\underline{S} \to \underline{B}$ for a light profinite set S be given. As A is metrisable compact Hausdorff, we find a surjection from the Cantor set $\mathfrak{C} \to A$. Then $T := \mathfrak{C} \times_B S$ together with the natural maps $T \to S$ and $T \to \mathfrak{C} \to A$ witness surjectivity of $\underline{f}$. $\square$

Remark 1.19. As in any category of sheaves, epimorphisms in $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ are effective. Hence, by Lemma 1.18, any surjection of metrisable compact Hausdorff spaces $A \to B$ induces an isomorphism $\mathrm{coeq}(\underline{A} \times_{\underline{B}} \underline{A} \rightrightarrows \underline{A}) \to \underline{B}$. Equivalently, if A is metrisable compact Hausdorff and $R \subseteq A \times A$ is a closed equivalence relation, then $\mathrm{coeq}(\underline{R} \rightrightarrows \underline{A}) \to \underline{A/R}$ is an isomorphism.

Lemma 1.18 suggests that condensification behaves well on metrisable compact Hausdorff spaces. To pursue this further, we introduce quasi-compact and quasi-separated light condensed sets.

Definition 1.20. A light condensed set X is

(1) **quasi-compact** if for any surjection $\coprod_{i \in I} X_i \to X$ there is a finite subset $J \subseteq I$ such that $\coprod_{j \in J} X_j \to X$ is surjective.
(2) **quasi-separated** if for all quasi-compact Y, Z with morphisms $Y \to X \leftarrow Z$, the fibre product $Y \times_X Z$ is quasi-compact.
(3) **qcqs** if it is both quasi-compact and quasi-separated.

By general sheaf theory, images and finite disjoint unions of quasi-compact light condensed sets remain quasi-compact. If X is quasi-compact, then we find a surjection $\underline{S} \to X$ from a light profinite set S by taking a finite subcover of $\coprod_{S \in \mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})} \coprod_{\alpha \in X(S)} \underline{S} \to X$.

Proposition 1.21. *Condensification restricts to an equivalence*

$$\{\textit{metrisable compact Hausdorff spaces}\} \xrightarrow{\sim} \{\textit{qcqs light condensed sets}\}$$

of full subcategories of $\mathcal{T}\mathrm{op}$ *and* $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$*, respectively.*

Proof. Let A be a metrisable compact Hausdorff space. We show that $\underline{A}$ is qcqs.

If A is a light profinite set, then $\underline{A}$ is quasi-compact by general sheaf theory as the Grothendieck topology is finitary. But any metrisable compact Hausdorff space A admits a surjection from the Cantor set, and this surjection stays surjective after condensification by Lemma 1.18, so quasi-compactness of $\underline{A}$ follows. To show that $\underline{A}$ is quasi-separated, let X and Y be quasi-compact with maps $X \to \underline{A} \leftarrow Y$. Take a surjection $\underline{T} \to X$ for a light profinite set T. Then also

$$\underline{T} \times_{\underline{A}} Y \to X \times_{\underline{A}} Y$$

is surjective as colimits in any category of sheaves are universal. This reduces to the case $X = \underline{T}$ for a light profinite set T. Analogously, we further reduce to the case $Y = \underline{U}$ for a light profinite set U. As A is Hausdorff, $T \times_A U$ is closed in $T \times U$, and so $\underline{T} \times_{\underline{A}} \underline{U} = \underline{T \times_A U}$ is quasi-compact.

The functor is fully faithful by Proposition 1.16. To show essential surjectivity, let X be a qcqs light condensed set. Then we find a surjection $\underline{S} \to X$ for a light profinite set S. As $\underline{S} \times_X \underline{S}$ is quasi-compact and injects into $\underline{S \times S}$, there exists a closed subset $R \subseteq S \times S$ such that $\underline{S} \times_X \underline{S}$ maps isomorphically to $\underline{R}$. Indeed, taking any surjection from a light profinite set $\underline{T} \to \underline{S} \times_X \underline{S}$, the composite $\underline{T} \to \underline{S} \times_X \underline{S} \to \underline{S \times S}$ comes from a map $T \to S \times S$. We deduce

$$X \cong \operatorname{coeq}(\underline{S} \times_X \underline{S} \rightrightarrows \underline{S}) \cong \operatorname{coeq}(\underline{R} \rightrightarrows \underline{S}) \cong \underline{S/R},$$

so X lies in the essential image. □

The upshot is that, thanks to the presence of the Cantor set, the topological compact Hausdorff notion translates to the correct sheaf-theoretic compact Hausdorff notion. We also remark that, by general sheaf theory, if X is a qcqs light condensed set, then X is categorically compact in the sense that $\operatorname{Mor}_{\operatorname{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(X, -)$ preserves filtered colimits.

Next, we identify the quasi-separated light condensed sets. By general sheaf theory, they are preserved under taking subsheaves, coproducts, and filtered colimits along injections. Let $\operatorname{Ind}(\mathrm{m}\mathcal{C}\mathrm{H})$ be the ind-category of the category of metrisable compact Hausdorff spaces and let $\operatorname{Ind}^{\mathrm{inj}}(\mathrm{m}\mathcal{C}\mathrm{H})$ be its full subcategory spanned by the diagrams with injective transition maps.

Proposition 1.22. *Condensification induces an equivalence*

$$\operatorname{Ind}^{\mathrm{inj}}(\mathrm{m}\mathcal{C}\mathrm{H}) \xrightarrow{\sim} \{\textit{quasi-separated light condensed sets}\}$$

of full subcategories of $\operatorname{Ind}(\mathrm{m}\mathcal{C}\mathrm{H})$ *and* $\operatorname{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$*, respectively.*

Proof. Note that "$\varinjlim_i S_i$" $\mapsto \varinjlim_i \underline{S_i}$ defines a functor from $\operatorname{Ind}^{\mathrm{inj}}(\mathrm{m}\mathcal{C}\mathrm{H})$ to the category of quasi-separated light condensed sets by Proposition 1.21 and the fact that quasi-separated sheaves are stable under filtered colimits along injections. For full faithfulness, we compute

$$\begin{aligned}
\operatorname{Mor}_{\operatorname{Ind}^{\mathrm{inj}}(\mathrm{m}\mathcal{C}\mathrm{H})}(\text{“}\varinjlim_i S_i\text{”}, \text{“}\varinjlim_j T_j\text{”}) &= \varprojlim_i \varinjlim_j \operatorname{Cont}(S_i, T_j) \\
&= \varprojlim_i \varinjlim_j \operatorname{Mor}_{\operatorname{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(\underline{S_i}, \underline{T_j}) \\
&= \varprojlim_i \operatorname{Mor}_{\operatorname{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(\underline{S_i}, \varinjlim_j \underline{T_j}) \\
&= \operatorname{Mor}_{\operatorname{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}(\varinjlim \underline{S_i}, \varinjlim_j \underline{T_j}),
\end{aligned}$$

using categorical compactness of all $\underline{S_i}$ in the third equality. For essential surjectivity, let X be a quasi-separated light condensed set. Write $X = \operatorname{colim}_{i \in I} \underline{S_i}$ for light profinite S_i and let X_i be the image of $\underline{S_i}$ in X. Then

$$X = \operatorname{colim}_{i \in I} X_i = \varinjlim_{J \subseteq I \text{ finite}} \operatorname{colim}_{j \in J} X_j.$$

Each $\operatorname{colim}_{j \in J} X_j = \bigcup_{j \in J} X_j \subseteq X$ is quasi-separated as a subsheaf of X and quasi-compact as it admits surjection from $\coprod_{j \in J} \underline{S_j}$. Thus $\operatorname{colim}_{j \in J} X_j = \underline{S_J}$ for metrisable compact S_J. □

If X is quasi-separated, then for light profinite sets S natural map $X(S) \to \prod_{s\in S} X(\{s\})$ is injective. Indeed, write $X = \varinjlim_i \underline{X_i}$ for metrisable compact X_i, then

$$X(S) = \varinjlim \underline{X_i}(S) = \varinjlim_i \mathrm{Cont}(S, X_i) \subseteq \mathrm{Cont}(S, \varinjlim_i X_i) = \mathrm{Cont}(S, X(*)_{\mathrm{top}}).$$

Intuitively, light condensed mathematics allows many "topologies" on the point (since $X(*) = *$ need not imply $X(S) = *$ for all light profinite sets S), but only a single quasi-separated one.

The quasi-separated light condensed sets that correspond to sequential colimits under the equivalence of Proposition 1.22 even come from topological spaces:

Lemma 1.23. *Let $\varinjlim_n S_n$ be a sequential colimit of metrisable compact Hausdorff spaces along injections. Then the natural map*

$$\varinjlim_n \underline{S_n} \to \underline{\varinjlim_n S_n}$$

is an isomorphism.

Proof. For compact T, the natural injection $\varinjlim_n \mathrm{Cont}(T, S_n) \to \mathrm{Cont}(T, \varinjlim_n S_n)$ is also surjective. Indeed, it suffices to show that any compact subset $K \subseteq S$ is contained in some S_n. If not, then, possibly passing to a subsequence of $(S_n)_{n=0}^{\infty}$, we find a sequence $(x_n)_{n=0}^{\infty}$ in S with $x_n \in (S_{n+1} \setminus S_n) \cap K$. If $A \subseteq \{x_n : n \in \mathbb{N}\}$ is any subset, then $A \cap S_n$ is finite for all n. Hence $\{x_n : n \in \mathbb{N}\}$ is an infinite discrete subset of K, which is absurd. Thus, for a light profinite set T,

$$(\underline{\varinjlim_n S_n})(T) \cong \mathrm{Cont}(T, \varinjlim_n S_n) \cong \varinjlim_n \mathrm{Cont}(T, S_n) \cong \varinjlim_n \underline{S_n}(T) \cong (\varinjlim_n \underline{S_n})(T)$$

and we conclude. □

Proposition 1.22 and Lemma 1.23 are key to specifying light condensed structures in practice. We can promote a set to a light condensed set by identifying it with a countable ascending union of metrisable compact Hausdorff spaces.

In general, the "underlying topological space" functor $(-)(*)_{\mathrm{top}}$ does not preserve limits. The following positive result, however, proves useful in practice.

Corollary 1.24. *Let X and Y be sequential colimits of qcqs light condensed sets along injections. Then the natural map $(X \times Y)(*)_{\mathrm{top}} \to X(*)_{\mathrm{top}} \times Y(*)_{\mathrm{top}}$ is a homeomorphism.*

Proof. For any topological space A we have $\underline{A}(*)_{\mathrm{top}} = A^s$. By Lemma 1.23, it thus suffices to prove that for sequential colimits of metrisable compact Hausdorff spaces along injections $\varinjlim_n X_n$ and $\varinjlim_n Y_n$, their product as topological spaces is sequential. Indeed, the natural map $\varinjlim_n (X_n \times Y_n) \to \varinjlim_n X_n \times \varinjlim_n Y_n$ is a homeomorphism (see [CS20, Lemma 1.3]). □

We conclude this subsection with an alternative way to think about light condensed sets. Let m$\mathcal{C}$H be the site of metrisable compact Hausdorff spaces with coverings given by finite families of jointly surjective maps. Restriction along $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in}) \hookrightarrow \mathrm{m}\mathcal{C}\mathrm{H}$ gives a functor

$$\mathrm{Shv}_{\mathcal{S}\mathrm{et}}(\mathrm{m}\mathcal{C}\mathrm{H}) \to \mathrm{Shv}_{\mathcal{S}\mathrm{et}}(\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})) = \mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et}).$$

Any metrisable compact Hausdorff space admits a surjection from the Cantor set, so $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ is a dense subsite of m$\mathcal{C}$H and this functor is an equivalence (see [Joh02, §C2.2 Theorem 2.2.3]).

Therefore, we can equivalently view light condensed sets as sheaves on the site of metrisable compact Hausdorff spaces. This viewpoint has the advantage that m$\mathcal{C}$H is a *pretopos*, allowing a more systematic approach to the results we discussed. For example, for any pretopos $\mathcal{C}$ with the Grothendieck topology of finite collections of jointly epic morphisms, the Yoneda embedding induces an equivalence between $\mathcal{C}$ and the qcqs sheaves on $\mathcal{C}$ (see [Lur18, §13 Proposition 5]).

The viewpoint through $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ emphasises that light condensed sets are algebraic at heart. They can be defined without any reference to topological spaces.

1.3 Light condensed Abelian groups

The promise of light condensed mathematics is that light condensed sets, unlike topological spaces, mix well with algebraic structure. Since light condensed sets are sheaves of sets, we can combine them with algebra by considering sheaves of Abelian groups, rings, modules, etc.

Definition 1.25. A **light condensed Abelian group** is a sheaf of Abelian groups on $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$. We denote the category of light condensed Abelian groups by $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$.

Equivalently, a light condensed Abelian group is an Abelian group object in $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$. As for sheaves on any site, the forgetful functor $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b}) \to \mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ has a left adjoint $\mathbb{Z}[-]$ that sends a light condensed set X to the sheafification of $\mathbb{Z}^{\mathrm{pre}}[X] \colon S \mapsto \mathbb{Z}[X(S)]$.

The main defect of topological Abelian groups is that their category is not Abelian. In contrast, $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ is a very nice Abelian category simply because it is a category of Abelian sheaves.

Theorem 1.26. *The category* $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ *is an Abelian category with all limits and colimits (AB3*, AB3) such that direct sums and filtered colimits are exact (AB4, AB5). It has a generator and enough injectives. In particular, it is a Grothendieck Abelian category.*

Proof. General sheaf theory. □

Condensification preserves limits, hence induces a functor $\mathcal{T}\mathrm{op}\mathcal{A}\mathrm{b} \to \mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$. It becomes fully faithful when restricted to sequential Abelian groups and further restricts to a fully faithful exact functor $\mathcal{A}\mathrm{b} \to \mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$.

Notation 1.27. From now on, we will denote $\underline{S}$ by S, considering everything as light condensed from the start. We will also write Mor for $\mathrm{Mor}_{\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})}$ and Hom for $\mathrm{Hom}_{\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})}$.

Proposition 1.28. *The category* $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ *enjoys the following properties:*

(1) For a sequence of surjections $\cdots \to X_1 \to X_0$, *each projection* $\varprojlim_n X_n \to X_n$ *is surjective.*

(2) Countable products are exact (countable AB4).*

(3) Products commute with filtered colimits (AB6).

Proof. (1): Set $X_\infty \coloneqq \varprojlim_n X_n$. To show that $X_\infty \to X_0$ is surjective, let S_0 be a light profinite set and let $S_0 \to X_0$ be given. Since each $X_{n+1} \to X_n$ is surjective, we inductively find light profinite sets S_{n+1} and surjections $S_{n+1} \to S_n$ such that we have a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, dashed] & S_2 \arrow[r, dashed, two heads] \arrow[d, dashed] & S_1 \arrow[r, dashed, two heads] \arrow[d, dashed] & S_0 \arrow[d] \\ \cdots \arrow[r] & X_2 \arrow[r, two heads] & X_1 \arrow[r, two heads] & X_0. \end{tikzcd}$$

Set $S_\infty \coloneqq \varprojlim_n S_n$, then the diagram induces a map $S_\infty \to X_\infty$. Now $S_\infty \to S_0$ is a surjection of light profinite sets, proving surjectivity of $X_\infty \to X_0$. To show that $X_\infty \to X_n$ is surjective for $n > 0$, given $S_n \to X_n$, we can take $S_0 = \cdots = S_n$ and proceed as above.

(2): We need to show that for surjections $f_n \colon M_n \to N_n$, also $\prod_n f_n \colon \prod_n M_n \to \prod_n N_n$ is surjective. Consider the sequence of surjections of light condensed Abelian groups

$$\cdots \twoheadrightarrow (\textstyle\prod_{n=0}^m M_n \times \prod_{n=m+1}^\infty N_n) \twoheadrightarrow \cdots \twoheadrightarrow (M_0 \times \prod_{n=1}^\infty N_n) \twoheadrightarrow (\prod_{n=0}^\infty N_n).$$

By part (1), the projection $\prod_{n=0}^\infty M_n = \varprojlim_m (\prod_{n=0}^m M_n \times \prod_{n=m+1}^\infty N_n) \to \prod_{n=0}^\infty N_n$ is surjective.

(3): As the Grothendieck topology is finitary, filtered colimits can be computed as presheaves. Given filtered systems $(M_{ij})_{i \in I}$ for $j \in J$, to show that $\varinjlim_I \prod_J M_{ij} \to \prod_J \varinjlim_I M_{ij}$ is an isomorphism it thus suffices to show that $\varinjlim_I \prod_J M_{ij}(S) \to \prod_J \varinjlim_I M_{ij}(S)$ is one for all light profinite sets S. Indeed, the category of Abelian groups satisfies AB6. □

Remark 1.29. Proposition 1.28(1) is already a property of light condensed sets. It tells us that the topos of light condensed sets is *replete* in the sense of [BS13].

Let us get some hands-on practice with the category $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$. The map $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{eucl}}, x \mapsto x$ from the discrete to the Euclidan reals is injective. The following description of its cokernel also follows from cohomological considerations (see Remark 1.31 below).

Lemma 1.30. *For any light profinite set S, we have $(\mathbb{R}_{\mathrm{eucl}}/\mathbb{R}_{\mathrm{disc}})(S) = \mathbb{R}_{\mathrm{eucl}}(S)/\mathbb{R}_{\mathrm{disc}}(S)$.*

Proof. We expand on the discussion in [Ked25, Lemma 4.1.8]. It suffices to show that the cokernel presheaf $S \mapsto \mathbb{R}_{\mathrm{eucl}}(S)/\mathbb{R}_{\mathrm{disc}}(S)$ is a sheaf. It is certainly trivial on the empty set and takes finite disjoint unions to products. It remains to show that for a surjection of light profinite sets $q\colon T \to S$, the following sequence is exact

$$0 \to \frac{\mathrm{Cont}(S, \mathbb{R}_{\mathrm{eucl}})}{\mathrm{Cont}(S, \mathbb{R}_{\mathrm{disc}})} \xrightarrow{q^*} \frac{\mathrm{Cont}(T, \mathbb{R}_{\mathrm{eucl}})}{\mathrm{Cont}(T, \mathbb{R}_{\mathrm{disc}})} \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} \frac{\mathrm{Cont}(T \times_S T, \mathbb{R}_{\mathrm{eucl}})}{\mathrm{Cont}(T \times_S T, \mathbb{R}_{\mathrm{disc}})}.$$

Injectivity of q^* is immediate as q is a quotient map, and the composite $(\mathrm{pr}_1^* - \mathrm{pr}_2^*) \circ q^*$ is zero. Writing $\mathbb{R}$ instead of $\mathbb{R}_{\mathrm{eucl}}$ for ease of notation, it remains to show the following.

Claim. *Let $q\colon T \to S$ be a surjection of light profinite sets and let $f\colon T \to \mathbb{R}$ be a continuous map such that $f\mathrm{pr}_1 - f\mathrm{pr}_2\colon T \times_S T \to \mathbb{R}$ is locally constant. Then there exists a locally constant map $g\colon T \to \mathbb{R}$ such that $f - g$ factors through S.*

Proof of the Claim. A basis of $T \times_S T$ is given by the opens of the form $V \times_S V'$ for open $V, V' \subseteq T$. We thus find a collection of non-empty opens $\{V_i\}_{i \in I}$ of T such that

$$T \times_S T = \bigcup\nolimits_{i,j \in I} V_i \times_S V_j$$

and $f\mathrm{pr}_1 - f\mathrm{pr}_2$ is constant on each $V_i \times_S V_j$. Then $T = \bigcup_{i \in I} V_i$ as $(t,t) \in T \times_S T$ for all $t \in T$. By Lemma 1.6, taking a refinement, we may assume that I is finite and the V_i are disjoint clopens. For $|I| = 1$, the claim holds with $g = 0$. Now assume $|I| \geq 2$ and that the claim holds if I has strictly smaller cardinality.

Case 1. Assume that $V_i \times_S V_j \times_S V_k \neq \emptyset$ for all $i, j, k \in I$. Then also $V_i \times_S V_j \neq \emptyset$ for all $i, j \in I$. Let c_{ij} be the value of $f\mathrm{pr}_1 - f\mathrm{pr}_2$ on $V_i \times_S V_j$. As the triple fibre products are non-empty,

$$c_{ij} + c_{jk} = c_{ik} \quad \text{for all } i, j, k \in I.$$

Pick $i_0 \in I$ and define a locally constant map $g\colon T \to \mathbb{R}$ by $g(t_i) \coloneqq c_{ii_0}$ for $t \in V_i$. We show that $f - g$ factors through S. Take $t, t' \in T$ with $q(t) = q(t')$. If, say, $t \in T_i$ and $t' \in T_j$, then

$$g(t) - g(t') = c_{ii_0} - c_{ji_0} = c_{ij} = f(t) - f(t')$$

and so $(f-g)(t) = (f-g)(t')$. Thus, in this case, the claim follows directly.

Case 2. Assume that there exist $i, j, k \in I$ such that $V_i \times_S V_j \times_S V_k = \emptyset$. We get an open cover

$$S = (S \setminus q(V_i)) \cup (S \setminus q(V_j)) \cup (S \setminus q(V_k)).$$

By Lemma 1.6, we find a finite disjoint clopen refinement $S = \bigsqcup_{\ell \in L} S_\ell$. It suffices to prove the claim for each $f|_{q^{-1}(S_\ell)}$ and $q|_{q^{-1}(S_\ell)}$. If, say, $S_\ell \subseteq S \setminus q(V_i)$, then $q^{-1}(S_\ell) = \bigsqcup_{m \in I \setminus \{i\}} q^{-1}(S_\ell) \cap V_m$ and we conclude by induction. The claim follows, hence so does the Lemma. $\square$

For instance, not all convergent sequences in $\mathbb{R}$ are eventually constant, so $(\mathbb{R}_{\mathrm{eucl}}/\mathbb{R}_{\mathrm{disc}})(\mathbb{N} \cup \infty) \neq 0$. The light condensed set $\mathbb{R}_{\mathrm{eucl}}/\mathbb{R}_{\mathrm{disc}}$ equips the point with a non-trivial "topology".

Remark 1.31. The cocycle condition in the proof of Lemma 1.30 hints that we are glimpsing at a cohomology group. Indeed, by [CS19, Theorem 3.2] we have $H^i(S, M) = 0$ for any light profinite set S, any discrete Abelian group M, and any $i > 0$.

Again, the following holds for $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ just because it is a category of Abelian sheaves.

Theorem 1.32. *The category* $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ *is symmetric monoidal with unit* $\mathbb{Z}$ *and tensor product* $A \otimes B$ *given by the sheafification of* $S \mapsto A(S) \otimes B(S)$*. This symmetric monoidal structure is closed (there is an internal hom-functor determined by* $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C))$*).*

Proof. General sheaf theory. □

The tensor product satisfies $\mathbb{Z}[X \times Y] = \mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ and preserves colimits in each entry. The internal hom can be given explicitly by $\underline{\mathrm{Hom}}(B, C)(S) = \mathrm{Hom}(\mathbb{Z}[S] \otimes B, C)$.

Remark 1.33. As for any Grothendieck Abelian category, the right derived functors $\mathrm{RHom}(-,-)$ and $\mathrm{R}\underline{\mathrm{Hom}}(-,-)$ exist on the unbounded derived category (see [Sta25, 079P,070K]). Since $\mathbb{Z}[X]$ is flat for any light condensed set X, the category $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$ has enough flat objects and we also get a derived tensor product $- \otimes^{\mathrm{L}} -$ on the unbounded derived category (see [Sta25, 0794]).

If S is locally compact Hausdorff, then we can equip $\mathrm{Cont}(S, B)$ for topological spaces B with the *compact open topology* so that $S \times -$ is left adjoint to $\mathrm{Cont}(S, -)$. If B is a topological Abelian group, then the pointwise group operations on $\mathrm{Cont}(S, B)$ are continuous for the compact open topology, turning it into a topological Abelian group. Now, if B is moreover sequential, then

$$\underline{\mathrm{Hom}}(\mathbb{Z}[S], B) = \underline{\mathrm{Cont}(S, B)}.$$

Indeed, $\underline{\mathrm{Hom}}(\mathbb{Z}[\underline{S}], \underline{B})(T) = \mathrm{Hom}(\mathbb{Z}[\underline{S \times T}], \underline{B}) = \mathrm{Mor}(\underline{S \times T}, \underline{B}) = \mathrm{Cont}((\underline{S \times T})(*)_{\mathrm{top}}, B) = \mathrm{Cont}(S \times T, B) = \mathrm{Cont}(T, \mathrm{Cont}(S, B))$. The following related result is stronger in practice.

Lemma 1.34. *If* A *and* B *be Hausdorff topological Abelian groups with* B *sequential, then*

$$\underline{\mathrm{Hom}}(A, B) = \underline{\mathrm{Hom}(A, B)}$$

for the compact-open topology on $\mathrm{Hom}(A, B)$*.*

Proof. See [CS19, Proposition 4.2] or [RC24a, Guided Exercise 1.2.5] □

Sheaves of rings on $\mathrm{Pro}_{\mathbb{N}}(\mathcal{F}\mathrm{in})$ are called *light condensed rings* and sheaves of modules over them are called *light condensed modules*. Note that the formal parts of our discussion extend from $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b}) = \mathrm{Mod}_{\mathbb{Z}}$ to Mod_R for any light condensed ring R by general sheaf theory.

1.4 The free Abelian group on a null sequence

In *light* condensed mathematics, one light condensed Abelian group is of unmatched importance.

Definition 1.35. The **free Abelian group on a null sequence**, denoted P, is $\mathbb{Z}[\mathbb{N} \cup \infty]/\mathbb{Z}[\infty]$.

If M is a topological Abelian group, then by adjunctions

$$\mathrm{Hom}(P, M) = \ker(\mathrm{Cont}(\mathbb{N} \cup \infty, M) \to M,\ f \mapsto f(\infty)).$$

This is the classical group of null sequences in M. The example $\mathbb{R}_{\mathrm{eucl}}/\mathbb{R}_{\mathrm{disc}}$ from Lemma 1.30 shows that the map $\mathrm{Hom}(P, M) \to \prod_{\mathbb{N}} M(*),\ f \mapsto (f_{\{*\}}(n))_n$ can fail to be injective.

