Axle Damping Calculation

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0.1 Laplacian

We start with the differential equation:

$$u(t) = m\ddot{x} + c\dot{x} + kx$$

We then take the Laplacian by breaking it down:

$$\mathcal{L}\{m\ddot{x} + c\dot{x} + kx\} = \mathcal{L}\{m\ddot{x}\} + \mathcal{L}\{c\dot{x}\} + \mathcal{L}\{kx\} = \mathcal{L}\{u\}$$

We have

$$\mathcal{L}\{m\ddot{x}\} = m(s^2X(s) - sx(0) - \dot{x}(0))$$
$$\mathcal{L}\{c\dot{x}\} = c(sX(s) + x(0))$$
$$\mathcal{L}\{kx\} = kX(s)$$
$$\mathcal{L}\{u\} = U(s)$$

Combining yields:

$$U(s) = m(s^2X(s) - sx(0) - \dot{x}(0)) + c(sX(s) + x(0)) + kX(s)$$

To solve for X it's easiest if we have U, which we know is a step response of unknown magnitude, and therefore is

$$U(s) = \mu\frac{1}{s}$$

where μ is the magnitude. Solving the equation yields:

$$X(s) = \frac{\mu + ms\dot{x}(0) + (ms^2 + ks)x(0)}{ms^3 + cs^2 + ks}$$

0.2 Inverse Laplacian

Now we use the inverse laplacian to solve for the step response in the time domain. First we break it up:

$$X(s) =$$
$$\frac{\mu}{ms^3 + cs^2 + ks}$$
$$\frac{ms\dot{x}(0)}{ms^3 + cs^2 + ks} = \frac{m\dot{x}(0)}{ms^2 + cs + k}$$
$$\frac{(ms^2 + ks)x(0)}{ms^3 + cs^2 + ks} = \frac{(ms + k)x(0)}{ms^2 + cs + k}$$

0.2.1 First Part

For the first part, it most-closely resembles this transform:

$$K\frac{a^2+b^2}{s[(s+a^2)+b^2]} = K - Ke^{-at}(\cos(bt) + \frac{a}{b}\sin(bt))$$

Setting

$$K\frac{a^2+b^2}{s[(s+a)^2+b^2]} = \frac{\mu}{s[ms^2+cs+k]}$$

We find that

$$a = \frac{c}{2m}$$
$$a^2+b^2 = \frac{k}{m}$$
$$b = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$
$$K = \frac{\mu}{k}$$

Solving back yields:

$$K\frac{a^2+b^2}{s[(s+a)^2+b^2]} = \frac{\frac{\mu}{k}\frac{k}{m}}{s[s^2+\frac{c}{m}s+\frac{c^2}{4m^2}+\frac{k}{m}-\frac{c^2}{4m^2}]} = \frac{\frac{\mu}{m}}{s[s^2+\frac{c}{m}s+\frac{k}{m}]} = \frac{\mu}{s[ms^2+cs+k]}$$

So in the time domain, it's

$$\frac{\mu}{k} - \frac{\mu}{k}e^{-\frac{c}{2m}t}(\cos(\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}t) + \frac{\frac{c}{2m}}{\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}}\sin(\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}t))$$

0.2.2 Second Part

The second resembles

$$K\frac{b}{(s+a)^2+b^2} = Ke^{-at}\sin(bt)$$

Setting

$$K\frac{b}{(s+a)^2+b^2} = \frac{m\dot{x}(0)}{ms^2+cs+k}$$

We find that

$$a = \frac{c}{2m}$$
$$a^2+b^2 = \frac{k}{m}$$
$$b = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$
$$K = \frac{\dot{x}(0)}{\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}}$$

Solving back yields:

$$K\frac{b}{(s+a)^2+b^2} = \frac{\frac{\dot{x}(0)}{\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}}\sqrt{\frac{k}{m}-\frac{c^2}{4m^2}}}{s^2+\frac{c}{m}+\frac{c^2}{4m^2}+\frac{k}{m}-\frac{c^2}{4m^2}} = \frac{\dot{x}(0)}{s^2+\frac{c}{m}s+\frac{k}{m}} = \frac{m\dot{x}(0)}{ms^2+cs+k}$$

So in the time domain, it's

$$\frac{\dot{x}(0)}{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}} e^{-\frac{c}{2m}t} \sin(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} t)$$

0.2.3 Third Part

The third resembles:

$$K \frac{s + a}{(s + a)^2 + b^2} = K e^{-} a t \cos(bt)$$

Setting

$$K \frac{s + a}{(s + a)^2 + b^2} = \frac{(ms + k)x(0)}{ms^2 + cs + k}$$

We find that

$$a = \frac{c}{2m}$$
$$a^2 + b^2 = \frac{k}{m}$$
$$b = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$
$$K = ????$$

Just realized, we don't actually need to know the forms of the constants, we just need to know that they exist.

0.3 Putting it together

Solving for the first two terms gives us this equation
We have this equation:

$$\frac{\mu}{k} - \frac{\mu}{k} e^{-\frac{c}{2m}t} (\cos(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} t) + \frac{\frac{c}{2m}}{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}} \sin(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} t)) + \frac{\dot{x}(0)}{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}} e^{-\frac{c}{2m}t} \sin(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} t)$$

But this is hard to fit to, so we'll use:

$$K_1 - K_1 e^{-at} (\cos(bt) + \frac{a}{b} \sin(bt)) + K_2 e^{-at} \sin(bt)$$

With

$$a = \frac{c}{2m}$$
$$a^2 + b^2 = \frac{k}{m}$$
$$b = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$
$$K_1 = \frac{\mu}{k}$$
$$K_2 = \frac{\dot{x}(0)}{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}}$$

Fitting in Matlab doesn't do well enough, so we use:

$$K_1 - K_1 e^{-at} (\cos(bt) + \frac{a}{b} \sin(bt)) + K_2 e^{-at} \sin(bt) + K_3 e^{-at} \cos(bt)$$