Assignment 1

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Question 1.

(a) P(A) = 0, so P(A$\bigcap$B) = P(A)$\cdot$P(B) = 0 , so, A and B are independent events.

(b) P(A) = 0.2, P(B) = 0.5, P(A$\bigcap$B) = 0.1 =>

P(A$\bigcup$B) = P(A) + P(B) - P(A$\bigcap$B) = 0.6 =>

P($(A\bigcap\bar{B})\bigcup(\bar{A})$) = P(A)P(1-P(B))+P(B)(1-P(A)) = 0.5

(c)

$$P(card = GG|see = green) = \frac{P(see = green|card = GG) * P(card = GG)}{P(see = green)}$$
$$= \frac{1 * \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Question 2.Bishop(1.11)

$$\frac{\alpha}{\alpha\mu} = \frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n - \mu) = \frac{1}{\sigma^2}\sum_{n=1}^{N}x_n - N\mu = 0$$

since $\mu_{ML} = \bar{x}$, where $\bar{x} = \frac{1}{N}\sum x_n$ is the sample mean.

$$\frac{\alpha}{\alpha\sigma} = -\frac{N}{\sigma} + \frac{1}{\sigma^3}\sum_{n=1}^{N}(x_n - \bar{x})^2 = 0$$

so,

$$\sigma^3 N = \sigma \sum_{n=1}^{N}(x_n - \bar{x})^2 => \sigma_{ML}^2 = \frac{1}{N}\sum_{n=1}^{N} x_n(x_n - \bar{x})^2$$

Question 3. Bishop(1.13)

$$E(\sigma_{ML}^2) = E(\frac{1}{N}\sum_{n=1}^{N}(x_n - \mu)^2) = \frac{1}{N}E(\sum_{n=1}^{N}(x_n - \mu)^2)$$

$$= \frac{1}{N}E(\sum_{n=1}^{N}(x_n^2 - 2x_n\mu + \mu^2)) = \frac{1}{N}E(\sum_{n=1}^{N} x_n^2) - \frac{1}{N}E(\sum_{n=1}^{N} 2x_n\mu) + \frac{1}{N}E(\sum_{n=1}^{N} \mu^2)$$

$$= \mu^2 + \sigma^2 \frac{2\mu}{N}E(\sum_{n=1}^{N} x_n) + \mu^2 = 2\mu^2 + \sigma^2 - 2\mu^2$$

$$= \sigma^2$$

Question 4.

if we let

$$w_{ij}^S = \frac{w_{ij} + w_{ji}}{2}, w_{ij}^A = \frac{w_{ij} - w_{ji}}{2}$$

we can get:

$$w_{ij} = w_{ij}^S + w_{ij}^A, w_{ij}^S = w_{ji}^S, w_{ij}^A = -w_{ji}^A$$

because

$$\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}x_i x_j = \sum_{i=1}^{D}\sum_{j=1}^{D}(w_{ij}^S + w_{ij}^A)x_i x_j = \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^S x_i x_j + \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^A x_i x_j$$

so we need to prove

$$\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^A x_i x_j = 0$$

$$=> 2\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{A}x_i x_j = \sum_{i=1}^{D}\sum_{j=1}^{D}(w_{ij}^{A} + w_{ij}^{A})x_i x_j$$

$$= \sum_{i=1}^{D}\sum_{j=1}^{D}(w_{ij}^{A} - w_{ji}^{A})x_i x_j$$

$$= \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{A}x_i x_j - \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ji}^{A}x_i x_j$$

$$= \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^{A}x_i x_j - \sum_{j=1}^{D}\sum_{i=1}^{D} w_{ji}^{A}x_i x_j = 0$$

Consider about the symmetry, we can see if and only if i= 1,2,...,D and $i <= j, w_{ij}$ is given, the whole matrix will be determined.So:

$$D + D - 1 + ... + 1 = \frac{D(D+1)}{2}$$

Question 5.