It turns out that P only depends on the open subset $\mathbb{N} \subseteq \mathbb{N} \cup \infty$ in the sense that the choice of a light profinite compactification of $\mathbb{N}$ does not matter:

Lemma 1.36. *Let $g\colon S' \to S$ be a surjection of light profinite sets, let $Z \subseteq S$ be a closed subset and set $Z' := g^{-1}(Z)$. If g restricts to a homeomorphism $S' \setminus Z' \to S \setminus Z$, then g induces an isomorphism $\mathbb{Z}[S']/\mathbb{Z}[Z'] \to \mathbb{Z}[S]/\mathbb{Z}[Z]$.*

Proof. The quotient map $S \to S/Z$ restricts to a homeomorphism $S \setminus Z \to (S/Z) \setminus \infty$, so we can assume without loss of generality that Z is a point ∞. Then $S \cong S'/Z'$ as light profinite sets, hence also as light condensed sets by the sheaf axiom applied to the surjection $S' \to S'/Z'$. Applying $\mathbb{Z}[-]$ to g gives a surjection $\mathbb{Z}[S'] \to \mathbb{Z}[S]$ which induces a surjection

$$p\colon \mathbb{Z}[S']/\mathbb{Z}[Z'] \to \mathbb{Z}[S]/\mathbb{Z}[\infty].$$

To construct an inverse, note that $S' \to \mathbb{Z}[S'] \to \mathbb{Z}[S']/\mathbb{Z}[Z']$ is zero when restricted to Z'. Hence this composite factors through $S' \twoheadrightarrow S'/Z' \cong S$, via a map $S = S'/Z' \to \mathbb{Z}[S']/\mathbb{Z}[Z']$. We obtain a map $f\colon \mathbb{Z}[S] \to \mathbb{Z}[S']/\mathbb{Z}[Z']$ making the following diagram commute

$$\begin{tikzcd}
\mathbb{Z}[S'] \arrow[r] \arrow[d, two heads] & \mathbb{Z}[S']/\mathbb{Z}[Z'] \arrow[d, two heads, "p"] \\
\mathbb{Z}[S] \arrow[r] \arrow[ur, dashed, "f"] & \mathbb{Z}[S]/\mathbb{Z}[\infty].
\end{tikzcd}$$

Let $\overline{f}\colon \mathbb{Z}[S]/\mathbb{Z}[\infty] \to \mathbb{Z}[S']/\mathbb{Z}[Z']$ be the map induced by f. Commutativity of the lower-right triangle in implies $p \circ \overline{f} = \mathrm{id}$, and commutativity of the upper-left triangle implies $\overline{f} \circ p = \mathrm{id}$. □

The following is a direct consequence.

Lemma 1.37. *We have $P \otimes P \cong P$.*

Proof. To avoid notational confusion, we write $\mathbb{N}^\infty$ for $\mathbb{N} \cup \infty$ in this proof. Choose any bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. It induces a continuous surjection $\mathbb{N}^\infty \times \mathbb{N}^\infty \to \mathbb{N}^\infty$ with fibre $(\mathbb{N}^\infty \times \infty) \cup (\infty \times \mathbb{N}^\infty)$ over ∞. Using $\mathbb{Z}[(\mathbb{N}^\infty \times \infty) \cup (\infty \times \mathbb{N}^\infty)] \cong \mathbb{Z}[\mathbb{N}^\infty \times \infty] + \mathbb{Z}[\infty \times \mathbb{N}^\infty]$ and Lemma 1.36, we obtain

$$\frac{\mathbb{Z}[\mathbb{N}^\infty]}{\mathbb{Z}[\infty]} \otimes \frac{\mathbb{Z}[\mathbb{N}^\infty]}{\mathbb{Z}[\infty]} \cong \frac{\mathbb{Z}[\mathbb{N}^\infty \times \mathbb{N}^\infty]}{\mathbb{Z}[(\mathbb{N}^\infty \times \infty) \cup (\infty \times \mathbb{N}^\infty)]} \cong \frac{\mathbb{Z}[\mathbb{N}^\infty]}{\mathbb{Z}[\infty]}$$

as claimed. □

Next, we prove the most important property of P. It is internally projective in $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$. It is instructive to give the argument for projectivity first. For a surjection $N' \to N$ and any map $P \to N$, we want to find a lift

$$\begin{tikzcd}
 & N' \arrow[d, two heads] \\
P \arrow[r] \arrow[ur, dashed] & N.
\end{tikzcd}$$

The map $P \to N$ corresponds to a map of light condensed sets $\mathbb{N} \cup \infty \to N$ that is zero when restricted to ∞. By the surjectivity of $N' \to N$, there exists a surjection $g\colon S \to \mathbb{N} \cup \infty$ from a light profinite set fitting into a commutative diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d, two heads] & N' \arrow[d, two heads] \\
\mathbb{N} \cup \infty \arrow[r] & N.
\end{tikzcd}$$

Replace S by its closed subset containing one point of each fibre $g^{-1}(n)$ for $n \in \mathbb{N}$ and all points of $g^{-1}(\infty)$. Then g is still surjective and, setting $S_\infty := g^{-1}(\infty)$, it restricts to a homeomorphism

$S \setminus S_\infty \to \mathbb{N}$. By Lemma 1.36, it induces an isomorphism $\overline{g}\colon \mathbb{Z}[S]/\mathbb{Z}[S_\infty] \xrightarrow{\sim} P$.

The inclusion $S_\infty \to S$ has a retraction by Lemma 1.10, hence the quotient map $\mathbb{Z}[S] \to \mathbb{Z}[S]/\mathbb{Z}[S_\infty]$ has a section s (note that here we make critical use of lightness). Then

$$P \xrightarrow{\overline{g}^{-1}} \mathbb{Z}[S]/\mathbb{Z}[S_\infty] \xrightarrow{s} \mathbb{Z}[S] \to N'$$

solves the lifting problem. We generalise this argument to prove internal projectivity.

Theorem 1.38. *The object P is internally projective in* $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$.

Proof. Let $N' \to N$ be a surjection of light condensed Abelian groups, let S be a light profinite set, and let $f \in \underline{\mathrm{Hom}}(P, N)(S) = \mathrm{Hom}(\mathbb{Z}[S] \otimes P, N)$. We have to find a surjection of light profinite sets $S' \to S$ whose induced map $\mathbb{Z}[S'] \otimes P \to \mathbb{Z}[S] \otimes P$ fits into a commutative square

$$\begin{tikzcd}
\mathbb{Z}[S'] \otimes P \arrow[r, dashed] \arrow[d, dashed] & N' \arrow[d, two heads] \\
\mathbb{Z}[S] \otimes P \arrow[r, "f"] & N.
\end{tikzcd} \tag{$\star$}$$

By right exactness of $\mathbb{Z}[S] \otimes -$, the universal property of the cokernel, and adjunctions,

$$\mathrm{Hom}(\mathbb{Z}[S] \otimes P, N) = \ker(\mathrm{Mor}(S \times (\mathbb{N} \cup \infty), N) \to \mathrm{Mor}(S \times \infty, N)).$$

So f corresponds to a map of light condensed sets $S \times (\mathbb{N} \cup \infty) \to N$ that is zero on $S \times \infty$. By surjectivity of $N' \to N$, we find a light profinite set T and a surjection $g\colon T \to S \times (\mathbb{N} \cup \infty)$ such that $T \xrightarrow{g} S \times (\mathbb{N} \cup \infty) \xrightarrow{f} N$ factors through $N' \to N$.

Claim. *There exists a surjection of light profinite sets $S' \to S$ such that the induced surjection*

$$g'\colon T' := T \times_{S \times (\mathbb{N} \cup \infty)} (S' \times (\mathbb{N} \cup \infty)) \to S' \times (\mathbb{N} \cup \infty)$$

restricts to split surjections $T'_n := g'^{-1}(S' \times \{n\}) \to S' \times \{n\}$ for all $n \in \mathbb{N}$.

Note that when $S = *$, which is the case of projectivity explained above, then we can take $S' = S$ and to split $T'_n \to *$ means to pick an element in the fibre of g over n.

Proof of the Claim. Set $T_n := g^{-1}(S \times \{n\})$ for $n \in \mathbb{N}$. We have $T'_n = T_n \times_{S \times \{n\}} (S' \times \{n\})$ by fibre product laws, so we need to guarantee that each surjection

$$T_n \times_{S \times \{n\}} (S' \times \{n\}) \to S' \times \{n\}$$

splits. We can take $S' := S \times_{\prod_{m \in \mathbb{N}} S \times \{m\}} \prod_{m \in \mathbb{N}} T_m$ together with the projection to S. $\square$(Claim)

Pick splittings $i_n\colon S' \times \{n\} \to T'_n$ for all $n \in \mathbb{N}$ and replace T' by its closed subset

$$\bigcup_{n \in \mathbb{N}} i_n(S' \times \{n\}) \cup T'_\infty$$

where $T'_\infty := g'^{-1}(S' \times \{\infty\})$. Then g' stays surjective and restricts to a homeomorphism $T' \setminus T'_\infty \to S' \times \mathbb{N}$. Hence, by Lemma 1.36, it induces an isomorphism $\overline{g}'\colon \mathbb{Z}[T']/\mathbb{Z}[T'_\infty] \xrightarrow{\sim} \mathbb{Z}[S'] \otimes P$. By Lemma 1.10, the quotient map $\mathbb{Z}[T'] \to \mathbb{Z}[T']/\mathbb{Z}[T'_\infty]$ has a section s. Then the composite

$$\mathbb{Z}[S'] \otimes P \xrightarrow{\overline{g}'^{-1}} \mathbb{Z}[T']/\mathbb{Z}[T'_\infty] \xrightarrow{s} \mathbb{Z}[T'] \to N'$$

fits as the desired top horizontal arrow in $(\star)$. $\square$

Remark 1.39. As $\mathbb{Z}[\mathbb{N}\cup\infty]\cong P\oplus\mathbb{Z}[\infty]$, the internal projectivity of P is equivalent to internal projectivity of $\mathbb{Z}[\mathbb{N}\cup\infty]$. We have at least two reasons to be surprised by Theorem 1.38.

(1) If $X\in\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$ is projective, then so is $\mathbb{Z}[X]\in\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$. But $\mathbb{N}\cup\infty$ is not projective in $\mathrm{Cond}^{\mathrm{light}}(\mathcal{S}\mathrm{et})$. The surjection $(2\mathbb{N}\cup\infty)\amalg(2\mathbb{N}+1\cup\infty)\to\mathbb{N}\cup\infty$ is not split.

(2) It is not true that $\mathbb{Z}[X]$ is projective for any light condensed set X. For example, $\mathbb{Z}[\mathfrak{C}]$ is not projective by a theorem of Amir (see [CS22, Appendix to Lecture 3]).

We conclude our study of P by describing it explicitly. We will use the following result.

Proposition 1.40. *Let $S=\varprojlim_i S_i$ be a light profinite set. Then the natural map $\mathbb{Z}[S]\to\varprojlim_i\mathbb{Z}[S_i]$ is injective and induces an isomorphism of light condensed Abelian groups*

$$\mathbb{Z}[S]\xrightarrow{\sim}\bigcup_{N=0}^{\infty}\varprojlim_i\mathbb{Z}[S_i]_{\leq N}$$

for the finite discrete subsets $\mathbb{Z}[S_i]_{\leq N}:=\{\sum_{s\in S_i}a_s[s]\in\mathbb{Z}[S_i]:\sum_{s\in S_i}|a_s|\leq N\}\subseteq\mathbb{Z}[S_i]$.

Proof. See [CS20, Proposition 2.1]. $\square$

Hence, $\mathbb{Z}[S]$ is quasi-sepearated by Proposition 1.22, identifies with the condensification of the corresponding colimit of topological spaces by Lemma 1.23, and in fact comes from the corresponding sequential Abelian group by Corollary 1.24.

Corollary 1.41. *The light condensed Abelian group P identifies with the sequential union*

$$\bigcup_{N=0}^{\infty}\Big\{\sum_{m=0}^{\infty}a_m[m]\in\mathbb{Z}[\mathbb{N}]:\sum_{m=0}^{\infty}|a_m|\leq N\Big\}$$

whose N-th member carries the light profinite subset structure from $\prod_{m=0}^{\infty}\mathbb{Z}\cap[-N,N]$.

Proof. By Proposition 1.40, a sequence $(a_\infty^{(n)}[\infty]+\sum_{m\in\mathbb{N}}a_m^{(n)}[m])_n$ in $\mathbb{Z}[\mathbb{N}\cup\infty](*)_{\mathrm{top}}$ is a null sequence if and only if

(a) there exists $N\in\mathbb{N}$ such that, for all $n\in\mathbb{N}$, we have $|a_\infty^{(n)}|+\sum_{m\in\mathbb{N}}|a_m^{(n)}|\leq N$, and

(b) for all $m\in\mathbb{N}$, the sequence $(a_m^{(n)})_n$ is eventually constantly 0 and, for all i, the sequence $(a_\infty^{(n)}+\sum_{m=i+1}^{\infty}a_m^{(n)})_n$ is eventually constantly 0.

The map $s\colon P\to\mathbb{Z}[\mathbb{N}\cup\infty]$, $[m]\mapsto[m]-[\infty]$ splits the quotient map $q\colon\mathbb{Z}[\mathbb{N}\cup\infty]\to P$. Thus, $P(*)_{\mathrm{top}}$ identifies with the image of $s_{\{*\}}$, giving $P(*)_{\mathrm{top}}$ the structure of a sequential Abelian group. Then $P\to\underline{P(*)_{\mathrm{top}}}$ is an isomorphism of light condensed Abelian groups with inverse

$$\underline{P(*)_{\mathrm{top}}}\xrightarrow{s_{\{*\}}}\underline{\mathbb{Z}[\mathbb{N}\cup\infty](*)_{\mathrm{top}}}\xrightarrow{\sim}\mathbb{Z}[\mathbb{N}\cup\infty]\xrightarrow{q}P.$$

Hence, P comes from the Abelian group $\mathbb{Z}[\mathbb{N}]$ with topology induced by the injection $\mathbb{Z}[\mathbb{N}]\to\mathbb{Z}[\mathbb{N}\cup\infty]$, $[m]\mapsto[m]-[\infty]$. Explicitly, $(\sum_{m=0}^{\infty}a_m^{(n)}[m])_n$ is a null sequence in $\mathbb{Z}[\mathbb{N}]$ if and only if $(\sum_{m=0}^{\infty}a_m^{(n)}[m]-(\sum_{m=0}^{\infty}a_m^{(n)})[\infty])_n$ is a null sequence in $\mathbb{Z}[\mathbb{N}\cup\infty]$, if and only if

(a′) there exists $N\in\mathbb{N}$ such that, for all $n\in\mathbb{N}$, we have $|\sum_{m=0}^{\infty}a_m^{(n)}|+\sum_{m=0}^{\infty}|a_m^{(n)}|\leq N$, and

(b′₁) for all $m\in\mathbb{N}$ the sequence $(a_m^{(n)})_n$ is eventually constantly zero, and

(b′₂) for all $i\in\mathbb{N}$ the sequence $(-\sum_{m=0}^{\infty}a_m^{(n)}+\sum_{m=i+1}^{\infty}a_m^{(n)})_n$ is eventually constantly zero.

But (b′₁) implies (b′₂), and (a’) is equivalent to uniform boundedness of $\sum_{m=0}^{\infty}|a_m^{(n)}|$ for all n. $\square$

For example, $([n])_n$ is a null sequence, but $(n[n])_n$ and $(\sum_{m=n}^{2n}[m])_n$ are not null sequences in P.

Remark 1.42. Addition extends to a continuous map $(\mathbb{N}\cup\infty)\times(\mathbb{N}\cup\infty)\to(\mathbb{N}\cup\infty)$ with fibre $(\mathbb{N}\times\infty)\cup(\infty\times\mathbb{N})$ over ∞. It induces a map $P\otimes P\to P$, turning P into a light condensed ring with underlying ring $\mathbb{Z}[q]$. We denote it by $\mathbb{Z}[\hat{q}]$. By Corollary 1.41, we have injections

$$(\mathbb{Z}[q], \text{discrete topology}) \to \mathbb{Z}[\hat{q}] \to (\mathbb{Z}[q], q\text{-adic topology})$$

but neither of them are isomorphisms.

2 Analytic rings

We now introduce *analytic rings*. Just like rings are the basic building blocks in algebraic geometry, analytic rings are the basic building blocks in the analytic geometry of Clausen-Scholze. In this sense, the study of analytic rings is a study of analytic commutative algebra.

We introduce the basic notions in Subsection 2.1, and then extend them to derived categories in Subsection 2.2. In the following two subsections we take a closer look at the *gaseous* case, which expresses completeness by the summability of null sequences against a ring element. In Subsection 2.3 we study to which extend the choice of this element matters, and in Subsection 2.4 we consider a "universal" example. We will present a selection from [CS24, Lectures 5 – 14].

2.1 First definitions and examples

We argued in the previous section that the best way to mix algebra and topology is not through topological rings but through light condensed rings. The main reason is that category of light condensed modules over a light condensed ring is an exceptionally well-behaved Abelian category. The analysis comes into play by singling out the modules we want to consider complete.

Definition 2.1. A **pre-analytic ring structure** on a light condensed ring $A^{\triangleright}$ is a full subcategory $\mathrm{Mod}_A \subseteq \mathrm{Mod}_{A^{\triangleright}}$ satisfying the following conditions:

(1) It is closed under all limits, colimits and extensions.

(2) It is closed under the functors $\underline{\mathrm{Ext}}^i_{A^{\triangleright}}(M, -)$ for all $M \in \mathrm{Mod}_{A^{\triangleright}}$ and all $i \geq 0$.

We call it an **analytic ring structure** if moreover $A^{\triangleright} \in \mathrm{Mod}_A$. A **(pre-)analytic ring** is a pair $(A^{\triangleright}, \mathrm{Mod}_A)$ consisting of a light condensed ring and a (pre-)analytic ring structure on it. A **map of (pre-)analytic rings** $(A^{\triangleright}, \mathrm{Mod}_A) \to (B^{\triangleright}, \mathrm{Mod}_B)$ is a map of light condensed rings $\varphi\colon A^{\triangleright} \to B^{\triangleright}$ such that restriction of scalars along φ carries Mod_B into Mod_A.

In particular, condition (1) guarantees that $\mathrm{Mod}_A \subseteq \mathrm{Mod}_{A^{\triangleright}}$ is an Abelian subcategory.

Definition 2.1 is best motivated by the most basic non-trivial example, the category of *solid Abelian groups*. We introduce a subcategory $\mathrm{Solid} \subseteq \mathrm{Mod}_{\mathbb{Z}}$ of non-Archimedean complete objects. Classically, completeness is defined using Cauchy sequences, but in the condensed world we cannot make sense of a convergent sequence without specifying its limit point. Moreover, to fulfil the promise of condensed mathematics, the complete objects should assemble into an Abelian category. As $\mathbb{Z}$ and $\mathbb{Z}_p$ are complete, so should be the "non-Hausdorff" quotient $\mathbb{Z}_p/\mathbb{Z}$, for which we could not require uniqueness of limits even if there was a way to make sense of them.

The key is to formulate non-Archimedean completeness in terms of summability of null sequences. If $(a_n)_n$ is a sequence in a valued field K, then convergence of $\sum_{n=0}^{\infty} a_n$ implies that $(a_n)_n$ is a null sequence. Prominently, the converse holds if K is non-Archimedean complete. As before, we set $P \coloneqq \mathbb{Z}[\mathbb{N} \cup \infty]/\mathbb{Z}[\infty]$. Let shift: $P \to P$ be induced by the map $n \mapsto n+1$ fixing ∞.

Definition 2.2. A light condensed Abelain group M is **solid** if the map $d \coloneqq \mathrm{id} - \mathrm{shift}\colon P \to P$ induces an isomorphism on $\underline{\mathrm{Hom}}(P, M)$. The **category of solid Abelian groups**, denoted Solid, is the resulting full subcategory of $\mathrm{Cond}^{\mathrm{light}}(\mathcal{A}\mathrm{b})$.

Intuitively, $\underline{\mathrm{Hom}}(P, M)$ consists of null sequences in M, and $\mathrm{id} - \mathrm{shift}$ acts by

$$(m_0, m_1, \ldots) \mapsto (m_0 - m_1, m_1 - m_2, \ldots).$$

If this map is an isomorphism, then $(m'_0, m'_1, \ldots) \mapsto (\sum_{i=0}^{\infty} m'_i, \sum_{i=1}^{\infty} m'_i, \ldots)$ should be its inverse, witnessing the summability of null sequences in M. By the internal projectivity of P we proved in Theorem 1.38, the category Solid enjoys the closure properties we hope for:

Proposition 2.3. *The pair* $(\mathbb{Z}, \mathrm{Solid})$ *is a pre-analytic ring.*

Proof. For closure under limits and colimits, it suffices to show that $\underline{\mathrm{Hom}}(P, -)$ preserves limits and colimits. It preserves limits as it is a right adjoint. By internal projectivity of P, it preserves finite colimits. Categorical compactness of $\mathbb{Z}[S] \otimes P = \mathbb{Z}[S \times (\mathbb{N} \cup \infty)]/\mathbb{Z}[S \times \infty]$ for any light profinite set S shows that it preserves filtered colimits. Hence it preserves all colimits. Exactness of $\underline{\mathrm{Hom}}(P, -)$ and the five lemma give closure under extensions.

Now let $M \in \mathrm{Mod}_{\mathbb{Z}}$ and $N \in \mathrm{Solid}$. Then $\underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(M, N)) = \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(P, N))$, so $\underline{\mathrm{Hom}}(M, N)$ is solid. For $i > 0$, since P is internally projective,

$$\underline{\mathrm{Hom}}(P, \underline{\mathrm{Ext}}^i(M, N)) = \underline{\mathrm{Ext}}^i(M, \underline{\mathrm{Hom}}(P, N)),$$

hence $\underline{\mathrm{Ext}}^i(M, N)$ is solid. □

A *non-Archimedean complete norm* on an Abelian group M is a map $\|\cdot\| \colon M \to \mathbb{R}_{\geq 0}$ such that $\|-m\| = \|m\|$, $\|m + n\| \leq \max\{\|m\|, \|n\|\}$ and $\|m\| = 0$ if and only if $m = 0$ for all $m, n \in M$, and such that all Cauchy sequences with respect to the metric $(m, n) \mapsto \|m - n\|$ converge. Null sequences with respect to such norms are summable, so the following is not surprising.

Lemma 2.4. *Let M be an Abelian group topologised by a non-Archimedean complete norm. Then the condensification of M is solid. In particular* $(\mathbb{Z}, \mathrm{Solid})$ *is an analytic ring.*

Proof. We have to show that $\mathrm{id} - \mathrm{shift}$ induces an isomorphism on $\underline{\mathrm{Hom}}(P, M)(S)$ for any light profinite set S. By adjunctions,

$$\underline{\mathrm{Hom}}(P, M)(S) = \mathrm{Hom}(\mathbb{Z}[S] \otimes P, M) = \mathrm{Hom}(P, \underline{\mathrm{Hom}}(\mathbb{Z}[S], M)) = \mathrm{Hom}(P, \mathrm{Cont}(S, M))$$

for the compact open topology on $\mathrm{Cont}(S, M)$. If M is topologised by the non-Archimedean complete norm $\|\cdot\|$, then $\mathrm{Cont}(S, M)$ is topologised by the corresponding sup-norm, which is also non-Archimedean complete. Upon replacing M by $\mathrm{Cont}(S, M)$ we reduce to the case $S = *$, in which $(\mathrm{id} - \mathrm{shift})^*$ has the inverse $(m_0', m_1', \ldots) \mapsto (\sum_{i=0}^{\infty} m_i', \sum_{i=1}^{\infty} m_i', \ldots)$. □

We can readily give a few examples of (pre-)analytic rings.

Examples 2.5.

(1) The pair $\mathbb{Z}_{\square} := (\mathbb{Z}, \mathrm{Solid})$ is an analytic ring, the analytic ring of **solid integers**.

(2) If $A^{\triangleright}$ is a light condensed ring, then $(A^{\triangleright}, \mathrm{Mod}_{A^{\triangleright}})$ is an analytic ring. We call $\mathrm{Mod}_{A^{\triangleright}}$ the **trivial analytic ring structure** on $A^{\triangleright}$.

(3) Let $A = (A^{\triangleright}, \mathrm{Mod}_A)$ be a pre-analytic ring, let $\varphi\colon A^{\triangleright} \to B^{\triangleright}$ be a map of light condensed rings, and denote restriction of scalars along φ by ${}_{A^{\triangleright}}(-)$. The **induced pre-analytic ring structure** on $B^{\triangleright}$ is $\mathrm{Mod}_{A/B^{\triangleright}} := \{M \in \mathrm{Mod}_{B^{\triangleright}} : {}_{A^{\triangleright}}M \in \mathrm{Mod}_A\}$.

(4) As a special case of (3), on any light condensed ring $B^{\triangleright}$ we get a pre-analytic ring structure $\mathrm{Mod}_{B^{\triangleright}\square} := \mathrm{Mod}_{\mathbb{Z}_{\square}/B^{\triangleright}}$ of modules whose underlying light condensed Abelian group is solid. For example, we get analytic ring structures on $\mathbb{Z}[\![t]\!]$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ by Lemma 2.4.

By a variant of the adjoint functor theorem, pre-analytic rings $(A^{\triangleright}, \mathrm{Mod}_A)$ come with a left adjoint to the inclusion $\mathrm{Mod}_A \hookrightarrow \mathrm{Mod}_{A^{\triangleright}}$. Indeed, this inclusion preserves all limits and colimits, so the *Reflection Theorem* of Adámek and Rosický applies: If $\mathcal{C}$ is a presentable category and $\mathcal{C}' \subseteq \mathcal{C}$ is a full subcategory closed under limits and κ-filtered colimits for some regular cardinal κ, then $\mathcal{C}'$ is presentable and the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint (see [AR89]).

Definition 2.6. For a pre-analytic ring $A = (A^\triangleright, \mathrm{Mod}_A)$, the left adjoint of $\mathrm{Mod}_A \hookrightarrow \mathrm{Mod}_{A^\triangleright}$ is called ***A*-completion** and denoted by $(-)^{\wedge_A}$.

The completion functor gives rise to a completed tensor product:

Proposition 2.7. *Let $A = (A^\triangleright, \mathrm{Mod}_A)$ be a pre-analytic ring. Then Mod_A has a unique symmetric monoidal structure making $(-)^{\wedge_A}$ symmetric monoidal. It has unit $\widehat{A^\triangleright}$ and tensor product $M \otimes_A^\wedge N := (M \otimes_{A^\triangleright} N)^{\wedge_A}$, preserving colimits in each variable.*

Proof. We write $(-)^\wedge$ for $(-)^{\wedge_A}$. Set $M \otimes_A^\wedge N := (M \otimes_{A^\triangleright} N)^\wedge$. Then $(\mathrm{Mod}_A, \otimes_A^\wedge, \widehat{A^\triangleright})$ is symmetric monoidal. To show that $(-)^\wedge$ is symmetric monoidal, we show that for $M, N \in \mathrm{Mod}_{A^\triangleright}$, the natural map $(M \otimes_{A^\triangleright} N)^\wedge \to (M^\wedge \otimes_{A^\triangleright} N^\wedge)^\wedge$ is an isomorphism. Indeed, for any $L \in \mathrm{Mod}_A$,

$$
\begin{aligned}
\mathrm{Hom}_{A^\triangleright}((M^\wedge \otimes_{A^\triangleright} N^\wedge)^\wedge, L) &= \mathrm{Hom}_{A^\triangleright}(M^\wedge \otimes_{A^\triangleright} N^\wedge, L) = \mathrm{Hom}_{A^\triangleright}(M^\wedge, \underline{\mathrm{Hom}}_{A^\triangleright}(N^\wedge, L)) \\
&= \mathrm{Hom}_{A^\triangleright}(M, \underline{\mathrm{Hom}}_{A^\triangleright}(N, L)) = \mathrm{Hom}_{A^\triangleright}(M \otimes_{A^\triangleright} N, L) \\
&= \mathrm{Hom}_{A^\triangleright}((M \otimes_{A^\triangleright} N)^\wedge, L),
\end{aligned}
$$

as the adjunction of completion and inclusion also holds internally. The completed tensor product preserves colimits because $(-)^\wedge$ is a left adjoint. Any symmetric monoidal structure on Mod_A making $(-)^\wedge$ symmetric monoidal must have tensor product $(M \otimes_{A^\triangleright} N)^\wedge$ and unit $\widehat{A^\triangleright}$. $\square$

We record some properties of $\mathbb{Z}_\square$. Its completion functor is called *solidification* and denoted by $(-)^\square$. Its completed tensor product is called the *solid tensor product* and denoted by $\otimes^\square$.

Proposition 2.8.

(1) If $M \in \mathrm{Mod}_\mathbb{Z}$ admits the structure of a light condensed $\mathbb{R}$-vector space, then $M^\square = 0$.

(2) The map $[n] \mapsto (\delta_{mn})_m$, for the Kronecker-delta δ_{mn}, induces an isomorphism $P^\square \cong \prod_\mathbb{N} \mathbb{Z}$.

(3) For any light profinite set $S = \varprojlim_n S_n$, the natural map $\mathbb{Z}[S] \to \varprojlim_n \mathbb{Z}[S_n]$ induces an isomorphism $\mathbb{Z}[S]^\square \cong \varprojlim_n \mathbb{Z}[S_n]$. In particular, if S is infinite, then abstractly $\mathbb{Z}[S]^\square \cong \prod_\mathbb{N} \mathbb{Z}$.

(4) The object $\prod_\mathbb{N} \mathbb{Z}$ is a compact internally projective generator of Solid. *Moreover, the bilinear map $((a_m)_m, (b_n)_n) \mapsto (a_m b_n)_{m,n}$ induces an isomorphism $\prod_\mathbb{N} \mathbb{Z} \otimes^\square \prod_\mathbb{N} \mathbb{Z} \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}$.*

Proof. See [CS24, Lecture 5, timestamp 50:00] for (2), (3), and (4). We only prove (1) in detail. Symmetric monoidal functors preserve algebras and modules over them. In particular, $(-)^\square$ gives $\mathbb{R}^\square$ the structure of an algebra in $(\mathrm{Solid}, \otimes^\square, \mathbb{Z})$ and the solidification of any $\mathbb{R}$-module the structure of an $\mathbb{R}^\square$-module. It thus suffices to show $\mathbb{R}^\square = 0$. We follow [RC24b, Proposition 3.2.5(4)]. Since $\mathbb{R}^\square$ is an algebra, it is enough to show $1 = 0$ as maps $\mathbb{Z} \to \mathbb{R}^\square$. The null sequence

$$
(2^{\lfloor -\log_2(n+1) \rfloor})_n = \left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots\right) \tag{$\star$}
$$

defines an element $x \in \mathrm{Hom}(P, \mathbb{R})$. Composing with $\mathbb{R} \to \mathbb{R}^\square$ gives an element $x' \in \mathrm{Hom}(P, \mathbb{R}^\square)$, and applying the inverse of d^* we obtain an element $x'' \in \mathrm{Hom}(P, \mathbb{R}^\square)$. So x'' is the unique dashed arrow making the following diagram commute

$$
\begin{tikzcd}
P \arrow[r, "x"] \arrow[d, "d"'] \arrow[dr, "x'"] & \mathbb{R} \arrow[d] \\
P \arrow[r, dashed, "x''"] & \mathbb{R}^\square.
\end{tikzcd} \tag{$\star\star$}
$$

For $i \in \mathbb{N}$, let x_i'' be the composite $\mathbb{Z} \to P \xrightarrow{x''} \mathbb{R}^\square$ where the first map is the inclusion of the i-th component $\mathbb{Z} \cong \mathbb{Z}[i] \to P$. We think of x' as the null sequence $(\star)$ in $\mathbb{R}^\square$ and of x_0'' as its sum.

We argue that the following rearrangement makes sense in $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}^{\square})$:

$$1 + \Big(\frac{1}{2} + \frac{1}{2}\Big) + \Big(\frac{1}{4} + \frac{1}{4}\Big) + \Big(\frac{1}{4} + \frac{1}{4}\Big) + \cdots = 1 + \Big(1 + \frac{1}{2} + \frac{1}{2} + \cdots\Big).$$

To make this precise, consider the map $f\colon P \to P$ induced by $[n] \mapsto [2n+1] + [2n+2]$. Then $x \circ f = x$ as null sequences in $\mathbb{R}$, that is,

$$\Big(\frac{1}{2} + \frac{1}{2}, \frac{1}{4} + \frac{1}{4}, \frac{1}{4} + \frac{1}{4}, \dots\Big) = \Big(1, \frac{1}{2}, \frac{1}{2}, \dots\Big).$$

Let $g\colon P \to P$ be the map induced by $[n] \mapsto [2n+1]$. Then $d \circ f = g \circ d$ and hence, by uniqueness of the dashed arrow in $(\star\star)$, $x'' \circ g = x''$. Thus the diagram

$$\begin{tikzcd}
 & \mathbb{Z}[0] \arrow[r] & P \arrow[d, "g"] \arrow[dr, "x''"] & \\
\mathbb{Z} \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & & \mathbb{R}^{\square} \\
 & \mathbb{Z}[1] \arrow[r] & P \arrow[ur, "x''"'] &
\end{tikzcd}$$

commutes and we get $x_0'' - x_1'' = 0$. On the other hand, by the definition of d, the map $x_0'' - x_1''$ equals the composite $\mathbb{Z} \cong \mathbb{Z}[0] \to P \xrightarrow{d} P \xrightarrow{x''} \mathbb{R}^{\square}$, which by commutativity of $(\star\star)$ equals $\mathbb{Z} \cong \mathbb{Z}[0] \to P \xrightarrow{x} \mathbb{R} \to \mathbb{R}^{\square}$. As $\mathbb{Z} \cong \mathbb{Z}[0] \to P \xrightarrow{x} \mathbb{R}$ is the unit map and $\mathbb{R} \to \mathbb{R}^{\square}$ is a map of ring objects, we deduce $x_0'' - x_1'' = 1$. In conclusion, $0 = x_0'' - x_1'' = 1$ and thus $\mathbb{R}^{\square} = 0$ as desired. $\square$

Identifying $\prod_{\mathbb{N}} \mathbb{Z}$ with $\mathbb{Z}[\![q]\!]$ for the q-adic topology, Proposition 2.8(4) shows $\mathbb{Z}[\![q_1]\!] \otimes^{\square} \mathbb{Z}[\![q_2]\!] = \mathbb{Z}[\![q_1, q_2]\!]$. This is what we expect from a non-Archimedean completed tensor product. A detailed discussion of solid Abelian groups can be found in [RC24b, Section 3].

Proposition 2.8(1) tells us that solidity is not useful to capture Archimedean completeness. In an Archimedean context, we cannot hope for summability of all null sequences, but we can hope for summability of null-sequences against certain elements. For example, if $(a_n)_n$ is a null sequence in $\mathbb{R}$ and $f \in (0,1)$, then the series $\sum_{n=0}^{\infty} a_n f^n$ converges.

Definition 2.9. Let $A^{\triangleright}$ be a light condensed ring and let $f \in A^{\triangleright}(*)$. An $A^{\triangleright}$-module M is ***f*-gaseous** if $d_f := \mathrm{id} - f\mathrm{shift}\colon A^{\triangleright} \otimes P \to A^{\triangleright} \otimes P$ induces an isomorphism on $\underline{\mathrm{Hom}}_{A^{\triangleright}}(A^{\triangleright} \otimes P, M)$. The **category of *f*-gaseous modules** is the resulting full subcategory $\mathrm{Mod}_{A^{\triangleright}, f\text{-gas}}$ of $\mathrm{Mod}_{A^{\triangleright}}$.

We still think of $\underline{\mathrm{Hom}}_{A^{\triangleright}}(A^{\triangleright} \otimes P, M) = \underline{\mathrm{Hom}}(P, M)$ as the group of null sequences in M, and $\mathrm{id} - f\mathrm{shift}$ acts by

$$(m_0, m_1, \dots) \mapsto (m_0 - f m_1, m_1 - f m_2, \dots).$$

If this map is an isomorphism, then $(m_0', m_1', \dots) \mapsto (\sum_{i=0}^{\infty} m_i' f^i, \sum_{i=1}^{\infty} m_i' f^{i-1}, \dots)$ should be its inverse, witnessing the summability of null sequences against f.

The following holds in perfect analogy with Proposition 2.3.

Proposition 2.10. *Let $A^{\triangleright}$ be a light condensed ring and let $f \in A^{\triangleright}(*)$. Then $(A^{\triangleright}, \mathrm{Mod}_{A^{\triangleright}, f\text{-gas}})$ is a pre-analytic ring.*

Proof. As $A^{\triangleright} \otimes P = A^{\triangleright}[\mathbb{N} \cup \infty]/A^{\triangleright}[\infty]$ is internally projective and internally compact in $\mathrm{Mod}_{A^{\triangleright}}$, the proof of Proposition 2.3 goes through with $\mathrm{Mod}_{A^{\triangleright}, f\text{-gas}}$ in place of $\mathrm{Solid} = \mathrm{Mod}_{\mathbb{Z}, 1\text{-gas}}$. $\square$

Definition 2.11. For a light condensed ring $A^{\triangleright}$ and $f \in A^{\triangleright}(*)$, the completion functor of $A^{\triangleright}_{f\text{-gas}} := (A^{\triangleright}, \mathrm{Mod}_{A^{\triangleright}, f\text{-gas}})$ is called ***f*-gaseous completion** and denoted by $(-)^{f\text{-gas}}$.

In analogy with Lemma 2.4, we have the following.

Lemma 2.12. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ with the Euclidean topology and let $f \in \mathbb{K}$ with $0 < |f| < 1$. If V is the underlying topological space of a $\mathbb{K}$-Banach space, then the condensification of V is f-gaseous. In particular, $(\mathbb{K}, \mathrm{Mod}_{\mathbb{K},f\text{-gas}})$ is an analytic ring.*

Proof. For light profinite S, we have to show that $d_f = \mathrm{id} - f\mathrm{shift}$ induces an isomorphism on

$$\underline{\mathrm{Hom}}(P, V)(S) = \mathrm{Hom}(P \otimes \mathbb{Z}[S], V) = \mathrm{Hom}(P, \underline{\mathrm{Hom}}(\mathbb{Z}[S], V)) = \mathrm{Hom}(P, \mathrm{Cont}(S, V))$$

for the compact-open topology on $\mathrm{Cont}(S, V)$. This topology is induced by the sup-norm, for which $\mathrm{Cont}(S, V)$ itself is a Banach space. Upon replacing V by $\mathrm{Cont}(S, V)$, we reduce to the case $S = *$. Then d_f^* is inverted by $(v_0, v_1, \ldots) \mapsto (\sum_{i=0}^{\infty} v_i f^i, \sum_{i=1}^{\infty} v_i f^{i-1}, \ldots)$. $\square$

Remark 2.13. For a light condensed ring $A^{\triangleright}$ and $S \subseteq A^{\triangleright}(*)$ an $A^{\triangleright}$-module is *S-gaseous* if it is f-gaseous for all $f \in S$. The category of S-gaseous modules $\mathrm{Mod}_{A^{\triangleright},S\text{-gas}} := \bigcap_{f \in S} \mathrm{Mod}_{A^{\triangleright},f\text{-gas}}$ is a pre-analytic ring structure on $A^{\triangleright}$. We call its completion *S-gaseous completion* and denote it by $(-)^{S\text{-gas}}$. For $S = \{f_1, \ldots, f_n\}$, it follows formally that $M^{S\text{-gas}} = (\cdots (M^{f_1\text{-gas}})^{f_2\text{-gas}} \ldots)^{f_n\text{-gas}}$.

We conclude this introductory subsection with an observation on maps of pre-analytic rings. By definition, these are maps light condensed rings such that restriction of scalars preserves complete modules. Formally, we then also get a completed extension of scalars functor:

Lemma 2.14. *Let $\varphi\colon (A^{\triangleright}, \mathrm{Mod}_A) \to (B^{\triangleright}, \mathrm{Mod}_B)$ be a map of pre-analytic rings. Then there is a functor $- \otimes_A B\colon \mathrm{Mod}_A \to \mathrm{Mod}_B$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Mod}_{A^{\triangleright}} \arrow[r, "-\otimes_{A^{\triangleright}} B^{\triangleright}"] \arrow[d, "(-)^{\wedge A}"] & \mathrm{Mod}_{B^{\triangleright}} \arrow[d, "(-)^{\wedge B}"] \\
\mathrm{Mod}_A \arrow[r, dashed, "-\otimes_A B"] & \mathrm{Mod}_B
\end{tikzcd}$$

The functor $- \otimes_A B$ can be given by $M \mapsto (M \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B}$, is left adjoint to the restriction of scalars functor ${}_A(-)\colon \mathrm{Mod}_B \to \mathrm{Mod}_A$ and symmetric monoidal.

Proof. As the left adjoint of a fully faithful functor, $(-)^{\wedge A}\colon \mathrm{Mod}_{A^{\triangleright}} \to \mathrm{Mod}_A$ is the localisation of $\mathrm{Mod}_{A^{\triangleright}}$ at the morphisms that $(-)^{\wedge A}$ sends to isomorphisms. Hence, for the first part it suffices to show that $(- \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B}$ sends such morphisms to isomorphisms in Mod_B. In fact, we have $(- \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B} \cong ((-)^{\wedge A} \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B}$. Indeed, for $M \in \mathrm{Mod}_{A^{\triangleright}}$ and $N \in \mathrm{Mod}_B$,

$$\begin{aligned}
\mathrm{Hom}_B((M \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B}, N) &= \mathrm{Hom}_{B^{\triangleright}}(M \otimes_{A^{\triangleright}} B^{\triangleright}, N) = \mathrm{Hom}_{A^{\triangleright}}(M, {}_{A^{\triangleright}}N) \overset{(\star)}{=} \mathrm{Hom}_{A^{\triangleright}}(M^{\wedge A}, {}_{A^{\triangleright}}N) \\
&= \mathrm{Hom}_{B^{\triangleright}}(M^{\wedge A} \otimes_{A^{\triangleright}} B^{\triangleright}, N) = \mathrm{Hom}_B((M^{\wedge A} \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B}, N)
\end{aligned}$$

where in $(\star)$ we use that φ is a map of pre-analytic rings. It follows directly from the definition that $- \otimes_A B := (- \otimes_{A^{\triangleright}} B^{\triangleright})^{\wedge B}$ is left adjoint to ${}_A(-)$ and that $- \otimes_A B$ is symmetric monoidal. $\square$

2.2 Derived completion

We now extend the notion of completeness from categories of modules to their derived categories. The derived picture can reveal information that is invisible at the level of ordinary modules, but there is also a conceptual argument for the derived viewpoint. Already in scheme theory, we face the problem that $\mathrm{Spec}(R[f^{-1}]) \mapsto \mathrm{Mod}_{R[f^{-1}]}$ might not define a sheaf of Abelian categories on the distinguished opens of $\mathrm{Spec}(R)$. This can be overcome by considering $\mathrm{Spec}(R[f^{-1}]) \mapsto \mathcal{D}(\mathrm{Mod}_{R[f^{-1}]})$ instead, which does define a sheaf of ∞-categories. Therefore, with a view towards geometry, it is natural to consider module categories as derived ∞-categories from the start.

For a light condensed ring $A^{\triangleright}$, let $\mathcal{D}(\mathrm{Mod}_{A^{\triangleright}})$ be the derived ∞-category of the Grothendieck Abelian category $\mathrm{Mod}_{A^{\triangleright}}$ as in [Lur17, Definition 1.3.5.8]. Its homotopy category $D(\mathrm{Mod}_{A^{\triangleright}})$ is the classical triangulated derived category of $\mathrm{Mod}_{A^{\triangleright}}$.

Definition 2.15. A **derived pre-analytic ring structure** on a light condensed ring $A^\triangleright$ is a full sub-∞-category $\mathcal{D}(A)$ of $\mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ satisfying the following conditions:

(1) It is closed under all limits and colimits.

(2) It is closed under the functors $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M, -)$ for all $M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright})$.

(3) It is closed under truncations with respect to the canonical t-structure on $\mathcal{D}(\mathrm{Mod}_{A^\triangleright})$.

We call it a **derived analytic ring structure** if moreover $A^\triangleright \in \mathcal{D}(A)$. A **static (pre-)analytic ring** is a pair $(A^\triangleright, \mathcal{D}(A))$ consisting of a light condensed ring and a derived (pre-)analytic ring structure on it. A **map of static (pre-)analytic rings** $(A^\triangleright, \mathcal{D}(A)) \to (B^\triangleright, \mathcal{D}(B))$ is a map of light condensed rings $A^\triangleright \to B^\triangleright$ such that restriction of scalars carries $\mathcal{D}(B)$ into $\mathcal{D}(A)$.

Remarks 2.16.

(1) Points (1) and (2) are derived analogues of conditions in Definition 2.1. Point (3) is necessary and sufficient for $(\mathcal{D}(A) \cap \mathcal{D}(\mathrm{Mod}_{A^\triangleright})^{\leq 0}, \mathcal{D}(A) \cap \mathcal{D}(\mathrm{Mod}_{A^\triangleright})^{\geq 0})$ to define a t-structure on $\mathcal{D}(A)$. Indeed, for any $M \in \mathcal{D}(A)$ there is a fibre sequence $\tau^{\leq 0}M \to M \to \tau^{\geq 1}M$ and, by (3), all terms lie in $\mathcal{D}(A)$. This proves the last of the three conditions in [Lur17, Definition 1.2.1.1], while the first two conditions are immediate.

(2) Point (2) is equivalent to the seemingly weaker condition that $\mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ is closed under the functors $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(M, -)$ for all $M \in \mathcal{D}(\mathrm{Mod}_{\mathbb{Z}})$. Indeed, it suffices to verify (2) on generators $A^\triangleright[S] = A^\triangleright \otimes \mathbb{Z}[S]$ for light profinite sets S. Also point (3) can be replaced by an equivalent condition (see Remark 2.18 below).

(3) To be able to define sheaves of analytic rings, we should also allow the light condensed ring to be derived. Replacing $A^\triangleright$ by a light condensed animated ring or a light condensed $\mathbb{E}_\infty$-ring in Definition 2.1, we get the notion of a **(pre-)analytic animated ring** or a **(pre-)analytic $\mathbb{E}_\infty$-ring**. We refer to [CS20, Lectures 11−12] and [RC24b, Section 4]. The analytic rings of interest in this thesis will all be static by definition or proposition.

The *Reflection Theorem* has an ∞-categorical analogue: If $\mathcal{C}$ is a presentable ∞-category and $\mathcal{C}' \subseteq \mathcal{C}$ is a full sub-∞-category closed under limits and κ-filtered colimits for some regular cardinal κ, then $\mathcal{C}'$ is presentable (see [RS22, Theorem 1.1]). In particular, the inclusion $\mathcal{C}' \to \mathcal{C}$ has a left adjoint by Lurie's adjoint functor theorem (see [Lur09, Corollary 5.5.2.9]).

Definition 2.17. For a derived pre-analytic ring structure $\mathcal{D}(A)$ on $A^\triangleright$, the left adjoint to the inclusion $i \colon \mathcal{D}(A) \hookrightarrow \mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ is called **derived A-completion** and denoted $(-)^{\mathrm{L}\wedge_A}$ (or simply called **derived completion** and denoted $(-)^{\mathrm{L}\wedge}$, leaving A implicit).

Remark 2.18. Assuming closure under limits and colimits, point (3) in Definition 2.15 is equivalent to the condition that $i \circ (-)^{\mathrm{L}\wedge}$ preserves $\mathcal{D}^{\leq 0}(\mathrm{Mod}_{A^\triangleright})$. This follows from universal properties of truncations and $(-)^{\mathrm{L}\wedge}$, see [BBD82, Proposition 1.4.12].

As for ordinary pre-analytic rings, one of the most important features of their derived analogues is that they come with a completed tensor product, which we will denote by $\otimes_A^{\mathrm{L}\wedge}$.

Proposition 2.19. *Let $\mathcal{D}(A)$ be a derived pre-analytic ring structure on $A^\triangleright$. Then $\mathcal{D}(A)$ has a unique symmetric monoidal structure making $(-)^{\mathrm{L}\wedge}$ symmetric monoidal.*

Proof. By [NS18, Theorem I.3.6], it suffices to show that the kernel of $(-)^{\mathrm{L}\wedge}$ is a $\otimes^{\mathrm{L}}_{A^\triangleright}$-ideal. Let $M, N \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ with $M^{\mathrm{L}\wedge} = 0$. We have to prove $(M \otimes_{A^\triangleright} N)^{\mathrm{L}\wedge} = 0$, or equivalently $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}_{A^\triangleright})}(M \otimes^{\mathrm{L}}_{A^\triangleright} N, L) = 0$ for all $L \in \mathcal{D}(A)$. Indeed,

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}_{A^\triangleright})}(M \otimes^{\mathrm{L}}_{A^\triangleright} N, L) = \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}_{A^\triangleright})}(M, \mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(N, L)) = 0,$$

since $M^{\mathrm{L}\wedge} = 0$ by assumption and $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(N, L) \in \mathcal{D}(A)$. □

It turns out that derived and underived pre-analytic ring structures on light condensed rings are in perfect correspondence. In particular, this will provide us with many examples of derived analytic ring structures.

Proposition 2.20. *Let $A^\triangleright$ be a light condensed ring. Then there is a bijection*

$$\{\textit{pre-analytic ring structures on } A^\triangleright\} \xrightarrow{\sim} \{\textit{derived pre-analytic ring structures on } A^\triangleright\}$$

respecting analytic ring structures. It sends Mod_A to the full subcategory of $\mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ spanned by those $M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ such that $H^i(M) \in \mathrm{Mod}_A$ for all $i \in \mathbb{Z}$. It is inverted by $\mathcal{D}(A) \mapsto \mathcal{D}(A)^\heartsuit$.

The hardest part of the proof of Proposition 2.20 is to show that the specified map is well-defined:

Lemma 2.21. *Let Mod_A be a pre-analytic ring structure on a light condensed ring $A^\triangleright$. Then*

$$\mathcal{D}(A) := \{M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright}) : H^i(M) \in \mathrm{Mod}_A \textit{ for all } i \in \mathbb{Z}\}$$

is a derived pre-analytic ring structure on $A^\triangleright$.

Proof. It is clear that $\mathcal{D}(A)$ is closed under truncations, so it remains to verify points (1) and (2) of Definition 2.15. We can do this on the level of the homotopy category, following [CS24, Lecture 8, timestamp 47:00]. We prove that the triangulated subcategory $D(A) \subseteq D(\mathrm{Mod}_{A^\triangleright})$ is closed under cones, direct sums, products, and the functors $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M, -)$ for $M \in D(\mathrm{Mod}_{A^\triangleright})$.

Certainly, $D(A) \subseteq D(\mathrm{Mod}_{A^\triangleright})$ is a full additive subcategory closed under shifts. Let

$$M' \to M \to M'' \xrightarrow{h} M'[1]$$

be a distinguished triangle in $D(\mathrm{Mod}_{A^\triangleright})$ with $M', M'' \in D(A)$. From the associated long exact sequence we can extract, for all $i \in \mathbb{Z}$, the short exact sequence $0 \to \mathrm{coker}(H^{i-1}h) \to H^i(M) \to \ker(H^i h) \to 0$. As Mod_A is closed under kernels, cokernels and extensions, $M', M'' \in \mathrm{Mod}_A$ implies $H^i(M) \in \mathrm{Mod}_A$. This holds for all $i \in \mathbb{Z}$, hence $M \in D(A)$.

Closure under direct sums holds as for any Grothendieck Abelian category (by [Sta25, 07D9] and as direct sums $\mathrm{Mod}_{A^\triangleright}$ are exact). Since countable products in $\mathrm{Mod}_{A^\triangleright}$ are exact by Proposition 1.28, $D(A)$ is also closed under countable products (see [Sta25, Tag 07KC]).

We can reduce the closure under arbitrary products to the condition on internal homs as follows. Let $(M_\alpha)_{\alpha\in A}$ be a collection of objects of $D(A)$. Set $M := \bigoplus_\alpha M_\alpha$, then $\prod_\alpha M_\alpha$ is a retract of $\prod_\alpha M$, so it suffices to show $H^n(\prod_\alpha M) \in \mathrm{Mod}_A$ for all $n \in \mathbb{Z}$. We have

$$\textstyle\prod_\alpha M = \mathrm{R}(\underline{\mathrm{Hom}}_{A^\triangleright}(\bigoplus_\alpha A^\triangleright, -))(M) = \mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(\bigoplus_\alpha A^\triangleright, M).$$

As $M \in D(A)$ by closure under direct sums, this indeed reduces to the condition on $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}$.

It remains to show that for $M \in D(\mathrm{Mod}_{A^\triangleright})$ and $N \in D(A)$, we have $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M, N) \in D(A)$.

We first use truncations to reduce to the case when M is bounded above and N is bounded below. As for any Grothendieck Abelian category, by exactness of filtered colimits we have $M \cong \mathrm{Lcolim}\, \tau^{\leq m} M$ (see [Sta25, 0949]). As moreover countable products are exact in $\mathrm{Mod}_{A^\triangleright}$, we have $N \cong \mathrm{Rlim}\, \tau^{\geq -n} N$ (see the proof of [Sta25, 090Y]). Therefore,

$$\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M, N) = \mathrm{Rlim}_n\, \mathrm{Rlim}_m\, \mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(\tau^{\leq m} M, \tau^{\geq -n} N).$$

By stability under countable products and hence under countable homotopy limits, it thus suffices to show $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(\tau^{\leq m} M, \tau^{\geq -n} N) \in D(A)$ for all $m, n \in \mathbb{N}$. As $D(A)$ is closed under truncations, we have reduced to the case when M is bounded above and N is bounded below.