(a)

$$E(x) = \frac{1}{100}\sum_{x=100}^{200} x = 150$$

(b)

$$jump : E(X_i) = (-1)p + 1(1-p) = 1 - 2p$$

$$E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n) = n(1-2p)$$

(c) Areas means S = X * Y, because X and Y are independent, so

$$E(XY) = E(X)E(Y) = 1/2 * 7 = 3.5$$

(d) because $V(x) = E(x^2) - E(x)^2$ so,

$$E(x(x-1)) = E(x^2 - x) = E(x^2) - E(x) = V(x) + E(x)^2 - \mu = \sigma^2 + \mu^2 - \mu$$

Question 6.(Bishop 2.8)

(2.270)

$$
\begin{aligned}
E_y[E_x[x|y]] &= \int E_x[x|y]p(y)d_y \\
&= \int (\int xp(x|y)d_x)p(y)d_y \\
= \int \int xp(x|y)p(y)d_xd_y &= \int \int xp(x,y)d_xd_y \\
&= \int xp(x)d_x = E[x]
\end{aligned}
$$

(2.271)

$$
\begin{aligned}
&E_y[var_x[x|y]] + var_y[E_x[x|y]] \\
&= E_y[E_x[x^2|y] - E_x[x|y]^2] + E_y[E_x[x|y]^2] - E_y[E_x[x|y]]^2 \\
&= E_y[E_x[x^2|y]] - E_y[E_x[x|y]^2] + E_y[E_x[x|y]^2] - E_y[E_x[x|y]]^2 \\
&= E_y[E_x[x^2|y]] - E_y[E_x[x|y]]^2
\end{aligned}
$$

because $E_y[E_x[x|y]] = E[x]$, we can get

$$E_y[E_x[x^2|y]] = E[x^2]$$

so

$$E_y[var_x[x|y]] + var_y[E_x[x|y]] = E_x[x^2] + E_x[x]^2 = var_x[x]$$

Question 7.(Bishop 2.12)

$$\int_a^b \frac{1}{b-a} = 1$$

$$E[x] = \int_a^b x\frac{1}{b-a}dx = \frac{[x^2]_a^b}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} = \frac{a-b}{2}$$

$$E[x^2] = \int_a^b x^2\frac{1}{b-a}dx = \frac{[x^3]_a^b}{3(b-a)} = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

$$var[x] = e[x^2] - E[x]^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} = \frac{(b-a)^2}{12}$$

Question 8.

since $X = N(1, 2^2)$,we have $Z = \frac{X-1}{2}$,so

$$P(X <= 3) = P(\frac{X-1}{2} <= 1) = P(Z <= 1) = 0.8413$$

$$P(|X| <= 2) = P(-2 <= X <= 2) = P(-1.5 <= Z <= 0.5) = 0.6247$$

Question 9.

Assume B: blue eyes; O: other color eyes; G: grey eyes; NF: not in the first

$$P(B|NF) = \frac{P(B)P(NF|B)}{P(B)P(NF|B) + P(O)P(NF|O) + P(G)P(NF|G)}$$

$$= \frac{0.5 * 0.18}{0.5 * 0.18 + 0.2 * 0.5 + 0.3 * 0.35} = 0.3051$$

Question 10.

(a)

$$P(Label = Yes|Home = Yes, Married, Income = 70K)$$

$$= \frac{P(Home = Yes, Married, Income = 70K|Label = Yes)P(Label = Yes)}{P(Home = Yes, Married, Income = 70K)}$$

$$= \frac{0 * \frac{3}{10}}{P(Home = Yes, Married, Income = 70K)} = 0$$

so the class label of instance X is NO

(b)

$$
\begin{aligned}
&P(Label = Yes|Home = No, Single, Income > 80K) \\
&= \frac{P(Home = No, Single, Income > 80K|Label = Yes)P(Label = Yes)}{P(Home = No, Single, Income > 80K)} \\
&= \frac{\frac{2}{3} * \frac{3}{10}}{\frac{2}{10}} = 1
\end{aligned}
$$

so the class label of instance X is YES