Next, we reduce to the case when M and N are concentrated in degree 0. Take a bounded below complex of injectives I representing to N so that $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M,N)$ is computed by $\underline{\mathrm{Hom}}^\bullet_{A^\triangleright}(M,I)$, the total complex of the double complex $K^{\bullet,\bullet}$ with $K^{i,j} = \underline{\mathrm{Hom}}_{A^\triangleright}(M^{-i}, I^j)$. As M is bounded above and I is bounded below, with notations as in [Sta25, 0132] there exists a convergent spectral sequence

$$E_2^{p,q} = H^p_{II} H^q_I(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(\mathrm{Tot}(K^{\bullet,\bullet})) = H^{p+q}(\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M,N)).$$

As Mod_A is closed under kernels, cokernels and extensions, it suffices to prove $E_2^{p,q} \in \mathrm{Mod}_A$ for all $p, q \in \mathbb{Z}$. Each I^p is injective, hence internally injective as $\underline{\mathrm{Hom}}_{A^\triangleright}(-, I^p)(S) = \mathrm{Hom}(- \otimes \mathbb{Z}[S], I^p)$. It follows that $H^q_I(K^{\bullet,\bullet}) = H^q(\underline{\mathrm{Hom}}_{A^\triangleright}(M^{-\bullet}, I^p))$ has degree p term $\underline{\mathrm{Hom}}_{A^\triangleright}(H^{-q}(M), I^p)$. Hence,

$$E_2^{p,q} = H^p \underline{\mathrm{Hom}}_{A^\triangleright}(H^{-q}(M), I^\bullet) = H^p(\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(H^{-q}(M), N)),$$

reducing to the case when M sits in degree 0. Then, as N is bounded below, the Grothendieck spectral sequence $E_2^{p,q} = R^p \underline{\mathrm{Hom}}_{A^\triangleright}(M, H^q(N)) \Rightarrow R^{p+q} \underline{\mathrm{Hom}}_{A^\triangleright}(M,N)$ reduces to the case when also N sits in degree 0.

If M and N sit in degree 0, then $H^i \mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(M,N) = \underline{\mathrm{Ext}}^i_{A^\triangleright}(M,N) \in \mathrm{Mod}_A$ for all $n \in \mathbb{Z}$. □

We deduce the proposition:

Proof of Proposition 2.20. The specified map is well-defined by Lemma 2.21. Also the candidate for an inverse $\mathcal{D}(A) \mapsto \mathcal{D}(A)^\heartsuit$ is well defined. Indeed, the closure properties of $\mathcal{D}(A)^\heartsuit$ follow from corresponding closure properties of $\mathcal{D}(A)$ by taking cohomology, where for closure under extensions we note that short exact sequences in $\mathrm{Mod}_{A^\triangleright}$ become fibre sequences in $\mathcal{D}(\mathrm{Mod}_{A^\triangleright})$.

We can certainly recover Mod_A as the heart of $\{M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright}) : H^i(M) \in \mathrm{Mod}_A \text{ for all } i \in \mathbb{Z}\}$. For the other direction, let $\mathcal{D}(A)$ be a derived pre-analytic ring structure on $A^\triangleright$ and let $M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright})$. We show that we have $M \in \mathcal{D}(A)$ if and only if $H^i(M) \in \mathcal{D}(A)^\heartsuit$ for all $i \in \mathbb{Z}$.

The direct implication is immediate because $H^i(M)$ is obtained from M by truncations and shifts, under which $\mathcal{D}(A)$ is closed. For the converse, we assume $H^i(M) \in \mathcal{D}(A)^\heartsuit$ for all $i \in \mathbb{Z}$ and show $M \in \mathcal{D}(A)$. We first reduce to the case when M is bounded. The fibre sequence

$$\tau^{\leq 0} M \to M \to \tau^{\geq 1} M$$

reduces to the cases when M is bounded above and when M is bounded below. Writing $M \cong \lim \tau^{\geq -n} M$ in the former case (using exactness of countable products) and $M \cong \operatorname{colim} \tau^{\leq m} M$ in the latter case, we reduce to the bounded case. Say M is concentrated in cohomological degrees $[m,n]$. We have the fibre sequence

$$\tau^{\leq m} M \to M \to \tau^{\geq m+1} M$$

in which both $\tau^{\leq m} M$ and $\tau^{\geq m+1} M$ are bounded of strictly smaller length. By induction and shifting, we reduce to the case when M is concentrated in degree 0 and conclude.

The correspondence of pre-analytic ring structures follows, and it is clear that analytic ring structures correspond to analytic ring structures. □

Proposition 2.20 produces many examples of derived (pre-)analytic ring structures.

Definition 2.22. Let $A^\triangleright$ be a light condensed ring and let $f \in A^\triangleright(*)$. The **category of derived f-gaseous modules** is the full sub-∞-category of $\mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ given by

$$\mathcal{D}(A^\triangleright_{f\text{-gas}}) := \{M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright}) : H^i(M) \in \mathrm{Mod}_{A^\triangleright, f\text{-gas}} \text{ for all } i \in \mathbb{Z}\}.$$

The completion functor of the static pre-analytic ring $(A^\triangleright, \mathcal{D}(A^\triangleright_{f\text{-gas}}))$ is called **derived f-gaseous completion** and denoted $(-)^{\mathrm{L}f\text{-gas}}$. Its completed tensor product is called the **derived f-gaseous tensor product** and denoted $\otimes^{\mathrm{L}f\text{-gas}}$.

By the internal projectivity of $A^\triangleright \otimes P$, an object $M \in \mathcal{D}(\mathrm{Mod}_{A^\triangleright})$ is derived f-gaseous if and only if $d_f = \mathrm{id} - f\mathrm{shift}$ induces an isomorphism on $\mathrm{R}\underline{\mathrm{Hom}}_{A^\triangleright}(A^\triangleright \otimes P, M)$.

The solid computations we recorded in Proposition 2.8 carry over to the derived solidification functor $(-)^{\mathrm{L}\square}$ and the derived solid tensor product $\otimes^{\mathrm{L}\square}$. The natural maps induce isomorphisms $P^{\mathrm{L}\square} \cong \prod_{\mathbb{N}} \mathbb{Z}$, $\prod_{\mathbb{N}} \mathbb{Z} \otimes^{\mathrm{L}\square} \prod_{\mathbb{N}} \mathbb{Z} \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}$ as well as $\mathbb{Z}[S]^{\mathrm{L}\square} \cong \varprojlim_i \mathbb{Z}[S_i]$ for light profinite sets $S = \varprojlim_i S_i$ (see [RC24b, Section 3]). In particular, these derived solidifications all sit degree 0.

We note that the most basic result generalises directly:

Lemma 2.23. *If $M \in \mathcal{D}(\mathrm{Mod}_{\mathbb{Z}})$ admits an $\mathbb{R}$-module structure, then $M^{\mathrm{L}\square} = 0$.*

Proof. As $(-)^{\mathrm{L}\square}$ is symmetric monoidal, it preserves ($\mathbb{E}_\infty$-)algebras and modules over them, so it suffices to show $\mathbb{R}^{\mathrm{L}\square} = 0$. But $1 = 0$ is a condition in $H^0(\mathbb{R}^{\mathrm{L}\square}) = \mathbb{R}^\square$, so we conclude. $\square$

We can identify $\mathcal{D}(\mathbb{Z}_\square)$ with the derived category of the Grothendieck Abelian category Solid, such that $(-)^{\mathrm{L}\square}$ and $\otimes^{\mathrm{L}\square}$ identify with the left derived functors of $(-)^\square$ and $\otimes^\square$, respectively (see [RC24b, Theorem 3.3.1 and Corollary 3.3.6]). However, this is not purely formal:

Warning 2.24. If $A = (A^\triangleright, \mathrm{Mod}_A)$ is a pre-analytic ring, then the inclusion $\mathrm{Mod}_A \to \mathrm{Mod}_{A^\triangleright}$ induces a functor $\mathcal{D}(\mathrm{Mod}_A) \to \mathcal{D}(A)$. However, it need not be an equivalence. When it is not one, then it makes no sense to ask if $(-)^{\mathrm{L}\wedge_A}$ is the left derived functor of $(-)^{\wedge_A}$ or if $\otimes_A^{\mathrm{L}\wedge}$ is the left derived functor of $\otimes_A^\wedge$. We should simply view these functors as derived analogues.

Derived completion preserves objects sitting in non-positive cohomological degrees, but the derived completion of an object concentrated in degree 0 might not stay in degree 0:

Example 2.25. We have a short exact sequence of light condensed Abelian groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, where exactness on the right holds as already $[0,1] \to \mathbb{R}/\mathbb{Z}$ is surjective. Then $\mathbb{Z}^{\mathrm{L}\square} = \mathbb{Z}$ and $\mathbb{R}^{\mathrm{L}\square} = 0$ shows that $(\mathbb{R}/\mathbb{Z})^{\mathrm{L}\square}$ equals $\mathbb{Z}$, concentrated in cohomological degree -1.

In fact, something curious is true. For a locally finite CW complex X, the derived solidification of $\mathbb{Z}[X]$ is quasi-isomorphic to the singular chain complex of X sitting in non-positive cohomological degrees (see [CS24, Lecture 6, timestamp 17:00]). We will not use this result, but it emphasises that important information might only be accessible through derived completion.

We have good reason to find the completion functor associated to a pre-analytic ring mysterious. Its existence is guaranteed by formal nonsense, and substantial effort can be required for basic computations. Fortunately, there is a quite explicit formula that is useful in many cases.

To motivate it, we revisit the solid case. Let $M \in \mathcal{D}(\mathrm{Mod}_{\mathbb{Z}})$ and set $d := \mathrm{id} - \mathrm{shift}$. The long exact sequence associated to $0 \to P \to P \to \mathrm{coker}(d) \to 0$ shows that M is solid if and only if

$$\mathrm{R}\underline{\mathrm{Hom}}(\mathrm{coker}(d), M) = 0.$$

With the long exact sequence associated to $0 \to \mathbb{Z} \to \mathrm{coker}(d) \to \mathrm{coker}(d)/\mathbb{Z} \to 0$ this shows that M is solid if and only if the connecting map $M = \mathrm{R}\underline{\mathrm{Hom}}(\mathbb{Z}, M) \to \mathrm{R}\underline{\mathrm{Hom}}((\mathrm{coker}(d)/\mathbb{Z})[-1], M)$

is an isomorphism. We might thus hope that $\mathrm{R}\underline{\mathrm{Hom}}((\mathrm{coker}(d)/\mathbb{Z})[-1], M)$ is the derived solidification of M. This is not true. If $\underline{\mathrm{Ext}}^1(\mathrm{coker}(d)/\mathbb{Z}, \mathbb{R})$ was zero, then $\underline{\mathrm{Hom}}(\mathrm{coker}(d), \mathbb{R})$ would surject onto $\mathbb{R}$, but we have $\underline{\mathrm{Hom}}(\mathrm{coker}(d), \mathbb{R}) = 0$ because constant null sequences in $\mathbb{R}$ are zero. However, $\mathrm{R}\underline{\mathrm{Hom}}((\mathrm{coker}(d)/\mathbb{Z})[-1], M)$ is a good first approximation of $M^{\mathrm{L}\square}$. A precise and general statement borrowed from [RC24b, Section 4.6] is as follows.

Proposition 2.26. *Let $\mathcal{C}$ be a closed symmetric monoidal stable ∞-category with all colimits. Let $A \in \mathcal{C}$ be an object admitting maps $m\colon A \otimes A \to A$ and $u\colon 1 \to A$ such that $m \circ (u \otimes \mathrm{id}_A) = \mathrm{id}_A$, and let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory of objects N that satisfy $\underline{\mathrm{Hom}}(A, N) = 0$.*

Let $C := \mathrm{fib}(u\colon 1 \to A)$ be the fibre, set $F := \underline{\mathrm{Hom}}(C, -)$, and let $\delta\colon \mathrm{id} \to F$ be the natural transformation induced by $C \to 1$. It gives rise to a sequence of natural transformations

$$\mathrm{id} \xrightarrow{\delta_0} F \xrightarrow{\delta_1} F^2 \xrightarrow{\delta_2} \cdots,$$

where $(\delta_n)_M := \delta_{F^n(M)}$. Assume that one of the following hypotheses is satisfied:

(H1) *For all $M \in \mathcal{C}$, the map $(\delta_n)_M$ is an isomorphism for $n \gg 0$.*

(H2) *The functor $\underline{\mathrm{Hom}}(A, -)$ preserves sequential colimits.*

Then the colimit $F^\infty := \mathrm{colim}(\mathrm{id} \to F \to F^2 \to \cdots)$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ factors through $\mathcal{D}$ such that the resulting functor $F^\infty\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$.

Proof. See [CS24, Lecture 13, timestamp 1:32:00] and also [RC24b, Section 4.6]. □

Of special interest is the case when A is an idempotent algebra in $\mathcal{C}$. Then an object $M \in \mathcal{C}$ admits an A-module structure if and only if the unit $1 \to A$ induces an isomorphism $M \cong M \otimes A$, so that the forgetful functor $i^*\colon \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ identifies with the full inclusion $\{M \in \mathcal{C} : M \cong M \otimes A\} \hookrightarrow \mathcal{C}$. This inclusion has a right adjoint ("coextension of scalars"), so by the yoga of admissible subcategories (see [Sta25, Tag 0H0P]) the Verdier quotient identifies with the full subcategory

$$\{M \in \mathcal{C} : \underline{\mathrm{Hom}}(A, M) = 0\}.$$

This is precisely the category $\mathcal{D}$. Now, because A is an idempotent algebra, $C = \mathrm{fib}(1 \to A)$ is an idempotent coalgebra. The sequence $\mathrm{id} \to F \to F^2 \to \cdots$ identifies with $\mathrm{id} \to \underline{\mathrm{Hom}}(C, -) \to \underline{\mathrm{Hom}}(C \otimes C, -) \to \cdots$, hence condition (H1) is satisfied and we have $F^\infty = \underline{\mathrm{Hom}}(C-)$. Summarising and adding auxiliary adjoints, we arrive at the picture

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathcal{C}) \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right=50, "i^*=A\otimes-"'] \arrow[l, bend left=50, "i^!=\underline{\mathrm{Hom}}(A,-)"] \arrow[r, "j^*=\underline{\mathrm{Hom}}(C,-)"] & \mathcal{D}, \arrow[l, bend right=50, "j_!=\mathrm{fib}(\mathrm{id}\to A\otimes-)"'] \arrow[l, bend left=50, "j_*"]
\end{tikzcd}$$

where i_* and j_* are the full inclusions and upper arrows are left adjoints of lower arrows. This mirrors the classical picture for derived categories of sheaves

$$\mathcal{D}(\mathrm{Sh}(Z)) \xrightarrow{i_*} \mathcal{D}(\mathrm{Sh}(X)) \xrightarrow{j^*} \mathcal{D}(\mathrm{Sh}(U))$$

on a topological space X, a closed subset $i\colon Z \subseteq X$, and its complementary open $j\colon U = X \setminus Z \subseteq X$. The categorical notion axiomatising this picture is that of a *recollement* (see [BBD82, Section 4.2] and [Lur17, Appendix A.8]). The upshot is that idempotent algebras in $\mathcal{C}$ should correspond to closed subsets, and we refer to [CS22, Lectures 5 – 6] for a detailed exploration of this idea. Let us only consider a concrete example from [CS24, Lecture 7].

Example 2.27. Consider the inclusion $\mathcal{D}(\mathbb{Z}[t]_{\{1,t\}\text{-gas}}) \subseteq \mathcal{D}(\mathbb{Z}[t]_{1\text{-gas}})$. Its left adjoint is derived t-gaseous completion of derived $\mathbb{Z}[t]$-modules which are already solid. By $P^{\mathrm{L}\square} = \mathbb{Z}[\![u]\!]$, the map $\mathrm{id} - t\mathrm{shift}$ induces an isomorphism on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[t]}(\mathbb{Z}[t] \otimes P, M)$ if and only if the map $1 - tu$ induces an isomorphism on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[t]}(\mathbb{Z}[\![u]\!][t], M)$. As $1 - tu$ is injective with $\mathbb{Z}[\![u]\!][t]/(1-tu) \cong \mathbb{Z}(\!(t^{-1})\!)$, this holds if and only if

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[t]}(\mathbb{Z}(\!(t^{-1})\!), M) = 0.$$

Now $\mathbb{Z}[\![t_1]\!] \otimes^{\mathrm{L}\square}_{\mathbb{Z}} \mathbb{Z}[\![t_2]\!] \cong \mathbb{Z}[\![t_1, t_2]\!]$ implies $\mathbb{Z}[\![t]\!] \otimes^{\mathrm{L}\square}_{\mathbb{Z}[t]} \mathbb{Z}[\![t]\!] = \mathbb{Z}[\![t]\!]$, which in turn gives idempotence of $\mathbb{Z}(\!(t^{-1})\!)$ in $\mathcal{D}(\mathbb{Z}[t]_{1\text{-gas}})$. Hence, the formal discussion from above applies.

We conclude this subsection by mentioning that the there exists a completion functor from the category of pre-analytic rings to the category of analytic rings. This result is most naturally proved in a derived setting. Analytic animated rings and pre-analytic animated rings form ∞-categories $\mathcal{A}\mathrm{nRing}$ and $\mathcal{P}\mathcal{A}\mathrm{nRing}$ with hom anima given by the full subanima of $\mathrm{Hom}_{\mathrm{CAlg}(\mathcal{D}(\mathrm{Mod}_{\mathbb{Z}}))}(A^{\triangleright}, B^{\triangleright})$ spanned by the maps of (pre-)analytic animated rings (given as in Definition 2.15).

Proposition 2.28. *The full inclusion $\mathcal{A}\mathrm{nRing} \hookrightarrow \mathcal{P}\mathcal{A}\mathrm{nRing}$ has a left adjoint. For a pre-analytic animated ring $(A^{\triangleright}, \mathcal{D}(A))$, the completion $(A^{\triangleright})^{\mathrm{L}\wedge} \in \mathcal{D}(A)$ has the structure of a light condensed animated $A^{\triangleright}$-algebra such that the left adjoint can be given by $(A^{\triangleright}, \mathcal{D}(A)) \mapsto ((A^{\triangleright})^{\mathrm{L}\wedge}, \mathcal{D}(A))$.*

Proof. See [Man22, Proposition 2.3.12] and, in the light setting, [RC24b, Section 4.2]. $\square$

Remark 2.29. This proposition is key to proving that the category of analytic rings has all colimits (see [RC24b, Section 4.2]).

The statement of Proposition 2.28 is also true for (undervied) analytic rings. In this case, the left adjoint $(A^{\triangleright}, \mathrm{Mod}_A) \mapsto ((A^{\triangleright})^{\wedge}, \mathrm{Mod}_A)$ might cut off important information if $(A^{\triangleright})^{\mathrm{L}\wedge}$ does not sit in degree 0. Curiously, Proposition 2.28 is most easily proved for analytic $\mathbb{E}_\infty$-rings, and this case implies the assertion for (underived) analytic rings (see [CS24, Lecture 13, timestamp 09:00]). The crux in the animated case is to show that not only has $(A^{\triangleright})^{\mathrm{L}\wedge}$ the structure of an $\mathbb{E}_\infty$-algebra, but also that of an animated algebra.

The upshot is that we can construct analytic rings by completing pre-analytic rings, and that this procedure does not change the categories of complete modules. In other words, for any pre-analytic ring structure there is a canonical choice of an underlying light condensed ring turning it into an analytic ring structure. The caveat is that it might not sit in degree 0, but if it does then Proposition 2.20 allows us to work on the Abelian level without loss of generality.

2.3 The f-independence of $\mathrm{Mod}_{R,f\text{-gas}}$

Let us now take a closer look at gaseous completion. In Lemma 2.12, we proved that for any $f \in \mathbb{R}$ with $0 < |f| < 1$ we get an analytic ring structure $\mathrm{Mod}_{\mathbb{R},f\text{-gas}}$ on $\mathbb{R}$. In this subsection, we show among other things that it is independent of the choice of f.

For $f, g \in \mathbb{R}$ with $0 < |g| \leq |f| < 1$, summability of null sequences against f should be stronger than summability of null sequences against g, so we expect an inclusion $\mathrm{Mod}_{\mathbb{R},f\text{-gas}} \subseteq \mathrm{Mod}_{\mathbb{R},g\text{-gas}}$. On the other hand, as $|f| < 1$, we have $|f^n| \leq |g|$ for $n \gg 0$ and then expect an inclusion $\mathrm{Mod}_{\mathbb{R},g\text{-gas}} \subseteq \mathrm{Mod}_{\mathbb{R},f^n\text{-gas}}$. We first study how $\mathrm{Mod}_{\mathbb{R},f\text{-gas}}$ and $\mathrm{Mod}_{\mathbb{R},f^n\text{-gas}}$ are related.

Lemma 2.30. *Let R be any light condensed ring, let $f \in R(*)$, and let n be a positive integer. Then $\mathrm{Mod}_{R,f^n\text{-gas}} = \mathrm{Mod}_{R,f\text{-gas}}$.*

As a motivation, for a classical null sequence $(a_m)_m$ we can write

$$\textstyle\sum_{m=0}^{\infty} a_m f^m = \sum_{i=0}^{n-1}\sum_{m=0}^{\infty}(a_{mn+i}f^i)(f^n)^m. \qquad (\star)$$

Thus $(a_m)_m$ is summable against f if $(a_{0+i}f^i, a_{n+i}f^i, \ldots)$ is summable against f^n for all $0 \le i < n$. We thus expect $\mathrm{Mod}_{R,f^n\text{-gas}} \subseteq \mathrm{Mod}_{R,f\text{-gas}}$. Conversely, $(a_0, a_1, \ldots)$ is summable against f^n if $(a_0, 0, \ldots, 0, a_1, 0, \ldots)$ is summable against f, so we expect $\mathrm{Mod}_{R,f\text{-gas}} \subseteq \mathrm{Mod}_{R,f^n\text{-gas}}$.

Proof of Lemma 2.30. For $\mathrm{Mod}_{R,f^n\text{-gas}} \subseteq \mathrm{Mod}_{R,f\text{-gas}}$, we follow [Ked25, Section 10.4]. Set $d_f := \mathrm{id} - f\mathrm{shift}$ as before and consider the map

$$g\colon R\otimes P \to R\otimes P, [m] \mapsto [m] + f[m+1] + \cdots + f^{n-1}[m+n-1].$$

Note that we have $d_f \circ g = g \circ d_f$, and that this map is given by $[m] \mapsto [m] - f^n[m+n]$. Indeed,

$$\begin{aligned} d_f g([m]) &= d_f([m] + \cdots + f^{n-1}[m+n-1]) \\ &= [m] - f[m+1] + \cdots + f^{n-1}[m+n-1] - f^n[m+n] = [m] - f^n[m+n] \\ &= [m] + \cdots + f^{n-1}[m+n-1] - (f[m+1] + \cdots + f^n[m+n]) = g([m] - f[m+1]) \\ &= g d_f([m]). \end{aligned}$$

Set $\overline{d_{f^n}} := g \circ d_f = d_f \circ g$. Motivated by $(\star)$, we consider the rearrangement map

$$P^{\oplus n} \to P,\ ([m_i])_{i=0}^{n-1} \mapsto \textstyle\sum_{i=0}^{n-1}[m_i n + i].$$

It is an isomorphism with inverse $P \to P^{\oplus n}$ given on the i-th factor by $[k] \mapsto [(k-i)/n]$ if $n \mid k-i$ and $[k] \mapsto 0$ otherwise. Extending scalars gives an isomorphism $e\colon (R\otimes P)^{\oplus n} \to R\otimes P$.

Next, note that we have $\overline{d_{f^n}} \circ e = e \circ (d_{f^n})^{\oplus n}$. Indeed,

$$\begin{aligned} \overline{d_{f^n}}(e(([m_i])_{i=0}^{n-1})) &= \overline{d_{f^n}}(\textstyle\sum_{i=0}^{n-1}[m_i n + i]) \\ &= \textstyle\sum_{i=0}^{n-1}[m_i n + i] - \sum_{i=0}^{n-1} f^n[m_i n + i + n] \\ &= \textstyle\sum_{i=0}^{n-1}[m_i n + i] - \sum_{i=0}^{n-1} f^n[(m_i+1)n + i] \\ &= e(([m_i] - f[m_i+1])_{i=0}^{n-1}) \\ &= e(d_f^{\oplus n}(([m_i])_{i=0}^{n-1})) \end{aligned}$$

Applying $\underline{\mathrm{Hom}}_R(-, M)$ for some $M \in \mathrm{Mod}_R$ gives a commutative square

$$\begin{tikzcd} \underline{\mathrm{Hom}}_R(R\otimes P, M)^{\oplus n} \arrow[r, "(d_{f^n}^*)^{\oplus n}"] & \underline{\mathrm{Hom}}_R(R\otimes P, M)^{\oplus n} \\ \underline{\mathrm{Hom}}_R(R\otimes P, M) \arrow[u, "\cong"', "e^*"] \arrow[r, "(\overline{d_{f^n}})^*"] & \underline{\mathrm{Hom}}_R(R\otimes P, M) \arrow[u, "\cong"', "e^*"] \end{tikzcd}$$

Hence, if $d_{f^n}^*$ is an isomorphism, then $(\overline{d_{f^n}})^* = d_f^* \circ g^* = g^* \circ d_f^*$ is an isomorphism, which implies in particular that d_f^* is an isomorphism. Thus $\mathrm{Mod}_{R,f^n\text{-gas}} \subseteq \mathrm{Mod}_{R,f\text{-gas}}$.

Next, we show $\mathrm{Mod}_{R,f\text{-gas}} \subseteq \mathrm{Mod}_{R,f^n\text{-gas}}$. Define maps $\varphi, \psi\colon R\otimes P \to R\otimes P$ by

$$\varphi\colon [m] \mapsto \begin{cases} [m/n] & \text{if } n \mid m \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \psi\colon [m] \mapsto f^{r_n(-m)}[\lceil m/n \rceil],$$

for the residue $r_n(-m) \in \{0, \ldots, n-1\}$ of $-m$ modulo n. We have $\psi = \varphi \circ g$ for the map g defined above, and a computation shows $d_{f^n} \circ \varphi = \psi \circ d_f$.

The maps φ and ψ are split surjective with the same section $s\colon [k] \mapsto [nk]$. Applying $\underline{\mathrm{Hom}}_R(-, M)$ for $M \in \mathrm{Mod}_{R,f\text{-gas}}$, we get the commutative square

$$\begin{tikzcd} \underline{\mathrm{Hom}}_R(R \otimes P, M) & \underline{\mathrm{Hom}}_R(R \otimes P, M) \arrow[l, hook', "\varphi^*"'] \\ \underline{\mathrm{Hom}}_R(R \otimes P, M) \arrow[u, "d_f^*"] & \underline{\mathrm{Hom}}_R(R \otimes P, M) \arrow[l, hook', "\psi^*"'] \arrow[u, "d_{f^n}^*"] \end{tikzcd}$$

in which φ^* and ψ^* are split injective with the same left inverse s^*. By $M \in \mathrm{Mod}_{R,f\text{-gas}}$, the map d_f^* is an isomorphism, and we can define

$$\rho \coloneqq s^* \circ d_f^{*-1} \circ \varphi^*.$$

By commutativity of the square, ρ is a left inverse of $d_{f^n}^*$. To show that it is a right inverse, first note that we have $d_{f^n}^* \circ s^* = s^* \circ (\overline{d_{f^n}})^*$. Therefore,

$$d_{f^n}^* \circ \rho = d_{f^n}^* \circ s^* \circ d_f^{*-1} \circ \varphi^* = s^* \circ (\overline{d_{f^n}})^* \circ d_f^{*-1} \circ \varphi^* = s^* \circ g^* \circ \varphi^* = s^* \circ \psi^* = \mathrm{id}.$$

We deduce that $d_{f^n}^*$ is an isomorphism, that is, $M \in \mathrm{Mod}_{R,f^n\text{-gas}}$. $\square$

In the example of $\mathrm{Mod}_{\mathbb{R},g\text{-gas}}$ for $g \in \mathbb{R}$ with $0 < |g| < 1$, the powers of g form a null sequence.

Definition 2.31. Let R be a light condensed ring. An element $g \in R(*)$ is **topologically nilpotent** if there is a map of light condensed Abelian groups $P \to R$ given on $*$ by $[m] \mapsto g^m$.

Beware that for non-quasi-separated R, a map $P \to R$ need not be determined by $P(*) \to R(*)$.

Lemma 2.32. *Let R be a light condensed ring and let $f, g \in R(*)$ with g topologically nilpotent. Then $\mathrm{Mod}_{R,f\text{-gas}} \subseteq \mathrm{Mod}_{R,fg\text{-gas}}$.*

Proof. As g is topologically nilpotent, we have a map of R-modules $R \otimes P \to R \otimes P$, $[n] \mapsto g^n[n]$. Tensoring with the identity of P and composing with $[m] \otimes [n] \mapsto [m+n]$ gives the map

$$\varphi\colon R \otimes P \otimes P \to R \otimes P,\ [m] \otimes [n] \mapsto g^n[m+n],$$

We have $d_{fg} \circ \varphi = \varphi \circ (\mathrm{id} \otimes d_f)$. Indeed,

$$\begin{aligned} d_{fg}\varphi([m] \otimes [n]) &= d_{fg}(g^n[m+n]) = g^n[m+n] - fg^{n+1}[m+n+1] \\ &= \varphi([m] \otimes [n] - f[m] \otimes [n+1]) = \varphi(\mathrm{id} \otimes d_f)([m] \otimes [n]). \end{aligned}$$

Also note that φ is split surjective with right inverse $s\colon [m] \mapsto [m] \otimes [0]$. Applying $\underline{\mathrm{Hom}}_R(-, M)$ for $M \in \mathrm{Mod}_{R,f\text{-gas}}$, we get the commutative square

$$\begin{tikzcd} \underline{\mathrm{Hom}}_R(R \otimes P \otimes P, M) & \underline{\mathrm{Hom}}_R(R \otimes P, M) \arrow[l, hook', "\varphi^*"'] \\ \underline{\mathrm{Hom}}_R(R \otimes P \otimes P, M) \arrow[u, "(\mathrm{id} \otimes d_f)^*"] & \underline{\mathrm{Hom}}_R(R \otimes P, M). \arrow[l, hook', "\varphi^*"'] \arrow[u, "d_{fg}^*"'] \end{tikzcd}$$

The map $(\mathrm{id} \otimes d_f)^*$ is an isomorphism by the tensor-hom adjunction, so we can define

$$\rho \coloneqq s^* \circ (\mathrm{id} \otimes d_f)^{*-1} \circ \varphi^*.$$

Then ρ is a left inverse of d_{fg}^* by commutativity of the square. Moreover, the identities

$$\begin{gathered} d_{fg}^* \circ s^* = s^* \circ (d_{fg} \otimes \mathrm{id})^*, \\ (d_{fg} \otimes \mathrm{id})^* \circ (\mathrm{id} \otimes d_f)^{*-1} = (\mathrm{id} \otimes d_f)^{*-1} \circ (d_{fg} \otimes \mathrm{id})^*, \text{ and} \\ (d_{fg} \otimes \mathrm{id})^* \circ \varphi^* = \varphi^* \circ d_{fg}^* \end{gathered}$$

imply $d_{fg}^* \circ \rho = \rho \circ d_{fg}^*$ by looking at two factors at a time. Hence $M \in \mathrm{Mod}_{R,fg\text{-gas}}$. $\square$

We obtain the desired independence result:

Proposition 2.33. *Let R be a quasi-separated light condensed ring and let $f, g \in R(*)$. If f is a unit and g is topologically nilpotent, then $\mathrm{Mod}_{R,f\text{-gas}} \subseteq \mathrm{Mod}_{R,g\text{-gas}}$. In particular, if f and g are both topologically nilpotent units, then $\mathrm{Mod}_{R,f\text{-gas}} = \mathrm{Mod}_{R,g\text{-gas}}$.*

Proof. As g is topologically nilpotent, so is $g^n f^{-1}$ for some $n \gg 0$. Indeed, because R is quasi-separated, topological nilpotence can be checked classically in $R(*)_{\mathrm{top}}$. Then

$$\mathrm{Mod}_{R,f\text{-gas}} \subseteq \mathrm{Mod}_{R,f(g^n f^{-1})\text{-gas}} = \mathrm{Mod}_{R,g^n\text{-gas}} = \mathrm{Mod}_{R,g\text{-gas}}$$

by Lemma 2.32 and Lemma 2.30. □

Corollary 2.34. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ with the Euclidean topology. Then the analytic ring structure $\mathrm{Mod}_{\mathbb{K},f\text{-gas}}$ on $\mathbb{K}$ is independent of the choice of $f \in \mathbb{K}$ with $0 < |f| < 1$.*

Thus, in the situation of Proposition 2.33, we might as well define "the" gaseous structure.

Definition 2.35. Let R be a quasi-separated light condensed ring that admits a topologically nilpotent unit $f \in R(*)$. Then we call $\mathrm{Mod}_{R,\mathrm{gas}} := \mathrm{Mod}_{R,f\text{-gas}}$ the **gaseous pre-analytic ring structure** on R and $(-)^{\mathrm{gas}} := (-)^{f\text{-gas}}$ the **gaseous completion functor**.

For example, we now obtain the gaseous reals $\mathbb{R}_{\mathrm{gas}} = (\mathbb{R}, \mathrm{Mod}_{\mathbb{R},\mathrm{gas}})$, the gaseous complex numbers $\mathbb{C}_{\mathrm{gas}} = (\mathbb{C}, \mathrm{Mod}_{\mathbb{C},\mathrm{gas}})$, and the gaseous p-adics $\mathbb{Q}_{p,\mathrm{gas}} = (\mathbb{Q}_p, \mathrm{Mod}_{\mathbb{Q}_p,\mathrm{gas}})$.

Our discussion directly carries over to the categories of derived f-gaseous modules $\mathcal{D}(R_{f\text{-gas}})$. In fact, the derived viewpoint reveals a more algebraic interpretation of the computations above:

Remark 2.36. Let R be a light condensed ring and let $f, g \in R(*)$. By the commutative algebra structrue on $P = \mathbb{Z}[\hat{x}]$, the inclusion $\mathrm{Mod}_{R,f\text{-gas}} \subseteq \mathrm{Mod}_{R,g\text{-gas}}$ holds if and only if

$$\mathrm{R}\underline{\mathrm{Hom}}_R(R[\hat{x}]/^{\mathrm{L}}(1-gx), M) = 0$$

for all $M \in \mathrm{Mod}_{R,f\text{-gas}}$. Equivalently, this holds if and only if the derived f-gaseous completion of $R[\hat{x}]/^{\mathrm{L}}(1-gx)$ vanishes. But $1 = 0$ is a condition on H^0, so this is equivalent to

$$R[\hat{x}]^{f\text{-gas}}/(1-gx) = 0.$$

In particular, proving that the action of $1 - gx$ on $\underline{\mathrm{Hom}}_R(R[\hat{x}], M)$ is left-invertible (resp. right-invertible) for all M amounts to finding a left-inverse (resp. right-inverse) of $1 - gx$ in $R[\hat{x}]^{f\text{-gas}}$. This is a a commutative algebra, so left-inverses are automatically right-inverses and vice versa.

Appendix: f-gaseous and g-gaseous implies fg-gaseous

In this appendix, we prove the following general result on gaseous modules that we will use in our application to tempered holomorphic functions in Section 3, Proposition 3.27.

Proposition 2.37. *Let R be a light condensed ring and let $f, g \in R(*)$. Then*

$$\mathrm{Mod}_{R,f\text{-gas}} \cap \mathrm{Mod}_{R,g\text{-gas}} \subseteq \mathrm{Mod}_{R,fg\text{-gas}}.$$

For example, we already know that this holds when $f = g$ by Lemma 2.30 and when f and g are topologically nilpotent by Lemma 2.32.

The proof will be technical, but the idea is simple. Classically, if null sequences are summable against f and g, then we can obtain the sum of a null sequence $(a_m)_m$ against fg as follows. Put the sequence $(a_m)_m$ on the main diagonal of a two-dimensional grid, perform the operations

$$\begin{pmatrix} a_0 & 0 & \cdots \\ 0 & a_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \xmapsto[\text{against } f]{\text{sum rows}} \begin{pmatrix} a_0 & fa_1 & \cdots \\ 0 & a_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \xmapsto[\text{against } g]{\text{sum columns}} \begin{pmatrix} a_0 + gfa_1 + \cdots & fa_1 + gf^2a_2 + \cdots & \cdots \\ ga_1 + g^2fa_2 + \cdots & a_1 + gfa_2 + \cdots & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

and then restrict to the main diagonal. We now turn this into a precise argument.

Proof of Proposition 2.37. Consider the maps $\mathrm{diag}\colon P \otimes P \to P$, $[m] \otimes [n] \mapsto \delta_{mn}[m]$, where δ_{mn} is the Kronecker-delta, and $\Delta\colon P \to P \otimes P$, $[m] \mapsto [m] \otimes [m]$. In the following, we also denote their base changes to R by diag and Δ, respectively, and generally suppress notations for base changes to R. We show that for $M \in \mathrm{Mod}_{R,f\text{-gas}} \cap \mathrm{Mod}_{R,g\text{-gas}}$, the composite

$$\underline{\mathrm{Hom}}(P, M) \xrightarrow{\mathrm{diag}^*} \underline{\mathrm{Hom}}(P \otimes P, M) \xrightarrow{(d_f \otimes d_g)^{*-1}} \underline{\mathrm{Hom}}(P \otimes P, M) \xrightarrow{\Delta^*} \underline{\mathrm{Hom}}(P, M)$$

is an inverse of d_{fg}. By Remark 2.36, it suffices to show that it is a left-inverse of d_{fg} (here, $\mathrm{Mod}_{R,f\text{-gas}} \cap \mathrm{Mod}_{R,g\text{-gas}} \subseteq \mathrm{Mod}_{R,fg\text{-gas}}$ is equivalent to $R[\hat{x}]^{\{f,g\}\text{-gas}}/(1 - gfx) = 0$). Define maps

$$\begin{aligned} \varphi &:= ((\mathrm{id} \otimes d_g)^*, (d_f \otimes \mathrm{id})^*, -(d_f \otimes d_g)^*)\colon\ (P \otimes P)^* \to ((P \otimes P)^*)^{\oplus 3}, \\ \psi &:= (d_f \otimes \mathrm{id})^{*-1} \oplus (\mathrm{id} \otimes d_g)^{*-1} \oplus \mathrm{id}\colon\ ((P \otimes P)^*)^{\oplus 3} \to ((P \otimes P)^*)^{\oplus 3} \end{aligned}$$

where $(-)^*$ denotes $\underline{\mathrm{Hom}}(-, M)$ also on objects. Then the following diagram commutes:

$$\begin{tikzcd} P^* \arrow[r, "\mathrm{diag}^*"] \arrow[dr, "{(\mathrm{diag}^*,\mathrm{diag}^*,-\mathrm{diag}^*)}"'] & (P \otimes P)^* \arrow[r, "(d_f \otimes d_g)^{*-1}"] \arrow[d, "{(\mathrm{id},\mathrm{id},\mathrm{id})}"] & (P \otimes P)^* \arrow[r, "\Delta^*"] \arrow[d, "\varphi"] & P^* \arrow[r, "d_{fg}^*"] & P^* \\ & ((P \otimes P)^*)^{\oplus 3} \arrow[r, "\psi"] & ((P \otimes P)^*)^{\oplus 3} \arrow[r, "(\Delta^*)^{\oplus 3}"] & (P^*)^{\oplus 3} \arrow[ur, "{(\mathrm{id},\mathrm{id},\mathrm{id})^*}"'] & \end{tikzcd}$$

Indeed, for the left triangle and the middle square this is immedate, and for the right pentagon it can be checked before applying $(-)^*$. The bottom composite is the sum of the three maps

$$\Delta^* \circ (d_f \otimes \mathrm{id})^{*-1} \circ \mathrm{diag}^*, \quad \Delta^* \circ (\mathrm{id} \otimes d_g)^{*-1} \circ \mathrm{diag}^* \quad \text{and} \quad \Delta^* \circ (-\mathrm{diag}^*).$$

The third map is $-\mathrm{id}$. If we show that the first two maps are both id, then the composite in question is $\mathrm{id} + \mathrm{id} - \mathrm{id} = \mathrm{id}$ and the proposition follows.

Claim. *We have* $\Delta^* \circ (d_f \otimes \mathrm{id})^{*-1} \circ \mathrm{diag}^* = \mathrm{id}$.

Proof of the Claim. We first prove that $(d_f \otimes \mathrm{id})^*$ restricts to an isomorphism on "upper triangular matrices". To this end, let $N \subseteq P \otimes P$ be the image of $\mathbb{Z}[\{(m, n) \in \mathbb{N} \cup \infty \times \mathbb{N} \cup \infty : m > n\}]$ under $\mathbb{Z}[\mathbb{N} \cup \infty \times \mathbb{N} \cup \infty] \to P \otimes P$, and set $B := (P \otimes P)/N$. Then N, $P \otimes P$ and B are all abstractly isomorphic to P by enumerating grids (see Lemma 1.37), and the inclusion $N \to P \otimes P$ admits a retraction r mapping $[m] \otimes [n]$ to itself if $m > n$ and to 0 otherwise.

As $d_f \otimes \mathrm{id}$ preserves N, it induces a self-map of the the short exact sequence $0 \to N \to P \otimes P \to B \to 0$. Applying $(-)^*$, we obtain a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & B^* \arrow[r] \arrow[d, "\rho"] & (P \otimes P)^* \arrow[r] \arrow[d, "(d_f \otimes \mathrm{id})^*", "\cong"'] & N^* \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & B^* \arrow[r] & (P \otimes P)^* \arrow[r] & N^* \arrow[r] & 0. \end{tikzcd}$$

We show that ρ is an isomorphism. By the Snake Lemma, it suffices to prove injectivity of τ.

The injective composite $(d_f \otimes \mathrm{id})^* \circ r^* \colon N^* \hookrightarrow (P \otimes P)^* \xrightarrow{\sim} (P \otimes P)^*$ can be viewed as the map

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ a_{10} & 0 & 0 & \cdots \\ a_{20} & a_{21} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \longmapsto \begin{pmatrix} -fa_{10} & 0 & 0 & \cdots \\ a_{10} - fa_{20} & -fa_{21} & 0 & \cdots \\ a_{20} - fa_{30} & a_{21} - fa_{31} & -fa_{32} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and τ can be viewed as this map followed by the map setting all main diagonal entries $(-fa_{10}, -fa_{21}, \ldots)$ to zero. Thus, classically, to prove injectivity of τ we need to show that these main diagonal entries can be recovered from the strictly lower triangular entries.

Precisely, to show that τ is injective it suffices to show that the injection $(d_f \otimes \mathrm{id})^* \circ r^*$ factors through τ. The map $(d_f \otimes \mathrm{id})^* \circ r^*$ has image in $N^* \oplus P^* \subseteq (P \otimes P)^*$, the "lower triangular matrices", where N^* is included via r^* and P^* is included via diag^*. We can take the desired factorisation $N^* \to N^* \oplus P^*$ to be the identity on N^*, so it remains to find a factorisation

$$\begin{tikzcd}
N^* \arrow[r, hook, "r^*"] \arrow[d, "\tau"] & (P \otimes P)^* \arrow[r, "(d_f \otimes \mathrm{id})^*"] & (P \otimes P)^* \arrow[r, "\Delta^*"] & P^* \\
N^*. \arrow[urrr, dashed] & & &
\end{tikzcd}$$

Consider the isomorphism $\vartheta \colon N^* \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(P, M))$ that identifies a strictly lower triangular matrix with the its sequence of strictly lower diagonals (explicitly, ϑ is induced by $P \otimes P \to N$, $[m] \otimes [n] \mapsto [m+n+1] \otimes [n]$ and tensor-hom). Let $\mathrm{ev}_0 \colon \underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(P, M)) \to P^*$ be the map that extracts the first sequence from a sequence of sequences (explicitly, ev_0 is induced by $P \to P \otimes P$, $[n] \mapsto [0] \otimes [n]$ and tensor-hom). Then the following diagram commutes:

$$\begin{tikzcd}
(P \otimes P)^* \arrow[r, "(d_f \otimes \mathrm{id})^*"] & (P \otimes P)^* \arrow[r, "\Delta^*"] & P^* \\
N^* \arrow[u, "r^*"] \arrow[r, "\vartheta"'] \arrow[d, "\tau"'] \arrow[rr, bend left, "\Delta^* \circ r^*"] & \underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(P, M)) \arrow[r, "\mathrm{ev}_0"'] \arrow[d, "d_f^*"] & P^* \arrow[u, "\cdot(-f)"'] \\
N^* \arrow[r, "\vartheta"] & \underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(P, M)) \arrow[r, "d_f^{*-1}"] & \underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(P, M)) \arrow[u, "\mathrm{ev}_0"]
\end{tikzcd}$$

This is immediate for the bottom right square and can be checked before applying $(-)^*$ everywhere else. This proves the existence of the desired factorisation, and hence that ρ^* is an isomorphism.

To deduce $\Delta^* \circ (d_f \otimes \mathrm{id})^{*-1} \circ \mathrm{diag}^* = \mathrm{id}$, note that $\mathrm{diag} \colon P^* \to (P \otimes P)^*$ has image in B^*, so it suffices to show $\Delta^*|_{B^*} \circ \rho^{-1} \circ \mathrm{diag}^* = \mathrm{id}$. Indeed, $\Delta^*|_{B*} = \Delta^*|_{B^*} \circ \rho$ follows before applying $(-)^*$, hence $\Delta^*|_{B^*} \circ \rho^{-1} \circ \mathrm{diag}^* = \Delta^*|_{B^*} \circ \mathrm{diag}^* = \mathrm{id}$ as wanted.

This concludes the proof of the claim and hence of the proposition. $\square$

Remark 2.38. Non-quasi-separated modules can admit counter-intuitive null sequences. For instance, by Lemma 1.30, any convergent sequence in $\mathbb{R}$ induces a null sequence in $\mathbb{R}/\mathbb{R}_{\mathrm{disc}}$ and it induces the zero null sequence if and only if it is eventually constant. Thus

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1/2 & 1/2 & \cdots \\ 1 & 1/2 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

defines a non-zero element of $\mathrm{Hom}(P \otimes P, \mathbb{R}/\mathbb{R}_{\mathrm{disc}})$ for which the restriction to any row or any column is the zero element of $\mathrm{Hom}(P, \mathbb{R}/\mathbb{R}_{\mathrm{disc}})$. Such examples hint why Proposition 2.37 is non-trivial, even though the corresponding statement for classical null sequences is fairly obvious.

2.4 The gaseous base ring

We saw in the previous subsection that for a quasi-separated light condensed ring R and a topologically nilpotent unit $f \in R(*)$, the pre-analytic ring structure $\mathrm{Mod}_{R,f\text{-gas}}$ is independent of the choice of f. This motivates us to look for a universal such pair (R, f).

Recall from Remark 1.42 that P is naturally a light condensed ring with underlying ring $\mathbb{Z}[q]$, which we denote by $\mathbb{Z}[\hat{q}]$.

The pair $(\mathbb{Z}[\hat{q}], q)$ is initial among quasi-separated light condensed rings equipped with a topologically nilpotent element. Indeed, let R be a quasiseparated light condensed ring and let $f \in R(*)$ be topologically nilpotent. Then there is a map of light condensed Abelian groups $\varphi\colon \mathbb{Z}[\hat{q}] \to R$ given on the point by $q^n \mapsto f^n$. As R is quasi-separated, the map

$$\mathrm{Hom}(\mathbb{Z}[\hat{q}], R) \to \textstyle\prod_{\mathbb{N}} R(*),\ \ \varphi \mapsto (\varphi_{\{*\}}(q^n))_n$$

is injective, so φ is the unique such map. Now φ is a map of light condensed rings by definition of the ring structure on $\mathbb{Z}[\hat{q}]$. On the other hand, any light condensed ring map $\psi\colon \mathbb{Z}[\hat{q}] \to R$ with $\psi_{\{*\}}(q) = f$ satisfies $\psi_{\{*\}}(q^n) = f^n$ and thus agrees with φ.

Inverting q with general sheaf theory, we obtain the light condensed ring $\mathbb{Z}[\hat{q}^{\pm 1}] := \mathbb{Z}[\hat{q}][q^{-1}]$. As a light condensed $\mathbb{Z}[\hat{q}]$-module, it identifies with the colimit along injections

$$\mathrm{colim}(\mathbb{Z}[\hat{q}] \xrightarrow{q} \mathbb{Z}[\hat{q}] \xrightarrow{q} \cdots) \cong \mathrm{colim}\Big(\mathbb{Z}[\hat{q}] \hookrightarrow \frac{1}{q}\mathbb{Z}[\hat{q}] \hookrightarrow \frac{1}{q^2}\mathbb{Z}[\hat{q}] \hookrightarrow \cdots\Big).$$

Hence, by the description of $\mathbb{Z}[\hat{q}]$ from Corollary 1.41, $\mathbb{Z}[\hat{q}^{\pm 1}]$ is quasi-separated and its light condensed structure is described by the union

$$\bigcup_{n=0}^{\infty} \frac{1}{q^n}\mathbb{Z}[\hat{q}] = \bigcup_{n=0}^{\infty}\bigcup_{N=0}^{\infty}\Big\{\sum_{m\geq -n} a_m q^m \in \mathbb{Z}[q][q^{-1}] : \sum_{m\geq -n} |a_m| \leq N\Big\}.$$

By the universal property of localisation, the pair $(\mathbb{Z}[\hat{q}^{\pm 1}], q)$ is initial among quasi-separated light condensed rings equipped with a topologically nilpotent unit.

Definition 2.39. The **gaseous base ring** is $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}} := (\mathbb{Z}[\hat{q}^{\pm 1}]^{q\text{-gas}}, \mathrm{Mod}_{\mathbb{Z}[\hat{q}^{\pm 1}],q\text{-gas}})$.

Note that with this definition we already anticipate that $\mathbb{Z}[\hat{q}^{\pm 1}]^{\mathrm{L}\,q\text{-gas}}$ is concentrated in degree 0 (Corollary 2.43 below). Otherwise, we should really consider $(\mathbb{Z}[\hat{q}^{\pm 1}]^{\mathrm{L}\,q\text{-gas}}, \mathcal{D}(\mathbb{Z}[\hat{q}^{\pm 1}]_{q\text{-gas}}))$.

For a quasi-separated light condensed ring R admitting a topologically nilpotent unit, the gaseous pre-analytic ring structure on R is induced by the gaseous pre-analytic ring structure on $\mathbb{Z}[\hat{q}^{\pm 1}]$. We thus think of $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}$ as a discretisation of $\mathbb{R}_{\mathrm{gas}}, \mathbb{C}_{\mathrm{gas}}$ and $\mathbb{Q}_{p,\mathrm{gas}}$.

We want to describe $\mathbb{Z}[\hat{q}^{\pm 1}]_{\mathrm{gas}}$ in terms of its underlying ring $\mathbb{Z}[\hat{q}^{\pm 1}]^{\mathrm{L}\,q\text{-gas}}$ and, more generally, in terms of the free completed modules $\mathbb{Z}[\hat{q}^{\pm 1}][S]^{\mathrm{L}\,q\text{-gas}}$ for light profinite sets S. To invert q means take a colimit commuting with completion, so in order to understand the completion of $\mathbb{Z}[\hat{q}^{\pm 1}]$ we really need to understand the completion of $\mathbb{Z}[\hat{q}]$. Let us make an educated guess, following the intuition that q-gaseous completion makes all null sequences summable against q.

Summing the null sequence $(q^m)_m$ in $\mathbb{Z}[\hat{q}]$ against q gives $\sum_{m=0}^{\infty} q^{2m}$, and we can expect more generally that $\mathbb{Z}[\hat{q}]^{q\text{-gas}}$ contains all power series with bounded coefficients. Then also $(\sum_{n=m}^{\infty} q^n)_m$ must be summable against q. Rearranging gives $\sum_{n=0}^{\infty}(\sum_{m=n}^{\infty} q^n)q^m = \sum_{\ell=0}^{\infty}\lceil\frac{\ell+1}{2}\rceil q^k$, so we can expect that $\mathbb{Z}[\hat{q}]^{q\text{-gas}}$ contains all power series with coefficients of at most linear growth. Iterating, this naturally leads to power series with coefficients of polynomial growth.

Definition 2.40. The **ring of tempered arithmetic power series** is the light condensed subring of $\mathbb{Z}[\![q]\!]$ defined by

$$\mathbb{Z}[\![q]\!]^{\mathrm{temp}} := \bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \Big\{ \sum_{m=0}^{\infty} a_m q^m \in \mathbb{Z}[\![q]\!] : |a_m| \leq N(m+1)^k \Big\}.$$

The (k, N)-th member of this union is the light profinite set $\prod_{m=0}^{\infty} \mathbb{Z} \cap [-N(m+1)^k, N(m+1)^k]$. As the multiplication on $\mathbb{Z}[\![q]\!]$ carries the (k_1, N_1)-th member times the (k_2, N_2)-th member to the $(k_1 + k_2 + 1, N_1 N_2)$-th member, $\mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ is indeed a light condensed subring of $\mathbb{Z}[\![q]\!]$.

Theorem 2.41.

(1) The natural map $\mathbb{Z}[\hat{q}] \to \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}} \cong \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$.

(2) For a light profinite set $S = \varprojlim_i S_i$, the derived q-gaseous completion of $\mathbb{Z}[\hat{q}][S]$ is concentrated in degree 0, given by

$$\bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \varprojlim_i \Big(\prod_{m=0}^{\infty} \mathbb{Z}[S_i]_{\leq N(m+1)^k} q^m \Big)$$

equipped with the $\mathbb{Z}[\hat{q}]$-submodule structure from $\varprojlim_i \mathbb{Z}[\![q]\!][S_i]$.

Here, $\mathbb{Z}[S_i]_{\leq M}$ is the finite discrete set $\{\sum_{s \in S_i} a_s[s] \in \mathbb{Z}[S_i] : \sum_{s \in S_i} |a_s| \leq M\}$ as before.

Proof. See [CS24, Lecture 14, timestamp 17:00]. □

The proof of Theorem 2.41 is by a quite subtle analysis of a Koszul complex arising from the formula for completion of Proposition 2.26. In [CS24], only the computation of $H^0(\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}})$ is sketched. The fact that this is hard is, in a sense, the price we pay for getting the gaseous analytic ring structure essentially for free. Compare this to the approach in [CS20], where the free completed modules are given and proving that they define an analytic ring structure is hard.

As a consequence of Theorem 2.41, we can compute the gaseous base ring.

Definition 2.42. The **ring of tempered arithmetic Laurent series** is the light condensed $\mathbb{Z}[\hat{q}^{\pm 1}]$-algebra $\mathbb{Z}(\!(q)\!)^{\mathrm{temp}} := \mathbb{Z}[\![q]\!]^{\mathrm{temp}}[q^{-1}]$.

By the explicit description of $\mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ in Definition 2.40, we can describe $\mathbb{Z}(\!(q)\!)^{\mathrm{temp}}$ explicitly as

$$\mathbb{Z}(\!(q)\!)^{\mathrm{temp}} = \bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \bigcup_{n=0}^{\infty} \{ \sum_{m \geq -n} a_m q^m \in \mathbb{Z}(\!(q)\!) : |a_m| \leq N(m+n+1)^k \}$$

equipped with the light condensed subring structure from $\mathbb{Z}(\!(q)\!) := \mathbb{Z}[\![q]\!][q^{-1}]$. By commuting completion and colimits, Theorem 2.41 gives the following description.

Corollary 2.43.

(1) The natural map $\mathbb{Z}[\hat{q}^{\pm 1}] \to \mathbb{Z}^{\mathrm{temp}}(\!(q)\!)$ induces an isomorphism $\mathbb{Z}[\hat{q}^{\pm 1}]^{\mathrm{L}\,q\text{-gas}} \cong \mathbb{Z}(\!(q)\!)^{\mathrm{temp}}$.

(2) For a light profinite set $S = \varprojlim_i S_i$, the derived q-gaseous completion of $\mathbb{Z}[\hat{q}^{\pm 1}][S]$ is concentrated in degree 0, given by the light condensed set

$$\bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \bigcup_{n=0}^{\infty} \varprojlim_i \Big(\prod_{m \geq -n} \mathbb{Z}[S_i]_{\leq N(m+n+1)^k} q^m \Big)$$

equipped with the $\mathbb{Z}[\hat{q}^{\pm 1}]$-submodule structure from $\varprojlim_i \mathbb{Z}[\![q]\!][S_i][q^{-1}]$.

We conclude this subsection by indicating how the computation of the gaseous base ring provides the foundation for gaseous complex geometry.

Let q act on $\mathbb{R}$ as multiplication by $1/2$, then we have an exact sequence of $\mathbb{Z}[\hat{q}^{\pm 1}]$-modules

$$0 \to \mathbb{Z}(\!(q)\!)^{\mathrm{temp}} \xrightarrow{(1-2q)} \mathbb{Z}(\!(q)\!)^{\mathrm{temp}} \to \mathbb{R} \to 0.$$

Here, the surjection is $\sum_{m \gg -\infty} a_m q^m \mapsto \sum_{m \gg -\infty} a_m 2^{-m}$. Exactness as light condensed modules is non-trivial because we need to keep track of growth conditions (compare this to [CS20, Proposition 7.2]).

The exact sequence together with the description of the free completed modules $\mathbb{Z}[\hat{q}^{\pm 1}][S]^{\mathrm{gas}}$ results in a quite complicated description of $\mathbb{R}[S]^{\mathrm{gas}}$ (see [CS24, Lecture 14, timestamp 1:15:00]). Passing to $\mathbb{R}[\hat{q}] = P \otimes \mathbb{R} = \mathbb{R}[\mathbb{N} \cup \infty]/\mathbb{R}[\infty]$ gives the more enlightening description

$$\mathbb{R}[\hat{q}]^{\mathrm{L\,gas}} = \bigcup_{N=1}^{\infty} \bigcup_{\varepsilon > 0} \Big\{ \sum_{m=0}^{\infty} a_m q^m \in \mathbb{R}[\![q]\!] : |a_m| \leq N \cdot 2^{-m^{\varepsilon}} \Big\}.$$

Base changing to $\mathbb{C}$, we see that $\mathbb{C}[\hat{q}]^{\mathrm{L\,gas}}$ has the analogous description with $\mathbb{C}$ in place of $\mathbb{R}$. The quasi-exponential decay condition on coefficients tells us that $\mathbb{C}[\hat{q}]_{\mathrm{gas}} = (\mathbb{C}[\hat{q}]^{\mathrm{gas}}, \mathrm{Mod}_{\mathbb{C}[\hat{q}],\mathrm{gas}})$ should be viewed as a ring of functions on a quite exotic disc sitting strictly in between the open and the closed unit disc.

Now let $\mathcal{O}(\overline{\mathbb{D}}^{\dagger})$ be the ring of overconvergent holomorphic functions on the closed unit disc. We would usually describe as

$$\mathcal{O}(\overline{\mathbb{D}}^{\dagger}) = \Big\{ \sum_{m=0}^{\infty} a_m q^m \in \mathbb{C}[\![q]\!] : \text{there exists } r > 1 \text{ such that } |a_m r^m| \to 0 \Big\}.$$

It turns out, however, that the exact condition on $|a_m r^m|$ does not matter since $r > 1$ is allowed to vary. As soon as there exists $r > 1$ such that $|a_m r^m|$ is uniformly bounded, we can pick any $1 < r' < r$ and then $|a_m (r')^m|$ converges to 0 with exponential decay. The conditions that $|a_m r^m|$ is a null sequence and that $|a_m r^m|$ is a quasi-exponentially decaying null sequence lie strictly in between these extremes, so it is equally valid to write

$$\mathcal{O}(\overline{\mathbb{D}}^{\dagger}) = \bigcup_{r>1} \Big\{ \sum_{m=0}^{\infty} a_m q^m \in \mathbb{C}[\![q]\!] : \text{there exist } N > 0 \text{ and } \varepsilon > 0 \text{ with } |a_m r^m| \leq N \cdot 2^{-m^{\varepsilon}} \Big\}.$$

In other words, $\mathcal{O}(\overline{\mathbb{D}}^{\dagger}) = \mathrm{colim}_{r>1} \mathbb{C}[\widehat{q_{/r}}]^{\mathrm{gas}}$, where the variables $q_{/r}$ indicate that the transition map $\mathbb{C}[\widehat{q_{/r'}}]^{\mathrm{gas}} \to \mathbb{C}[\widehat{q_{/r}}]^{\mathrm{gas}}$ for $r < r'$ multiplies coefficients by $(r'/r)^m$.

Hence $\mathcal{O}(\overline{\mathbb{D}}^{\dagger})$ has the structure of a gaseous light condensed algebra over $\mathbb{C}[q]$. We can check that it is idempotent in $\mathcal{D}(\mathbb{C}[q]_{\mathrm{gas}})$ and proceed as in [CS22, Lecture 5] to get a structure sheaf $U \mapsto \mathcal{O}(U)_{\mathrm{gas}}$ of analytic rings of holomorphic functions on $\mathbb{C}$. The upshot is that the complex geometry developed in [CS22] works perfectly well with the gaseous theory.

3 Tempered holomorphic functions

Let us now apply the theory of analytic rings to rediscover the notion of a tempered holomorphic function. Recall the definition that we already gave in the introduction of this thesis.

Definition 3.1. Let $U \subseteq \mathbb{C}$ be bounded and open with boundary ∂U. A holomorphic function $f\colon U \to \mathbb{C}$ is **tempered** if there exists an integer $N \geq 0$ such that $\sup_{z\in U} \mathrm{dist}(z, \partial U)^N |f(z)| < \infty$.

We give more standard and equivalent definition below (see Definition 3.5 and Lemma 3.6). Our main goal is to prove the following result.

Theorem A. *The derived q-gaseous completion of* $\mathbb{C}[q]$ *is concentrated in degree* 0*, given by the ring of tempered holomorphic functions on the open unit disc in* $\mathbb{C}$ *(equipped with a suitable light condensed structure, see Definition 3.13 below).*

Here, we simply have $\mathbb{C}[q] = \mathbb{C} \otimes \mathbb{Z}[q]$ for the discrete ring $\mathbb{Z}[q]$, so our result shows that tempered holomorphic functions arise really naturally in the gaseous theory. Although our discussion is located in Archimedean geometry, it will also give a new proof of Theorem 2.41(1).

Theorem B. *The natural map* $\mathbb{Z}[\hat{q}] \to \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ *induces an isomorphism* $\mathbb{Z}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ *(for the definition of* $\mathbb{Z}[\![q]\!]^{\mathrm{temp}}$*, see Definition 2.40 above).*

As we saw in the previous section, this computation is of independent interest because it captures a universal example of gaseous completion.

We start with a classical introduction to tempered holomorphic functions in Subsection 3.1. This subsection is included only for motivation and is not logically required for the proofs of our results. In Subsection 3.2, we make preliminary observations for the proof of Theorem A and give an outline of the argument. The proof proceeds in two main steps, discussed in Subsections 3.3 and 3.4. A digression in between the two subsections is devoted to the proof of Theorem B.

3.1 Tempered holomorphic functions, classically

In the introduction of this thesis, we recalled Hilbert's 21st problem. In modern terms, it asks for an equivalence between an "analytic" category of differential equations and a "topological" category of solutions. Nowadays, such equivalences are called *Riemann-Hilbert correspondences* and the quest to find them is called the *Riemann-Hilbert problem*.

Let us explicit the complex-analytic Riemann-Hilbert problem, give a more detailed introduction to tempered holomorphic functions, and to indicate how they are used in practice. We follow the exposition in [Hoh14].

For a complex manifold X, let $\mathcal{O}_X$ be its sheaf of holomorphic functions and let $\mathbb{C}_X$ be its constant sheaf with stalk $\mathbb{C}$. The *sheaf of differential operators on* X is defined classically as the $\mathbb{C}_X$-subalgebra $\mathcal{D}_X$ of $\underline{\mathrm{Hom}}_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the sheaf of holomorphic vector fields on X. For an open chart $U \subseteq X$ with local coordinates $(z_1, \dots, z_n)$, we have

$$\mathcal{D}_X|_U \cong \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial_z^\alpha, \quad \text{where} \quad \partial_z^{(\alpha_1,\dots,\alpha_n)} := \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}},$$

with non-commutative ring structure dictated by the rules of complex differentiation. A $\mathcal{D}_X$-*module* is a sheaf of left modules over $\mathcal{D}_X$. For example, $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, and for any $P \in \mathcal{D}_X(X)$ there is a $\mathcal{D}_X$-module $\mathcal{M}_P := \mathcal{D}_X / \mathcal{D}_X P$. For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{D}_X}$ we then have

$$\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}_P, \mathcal{F}) = \ker(P\colon \mathcal{F} \to \mathcal{F}),$$

so $\mathcal{M}_P$ classifies solutions of the differential equation $Pu = 0$.

Definition 3.2. Let X be a complex manifold and let $\mathcal{M}$ be a $\mathcal{D}_X$-module. The **holomorphic solution complex** of $\mathcal{M}$ is $\mathcal{S}\mathrm{ol}(\mathcal{M}) := \mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$.

The functor $\mathcal{S}\mathrm{ol}$ extends to the bounded derived category of $\mathrm{Mod}_{\mathcal{D}_X}$, and we can explicit the complex-analytic Riemann-Hilbert problem as follows.

Problem 3.3. *Find suitable subcategories of $D^b(\mathrm{Mod}_{\mathcal{D}_X})$ and $D^b(\mathrm{Mod}_{\mathbb{C}_X})$ for which $\mathcal{S}\mathrm{ol}$ restricts to an equivalence.*

Example 3.4. As in the introduction, consider $P := z^2\partial_z + z$ and $Q := z^2\partial_z + 1$ on $X = \mathbb{C}$. Solving $Pu = 0$ and $Qu = 0$ classically gives

$$H^0\mathcal{S}\mathrm{ol}(\mathcal{M}_P) = \mathbb{C}_{X\setminus\{0\}} \cdot \frac{1}{z} \quad \text{and} \quad H^0\mathcal{S}\mathrm{ol}(\mathcal{M}_Q) = \mathbb{C}_{X\setminus\{0\}} \cdot \exp(\frac{1}{z}),$$

so we have $H^0\mathcal{S}\mathrm{ol}(\mathcal{M}_P) \cong H^0\mathcal{S}\mathrm{ol}(\mathcal{M}_Q)$. By computing the cokernels of P and Q, it can be checked that we have $\mathcal{S}\mathrm{ol}(\mathcal{M}_P) \cong \mathcal{S}\mathrm{ol}(\mathcal{M}_Q)$. However, $\mathcal{M}_P$ and $\mathcal{M}_Q$ are not isomorphic as $\mathcal{D}_{\mathbb{C}}$-modules, for example because they have non-isomorphic meromorphic solutions.

Thus, $\mathcal{D}_{\mathbb{C}}$-modules need not be determined by their holomorphic solution complexes. As mentioned before, this is related to the fact that $\exp(1/z)$ is not tempered. Let us explain how to define a *tempered holomorphic solution complex* that can distinguish P and Q.

The correct definition of temperance for this purpose is a bit more subtle than Definition 3.1. Primarily, temperance is a property of smooth functions that can be defined without reference to a complex structure. The following is [KS01, Definition 7.2.3].

Definition 3.5. Let X be a real manifold, let $U \subseteq X$ be open, and let $f\colon U \to \mathbb{C}$ be smooth. For $p \in X$, f is said to have **polynomial growth at p** if for a local coordinate system $(x_1, \dots, x_n)$ around p there exists a compact neighbourhood K of p and a positive integer N such that

$$\sup_{x\in K\cap U} \mathrm{dist}(x, K \setminus U)^N |f(x)| < \infty.$$

The function f is said to have **polynomial growth** if it has polynomial growth at all points of X. We say that f is **tempered** if f and all its partial derivatives have polynomial growth. Let $\mathcal{C}_X^{\infty,\mathrm{temp}}(U) \subseteq \mathcal{C}_X^{\infty}(U)$ denote the sub-$\mathbb{C}$-algebra of tempered smooth functions on U.

Note that a smooth function on U has polynomial growth at all $p \in U$ and all $p \notin \overline{U}$. Indeed, if $p \in U$, then we can arrange that K is contained in U and take $N = 0$. If $p \notin \overline{U}$, then we can arrange $K \cap U = \emptyset$ so that the supremum is $-\infty$. Hence, polynomial growth is a condition at the boundary. This is further clarified by the following result, which also justifies Definition 3.1.

Lemma 3.6. *Let $U \subseteq \mathbb{R}^n$ be a bounded open and let $f\colon U \to \mathbb{C}$ be a smooth function. Then the following are equivalent:*

(1) The function f has polynomial growth.

(2) There exists an integer $N \geq 0$ such that $\sup_{x\in U} \mathrm{dist}(x, \partial U)^N |f(z)| < \infty$.

If $n = 2m$ such that f identifies with a holomorphic function on $U \subseteq \mathbb{C}^m$, then these equivalent conditions are satisfied if and only if f is tempered.

Proof. The equivalence of (1) and (2) follows directly from definitions, see [Mor07, Proposition 2.2.3]. If f is holomorphic, then the condition (2) for f already implies the condition (2) for all derivatives of f by Cauchy's integral formula, see [Siu70, Lemma 3]. $\square$

To define tempered holomorphic solution complexes, we first promote tempered smooth functions to a sheaf. Note that $U \mapsto \mathcal{C}_X^{\infty,\mathrm{temp}}(U)$ need not define sheaf on the topological space of a real manifold X. For example, $\exp(1/x)$ is tempered on all open intervals (ε, ∞) for $\varepsilon > 0$ but not on their union $\mathbb{R}_{>0}$. We should thus only allow finite covers. It turns out that, moreover, we should only allow opens that are "tame" in a suitable sense.

Definition 3.7. Let X be a real manifold.

(1) A subset $A \subseteq X$ is **semianalytic** if each $x \in X$ has an open neighbourhood U in X such that $A \cap U$ is a finite union of sets of the form

$$\{x \in U : f_1(x) > 0, \dots, f_r(x) > 0, g_1(x) = 0, \dots, g_s(x) = 0\}$$

for real analytic functions $f_1, \dots, f_r, g_1, \dots, g_r \colon U \to \mathbb{R}$.

(2) A subset $A \subseteq X$ is **subanalytic** if each $x \in X$ has an open neighbourhood U in X such that there exists a real manifold N and a relatively compact semi-analytic set $B \subseteq M \times N$ such that $A \cap U = \mathrm{pr}_M(B)$.

(3) The **subanalytic site** of X, denoted X_{sa}, is the site of relatively compact subanalytic open subsets of X with coverings given by finite ordinary coverings.

Here, a subset of X is called relatively compact if its closure is compact.

A detailed introduction to subanalytic sets can be found in [BM88]. The Łojasiewicz inequality (Theorem 6.4 in [BM88]) implies the following.

Proposition 3.8. *Let X be a real manifold. Then $U \mapsto \mathcal{C}_X^{\infty,\mathrm{temp}}(U)$ defines a sheaf of $\mathbb{C}$-algebras on the subanalytic site X_{sa}. We denote it by $\mathcal{C}_{X_{\mathrm{sa}}}^{\infty,\mathrm{temp}}$.*

Proof. See [KS01, Lemma 7.2.2 and Lemma 7.2.4]. □

To define tempered holomorphic solution complexes, some sheaf yoga is required. Denoting the usual site of open subsets of X by X_{top}, the inclusion defines a natural morphism of sites

$$\rho \colon X_{\mathrm{top}} \to X_{\mathrm{sa}}.$$

We get adjoint functors $\rho^{-1} : \mathrm{Mod}_{\mathbb{C}_{X_{\mathrm{sa}}}} \rightleftarrows \mathrm{Mod}_{\mathbb{C}_X} : \rho_*$ with $(\rho_*\mathcal{F})(V) = \mathcal{F}(V)$. It turns out that ρ^{-1} also has a left adjoint $\rho_!$ which sends $\mathcal{F}$ to the sheafification of $U \mapsto \varinjlim_{\overline{U} \subseteq V \in X_{\mathrm{top}}} \mathcal{F}(V)$. The functors ρ_* and $\rho_!$ are fully faithful, and $\rho_!$ is exact and symmetric monoidal. This follows from the theory of ind-sheaves (see [KS01]) but can also be checked directly (see [Pre08]).

Returning to complex geometry, let X be a complex manifold and identify its underlying real manifold with the diagonal in $X \times \overline{X}$, where $\overline{X}$ is the complex conjugate manifold. The holomorphic functions on X are singled out among smooth functions by the equation $\overline{\partial} f = 0$ for the Cauchy-Riemann operator $\overline{\partial}$ on $\overline{X}$. Let $\mathcal{O}_{\overline{X}}$ denote the sheaf of antiholomorphic functions $\mathcal{D}_{\overline{X}}/\mathcal{D}_{\overline{X}}\overline{\partial}$. While $\mathcal{C}_{X_{\mathrm{sa}}}^{\infty,\mathrm{temp}}$ can fail to be a $\rho_*\mathcal{D}_{\overline{X}}$ module, it is a $\rho_!\mathcal{D}_{\overline{X}}$-module.

Definition 3.9. Let X be a complex manifold. The **sheaf of tempered holomorphic functions** on X is $\mathcal{O}_{X_{\mathrm{sa}}}^{\mathrm{temp}} := \mathrm{R}\underline{\mathrm{Hom}}_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}},, \mathcal{C}_{X_{\mathrm{sa}}}^{\infty,\mathrm{temp}})$.

Remark 3.10. Some authors take a different but equivalent definition, see [KS01, Section 7.3].

We have $H^0\mathcal{O}_{X_{\mathrm{sa}}}^{\mathrm{temp}}(U) = \mathcal{O}_X(U) \cap \mathcal{C}_{X_{\mathrm{sa}}}^{\infty,\mathrm{temp}}(U)$. If X is a curve, then $\mathcal{O}_{X_{\mathrm{sa}}}^{\mathrm{temp}}$ is concentrated in degree 0, but this fails in higher dimensions (see [KS01, Remark 7.3.4]).

Definition 3.11. Let X be a complex manifold and let $\mathcal{M}$ be a $\mathcal{D}_X$-module. The **tempered holomorphic solution complex** of $\mathcal{M}$ is $\mathcal{S}\mathrm{ol}^{\mathrm{temp}}(\mathcal{M}) \coloneqq \mathrm{R}\underline{\mathrm{Hom}}_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathcal{O}^{\mathrm{temp}}_{X_{\mathrm{sa}}})$.

The following example is discussed in [KS03, Section 7].

Example 3.12. As before, consider $P \coloneqq z^2\partial_z + z$ and $Q \coloneqq z^2\partial_z + 1$ on $X = \mathbb{C}$. Since $1/z$ is tempered, by Example 3.4 we find

$$H^0\mathcal{S}\mathrm{ol}^{\mathrm{temp}}(\mathcal{M}_P) \cong \rho_*(\mathbb{C}_{X\setminus\{0\}} \cdot \frac{1}{z}).$$

For positive real numbers A, we set $U_A \coloneqq \{z \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(1/z) < A\}$, which equals the complement of the closed ball with radius $1/(2A)$ centred at $2A$ in $\mathbb{C}$. In [KS03, Section 7], it is proved that for $U \in X_{\mathrm{sa}}$, the function $\exp(1/z)$ is tempered on U if and only if $U \subseteq U_A$. Hence,

$$H^0\mathcal{S}\mathrm{ol}^{\mathrm{temp}}(\mathcal{M}_Q) \cong \underrightarrow{\mathrm{colim}}_{A>0}\, \rho_*(\mathbb{C}_{U_A} \cdot \exp(\frac{1}{z}))$$

where the colimit is taken in $\mathrm{Mod}_{\mathbb{C}_{\mathrm{sa}}}$. We find $H^0\mathcal{S}\mathrm{ol}^{\mathrm{temp}}(\mathcal{M}_P) \neq H^0\mathcal{S}\mathrm{ol}^{\mathrm{temp}}(\mathcal{M}_Q)$.

In [Kas80, Kas84], Kashiwara solves Problem 3.3. For suitable subcategories $D^b_{\mathrm{rh}}(\mathrm{Mod}_{\mathcal{D}_X}) \subseteq D^b(\mathrm{Mod}_{\mathcal{D}_X})$ of *regular holonomic* complexes and $D^b_{\mathbb{C}\text{-c}}(\mathrm{Mod}_{\mathbb{C}_X}) \subseteq D^b(\mathrm{Mod}_{\mathbb{C}_X})$ of $\mathbb{C}$-*constructible* complexes, the holomorphic solution functor restricts to an equivalence

$$D^b_{\mathrm{rh}}(\mathrm{Mod}_{\mathcal{D}_X})^{\mathrm{op}} \xrightarrow{\sim} D^b_{\mathbb{C}\text{-c}}(\mathrm{Mod}_{\mathbb{C}_X}).$$

The sheaf of tempered holomorphic functions plays a key role in defining the quasi-inverse, and on regular holonomic $\mathcal{D}_X$-modules we have $\rho_*\mathcal{S}\mathrm{ol}^{\mathrm{temp}}(-) \cong \mathcal{S}\mathrm{ol}(-)$ (see [KS03, Theorem 6.3]).

3.2 Statement and outline of the proof

Theorem A asserts that the derived q-gaseous completion of $\mathbb{C}[q]$ is concentrated in degree 0, given by the ring $\mathcal{O}^{\mathrm{temp}}(\mathbb{D})$ of tempered holomorphic functions on the open unit disc. Let us specify the light condensed structure on $\mathcal{O}^{\mathrm{temp}}(\mathbb{D})$ and give an outline of the proof. We can work with the hands-on definition of temperance for bounded opens $U \subseteq \mathbb{C}$ from Definition 3.1,

$$\mathcal{O}^{\mathrm{temp}}(U) \coloneqq \{f \in \mathcal{O}_{\mathbb{C}}(U) : \text{there exists } N \geq 0 \text{ such that } \sup_{z\in U} \mathrm{dist}(z, \partial U)^N |f(z)| < \infty\}.$$

On the open unit disc, this rational growth condition near the boundary is equivalent to a polynomial growth condition on Taylor coefficients:

$$\mathcal{O}^{\mathrm{temp}}(\mathbb{D}) = \Big\{ \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[\![z]\!] : \text{there exists } k \in \mathbb{N} \text{ such that } |a_n| \in O(n^k) \Big\}.$$

Indeed, let $f\colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function with Taylor expansion $\sum_{n=0}^{\infty} a_n z^n$. If f is tempered, then there exist $N \geq 0$ and $c > 0$ such that for all $z \in \mathbb{D}$ with $|z| = r > 0$ we have $|f(z)| \leq c/(1-r)^N$. Hence, by Cauchy's estimate,

$$|a_n| = \Big|\frac{f^{(n)}(0)}{n!}\Big| \leq \frac{\sup_{|z|=r} |f(z)|}{r^n} \leq \frac{c}{(1-r)^N r^n}.$$

For $r = 1 - 1/n$, we have $r^n \sim e$ while $1 - r = 1/n$, so $|a_n| \in O(n^N)$. Conversely, assume that we find $k \in \mathbb{N}$ and $C > 0$ such that $|a_n| \leq Cn^k$ for all n. Then, for all $z \in \mathbb{D}$ with $|z| = r > 0$,

$$|f(z)| \leq C \sum_{n=0}^{\infty} n^k r^n \leq C \frac{|P_k(r)|}{(1-r)^{k+1}}$$

for a polynomial P_k by differentiating geometric series. Hence $\sup_{z\in\mathbb{D}} |f(z)| \mathrm{dist}(z, \partial\mathbb{D})^{k+1} < \infty$. This description tells us what the light condensed structure on $\mathcal{O}^{\mathrm{temp}}(\mathbb{D})$ should be.

Definition 3.13. The **ring of tempered complex power series**, denoted $\mathbb{C}[\![q]\!]^{\mathrm{temp}}$, is the light condensed subring of $\mathbb{C}[\![q]\!]$ defined by

$$\mathbb{C}[\![q]\!]^{\mathrm{temp}} := \bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \Big\{ \sum_{m=0}^{\infty} a_m q^m \in \mathbb{C}[\![q]\!] : |a_m| \leq N(m+1)^k \Big\}.$$

Define the **ring of tempered real power series**, denoted $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$, analogously with $\mathbb{R}$ in place of $\mathbb{C}$ (and note that we also defined the **ring of tempered arithmetic power series**, denoted $\mathbb{Z}[\![q]\!]^{\mathrm{temp}}$, analogously in Definition 2.40).

The (k, N)-th member of the union defining $\mathbb{C}[\![q]\!]^{\mathrm{temp}}$ is a metrisable compact subspace of $\prod_{m=0}^{\infty} \mathbb{C}$, giving $\mathbb{C}[\![q]\!]^{\mathrm{temp}}$ its light condensed structure. As multiplication on $\mathbb{C}[\![q]\!]$ carries the (k_1, N_1)-th member times the (k_2, N_2)-th member to the $(k_1 + k_2 + 1, N_1 N_2)$-th member, it is indeed a light condensed subring of $\mathbb{C}[\![q]\!]$. The same applies to $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$.

According to the discussion above, the underlying ring of $\mathbb{C}[\![q]\!]^{\mathrm{temp}}$ is $\mathcal{O}^{\mathrm{temp}}(\mathbb{D})$, and a more refined version of our main result is as follows.

Theorem A. *The natural map $\mathbb{C}[q] \to \mathbb{C}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{C}[q]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{C}[\![q]\!]^{\mathrm{temp}}$.*

We structure the proof in two steps. Let $\mathbb{R}[\hat{q}]$ be the free $\mathbb{R}$-vector space on a null sequence $\mathbb{R} \otimes P$, equipped with its natural ring structure.

(1) The natural map $\mathbb{R}[\hat{q}] \to \mathbb{R}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[\![q]\!]^{\mathrm{temp}}$.

(2) The natural map $\mathbb{R}[q] \to \mathbb{R}[\hat{q}]$ induces an isomorphism $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[q]^{\mathrm{L}\, q\text{-gas}}$.

Together, these steps give $\mathbb{R}[q]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[\![q]\!]^{\mathrm{temp}}$. Extending scalars from $\mathbb{R}$ to $\mathbb{C}$ will cause no difficulties, completing the proof of Theorem A.

We handle the computation of $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}}$ in Subsection 3.3. Our initial plan for this step was to use the description of $\mathbb{Z}[\hat{q}]^{\mathrm{L}\, q\text{-gas}}$ as a blackbox. As a welcome surprise, however, we can also compute $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}}$ directly and reverse the implications in our initial plan. By comparison with $(\mathbb{R}/\mathbb{Z})[\hat{q}]^{\mathrm{L}\, q\text{-gas}}$, we will obtain a proof of the following result, stated earlier as Theorem 2.41(1), that does not rely on the abstract formula for completion from Proposition 2.26.

Theorem B. *The natural map $\mathbb{Z}[\hat{q}] \to \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{Z}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$.*

A brief digression after Subsection 3.3 is devoted to the proof of Theorem B. Afterwards, in Subsection 3.4, we prove $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[q]^{\mathrm{L}\, q\text{-gas}}$ and conclude the proof of Theorem A.

Let us get one possible source of confusion out of the way.

Remark 3.14. When speaking of the derived q-gaseous completion $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}}$ of $\mathbb{R}[\hat{q}]$, we could be referring to $\mathbb{R}[\hat{q}]$ as a module over $\mathbb{Z}[q]$, $\mathbb{Z}[\hat{q}]$, $\mathbb{R}[q]$ or $\mathbb{R}[\hat{q}]$. Our abuse of notation is justified because for any map of light condensed rings $\varphi \colon \mathbb{Z}[q] \to A^{\triangleright}$ and any $M \in \mathcal{D}(\mathrm{Mod}_{A^{\triangleright}})$, we have

$$_{\mathbb{Z}[q]}(M^{\mathrm{L}\, \varphi(q)\text{-gas}}) \cong (_{\mathbb{Z}[q]}M)^{\mathrm{L}\, q\text{-gas}}.$$

Indeed, for any $N \in \mathcal{D}(A^{\triangleright}_{\varphi(q)\text{-gas}})$, the fact that the $\varphi(q)$-gaseous pre-analytic ring structure on $A^{\triangleright}$ is induced by the q-gaseous pre-analytic ring structure on $\mathbb{Z}[q]$ gives

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[q]}(_{\mathbb{Z}[q]}(M^{\mathrm{L}\, \varphi(q)\text{-gas}}), N) &= \mathrm{Hom}_{A^{\triangleright}}(M^{\mathrm{L}\, \varphi(q)\text{-gas}}, \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(A^{\triangleright}, N)) \\ &= \mathrm{Hom}_{A^{\triangleright}}(M, \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(A^{\triangleright}, N)) \\ &= \mathrm{Hom}_{\mathbb{Z}[q]}(_{\mathbb{Z}[q]}M, N). \end{aligned}$$

Therefore, as restriction of scalars is conservative, the choice of base ring does not matter for the question whether $\mathbb{R}[\hat{q}] \to \mathbb{R}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[\![q]\!]^{\mathrm{temp}}$.

3.3 The q-gaseous completion of $\mathbb{R}[\hat{q}]$

We start the proof of Theorem A by understanding the derived q-gaseous completion of $\mathbb{R}[\hat{q}]$.

Proposition 3.15. *The natural map $\mathbb{R}[\hat{q}] \to \mathbb{R}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism*

$$\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[\![q]\!]^{\mathrm{temp}}.$$

The idea is to use a telescoping argument. It is natural to consider a filtration as follows.

Definition 3.16. For $k \in \mathbb{N}$, the **module of k-tempered real power series** is the light condensed $\mathbb{Z}[\hat{q}]$-module $\mathbb{R}[\![q]\!]^{k\text{-temp}} := \bigcup_{N=0}^{\infty}\{\sum_{m=0}^{\infty} a_m q^m \in \mathbb{R}[\![q]\!] : |a_m| \leq N(m+1)^k\}$.

So $\mathbb{R}[\![q]\!]^{0\text{-temp}}$ consists of power series with bounded coefficients, $\mathbb{R}[\![q]\!]^{1\text{-temp}}$ consists of power series with coefficients of at most linear growth, and so on. The ascending union $\bigcup_{k=0}^{\infty} \mathbb{R}[\![q]\!]^{k\text{-temp}}$ is $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$, and we will prove that each of the natural maps

$$\mathbb{R}[\hat{q}] \to \mathbb{R}[\![q]\!]^{0\text{-temp}} \to \mathbb{R}[\![q]\!]^{1\text{-temp}} \to \cdots$$

induces an isomorphism on derived q-gaseous completions.

We can readily check that $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$ is q-gaseous. Indeed, by the explicit description it is enough to show that classical null sequences in $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$ are summable against q. Let $(\sum_{m=0}^{\infty} a_{m,n} q^m)_n$ be such a null sequence. Then we find $k, N \in \mathbb{N}$ such that $|a_{m,n}| \leq N(m+1)^k$ for all $m, n \in \mathbb{N}$. The ℓ-th coefficient of $\sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} a_{m,n} q^m) q^n = \sum_{\ell=0}^{\infty}(\sum_{n=0}^{\ell} a_{\ell-n,n}) q^\ell$ satisfies

$$\Big|\sum_{n=0}^{\ell} a_{\ell-n,n}\Big| \leq \sum_{n=0}^{\ell} N(\ell-n+1)^k \leq N(\ell+1)^{k+1},$$

hence the sum against q lies in $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$. In fact, this shows that the sum of a null sequence in $\mathbb{R}[\![q]\!]^{k\text{-temp}}$ against q lies in $\mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}$, providing further motivation for the filtration above.

We will need an explicit description of $\mathbb{R}[\hat{q}]$.

Lemma 3.17. *The free $\mathbb{R}$-vector space on a null sequence $\mathbb{R}[\hat{q}]$ identifies with the union*

$$\bigcup_{N=0}^{\infty}\Big\{\sum_{m=0}^{\infty} r_m q^m \in \mathbb{R}[q] : \ \textit{at most } N \textit{ of the } r_m \textit{ are non-zero and } \sum_{m=0}^{\infty} |r_m| \leq N\Big\}$$

whose N-th member carries the metrisable compact subspace topology from $\prod_{m=0}^{\infty}[-N, N]$.

Proof. As noted in [CS22, Remark 3.10], the free $\mathbb{R}$-vector space on any light profinite set $S = \varprojlim_i S_i$ can be described as the sequential union

$$\bigcup_{N=0}^{\infty} \varprojlim_i \mathbb{R}[S_i]_{\ell^0 \leq N}.$$

where $\mathbb{R}[S_i]_{\ell^0 \leq N} := \{\sum_{s \in S_i} r_s[s] : \text{at most } N \text{ of the } r_s \text{ are non-zero and } \sum_{s \in S_i} |r_s| \leq N\}$ carries the metrisable compact subspace topology from $\prod_{s \in S_i}[-N, N]$. We give a proof of this in an appendix to this subsection (see Lemma 3.25). Specialising to $S = \mathbb{N} \cup \infty$ and splitting the quotient map $\mathbb{R}[\mathbb{N} \cup \infty] \to \mathbb{R}[\mathbb{N} \cup \infty]/\mathbb{R}[\infty]$ as in Corollary 1.41, we obtain the description. $\square$

We set $\mathbb{R}[\![q]\!]^{\mathrm{bdd}} := \mathbb{R}[\![q]\!]^{0\text{-temp}}$ for ease of notation. The proof of the following result is similar to the computation of $\mathbb{Z}[\hat{q}]^{\square}$ in [CS24, Lecture 5, timestamp 50:00].

Lemma 3.18. *The natural map $\mathbb{R}[\hat{q}] \to \mathbb{R}[\![q]\!]^{\mathrm{bdd}}$ induces an isomorphism on derived q-gaseous completions.*

Proof. We construct a commutative diagram of $\mathbb{Z}[\hat{q}]$-modules

$$\begin{tikzcd}
\mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\![q]\!]^{\mathrm{bdd}} \arrow[r, "g"] \arrow[d, "d_q\otimes\mathrm{id}"] & \mathbb{R}[\hat{q}] \arrow[d, "c"] \\
\mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\![q]\!]^{\mathrm{bdd}} \arrow[r, "h"] & \mathbb{R}[\![q]\!]^{\mathrm{bdd}},
\end{tikzcd} \tag{$\star$}$$

where d_q is $1-qx$ and c is the natural map. Over the point, g and h are given by

$$g_{\{*\}}\colon x^n \otimes \sum_{m=0}^{\infty} r_m q^m \mapsto r_{2n}q^n + r_{2n+1}q^{n+1} \quad \text{and} \quad h_{\{*\}}\colon x^n \otimes \sum_{m=0}^{\infty} r_m q^m \mapsto \sum_{m=2n}^{\infty} r_m q^{m-n}.$$

Intuitively, the map g decomposes a power series $\sum_{m=0}^{\infty} r_m q^m \in \mathbb{R}[\![q]\!]^{\mathrm{bdd}}$ into a null sequence in $\mathbb{R}[\hat{q}]$ such that taking the sum of this null sequence against q recovers $\sum_{m=0}^{\infty} r_m q^m$.

These maps lift uniquely to maps of light condensed $\mathbb{Z}[\hat{q}]$-modules by the explicit descriptions of $\mathbb{Z}[\hat{x}]$, $\mathbb{R}[\hat{q}]$ and $\mathbb{R}[\![q]\!]^{\mathrm{bdd}}$. For example, for g we can note that $(\sum_{m=0}^{\infty} r_m q^m \mapsto r_{2n}q^n + r_{2n+1}q^{n+1})_n$ defines a null sequence in $\underline{\mathrm{Hom}}(\mathbb{R}[\![q]\!]^{\mathrm{bdd}}, \mathbb{R}[\hat{q}])$ by Lemma 1.34. The same reasoning applies to h. To check that the square commutes, we compute

$$\begin{aligned}
h((d_q \otimes \mathrm{id})(x^n \otimes \sum_{m=0}^{\infty} r_m q^m)) &= h(x^n \otimes \sum_{m=0}^{\infty} r_m q^m - x^{n+1} q \sum_{m=0}^{\infty} r_m q^m) \\
&= \sum_{m=2n}^{\infty} r_m q^{m-n} - q \sum_{m=2n+2}^{\infty} r_m q^{m-n-1} \\
&= r_{2n}q^n + r_{2n+1}q^{n+1}.
\end{aligned}$$

Now h is split surjective with right inverse $s\colon \sum_{m=0}^{\infty} r_m q^m \mapsto x^0 \otimes \sum_{m=0}^{\infty} r_m q^m$. Applying the derived q-gaseous completion to ($\star$) gives the commutative square

$$\begin{tikzcd}
(\mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\![q]\!]^{\mathrm{bdd}})^{\mathrm{L}\,q\text{-gas}} \arrow[r, "g^{\mathrm{L}\,q\text{-gas}}"] \arrow[d, "(d_q\otimes\mathrm{id})^{\mathrm{L}\,q\text{-gas}}", "\cong"'] & \mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}} \arrow[d, "c^{\mathrm{L}\,q\text{-gas}}"] \\
(\mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\![q]\!]^{\mathrm{bdd}})^{\mathrm{L}\,q\text{-gas}} \arrow[r, "h^{\mathrm{L}\,q\text{-gas}}"] & (\mathbb{R}[\![q]\!]^{\mathrm{bdd}})^{\mathrm{L}\,q\text{-gas}} \arrow[l, dashed, bend left, "s^{\mathrm{L}\,q\text{-gas}}"]
\end{tikzcd}$$

in which $h^{\mathrm{L}\,q\text{-gas}}$ is split surjective with right inverse $s^{\mathrm{L}\,q\text{-gas}}$ and $(d_q \otimes \mathrm{id})^{\mathrm{L}\,q\text{-gas}}$ is an isomorphism because completion is symmetric monoidal. Define

$$\tau \coloneqq g^{\mathrm{L}\,q\text{-gas}} \circ ((d_q \otimes \mathrm{id})^{\mathrm{L}\,q\text{-gas}})^{-1} \circ s^{\mathrm{L}\,q\text{-gas}}.$$

Then $c^{\mathrm{L}\,q\text{-gas}} \circ \tau = \mathrm{id}$ by commutativity of the square. To prove $\tau \circ c^{\mathrm{L}\,q\text{-gas}} = \mathrm{id}$, we note that the square ($\star$) fits as the outer square in a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\![q]\!]^{\mathrm{bdd}} \arrow[rrr, "g"] \arrow[ddd, "d_q\otimes\mathrm{id}"] & & & \mathbb{R}[\hat{q}] \arrow[ddd, "c"] \\
& \mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\hat{q}] \arrow[ul, "\mathrm{id}\otimes c"'] \arrow[r, "g'"] \arrow[d, "d_q\otimes\mathrm{id}"] & \mathbb{R}[\hat{q}] \arrow[ur, equal] \arrow[d, equal] & \\
& \mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\hat{q}] \arrow[dl, "\mathrm{id}\otimes c"'] \arrow[r, "h'"] & \mathbb{R}[\hat{q}] \arrow[l, dashed, bend left, "s'"] \arrow[dr, "c"] & \\
\mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{R}[\![q]\!]^{\mathrm{bdd}} \arrow[rrr, "h"] & & & \mathbb{R}[\![q]\!]^{\mathrm{bdd}} \arrow[lll, dashed, bend left, "s"]
\end{tikzcd} \tag{$\star\star$}$$

Here, g', h' and s' are given by the same rules as g, h and s. In particular, s' is a right inverse of

h' satisfying $(\mathrm{id}\otimes c)\circ s' = s\circ c$. Taking derived q-gaseous completions, we see that

$$\tau' := (g')^{\mathrm{L}\,q\text{-gas}} \circ ((d_q\otimes \mathrm{id})^{\mathrm{L}\,q\text{-gas}})^{-1} \circ (s')^{\mathrm{L}\,q\text{-gas}}$$

is right inverse to the identity, hence itself the identity. Chasing the diagram $(\star\star)$ after completion, we find $\tau\circ c^{\mathrm{L}\,q\text{-gas}} = \tau' = \mathrm{id}$, thus c induces an isomorphism on derived q-gaseous completions. $\square$

Lemma 3.19. *Each of the natural maps $\mathbb{R}[\![q]\!]^{k\text{-temp}} \to \mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}$ induces an isomorphism on derived q-gaseous completions.*

Proof. We find suitable maps g and h fitting into a commutative diagram of $\mathbb{Z}[\hat{q}]$-modules

$$\begin{tikzcd}
\mathbb{Z}[\hat{x},\hat{q}]\otimes_{\mathbb{Z}[\hat{q}]}\mathbb{R}[\![q]\!]^{(k+1)\text{-temp}} \arrow[r, "g"] \arrow[d, "d_q\otimes\mathrm{id}"] & \mathbb{R}[\![q]\!]^{k\text{-temp}} \arrow[d, "c"] \\
\mathbb{Z}[\hat{x},\hat{q}]\otimes_{\mathbb{Z}[\hat{q}]}\mathbb{R}[\![q]\!]^{(k+1)\text{-temp}} \arrow[r, "h"] & \mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}.
\end{tikzcd} \tag{$\star$}$$

Again, intuitively, the map g should decompose a power series $\sum_{m=0}^\infty r_m q^m \in \mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}$ into a null sequence in $\mathbb{R}[\![q]\!]^{k\text{-temp}}$ such that taking the sum of this null sequence against q recovers $\sum_{m=0}^\infty r_m q^m$. Also note that the map h in Lemma 3.18 arises from g by $h(x^n\otimes\sum_{m=0}^\infty r_m q^m) = \sum_{\ell=n}^\infty g(x^\ell\otimes \sum_{m=0}^\infty a_m q^m)q^{\ell-n}$. This leads us to specifying g and h over the point by

$$g_{\{*\}}\colon x^n\otimes\sum_{m=0}^\infty r_m q^m \mapsto \sum_{m=n}^\infty \frac{r_{m+n}}{m+n+1}q^m + \sum_{m=n}^\infty \frac{r_{m+n+1}}{m+n+2}q^{m+1} \quad \text{and}$$

$$h_{\{*\}}\colon x^n\otimes\sum_{m=0}^\infty r_m q^m \mapsto \sum_{\ell=2n}^\infty\sum_{m=0}^\infty \frac{r_{m+\ell}}{m+\ell+1}q^{m+\ell-n}.$$

Note that $g_{\{*\}}$ maps to $\mathbb{R}[\![q]\!]^{k\text{-temp}}$. Indeed, if $|r_\ell|\le N(\ell+1)^{k+1}$ for all m, then for all $m\ge n$,

$$\left|\frac{r_{m+n}}{m+n+1}\right| \le N(m+n+1)^k \le N2^k(m+1)^k,$$

and the same estimate applies to the second summand. Moreover, $h_{\{*\}}$ maps to $\mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}$ as the sum of a null sequence in $\mathbb{R}[\![q]\!]^{k\text{-temp}}$ against q lies in $\mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}$. These maps lift uniquely to maps of light condensed modules by the explicit descriptions of $\mathbb{Z}[\hat{x}]$, $\mathbb{R}[\![q]\!]^{k\text{-temp}}$ and $\mathbb{R}[\![q]\!]^{(k+1)\text{-temp}}$. To check that the diagram $(\star)$ commutes, we compute

$$\begin{aligned}
h((d_q\otimes\mathrm{id})(x^n\otimes\sum_{m=0}^\infty r_m q^m)) &= h(x^n\otimes\sum_{m=0}^\infty r_m q^m - x^{n+1}\otimes q\sum_{m=0}^\infty r_m q^m)\\
&= \sum_{\ell=2n}^\infty\sum_{m=0}^\infty \frac{r_{m+\ell}}{m+\ell+1}q^{m+\ell-n} - \sum_{\ell=2n+2}^\infty\sum_{m=0}^\infty \frac{r_{m+\ell}}{m+\ell+1}q^{m+\ell-n}\\
&= \sum_{m=0}^\infty \frac{r_{m+2n}}{m+2n+1}q^{m+n} + \sum_{m=0}^\infty \frac{r_{m+2n+1}}{m+2n+2}q^{m+n+1}\\
&= \sum_{m=n}^\infty \frac{r_{m+n}}{m+n+1}q^{m} + \sum_{m=n}^\infty \frac{r_{m+n+1}}{m+n+2}q^{m+1}.
\end{aligned}$$

The map h is split surjective with right inverse $s\colon \sum_{m=0}^\infty r_m q^m \mapsto x^0\otimes\sum_{m=0}^\infty r_m q^m$, because

$$\sum_{\ell=0}^\infty\sum_{m=0}^\infty \frac{r_{m+\ell}}{m+\ell+1}q^{m+\ell} = \sum_{n=0}^\infty(\sum_{m=0}^n\frac{r_n}{n+1})q^n = \sum_{n=0}^\infty r_n q^n.$$

Now the argument for Lemma 3.18 goes through word by word. $\square$

Remark 3.20. Similar decompositions are used in [Ked25, Section 10] to give a computation of $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$, but it seems that the use of floor functions in [Ked25] makes the maps witnessing these decompositions non-linear. We will resolve this issue below by comparison with $\mathbb{R}/\mathbb{Z}$.

We deduce the Proposition.

Proof of Proposition 3.15. We have

$$\begin{aligned}\mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}} &\cong (\mathbb{R}[\![q]\!]^{0\text{-temp}})^{\mathrm{L}\,q\text{-gas}}\\ &\cong \operatorname{colim}((\mathbb{R}[\![q]\!]^{0\text{-temp}})^{\mathrm{L}\,q\text{-gas}} \xrightarrow{\sim} (\mathbb{R}[\![q]\!]^{1\text{-temp}})^{\mathrm{L}\,q\text{-gas}} \xrightarrow{\sim} \cdots)\\ &\cong \operatorname{colim}(\mathbb{R}[\![q]\!]^{0\text{-temp}} \hookrightarrow \mathbb{R}[\![q]\!]^{1\text{-temp}} \hookrightarrow \cdots)^{\mathrm{L}\,q\text{-gas}}\\ &\cong \mathbb{R}[\![q]\!]^{\mathrm{temp}},\end{aligned}$$

using Lemma 3.18 and Lemma 3.19, the fact that the left adjoint $(-)^{\mathrm{L}\,q\text{-gas}}$ commutes with colimits, and the fact that $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$ is q-gaseous. □

Digression: The gaseous base ring revisited

We digress from our goal of proving Theorem A for a moment to note that the description of $\mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ also gives a direct proof of the following description of $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$.

Theorem B. *The natural map $\mathbb{Z}[\hat{q}] \to \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}} \cong \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$.*

In particular, upon inverting q, this also implies the description of the underlying light condensed ring of the gaseous base ring as $\mathbb{Z}(\!(q)\!)^{\mathrm{temp}}$ (see Corollary 2.43).

To relate $\mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ and $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$, we compute $(\mathbb{R}/\mathbb{Z})[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ for $(\mathbb{R}/\mathbb{Z})[\hat{q}] := \mathbb{Z}[\hat{q}] \otimes \mathbb{R}/\mathbb{Z}$. The compactness of $\mathbb{R}/\mathbb{Z}$ makes all growth conditions superfluous, so the following is not surprising.

Proposition 3.21. *The natural map $\mathbb{Z}[\hat{q}] \otimes \mathbb{R}/\mathbb{Z} \to (\mathbb{R}/\mathbb{Z})[\![q]\!]$, $q^n \otimes r \mapsto rq^n$ induces an isomorphism $(\mathbb{Z}[\hat{q}] \otimes \mathbb{R}/\mathbb{Z})^{\mathrm{L}\,q\text{-gas}} \cong (\mathbb{R}/\mathbb{Z})[\![q]\!]$.*

Proof. Again, we proceed as in Lemma 3.18 and construct a commutative diagram of $\mathbb{Z}[\hat{q}]$-modules

$$\begin{tikzcd} \mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} (\mathbb{R}/\mathbb{Z})[\![q]\!] \arrow[r, "g"] \arrow[d, "d_q\otimes\mathrm{id}"] & \mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \arrow[d] \\ \mathbb{Z}[\hat{x},\hat{q}] \otimes_{\mathbb{Z}[\hat{q}]} (\mathbb{R}/\mathbb{Z})[\![q]\!] \arrow[r, "h"] & (\mathbb{R}/\mathbb{Z})[\![q]\!]. \end{tikzcd} \qquad (\star)$$

Over the point, g and h will be given by the same formulas we chose in the proof of Lemma 3.18:

$$g_{\{*\}}\colon x^n \otimes \sum_{m=0}^{\infty} r_m q^m \mapsto r_{2n}q^n + r_{2n+1}q^{n+1} \quad \text{and} \quad h_{\{*\}}\colon x^n \otimes \sum_{m=0}^{\infty} r_m q^m \mapsto \sum_{m=2n}^{\infty} r_m q^{m-n}.$$

For h, the specified map lifts uniquely to a map of light condensed modules because $(\sum_{m=0}^{\infty} r_m q^m \mapsto \sum_{m=2n}^{\infty} r_m q^{m-n})_n$ is a null sequence in $\underline{\mathrm{Hom}}((\mathbb{R}/\mathbb{Z})[\![q]\!], (\mathbb{R}/\mathbb{Z})[\![q]\!])$ by Lemma 1.34. We specify g as the map induced on cokernels

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}[\hat{x}] \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!]^{\mathrm{bdd}} \arrow[r] \arrow[d, "g_1"] & \mathbb{Z}[\hat{x}] \otimes_{\mathbb{Z}} \mathbb{R}[\![q]\!]^{\mathrm{bdd}} \arrow[r] \arrow[d, "g_2"] & \mathbb{Z}[\hat{x}] \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z})[\![q]\!] \arrow[r] \arrow[d, dashed, "g"] & 0 \\ 0 \arrow[r] & \mathbb{Z}[\hat{q}] \arrow[r] & \mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} \mathbb{R} \arrow[r] & \mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

for suitable maps g_1 and g_2. Here, we have $\mathbb{Z}[\![q]\!]^{\mathrm{bdd}} = \bigcup_{N=0}^{\infty} \prod_{\mathbb{N}} \mathbb{Z} \cap [-N, N]$ and the cokernel of the map $\mathbb{Z}[\![q]\!]^{\mathrm{bdd}} \hookrightarrow \mathbb{R}[\![q]\!]^{\mathrm{bdd}}$ is $(\mathbb{R}/\mathbb{Z})[\![q]\!] = \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z}$ because surjectivity of the map of metrisable compact spaces $\prod_{\mathbb{N}}[0,1] \to \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z}$ implies surjectivity of $\mathbb{R}[\![q]\!]^{\mathrm{bdd}} \to \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z}$.

By the descriptions of $\mathbb{Z}[\hat{q}]$ and $\mathbb{Z}[\![q]\!]^{\mathrm{bdd}}$, $(\sum_{m=0}^{\infty} a_m q^m \mapsto a_{2n}q^n + a_{2n+1}q^{n+1})_n$ is a null sequence in $\underline{\mathrm{Hom}}(\mathbb{Z}[\![q]\!]^{\mathrm{bdd}}, \mathbb{Z}[\hat{q}])$, giving the desired map g_1. Analogously, we obtain the map g_2, which is also the map we called g in the proof of Lemma 3.18.

For the map induced on cokernels g, the diagram $(\star)$ commutes because the diagrams for g_1 and g_2 commute if we replace $\mathbb{R}/\mathbb{Z}$ by $\mathbb{Z}$ and $\mathbb{R}$, respectively. Also note that h is split surjective with right inverse $s\colon \sum_{m=0}^{\infty} r_m q^m \mapsto x^0 \otimes \sum_{m=0}^{\infty} r_m q^m$.

Now the argument from the proof of Lemma 3.18 goes through. □

Remark 3.22. Proposition 3.21 is a gaseous analogue of the liquid result [CS20, Proposition 10.1]. The latter asserts that the derived r-liquid completion of $A[t^{-1}]$ is given by $A(\!(t)\!)$ for any $0 < r < 1$ and any compact Abelian group A. It is proved using a Breen-Deligne resolution.

Question 3.23. Is the derived q-gaseous completion of $A[\hat{q}]$ given by $A[\![q]\!]$ for any metrisable compact Abelian group A?

We deduce the description of $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$. Note that $\mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ is q-gaseous by the argument we gave for $\mathbb{R}[\![q]\!]^{\mathrm{temp}}$ at the start of this subsection.

Proof of Theorem B. The natural maps give a map of short exact sequences of $\mathbb{Z}[\hat{q}]$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}[\hat{q}] \arrow[r] \arrow[d] & \mathbb{R}[\hat{q}] \arrow[r] \arrow[d] & (\mathbb{R}/\mathbb{Z})[\hat{q}] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}[\![q]\!]^{\mathrm{temp}} \arrow[r] & \mathbb{R}[\![q]\!]^{\mathrm{temp}} \arrow[r] & (\mathbb{R}/\mathbb{Z})[\![q]\!] \arrow[r] & 0. \end{tikzcd}$$

The bottom row is exact by exactness of filtered colimits and as already $\prod_{\mathbb{N}}[-N, N] \subseteq \mathbb{R}[\![q]\!]^{\mathrm{temp}}$ surjects onto $(\mathbb{R}/\mathbb{Z})[\![q]\!]$. By Proposition 3.15 and Proposition 3.21, the centre and right vertical maps induce isomorphisms on derived q-gaseous completions. Hence also $\mathbb{Z}[\hat{q}] \to \mathbb{Z}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism on derived q-gaseous completions. □

Remark 3.24. The computation of the free completed modules $\mathbb{Z}[\hat{q}][S]^{\mathrm{L}\,q\text{-gas}}$ remains difficult. It seems that the analysis of the Koszul complex arising from Proposition 2.26 cannot be avoided.

Appendix: Description of free $\mathbb{R}$-vector spaces

The following is only remarked in [CS22, Remark 3.10]. Since it was critical to our arguments in Lemma 3.18 and Proposition 3.21, we include a proof.

Lemma 3.25. *Let $S = \varprojlim_i S_i$ be a light profinite set. Then the natural map $\mathbb{R}[S] \to \varprojlim_i \mathbb{R}[S_i]$ is injective and identifies $\mathbb{R}[S]$ with the light condensed $\mathbb{R}$-subspace given by the sequential union*

$$\bigcup_{N=0}^{\infty} \varprojlim_i \mathbb{R}[S_i]_{\ell^0 \leq N},$$

where $\mathbb{R}[S_i]_{\ell^0 \leq N} \coloneqq \{\sum_{s \in S_i} r_s[s] : \text{at most } N \text{ of the } r_s \text{ are non-zero and } \sum_{s \in S_i} |r_s| \leq N\}$ carries the metrisable compact subspace topology from $\prod_{s \in S_i}[-N, N]$.

Proof. We proceed as for the description of free Abelian groups $\mathbb{Z}[S]$ in [CS20, Proposition 2.1]. In fact, for injectivity of $\mathbb{R}[S] \to \varprojlim_i \mathbb{R}[S_i]$, the argument in [CS20, Proposition 2.1] goes through word by word with $\mathbb{R}$ in place of $\mathbb{Z}$. It remains to identify the image.

Addition carries $\varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N_1} \times \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N_2}$ into $\varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N_1+N_2}$ and multiplication by $\lambda \in \mathbb{R}$ carries $\varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N}$ into $\varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq \lceil\lambda\rceil N}$, thus

$$\bigcup_{N=0}^{\infty} \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N} \subseteq \varprojlim_i \mathbb{R}[S_i]$$

is a light condensed $\mathbb{R}$-subspace. The map of light condensed sets $S \to \varprojlim_i \mathbb{R}[S_i]$ factors through $\varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq 1}$ and thus induces map of light condensed $\mathbb{R}$-vector spaces

$$\mathbb{R}[S] \to \bigcup_N \varprojlim_i \mathbb{Z}[S_i]_{\ell^0\leq N}.$$

It remains to show that this map is surjective. Let $T \to \bigcup_N \varprojlim_i \mathbb{Z}[S_i]_{\ell^0\leq N}$ be a map from a light profinite set. We find N such that it factors through a map

$$\alpha\colon T \to \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N}.$$

Define the metrisable compact Hausdorff space $(\mathbb{R}^N)_{\ell^0\leq N} \coloneqq \{(r_i)_{i=1}^N \in \mathbb{R}^N : \sum_{i=1}^N |r_i| \leq N\} \subseteq [-N,N]^N$. Then, for all i, we have a surjection of metrisable compact Hausdorff spaces

$$S_i^N \times (\mathbb{R}^N)_{\ell^0\leq N} \to \mathbb{R}[S_i]_{\ell^0\leq N},\ (s_1,\dots,s_N,r_1,\dots,r_N) \mapsto r_1[s_1]+\cdots+r_N[s_N].$$

Taking sequential limits and condensifying gives a surjection

$$\beta\colon S^N \times (\mathbb{R}^N)_{\ell^0\leq N} \to \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N}.$$

The inclusion $[-N,N] \to \mathbb{R}$ induces a map $\mathbb{R}[[-N,N]] \to \mathbb{R}$, which gives a map $\mathbb{R}[S\times[-N,N]] \to \mathbb{R}[S]$ upon applying $\mathbb{R}[S]\otimes_{\mathbb{R}} -$. Take the sum of the pullbacks along the projections of this map to obtain a map $\mathbb{R}[S^N \times [-N,N]^N] \to \mathbb{R}[S]$. It corresponds to a map

$$\gamma\colon S^N \times (\mathbb{R}^N)_{\ell^0\leq N} \to \mathbb{R}[S].$$

Then, we find a surjection $T' \to T$ from a light profinite set such that the diagram

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & S^N \times (\mathbb{R}^N)_{\ell^0\leq N} \arrow[r, "\gamma"] \arrow[d, "\beta"] & \mathbb{R}[S] \arrow[d] \\
T \arrow[r, "\alpha"] & \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N} \arrow[r] & \bigcup_{N=0}^{\infty} \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N},
\end{tikzcd}$$

commutes, witnessing surjectivity of $\mathbb{R}[S] \to \bigcup_{N=0}^{\infty} \varprojlim_i \mathbb{R}[S_i]_{\ell^0\leq N}$. $\square$

3.4 Conclusion of the proof

Before proceeding with the proof of Theorem A, we remark that our discussion so far already recovers the ring of tempered holomorphic functions on $\mathbb{D}$.

Remark 3.26. As $\mathbb{Z}[\hat{q}]$-modules, we have $\mathbb{C}[\hat{q}] \cong \mathbb{R}[\hat{q}]^{\oplus 2}$ and $\mathbb{C}[\![q]\!]^{\mathrm{temp}} \cong (\mathbb{R}[\![q]\!]^{\mathrm{temp}})^{\oplus 2}$, hence, by Proposition 3.15, $\mathbb{C}[\hat{q}]^{\mathrm{L}\,q\text{-gas}} \cong \mathbb{C}[\![q]\!]^{\mathrm{temp}}$.

The final missing ingredient for the proof of Theorem A is the following result.

Proposition 3.27. *The natural map $\mathbb{R}[q] \to \mathbb{R}[\hat{q}]$ induces an isomorphism on derived q-gaseous completions.*

It is not surprising that making all null sequences summable against q forces q to become topologically nilpotent. If q was a real number against which $(\frac{1}{m+1})_m$ is summable, then it should better satisfy $|q| < 1$. Nevertheless, we will prove Proposition 3.27 quite indirectly. The idea is to show that there is a Verdier localisation sequence

$$\mathcal{D}(\mathbb{R}[\hat{q}]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{R}[q]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{R}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}})$$

in which $\mathcal{D}(\mathbb{R}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}})$ vanishes.

A slight variant of the following result is mentioned in [CS24, Lecture 20, timestamp 1:41:00].

Lemma 3.28.

(1) The algebra $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ is idempotent in $\mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}})$.

(2) The algebra $\mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ is idempotent in $\mathcal{D}(\mathbb{R}[q]_{q\text{-gas}})$.

Proof. First, note that (2) follows from (1) by extension of scalars. Indeed, if (1) holds, then

$$\begin{aligned}
(\mathbb{R}[\hat{q}] \otimes_{\mathbb{R}[q]} \mathbb{R}[\hat{q}])^{\mathrm{L}\,q\text{-gas}} &\cong (\mathbb{R}[q] \otimes_{\mathbb{Z}[q]} (\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}[q]} \mathbb{Z}[\hat{q}]))^{\mathrm{L}\,q\text{-gas}} \\
&\cong (\mathbb{R}[q]^{\mathrm{L}\,q\text{-gas}} \otimes_{\mathbb{Z}[q]} (\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}[q]} \mathbb{Z}[\hat{q}])^{\mathrm{L}\,q\text{-gas}})^{\mathrm{L}\,q\text{-gas}} \\
&\cong (\mathbb{R}[q]^{\mathrm{L}\,q\text{-gas}} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}})^{\mathrm{L}\,q\text{-gas}} \\
&\cong \mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}.
\end{aligned}$$

We prove part (1). We have

$$(\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}[q]} \mathbb{Z}[\hat{q}])^{\mathrm{L}\,q\text{-gas}} \cong ((\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} \mathbb{Z}[\hat{x}])/(x-q))^{\mathrm{L}\,q\text{-gas}} \cong (\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} \mathbb{Z}[\hat{x}])^{\mathrm{L}\,q\text{-gas}}/^{\mathrm{L}}(x-q),$$

so let us identify $(\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} \mathbb{Z}[\hat{x}])^{\mathrm{L}\,q\text{-gas}}$ first. We compute

$$(\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} \mathbb{Z}[\hat{x}])^{\mathrm{L}\,q\text{-gas}} = (\mathbb{Z}[\hat{q}] \otimes_{\mathbb{Z}} \mathbb{Z}[\{x^0, x^1, \ldots\} \cup \infty]/\mathbb{Z}[\infty])^{\mathrm{L}\,q\text{-gas}} \cong \frac{\mathbb{Z}[\hat{q}][\{x^0, x^1, \ldots\} \cup x^\infty]^{\mathrm{L}\,q\text{-gas}}}{\mathbb{Z}[\hat{q}][x^\infty]^{\mathrm{L}\,q\text{-gas}}}\mathrm{L}.$$

For a light profinite set $S = \varprojlim_i S_i$, Theorem 2.41(2) gives the description

$$\mathbb{Z}[\hat{q}][S]^{\mathrm{L}\,q\text{-gas}} \cong \bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \varprojlim_i \Big(\prod_{m=0}^{\infty} \mathbb{Z}[S_i]_{\le N(m+1)^k} q^m\Big).$$

For $\mathbb{N} \cup \infty = \varprojlim_i \{x^0, \ldots, x^i, x^\infty\}$ we can split the quotient map as in Corollary 1.41 and obtain

$$\frac{\mathbb{Z}[\hat{q}][\{x^0, x^1, \ldots\} \cup x^\infty]^{\mathrm{L}\,q\text{-gas}}}{\mathbb{Z}[\hat{q}][x^\infty]^{\mathrm{L}\,q\text{-gas}}}\mathrm{L} \cong \bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \Big\{ \sum_{m,n=0}^{\infty} a_{m,n} x^n q^m : \sum_{n=0}^{\infty} |a_{m,n}| \le N(m+1)^k \Big\}.$$

It remains to show that taking the quotient by $(x-q)$ recovers $\mathbb{Z}[\![q]\!]^{\mathrm{temp}}$. By the rearrangement $\sum_{m,n=0}^{\infty} a_{m,n} q^{m+n} = \sum_{m=0}^{\infty} (\sum_{k=0}^{m} a_{m,m-k}) q^m$, we expect that the map

$$\bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \Big\{ \sum_{m,n=0}^{\infty} a_{m,n} x^n q^m : \sum_{n=0}^{\infty} |a_{m,n}| \le N(m+1)^k \Big\} \to \bigcup_{k=0}^{\infty} \bigcup_{N=0}^{\infty} \Big\{ \sum_{m=0}^{\infty} a_m q^m : |b_m| \le N(m+1)^k \Big\}$$

sending $(a_{m,n})_{m,n}$ to $(\sum_{k=0}^{m} a_{m,m-k})_m$ induces the desired isomorphism. This is a map of light condensed modules because $\sum_{n=0}^{\infty} |a_{m,n}| \le N(m+1)^k$ implies $\sum_{k=0}^{m} |a_{m,m-k}| \le N(m+1)^{k+1}$. Let us verify the exactness in question.

Over the point, a preimage of $(a_m)_m$ can be given by $(\delta_{m,n}a_m)_{m,n}$, where $\delta_{m,n}$ is the Kronecker-delta. Since $|a_m| \le N(m+1)^k$ implies $\sum_{n=0}^{\infty} |\delta_{m,n}a_n| = |a_m| \le N(m+1)^k$, this shows surjectivity.

Multiplication by $(x-q)$ followed by the displayed map is zero because this composite is a map of quasi-separated light condensed sets that is zero over the point.

It remains to show that multiplication by $(x-q)$ surjects onto the kernel. Take a power series $\sum_{n=0}^{\infty} a_{m,n}x^n q^m$ with $\sum_{k=0}^{m} a_{m,m-k} = 0$ for all m. Comparing coefficients, we see that $(x-q)\sum_{m,n=0}^{\infty} b_{m,n}x^n q^m = \sum_{m,n=0}^{\infty} a_{m,n}x^n q^m$ is satisfied for

$$b_{m,n} := \sum_{k=0}^{m} a_{m-k,n+k+1}.$$

If $\sum_{n=0}^{\infty} |a_{m,n}| \le N(m+1)^k$ for all m, then

$$\sum_{n=0}^{\infty} |b_{m,n}| \le \sum_{n=0}^{\infty}\sum_{k=0}^{m} |a_{m-k,n+k+1}| = \sum_{k=0}^{m}\sum_{n=0}^{\infty} |a_{m-k,n+k+1}| \le N(m+1)^{k+1}$$

for all m, concluding the proof of exactness as light condensed modules. □

By the categorical yoga we explained after Proposition 2.26, the idempotence of $\mathbb{Z}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ in $\mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}})$ shows that the forgetful functor $\mathcal{D}(\mathbb{Z}[\hat{q}]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}})$ is fully faithful and that the corresponding Verdier quotient identifies with the full subcategory

$$\{N \in \mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}}) : \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{q}], N) = 0)\} \subseteq \mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}}).$$

The same formal discussion applies to the idempotent algebra $\mathbb{R}[\hat{q}]^{\mathrm{L}\,q\text{-gas}}$ in $\mathcal{D}(\mathbb{R}[q]_{q\text{-gas}})$.

Lemma 3.29.

(1) The Verdier quotient of $\mathcal{D}(\mathbb{Z}[\hat{q}]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}})$ *is* $\mathcal{D}(\mathbb{Z}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}})$.

(2) The Verdier quotient of $\mathcal{D}(\mathbb{R}[\hat{q}]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{R}[q]_{q\text{-gas}})$ *is* $\mathcal{D}(\mathbb{R}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}})$.

Proof. Note that (1) implies (2) by extension of scalars. Indeed, if (1) holds, then

$$\begin{aligned}
&\{N \in \mathcal{D}(\mathbb{R}[q]_{q\text{-gas}}) : \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{R}[q]}(\mathbb{R}[\hat{q}], N) = 0\} \\
&= \mathcal{D}(\mathrm{Mod}_{\mathbb{R}[q]}) \times_{\mathcal{D}(\mathrm{Mod}_{\mathbb{Z}[q]})} \{N \in \mathcal{D}(\mathbb{Z}[q]_{q\text{-gas}}) : \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{q}], N) = 0\} \\
&= \mathcal{D}(\mathrm{Mod}_{\mathbb{R}[q]}) \times_{\mathcal{D}(\mathrm{Mod}_{\mathbb{Z}[q]})} \mathcal{D}(\mathbb{Z}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}}) \\
&= \mathcal{D}(\mathbb{R}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}}).
\end{aligned}$$

It remains to prove part (1). Let us write $\mathbb{Z}[\hat{x}]$ for $\mathbb{Z}[\hat{q}]$ so that we can denote the actions of q on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{x}], N)$ through source and target by x and q, respectively. Then we have

$$\begin{aligned}
\mathbb{Z}[\hat{x}] &\cong \mathbb{Z}[\hat{x}] \otimes^{\mathrm{L}}_{\mathbb{Z}[q]} \mathbb{Z}[q] \cong \mathbb{Z}[\hat{x}] \otimes^{\mathrm{L}}_{\mathbb{Z}[q]} \mathrm{cofib}(\mathbb{Z}[x,q] \xrightarrow{x-q} \mathbb{Z}[x,q]) \\
&\cong \mathrm{cofib}(\mathbb{Z}[\hat{x}] \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}[q] \xrightarrow{x-q} \mathbb{Z}[\hat{x}] \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}[q]).
\end{aligned}$$

Therefore, for $N \in \mathcal{D}(\mathrm{Mod}_{\mathbb{Z}[q]})$, we get

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{q}], N) = \mathrm{fib}(\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[\hat{x}], N) \xrightarrow{q-x} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[\hat{x}], N)).$$

Hence, the Verdier quotient identifies with the full subcategory of $\mathcal{D}(\mathrm{Mod}_{\mathbb{Z}[q]})$ spanned by those N such that multiplication by $1-qx$ and by $q-x$ induce isomorphisms on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{x}] \otimes \mathbb{Z}[q], N)$.

We show that if $q-x$ acts isomorphically on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{x}] \otimes \mathbb{Z}[q], N)$, then already q acts isomorphically on N. We have a commutative diagram

$$\begin{tikzcd}
N = \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[q], N) \arrow[r, "p_0^*"] \arrow[d, "q^*"] & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[q] \otimes^{\mathrm{L}} \mathbb{Z}[\hat{x}], N) \arrow[d, "(q-x)^*"] \\
N = \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[q], N) \arrow[r, "p_0^*"] & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[q] \otimes^{\mathrm{L}} \mathbb{Z}[\hat{x}], N)
\end{tikzcd}$$

where p_0 is the quotient map $\mathbb{Z}[q] \otimes \mathbb{Z}[\hat{x}] \to \mathbb{Z}[q] \otimes \mathbb{Z}[\{x^0\}]$ to the 0-th coordinate. Intuitively, commutativity records that $(n_0, n_1, \ldots) \mapsto (qn_0 - n_1, qn_1 - n_2, \ldots)$ sends $(n_0, 0, \ldots)$ to $(qn_0, 0, \ldots)$. Now p_0^* is split injective with retraction c_0^* for $c_0 \colon \mathbb{Z}[q] \to \mathbb{Z}[q] \otimes \mathbb{Z}[\hat{x}]$, $q^m \mapsto q^m \otimes x^0$. Set

$$\tau := c_0^* \circ (q-x)^{*-1} \circ p_0^*,$$

then commutativity of the square shows $\tau \circ q^* = \mathrm{id}$. Let $\overline{q}^*$ denote the the map induced by multiplication by q on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[q] \otimes \mathbb{Z}[\hat{x}], N)$. Then

$$\begin{aligned}
q^* \circ c_0^* \circ (q-x)^{*-1} \circ p_0^* &= c_0^* \circ \overline{q}^* \circ (q-x)^{*-1} \circ p_0^* \\
&= c_0^* \circ (q-x)^{*-1} \circ \overline{q}^* \circ p_0^* \\
&= c_0^* \circ (q-x)^{*-1} \circ p_0^* \circ q^*.
\end{aligned}$$

Hence $q^* \circ \tau = \tau \circ q^*$ and q^* is an isomorphism.

Thus, $q-x$ acts isomorphically on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q]}(\mathbb{Z}[\hat{x}] \otimes \mathbb{Z}[q], N)$ if and only if q acts isomorphically on N and $1-q^{-1}x$ acts isomorphically on $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[q^{\pm 1}]}(\mathbb{Z}[\hat{x}] \otimes \mathbb{Z}[q^{\pm 1}], N)$. Together with the condition that $1-qx$ acts isomorphically, we identify the Verdier quotient as $\mathcal{D}(\mathbb{Z}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}})$. □

We can deduce the proposition.

Proof of Proposition 3.27. By Lemma 3.28 and Lemma 2.23, we have a recollement

$$\mathcal{D}(\mathbb{R}[\hat{q}]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{R}[q]_{q\text{-gas}}) \to \mathcal{D}(\mathbb{R}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}}).$$

Any $\mathbb{R}$-module that is q-gaseous and q^{-1}-gaseous is also 1-gaseous, that is, solid by Proposition 2.37. But all solid $\mathbb{R}$-vector spaces are zero by Lemma 2.23, so we obtain $\mathcal{D}(\mathbb{R}[q^{\pm 1}]_{\{q,q^{-1}\}\text{-gas}}) = 0$. It follows that restriction of scalars induces an equivalence

$$\mathcal{D}(\mathbb{R}[\hat{q}]_{q\text{-gas}}) \xrightarrow{\sim} \mathcal{D}(\mathbb{R}[q]_{q\text{-gas}}).$$

In particular, the natural map $\mathbb{R}[q]^{\mathrm{L}\, q\text{-gas}} \to {}_{\mathbb{R}[q]}\mathbb{R}[\hat{q}]^{\mathrm{L}\, q\text{-gas}}$ is an isomorphism. □

As claimed, this was the final step required for the proof of Theorem A.

Proof of Theorem A. The natural map $\mathbb{R}[q] \to \mathbb{R}[\![q]\!]^{\mathrm{temp}}$ factors as $\mathbb{R}[q] \to \mathbb{R}[\hat{q}] \to \mathbb{R}[\![q]\!]^{\mathrm{temp}}$. By Proposition 3.15 and Proposition 3.27, it induces an isomorphism

$$\mathbb{R}[q]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{R}[\![q]\!]^{\mathrm{temp}}.$$

As $\mathbb{Z}[q]$-modules, we have $\mathbb{C}[q] \cong \mathbb{R}[q]^{\oplus 2}$ and $\mathbb{C}[\![q]\!]^{\mathrm{temp}} \cong (\mathbb{R}[\![q]\!]^{\mathrm{temp}})^{\oplus 2}$, hence also the natural map $\mathbb{C}[q] \to \mathbb{C}[\![q]\!]^{\mathrm{temp}}$ induces an isomorphism $\mathbb{C}[q]^{\mathrm{L}\, q\text{-gas}} \cong \mathbb{C}[\![q]\!]^{\mathrm{temp}}$. □

Noting that $\mathbb{C}[\![q]\!]^{\mathrm{temp}}$ is already (1/2-)gaseous, we can now define the static analytic ring of tempered holomorphic functions on $\mathbb{D}$,

$$\mathcal{O}^{\mathrm{temp}}(\mathbb{D})_{\mathrm{gas}} := (\mathbb{C}[\![q]\!]^{\mathrm{temp}}, \mathcal{D}(\mathbb{C}[q]_{\{1/2,q\}\text{-gas}})).$$

The striking question is if we can proceed similarly for more general opens in $\mathbb{C}$, with the goal of obtaining a sheaf of analytic rings $\mathcal{O}^{\mathrm{temp}}(-)_{\mathrm{gas}}$ on the subanalytic site $\mathbb{C}_{\mathrm{sa}}$ (see Definition 3.7).

We conclude this thesis by posing a concrete question towards this goal. Recall that we can equip the ring of holomorphic functions $\mathcal{O}(V)$ on an open $V \subseteq \mathbb{C}$ with a natural light condensed structure (see [CS22, Lecture 5] and our discussion at the end of Subsection 2.4).

Question 3.30. Let $U \subseteq \mathbb{C}$ be a bounded open such that there exist an open $V \supseteq \overline{U}$ and holomorphic functions $f_1, \dots, f_r \colon V \to \mathbb{C}$ with

$$U = \{z \in V : |f_1(z)| < 1, \dots, |f_r(z)| < 1\}.$$

Is $\mathcal{O}(V)^{\mathrm{L}\,\{f_1,\dots,f_r\}\text{-gas}}$ concentrated in degree 0 with underlying ring $\mathcal{O}^{\mathrm{temp}}(U)$?

We expect in particular that $\mathcal{O}(V)^{\mathrm{L}\,\{f_1,\dots,f_r\}\text{-gas}}$ only depends on U and not on the choice of the f_i. A first step would be to show that given another holomorphic function $h \colon V \to \mathbb{C}$, we have

$$\mathcal{O}(V)^{\mathrm{L}\,\{f_1,\dots,f_r\}\text{-gas}} \cong \mathcal{O}(V)^{\mathrm{L}\,\{f_1,\dots,f_r,h\}\text{-gas}},$$

provided that $|h(z)| < 1$ for all $z \in U$.

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